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Spherical Steiner Symmetrizations

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Abstract: In this paper, we primarily investigate and establish several properties of spherical Steiner symmetrizations, along with the isoperimetric property of the spherical cap in $\mathbb{S}^n$. Specifically, we study the monotonically decreasing property of the measure of the symmetric difference of two spherical compact sets, the monotonically decreasing property of the spherical diameter of a spherical compact set, the convergence of iterative spherical Steiner symmetrizations, and so on. In particular, we prove that the sequence of iterative spherical Steiner symmetrizations of $K \subset \mathbb{S}^n$, which follow sequences selected from a finite set of directions, converges to a spherical cap with the same measure as K, extending the result from $\mathbb{R}^n$ to $\mathbb{S}^n$ on Steiner symmetrizations. It provides us with valuable insights for studying the relevant applications and conclusions of spherical Steiner symmetrizations.

Keywords: spherical geometry; spherical Steiner symmetrization; outer Minkowski content; spherical distance

MSC: 53A35; 52A55

1. Introduction

As an essential tool for exploring inquiries into isoperimetry and its related geometric inequalities, Steiner symmetrizations were provided in 1838 by Swiss mathematician J. Steiner (1796–1863). Steiner symmetrizations are clearly referenced in many scholarly publications (see [1–6]) and play a significant role in contemporary research (see [7–10]). In detail, the applications of Steiner symmetrizations in geometric inequalities can be categorized into the following aspects.

The application of Steiner symmetrizations allows for the proof of the Brezis inequality, which is a fundamental result in the context of Sobolev spaces (see [11–13]). In [14], by employing symmetrizations, the proof of the Hölder inequality can be simplified, making the form of the inequality more intuitive. In many cases, proving the Cauchy–Schwarz inequality requires the use of symmetry, and Steiner symmetrizations can preserve the properties of the inequality, allowing for effective comparisons even after symmetrizations, thereby ensuring the inequality holds. Additionally, through symmetrizations, one can construct functions suitable for applying the Cauchy–Schwarz inequality, making the proof more intuitive and clear (for further details, please refer to the article [15–19]). In addition to their various applications, Steiner symmetrizations are also a valuable tool for both demonstrating the geometric mean inequality and exploring the relationship between the arithmetic mean and the geometric mean (as mentioned in [20–23]). Additionally, Steiner symmetrizations provide significant assistance in proving the Torelli inequality under specific conditions, particularly in contexts involving the symmetry of functions. By introducing the concept of symmetrizations, the structure of the inequality can be simplified, thereby clarifying the relationships among different variables (see [24–26]). Moreover, the Minkowski inequality, a key type of integral inequality, is proved using Steiner symmetrizations. By utilizing Steiner symmetrizations, one can effectively address problems within L^p spaces, also revealing the profound connections between the arithmetic mean and the geometric mean (see [27–30]).

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Citation: Lin, Y.; Deng, Z. Spherical Steiner Symmetrizations. *Axioms* **2024**, *13*, 751. https://doi.org/10.3390/axioms13110751

Academic Editors: Baocheng Zhu, Senlin Wu, Jingbo Dou and Wenxue Xu

Received: 7 September 2024
Revised: 26 October 2024
Accepted: 27 October 2024
Published: 31 October 2024

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In spherical geometry, there are many similar isoperimetric inequalities for Euclidean space. Thus, it is necessary to study the spherical symmetrical rearrangement on $\mathbb{S}^n$. Two-point symmetrizations, as prominent spherical rearrangements, have long been utilized as an effective tool for analysis in $\mathbb{S}^n$ for isoperimetric inequalities and multiple integral rearrangement inequalities (see [31–34]). However, the spherical Steiner symmetrizations appear to have received relatively little attention, and even the definition itself is unclear. In [35], to prove isoperimetric inequalities for simplices and polytopes with $n+1$ vertices in n-spherical space, B. Basit and Z. Lángi provided a definition of spherical Steiner symmetrizations. However, the spherical Steiner symmetrizations presented in [35] are unable to meet the volume invariance, which is a fundamental property of rearrangement. In [36], K. Leichtweiss introduced the notion of Steiner symmetrizations in the two-dimensional spherical geometry. The symmetrizations given in [36] preserves volume but usually not convexity.

In this study, we define spherical Steiner symmetrizations on a non-empty compact set on $\mathbb{S}^n$. The definition is consistent with that in [36] for the case $n = 2$ and keeps the n-Hausdorff measure invariant (see Proposition 1). Moreover, we prove some fundamental properties of the spherical Steiner symmetrizations, e.g., the monotonically decreasing property of the measure of symmetric difference of two spherical compact sets (see Proposition 2), the monotonically decreasing property of spherical diameter of spherical compact set (see Proposition 3), the convergence of iterative spherical Steiner symmetrizations (see Proposition 4), and the isoperimetric property of the spherical cap on outer Minkowski content (see Proposition 5). In particular, in Section 4, we prove that the sequence of iterative spherical Steiner symmetrizations of $K \subset \mathbb{S}^n$, which follow sequences selected from a finite set of directions, converges to a spherical cap with the same measure as K, extending a result of Klain [10] from $\mathbb{R}^n$ to $\mathbb{S}^n$ on Steiner symmetrizations.

2. Preliminaries

Define $\mathbb{N}$ to be the set of all natural numbers. Let $\mathbb{R}^{n+1}$ denote the $(n+1)$-dimensional Euclidean space, where $n \geq 1$. Let o represent the origin. We define $e_1, \ldots, e_n, e_{n+1}$ as the orthonormal vector basis of $\mathbb{R}^{n+1}$. The notation $\mathbb{S}^n$ is used to represent the Euclidean unit sphere in $\mathbb{R}^{n+1}$, i.e.,

$$\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : \ \|x\| = 1\}.$$

Let $\mathbb{S}^n_+$ denote the upper hemisphere of $\mathbb{S}^n$, i.e., $\mathbb{S}^n_+ := \{x \in \mathbb{S}^n : \ x \cdot e_{n+1} \geq 0\}$. Let $\mathbb{S}^{n-1} := \{u \in \mathbb{S}^n : \ u \cdot e_{n+1} = 0\}$. Let $u \in \mathbb{S}^{n-1}$ and let $u^\perp$ denote its orthogonal complementary space of the direction u in $\mathbb{R}^{n+1}$. A subset K of the sphere $\mathbb{S}^n$ is termed (spherical) convex if its radial extension

$$\operatorname{rad} K = \left\{ rv \in \mathbb{R}^{n+1} : r \geq 0 \text{ and } v \in K \right\} \tag{1}$$

exhibits convexity in $\mathbb{R}^{n+1}$. A closed convex subset of $\mathbb{S}^n$ is called a (spherical) convex body.

For any $x, y \in \mathbb{S}^n$, their *spherical distance* is described as follows:

$$d_s(x,y) := \arccos(x \cdot y) = \arccos\left(\sum_{i=1}^{n+1} x_i y_i\right). \tag{2}$$

For any $x \in \mathbb{S}^n$ and compact set $K \subset \mathbb{S}^n$, their spherical distance is described as follows:

$$d_s(x,K) := \min\{d_s(x,y) : \ y \in K\}. \tag{3}$$

For $x \in \mathbb{S}^n$ and $r > 0$, We define $B_r(x)$ as the spherical cap with center at x with radius r, i.e.,

$$B_r(x) := \{y \in \mathbb{S}^n : \ d_s(x,y) \leq r\}.$$

To reduce notation complexity, we use B_r instead of $B_r(e_{n+1})$. We say that $K \subset \mathbb{S}^n$ has a non-empty interior if $x \in \mathbb{S}^n$ and there is a positive radius $r > 0$ such that the spherical cap $B_r(x)$ is included in K; x qualifies as a interior point of K. The set of all interior points of K is called the interior of K and is denoted by int K.

For a compact set $K \subset \mathbb{S}^n$ and $\delta > 0$, the definition of the spherical parallel set of K is expressed as

$$K_\delta := \{x \in \mathbb{S}^n : \text{ there exists a } y \in K \text{ such that } d_s(x,y) \leq \delta\}. \tag{4}$$

Let K and L be compact sets contained in S^n. The definition of their spherical Hausdorff distance is the following:

$$d_H(K,L) := \inf\{\delta > 0 : \ K \subset L_\delta, \ \ L \subset K_\delta\}. \tag{5}$$

Let K and L be compact sets contained in S^n. The definition of their spherical symmetric difference is the following:

$$K \triangle L := (K \backslash L) \cup (L \backslash K), \tag{6}$$

where

$$K \backslash L := \{x \in \mathbb{S}^n : \ x \in K \ \text{ and } \ x \notin L\}.$$

For $u \in \mathbb{S}^{n-1}$ and $x' \in u^\perp \cap \mathbb{S}^n$, let

$$K_{x',u} := K \cap \{x' + ru + te_{n+1} : \ r,t \in \mathbb{R}\}. \tag{7}$$

Let K'_u denote the projection of K onto $u^\perp \cap \mathbb{S}^n$, i.e.,

$$K'_u := \{x' \in u^\perp \cap \mathbb{S}^n : \text{ there exists } x \in K_{x',u} \ \}. \tag{8}$$

For abbreviation, we let $K_{x'}$ stand for K_{x',e_n} and K' stand for K'_{e_n}.

For a Lebesgue measurable set $K \subset \mathbb{S}^n$, if the outer Minkowski content exists for the boundary ∂K, the outer Minkowski content $\mathcal{M}_+(K)$ can be characterized as

$$\mathcal{M}_+(K) = \lim_{\delta \to 0^+} \frac{\mathcal{H}^n(K_\delta) - \mathcal{H}^n(K)}{\delta}. \tag{9}$$

3. Spherical Steiner Symmetrizations and Some Basic Properties

Definition 1. *For $K \subset \mathbb{S}^n_+$ and $u \in \mathbb{S}^{n-1}$. Denote the Steiner symmetrizations S_uK, of K along u is characterized as*

$$S_uK := \cup_{x' \in K'_u} B_{r(x')}(x') \cap (\mathbb{S}^n)_{x',u}, \tag{10}$$

where radius $r(x')$ satisfies that

$$\mathcal{H}^1\Big(B_{r(x')}(x') \cap (\mathbb{S}^n)_{x',u}\Big) = \mathcal{H}^1\big(K_{x',u}\big), \tag{11}$$

and where $K_{x',u} := K \cap \{x' + ru + te_{n+1} : \ r,t \in \mathbb{R}\}$ given in (7) To enhance notation simplicity, we write SK instead of $S_{e_n}K$.

Proposition 1. *For $K \subset \mathbb{S}^n$, the n-dimensional Hausdorff measure remains invariant under spherical Steiner symmetrizations, i.e.,*

$$\mathcal{H}^n(SK) = \mathcal{H}^n(K). \tag{12}$$

Proof. For $x'' \in \mathbb{R}^{n-1}$, let

$$K_{x''} := K \cap \{x \in \mathbb{R}^{n+1} : \ x = (x'', x_n, x_{n+1})\}.$$

By Fibini's theorem, (11) and area formula (see [29], Theorem 3.8), we have

$$\mathcal{H}^n(K) = \int_{\mathbb{R}^{n-1}} \mathcal{H}^1(K_{x''}) \frac{1}{\sqrt{1-\|x''\|^2}} dx'' = \int_{\mathbb{R}^{n-1}} \mathcal{H}^1((SK)_{x''}) \frac{1}{\sqrt{1-\|x''\|^2}} dx'' = \mathcal{H}^n(SK).$$

This is the desired conclusion. □

Proposition 2. *For compact sets $K, L \subset \mathbb{S}^n$ and $u \in \mathbb{S}^{n-1}$, we can obtain*

$$\mathcal{H}^n(S_u K \triangle S_u L) \leq \mathcal{H}^n(K \triangle L). \tag{13}$$

Proof. By Definition 1, for compact sets $K, L \subset \mathbb{S}^n$, we have

$$S_u(K \cap L) \subseteq S_u K \cap S_u L. \tag{14}$$

By Proposition 1 and (14),

$$\begin{aligned} \mathcal{H}^n(S_u K \backslash S_u L) &= \mathcal{H}^n(S_u K) - \mathcal{H}^n(S_u K \cap S_u L) \\ &\leq \mathcal{H}^n(S_u K) - \mathcal{H}^n(S_u(K \cap L)) \\ &= \mathcal{H}^n(K) - \mathcal{H}^n(K \cap L) \\ &= \mathcal{H}^n(K \backslash L). \end{aligned} \tag{15}$$

By (15) and the definition of spherical symmetric difference (6), we have

$$\begin{aligned} \mathcal{H}^n(S_u K \triangle S_u L) &= \mathcal{H}^n(S_u K \backslash S_u L) + \mathcal{H}^n(S_u L \backslash S_u K) \\ &\leq \mathcal{H}^n(K \backslash L) + \mathcal{H}^n(L \backslash K) \\ &= \mathcal{H}^n(K \triangle L). \end{aligned}$$

This completes the proof. □

Let $R(K) := \sup_{x,y \in K} d_s(x,y)$ denote the size of K in terms of its diameter and $K \subset \mathbb{S}^n$, where $d_s(x,y)$ is the spherical distance given in (2).

Proposition 3. *For the compact set K on $\mathbb{S}^n$,*

$$R(SK) \leq R(K). \tag{16}$$

Proof. For fixed $x'', y'' \in \mathbb{R}^{n-1}$, let $O_1 := (x'', 0, 0)$ and $O_2 := (y'', 0, 0)$. Let

$$A' := \left(x'', 0, \sqrt{1-\|x''\|^2}\right) \quad \text{and} \quad B' := \left(y'', 0, \sqrt{1-\|y''\|^2}\right).$$

Then A' and B' are on a great circle perpendicular to e_n. Moreover, let

$$A := \left(x'', \sqrt{1-\|x''\|^2}\sin\theta_A, \sqrt{1-\|x''\|^2}\cos\theta_A\right)$$

and

$$B := \left(y'', \sqrt{1-\|y''\|^2}\sin\theta_B, \sqrt{1-\|y''\|^2}\cos\theta_B\right),$$

where $\theta_A = \angle_{AO_1A'}$ and $\theta_B = \angle_{BO_2B'}$. ($\theta_A$ shows that O_1 is the vertex, and the angle is formed by the rays O_1 A and O_1A', θ_B shows that O_2 is the vertex, and the angle is formed by the rays O_1 B and O_1B') Then

$$\overrightarrow{OA} \cdot \overrightarrow{OB} = x'' \cdot y'' + \sqrt{1 - \|x''\|^2}\sqrt{1 - \|y''\|^2}\cos(\theta_A - \theta_B).$$

One can easily observe that $\overrightarrow{OA} \cdot \overrightarrow{OB}$ increases when $|\theta_A - \theta_B|$ decreases. For fixed x'' and y'', the maximum value

$$\max\{|\theta_A - \theta_B| : \ A \in K_{x''} \ \ B \in K_{y''}\}$$

decreases after spherical Steiner symmetrizations are performed. Thus, for fixed $x'', y'' \in \mathbb{R}^{n-1}$,

$$\max\left\{d_s(x,y) : \ x \in K_{x''}, \ y \in K_{y''}\right\}$$

shows a decline after the implementation of spherical Steiner symmetrizations. By the arbitrariness of x'' and y'', the diameter of SK is less than or equal to the size of K in terms of its diameter. □

The following proposition proves that the sequence of iterative spherical Steiner symmetrizations on a spherical compact set converges to a spherical cap with the same measure. The proof of the proposition is placed at the conclusion of Section 4. As a matter of fact, we can demonstrate a stronger conclusion: only one finite set of directions may be used to select the directions of Steiner symmetrizations.

Proposition 4. *Let $K \subset \mathbb{S}^n_+$ be a compact set that has a non-empty interior and let B_{r_0} satisfy $\mathcal{H}^n(K) = \mathcal{H}^n(B_{r_0})$. Then there is a sequence of directions $\{u_i\}_{i\in\mathbb{N}} \subset \mathbb{S}^{n-1}$ such that*

$$K_i := S_{u_i} \cdots S_{u_1} K \to B_{r_0}$$

in spherical Hausdorff distance when $i \to \infty$.

The following proposition illustrates the isoperimetric property of the spherical cap in $\mathbb{S}^n$ (its proof is given in Section 5).

Proposition 5. *Let $K \subset \mathbb{S}^n_+$ be a compact set that has a non-empty interior and the spherical cap B_{r_0} has the same measure as the set K. Then*

$$\mathcal{M}_+(K) \geq \mathcal{M}_+(B_{r_0}). \tag{17}$$

According to Propositions 1–5, we can obtain spherical Steiner symmetrizations, and Euclidean Steiner symmetrizations have many similar properties. However, the convexity is not maintained by the spherical Steiner symmetrizations, i.e., a spherical convex body's Steiner symmetrical is not necessarily convex.

For example, let $K \subset \mathbb{S}^2$ be a spherical convex body and $K \subset \{x \in \mathbb{S}^2 : \ x \cdot u > 0\}$, where $u \in \mathbb{S}^2$ and $u \cdot e_3 = 0$. Let $H := \{x \in \mathbb{R}^3 : \ x \cdot u = 1\}$, i.e., H is a tangent plane of $\mathbb{S}^2$ passing through the point u. Suppose K is a spherical convex body in accordance with the definition of the projection of K onto $u^\perp \cap \mathbb{S}^n$ (8). Then the projection of K onto H is also a convex body, i.e., $\bar{K} := \operatorname{rad} K \cap H$ is convex, where rad K is the radial extension of K given in (1). Let $v \in \mathbb{S}^2$ satisfy $v \cdot e_3 = 0$ and $v \cdot u = 0$. By Definition 1, the spherical Steiner symmetral $S_v K$ of K corresponds to the circular symmetrial of $\bar{K}$ in H around point u (see ([26], 1.5) for the definition of circular symmetrization). Figure 1 Confirms that the circle symmetrial of a convex body in a plane is not necessarily convex.

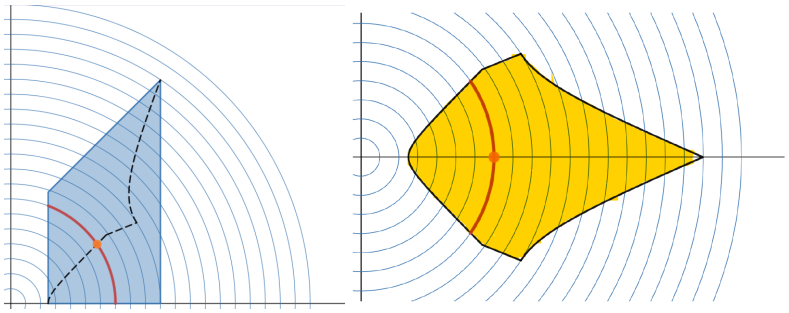


Figure 1. Circular symmetrization.

4. Spherical Steiner Processes Using a Finite Set of Directions

In this section, we prove that for any non-empty compact set $K \subset \mathbb{S}^n$, A sequence of unit vectors $\{u_i\}_{i=1}^{\infty}$, which are chosen from a finite set of unit vectors, such that iterative spherical Steiner symmetrizations $S_{u_i} \cdots S_{u_1} K$ converges to a spherical cap B_{r_0} with the same measure as K. Our proof aligns with the methodology used in [8] on the convergence of symmetrizations processes in $\mathbb{R}^n$.

Lemma 1. *Let $K \subset \mathbb{S}^n$ be a spherical compact set. Then, for $\delta > 0$ and $r > 0$, we can imply that*

$$(K_\delta)_r = K_{\delta+r}. \tag{18}$$

Proof. From one perspective, suppose $x \in (K_\delta)_r$, we can simply imply that there is $z \in K_\delta$ and $y \in K$ such that

$$d_s(z, y) \leq \delta$$

and

$$d_s(x, z) < r,$$

where K_δ is the spherical parallel set, which is given in (4). Then, by the spherical triangle inequality, we can obtain

$$d_s(x, y) \leq d_s(x, z) + d_s(z, y) \leq \delta + r.$$

Thus, $x \in (K)_{\delta+r}$. Then again, if $x \in (K)_{\delta+r}$, we can also simply imply there is $z \in K$ such that

$$d_s(z, x) \leq \delta + r.$$

Due to the continuity of spherical distance, y can be found on the great circle through x and z such that

$$d_s(z, y) \leq \delta$$

and

$$d_s(y, x) \leq r.$$

We can imply that $y \in K_\delta$, and thus $x \in (K_\delta)_r$. □

Lemma 2. *If K is a spherical compact set on $\mathbb{S}^n$, then $\mathcal{H}^n(K_\delta)$ is continuous in $\delta > 0$, and*

$$\lim_{\delta \to 0} \mathcal{H}^n(K_\delta) = \mathcal{H}^n(K). \tag{19}$$

Proof. On the one hand, since the spherical parallel sets are compact sets arranged in a nested manner and

$$\bigcap_{\rho > \delta} K_\rho = K_\delta, \quad \bigcap_{\delta > 0} K_\delta = K,$$

we can imply that the measure exhibits continuity from the right and converges to $\mathcal{H}^n(K)$ when δ approaches zero.

On the other hand, since

$$\bigcup_{\rho<\delta} K_\rho = \{x : \ d_s(x,K) < \delta\} =: \tilde{K}_\delta,$$

where $d_s(x,K)$ is the spherical distance given in (3), we have

$$\lim_{\rho\to\delta^-} \mathcal{H}^n(K_\rho) = \mathcal{H}^n(\tilde{K}_\delta).$$

Next, we show that $\mathcal{H}^n(K_\delta \backslash \tilde{K}_\delta) = 0$. For any $y \in K_\delta$, there exists $z \in K$ such that

$$d_s(z,y) \le \delta.$$

Since

$$\{x \in \mathbb{S}^n : \ d_s(x,z) < \delta\} \subset \tilde{K}_\delta,$$

the density of $\tilde{K}_\delta$ at y on the spherical Lebesgue measure is more than or equal to $1/2$. According to the spherical Lebesgue differentiation theorem, almost every point in $\mathbb{S}^n \backslash \tilde{K}_\delta$ exhibits zero density concerning $\tilde{K}_\delta$, thus

$$\mathcal{H}^n(K_\delta \backslash \tilde{K}_\delta) = 0.$$

□

Lemma 3. *Suppose the sequence $\{K_m\}$ of spherical compact sets in $\mathbb{S}^n$ converges to a limit L, which has an interior that is not empty. Thus, for any $\delta > 0$, $\{(K_m)_\delta\}$ converges in spherical Hausdorff distance to L_δ. Moreover, for any $r \in (0,\delta)$, there is an integer $N > 0$ such that*

$$L_{\delta-r} \subseteq (K_m)_\delta \subseteq L_{\delta+r}$$

for all $m \ge N$.

Proof. Assuming that the sequence $\{K_m\}$ converges to L with respect to the spherical Hausdorff distance, thus for all values of $\rho > 0$, there is $N > 0$, when $m > N$, $K_m \subset L_\rho$ and $L \subset (K_m)_\rho$. By Lemma 1,

$$(K_m)_\delta \subset (L_\rho)_\delta = L_{\delta+\rho} = (L_\delta)_\rho$$

and

$$L_\delta \subset (K_m)_{\rho+\delta} = [(K_m)_\delta]_\rho.$$

By combining with the definition of spherical symmetric difference (5), this implies that $\{(K_m)_\delta\}$ converges in spherical Hausdorff distance to L_δ.

Let $r' = \delta - r$. Then $\{(K_m)_{r'}\}$ converges in spherical Hausdorff distance to $L_{r'}$. Thus, there exists $N_1 > 0$ such that when $m > N_1$,

$$L_{r'} \subset (K_m)_{r'+r}.$$

Thus, when $m > N_1$,

$$L_{\delta-r} \subseteq (K_m)_\delta. \tag{20}$$

Moreover, there exists $N_2 > 0$ such that when $m > N_2$,

$$(K_m)_\delta \subseteq L_{\delta+r}. \tag{21}$$

Let $N = \max\{N_1, N_2\}$. So, we obtain the desired conclusion. □

Lemma 4. *Let L and K_m, $m \geq 1$, be compact sets on $\mathbb{S}^n$ that are also non-empty. The convergence of the sequence $\{K_m\}$ to L in spherical Hausdorff distance holds if and only if*

$$\lim_{m\to\infty} \mathcal{H}^n((K_m)_\delta \triangle L_\delta) = 0 \tag{22}$$

for all values of $\delta > 0$.

Proof. First, we prove the necessity. By Lemma 2, for any fixed $\delta > 0$ and for all values of $\varepsilon > 0$, there is $0 < \varepsilon' < \delta$ such that for all $r \in (0, \varepsilon']$,

$$\mathcal{H}^n(L_{\delta+r} \backslash L_\delta) = \mathcal{H}^n(L_{\delta+r}) - \mathcal{H}^n(L_\delta) < \varepsilon \tag{23}$$

$$\mathcal{H}^n(L_\delta \backslash L_{\delta-r}) = \mathcal{H}^n(L_\delta) - \mathcal{H}^n(L_{\delta-r}) < \varepsilon. \tag{24}$$

Thus, by Lemma 3, for all values of above ε', there is $M > 0$ such that when $m > M$,

$$(K_m)_\delta \subset L_{\delta+\varepsilon'} \quad \text{and} \quad L_{\delta-\varepsilon'} \subset (K_m)_\delta. \tag{25}$$

Therefore, combining (23) and (25)

$$\mathcal{H}^n((K_m)_\delta \backslash L_\delta) \leq \mathcal{H}^n(L_{\delta+\varepsilon'} \backslash L_\delta) \leq \varepsilon$$

and

$$\mathcal{H}^n(L_\delta \backslash (K_m)_\delta) \leq \mathcal{H}^n(L_\delta \backslash L_{\delta-\varepsilon'}) \leq \varepsilon.$$

This shows that (22) is established for each $\delta > 0$.

Next, we prove the sufficiency. If (22) holds for each $\delta > 0$ but there exists $\delta_0 > 0$ and a subsequence of $\{m_j\}$ of $\{m\}$ such that $K_{m_j} \backslash L_{\delta_0} \neq \emptyset$ for any j. Then, for any $P_j \in K_{m_j} \backslash L_{\delta_0}$, we have $B(P_j, \delta_0/3) \cap L_{\delta_0/3} = \emptyset$. Therefore,

$$\mathcal{H}^n\left(\left(K_{m_j}\right)_{\delta_0/3} \backslash L_{\delta_0/3}\right) > \mathcal{H}^n(B(P_j, \delta_0/3)).$$

This leads to a contradiction with (22). □

Lemma 5. *Let K be a non-empty subset on $\mathbb{S}^n$ that is compact. Then for any $\delta > 0$,*

$$(SK)_\delta \subseteq S(K_\delta). \tag{26}$$

Proof. Let $x' \in e_n^\perp \cap \mathbb{S}^n_+$. First, we prove that if $\mathcal{H}^0(K_{x'}) = 2$, then for $\delta > 0$,

$$((SK)_{x'})_\delta = S(K_{x'})_\delta; \tag{27}$$

if $\mathcal{H}^0(K_{x'}) > 2$, then

$$((SK)_{x'})_\delta \subseteq S(K_{x'})_\delta. \tag{28}$$

For any $x \in K_{x'}$, $(B_\delta(x))'$ is the same subset of $\mathbb{S}^n \cap e_n^\perp$. If $\mathcal{H}^0(K_{x'}) = 2$, then by the definition of spherical Steiner symmetrizations (10), $(SK)_{x'}$ is the rotation of $K_{x'}$ on the circle $(\mathbb{S}^n)_{x',e_n}$. Thus, (27) is established. If $\mathcal{H}^0(K_{x'}) > 2$, then for any $y' \in (B_\delta(x))'$,

$$\mathcal{H}^1\Big(((K_{x'})_\delta)_{y'}\Big) > \mathcal{H}^1\Big(((SK)_{x'})_\delta)_{y'}\Big).$$

Thus, by the definition of spherical Steiner symmetrizations (10), (28) is established.

By (27), (28) and Definition 1,

$$(SK)_\delta = \cup_{x' \in u^\perp}((SK)_{x'})_\delta \subseteq \cup_{x' \in u^\perp} S((K_{x'})_\delta) \subseteq S\big(\cup_{x' \in u^\perp}(K_{x'})_\delta\big) \subseteq S\big(\big(\cup_{x' \in u^\perp} K_{x'}\big)_\delta\big) = S(K_\delta).$$

This is the desired conclusion. □

Lemma 6. *Let $u \in \mathbb{S}^{n-1}$, and let K be a non-empty subset on $\mathbb{S}^n$ that is compact. If there is a point $p \in \mathbb{S}^n \cap u^\perp$ such that*

$$\mathcal{H}^n((S_uK)_\delta \backslash B_r(p)) = \mathcal{H}^n(K_\delta \backslash B_r(p)) \tag{29}$$

for all elements of $\delta, r > 0$, then $S_uK = K$.

Proof. Assume that $S_uK \neq K$, since $\mathcal{H}^n(S_uK) = \mathcal{H}^n(K)$, there exists $q \in K \backslash S_uK$. By the compactness of S_uK, one can choose a sufficiently small $\delta_0 > 0$ such that

$$B_{\delta_0}(q) \cap (S_uK)_{\delta_0} = \emptyset. \tag{30}$$

Let ℓ_q be the one-dimensional section of $\mathbb{S}^{n-1} \cap H_q$, where H_q is a hyperplane parallel to e_{n+1} and u through q. Since $p \in \mathbb{S}^n \cap u^\perp$ and S_uK is spherical symmetric with respect to $\mathbb{S}^n \cap u^\perp$, there exists some $r > 0$ and sufficiently small $\delta_0 > 0$ such that

$$(\ell_q)_{\delta_0} \cap (S_uK)_{\delta_0} \subset B_r(p) \tag{31}$$

and

$$B_{\delta_0}(q) \cap B_r(p) = \emptyset. \tag{32}$$

Therefore,

$$\mathcal{H}^n\big(((\ell_q)_{\delta_0} \cap (S_uK)_{\delta_0}) \backslash B_r(p)\big) < \mathcal{H}^n\big(((\ell_q)_{\delta_0} \cap K_{\delta_0}) \backslash B_r(p)\big), \tag{33}$$

where the left side equals zero by (31), and the right side is bigger to zero by (32).

Moreover, let $C := \mathbb{S}^n \backslash (\ell_q)_{\delta_0}$, then by (15)

$$\mathcal{H}^n\big((C \cap (S_uK)_{\delta_0}) \backslash S_uB_r(p)\big) \leq \mathcal{H}^n\big((C \cap K_{\delta_0}) \backslash B_r(p)\big). \tag{34}$$

Combining (33) and (34), we have

$$\mathcal{H}^n\big((S_uK)_{\delta_0} \backslash S_uB_r(p)\big) < \mathcal{H}^n\big(K_{\delta_0} \backslash B_r(p)\big),$$

which is contradiction with (29). □

Theorem 1. *Let $\{u_m\}$ be a sequence of directions that are elements of a finite set $\mathcal{F} := \{u_1, \ldots, u_k\} \subset \mathbb{S}^{n-1}$. Thus, for all compact set $K \subset \mathbb{S}^n$, the symmetrals*

$$K_m := S_{u_m} \ldots S_{u_1} K$$

tend toward a compact set $L \subset \mathbb{S}^n$. In addition, L is symmetric w.r.t. each subspace $u_i^\perp$, where $u_i \in \mathcal{F}$ occur in the sequence an infinite number of times.

Proof. By omitting an initial segment $\{K_m\}_{m \leq N}$ of the sequence and potentially replacing $\mathcal{F}$ with one of its subsets without affecting the general case, we can assume that every direction in $\mathcal{F}$ occurs infinitely many times in the sequence $\{u_m\}$.

The primary objective is to develop a subsequence that arranges the directions $u_i \in \mathcal{F}$ that appear in a particular arrangement. For any given index m, we construct a permutation

π_m that describes the order of first encounters of the directions $1, \ldots, k$ among the directions $u_1, \ldots, u_k$ are encountered for the first time among the directions u_i with $i \geq m$.

Since there are a limited number of permutations, it is possible to choose a subsequence $\{u_{j_m}\}$ where the permutation π_{j_m} is identical for all m. By renaming the directions, we can treat the permutation as the identity. Transitioning to a subsequent subsequence allows us to assume that all directions in $\mathcal{F}$ occur in every segment $u_{j_m}, u_{j_m+1}, \ldots, u_{j_{m+1}}$.

Through the use of the employing the Blaschke selection principle, we obtain a subsequence, also referred to as $\{K_{j_m}\}$, that converges to a compact set L, in terms of Hausdorff distance. Notably, for every $\delta > 0$, the entire sequence $\{\mathcal{H}^n((K_m)_\delta)\}$ is decreasing, as established in (26), leading to its convergence. By Lemma 22, it follows that the limit is provided by

$$\inf_m \mathcal{H}^n((K_m)_\delta) = \lim_{m\to\infty} \mathcal{H}^n\left(\left(K_{j_m}\right)_\delta\right) = \mathcal{H}^n(L_\delta). \tag{35}$$

We illustrate by means of induction that

$$S_{u_i} L = L for i = 1, \ldots, k.$$

When $i = 1$, it can be seen that $u_{j_m} = u_1$. Consequently, $\{K_{j_m}\}$ is rotationally symmetric around u_1, and this property also holds for L. It is assumed that L maintains its invariance under Blaschke symmetrization with respect to the subspaces $u_1, \ldots u_{i-1}$. Assuming j'_m is the index of the first occurrence of u_i after j_m, the hypothesis of induction indicates

$$S_{u_{j'_m-1}} \ldots S_{u_{j_m+1}} L = L.$$

By (15) and (26), for every value of $\delta > 0$, we can imply that

$$\begin{aligned} \mathcal{H}^n\left(\left(K_{j_m}\right)_\delta \backslash L_\delta\right) &\geq \mathcal{H}^n\left(S_{u_{j'_m-1}} \ldots S_{u_{j_m+1}} \left(K_{j_m}\right)_\delta \backslash L_\delta\right) \\ &\geq \mathcal{H}^n\left(\left(S_{u_{j'_m-1}} \ldots S_{u_{j_m+1}} K_{j_m}\right)_\delta \backslash L_\delta\right) \\ &= \mathcal{H}^n\left(\left(K_{j'_m-1}\right)_\delta \backslash L_\delta\right). \end{aligned} \tag{36}$$

Since $\left\{\left(K_{j_m}\right)_\delta\right\}$ approaches L_δ with respect to symmetric difference, the left part of the inequality (36) tends toward 0. Thus, the right part of the inequality (36) approaches zero, and from (35), the sequence $\left(K_{j'_m-1}\right)_\delta$ approaches L_δ in the sense of symmetric difference as well. We shall discuss and estimate

$$\begin{aligned} \mathcal{H}^n(L_\delta \backslash B_r) &\geq \mathcal{H}^n(S_{u_i} L_\delta \backslash B_r) \\ &= \lim_{m\to\infty} \mathcal{H}^n\left(S_{u_i}\left(K_{j'_m-1}\right)_\delta \backslash B_r\right) \\ &\geq \liminf_{m\to\infty} \mathcal{H}^n\left(\left(S_{u_i} K_{j'_m-1}\right)_\delta \backslash B_r\right) \\ &= \liminf_{m\to\infty} \mathcal{H}^n\left(\left(K_{j'_m}\right)_\delta \backslash B_r\right). \end{aligned} \tag{37}$$

The first line's inequality follows from (15). The second line incorporates the convergence of $\left(K_{j'_m-1}\right)_\delta$ to L_δ in symmetric difference, as well as the continuity of the Steiner symmetrizations relative to the symmetric difference distance. The third line and the fourth line from (26) as well as the equality $S_{u_i} K_{j'_m-1} = K_{j'_m}$.

Since $\{\mathcal{H}^n((K_m)_\delta \backslash B_r)\}$ forms a decreasing sequence according to (15), and since it includes the subsequence $\left\{\mathcal{H}^n\left(\left(K_{j_m}\right)_\delta \backslash B_r\right)\right\}$ converging to $\mathcal{H}^n(L_\delta \backslash B_r)$, it follows that the first and last terms in (37) are equal. The first line, in fact, holds with equality for any $\delta > 0$. By Lemma 6, we can imply $L = S_{u_i} L$, this marks the completion of the inductive.

Verification of the convergence of the entire sequence is required. Since L is preserved under Steiner symmetrizations for every direction $u_1, \ldots, u_k$ in $\mathcal{F}$, it can follow the same logic as in (36) and the lines that follow that

$$\mathcal{H}^n\Big(\big(K_{j_m}\big)_\delta \backslash L_\delta\Big) \geq \mathcal{H}^n\Big(\big(K_j\big)_\delta \backslash L_\delta\Big)$$

when $j \geq j_m$. Therefore, we can imply that the sequence $\{K_m\}_\delta$ converges to L_δ in symmetric difference for all $\delta \geq 0$. According to Lemma 22, The sequence $\{K_m\}$ is shown to converge to L in both Hausdorff distance and the symmetric difference. □

Next, using Theorem 1 and the following two lemmas, we prove the convergence of iterative spherical Steiner symmetrizations.

Let $u \in \mathbb{S}^{n-1}$ denote a direction, and define the reflection of a point $x \in \mathbb{R}^n$ across the orthogonal hyperplane by

$$R_u x = x - (2x \cdot u)u.$$

It is obviously that $R_u = R_{-u}$. We denote the open positive half-space corresponding to u as $H_u^+ = \{x \in \mathbb{R}^n : \ x \cdot u > 0\}$, and we can define

$$F_u(x) := \begin{cases} x, & \text{if } x \in H_u^+ \\ R_u x, & \text{otherwise.} \end{cases}$$

We denote F_u as the *folding map* oriented along u, which folds $\mathbb{R}^n$ over the crease at $u^\perp$ onto the closed nonnegative half-space. The *trajectory* of a point x in $\mathbb{S}^{n-1}$ resulting from the corresponding sequence of reflections is described by

$$x_0 = x, \quad x_i = F_{u_i} x_{i-1} \text{ for } i \geq 1. \tag{38}$$

Define $O(n)$ to be the orthogonal group of $\mathbb{R}^n$. Denote by

$$\langle G \rangle := \{R_{u_k} \ldots R_{u_1} \in O(n) : \ u_1, \ \ldots, u_k \in G\}$$

the subgroup consisting of reflections R_u across hyperplanes whose unit normals are represented by $u \in G$.

Lemma 7 ([28], Proposition 4.2)**.** *Let G denote a subset of $\mathbb{S}^{n-1}$. Suppose that the following holds: (1) G spans $\mathbb{R}^n$; (2) G cannot be divided into two non-empty orthogonal subsets; (3) The angles formed by directions in G are not all commensurable with π. Under these assumptions, the subgroup $\langle G \rangle$ is dense in $O(n)$.*

Lemma 8 ([28], Theorem 1)**.** *Let G as a subset of $\mathbb{S}^{n-1}$ representing a collection of directions. Suppose that (C1) the open half-spaces H_u^+ with $u \in G$ cover $\mathbb{S}^{n-1}$ and (C2) $\langle G \rangle$ is dense in $O(n)$, thus this implies the existence of sequence $\{u_i\}_{i \geq 1}$ within G such that for any initial point $x \in \mathbb{S}^{n-1}$, the trajectory given by Equation (38) becomes dense in $\mathbb{S}^{n-1}$.*

Proof of Proposition 4. Choose a finite set of directions $\mathcal{F} := \{u_1, u_2, \ldots, u_k\} \subset \mathbb{S}^{n-1}$ such that $\mathcal{F}$ satisfies condition (C1) in Lemma 8 and the three conditions specified as (1), (2) and (3) in Lemma 7. By Theorem 1, we can proceed as follows. Let $\{u_m\}$ be a sequence of directions chosen from $\mathcal{F}$ such that every occurs in the sequence infinitely often, we can imply that the sequence $K_m := S_{u_m} \ldots S_{u_1} K$ tends to a compact set $L \subset \mathbb{S}^n$ and L exhibits symmetry with respect to every subspace $u_i^\perp$. Next, we prove $L = B_{r_0}$. For $t \in (0, 1)$, let

$$\mathbb{S}_t^{n-1} := \{x \in \mathbb{S}^n : \ x \cdot e_{n+1} = t\}$$

and

$$L_t := L \cap \mathbb{S}_t^{n-1}.$$

If $L_t \neq \emptyset$, then for any $x_0 \in L_t - te_{n+1}$, the reflection of x_0 about $u_i^{\perp}$ in $\mathbb{R}^n$ still belongs to $L_t - te_{n+1}$ for any $u_i \in \mathcal{F}$. By Lemmas 7 and 8 and compactness of L_t, $L_t = \mathbb{S}_t^{n-1}$. Moreover, the definition of spherical Steiner symmetrizations ensures that the connectivity of $T := \{t \in (0,1) : \ L_t \neq \emptyset\}$ and there is some $h_0 \in (0,1)$ such that $T = [h_0, 1]$. Thus, the spherical cap L entered at e_{n+1} and its measure is the same as K, i.e., $L = B_{r_0}$. □

5. Isoperimetric Property of the Spherical Cap

In this section, we will prove Proposition 5, i.e., the spherical cap has the smallest outer Minkowski content among all spherical compact sets with the same measure.

Lemma 9. *Let $K \subset \mathbb{S}^n$ be a compact set that has a non-empty interior, and $\{u_i\}_{i=1}^{\infty} \subset \mathbb{S}^{n-1}$. If the sequence of $K_i := S_{u_i} \cdots S_{u_1} K$ converges to the set K' in Hausdorff distance, then*

$$\mathcal{H}^n(K) = \mathcal{H}^n(K') \tag{39}$$

and for any $\delta > 0$

$$\mathcal{H}^n(K'_\delta) \leq \mathcal{H}^n(K_\delta). \tag{40}$$

Proof. If $K \subset \mathbb{S}_+^n$ is a compact set that has a non-empty interior, we can know K_i is also a compact set that has a non-empty interior and $\mathcal{H}^n(K_i) = \mathcal{H}^n(K)$. By (26), for any $\delta > 0$,

$$(K_i)_\delta = (S_{u_i} \cdots S_{u_1} K)_\delta \subset S_{u_i}\big(S_{u_{i-1}} \cdots S_{u_1} K\big)_\delta \subset \cdots \subset S_{u_i} \cdots S_{u_1}(K_\delta).$$

By the volume invariance of spherical Steiner symmetrizations, we have

$$\mathcal{H}^n((K_i)_\delta) \leq \mathcal{H}^n(K_\delta). \tag{41}$$

Since $\{K_i\}_{i=1}^{\infty}$ converges to K' in Hausdorff distance, it follows that, for all values of $\delta' \in (0, \delta)$, there exists $M > 0$ such that $K'_{\delta'} \subset (K_i)_\delta$ for any $i > M$. By (41), we have

$$\mathcal{H}^n\big(K'_{\delta'}\big) \leq \mathcal{H}^n(K_\delta). \tag{42}$$

The desired inequality (40) is followed when $\delta' \to \delta$.

Moreover, since the volume is an outer measure,

$$\mathcal{H}^n(K') \geq \limsup_{i \to \infty} \mathcal{H}^n(K_i) = \mathcal{H}^n(K). \tag{43}$$

Therefore, combining (42) and (43),

$$\mathcal{H}^n(K) \leq \mathcal{H}^n(K') \leq \mathcal{H}^n\big(K'_{\delta'}\big) \leq \mathcal{H}^n(K_\delta). \tag{44}$$

The desired equality (39) is followed when $\delta \to 0^+$ in (44). □

The following theorem implies that the outer Minkowski content is monotonically decreasing in relation to spherical Steiner symmetrizations.

Theorem 2. *For non-empty compact set K on $\mathbb{S}^n$, the outer Minkowski content of K and SK satisfy*

$$\mathcal{M}_+(SK) \leq \mathcal{M}_+(K). \tag{45}$$

Proof. By the definition of outer Minkowski content (9) and Lemma 5,

$$\begin{aligned}\mathcal{M}_+(SK) &= \lim_{\delta\to 0^+}\frac{\mathcal{H}^n((SK)_\delta)-\mathcal{H}^n(SK)}{\delta}\\ &\le \lim_{\delta\to 0^+}\frac{\mathcal{H}^n(S(K_\delta))-\mathcal{H}^n(SK)}{\delta}\\ &= \lim_{\delta\to 0^+}\frac{\mathcal{H}^n(K_\delta)-\mathcal{H}^n(K)}{\delta}\\ &= \mathcal{M}_+(K),\end{aligned} \tag{46}$$

where the inequality is due to (26), the next equality follows from (12), and the first equality and last equality are due to the definition of outer Minkowski content (9). □

Proof of Proposition 5. By Proposition 4, a sequence of directions $\{u_i\}_{i\in\mathbb{N}} \subset \mathbb{S}^{n-1}$ exists such that $K_i := S_{u_i}\cdots S_{u_1}K$ approaches B_{r_0} in spherical Hausdorff distance. By Lemma 9,

$$\mathcal{H}^n(B_{r_0}) = \mathcal{H}^n(K)$$

and

$$\mathcal{H}^n((B_{r_0})_\delta) \le \mathcal{H}^n(K_\delta).$$

Thus, by the definition of outer Minkowski content,

$$\mathcal{M}_+(K) = \lim_{\delta\to 0^+}\frac{\mathcal{H}^n(K_\delta)-\mathcal{H}^n(K)}{\delta} \ge \lim_{\delta\to 0^+}\frac{\mathcal{H}^n((B_{r_0})_\delta)-\mathcal{H}^n(B_{r_0})}{\delta} = \mathcal{M}_+(B_{r_0}).$$

This completes the proof. □

6. Implications and Future Directions

In this paper, We have proved several properties of spherical Steiner symmetrizations, along with the isoperimetric property of the spherical cap in $\mathbb{S}^n$. In particular, we prove that the sequence of iterative spherical Steiner symmetrizations of $K \subset \mathbb{S}^n$ along sequences chosen from a finite set of directions converges to a spherical cap with the same measure as K. In our future research, this is a great idea to study the Steiner symmetrizations of functions defined on $\mathbb{S}^n$. By utilizing Steiner symmetrizations of these functions, we aim to prove related inequalities.

For instance, we will establish the concept of Steiner symmetrizations for functions defined on $\mathbb{S}^n$. Utilizing this definition of Steiner symmetrizations on the sphere, attempting to demonstrate the Pólya–Szegö inequality for spherical functions. Furthermore, we can utilize Steiner symmetrizations on the sphere to establish the Petty projection inequality as well as the Sobolev inequality for functions defined on the sphere. Additionally, it is also beneficial to study the convergence of iterative Steiner symmetrizations on the sphere to a cap of the same volume, analyzing the properties of the direction set needed for Steiner symmetrizations.

Author Contributions: Conceptualization, Y.L.; formal analysis, Y.L. and Z.D.; writing— original draft preparation, Y.L.; writing—review and editing, Y.L. and Z.D.; funding acquisition, Y.L. All authors have read and agreed to the published version of the manuscript.

Funding: Research of the author is supported by the National Natural Science Foundation of China NSFC 11971080 and NSFC 12371137.

Data Availability Statement: This manuscript has no associated data.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

1. Almgren, F.J., Jr.; Lieb, E.H. Symmetric rearrangement is sometimes continuous. *J. Am. Math. Soc.* **1989**, *2*, 683–773. [CrossRef]
2. Ambrosio, L.; Fusco, N.; Pallara, D. *Functions of Bounded Variation and Free Discontinuity Problems*; Oxford Mathematical Monographs; The Clarendon Press: New York, NY, USA, 2000.
3. Burago, Y.D.; Zalgaller, V.A. *Geometric Inequalities*; Springer: Berlin, Germany, 1988.
4. Klimov, V.S. Imbedding theorems and geometric inequalities. *Math. USSR-Izv.* **1976**, *10*, 615–638. [CrossRef]
5. Schmidt, E. Beweis der isoperimetrischen Eigenschaft der Kugel im hyperbolischen und sphärischen Raumjeder Dimensionszahl. *Math. Z.* **1943**, *49*, 1–109. [CrossRef]
6. Schneider, R. *Convex Bodies: The Brunn-Minkowski Theory*, expanded ed.; Encyclopedia of Mathematics and its Applications; Cambridge University Press: Cambridge, UK, 2014; Volume 151.
7. Bianchi, G.; Gardner, R.J.; Gronchi, P. Symmetrization in geometry. *Adv. Math.* **2017**, *306*, 51–88. [CrossRef]
8. Bianchi, G.; Gardner, R.J.; Gronchi, P. Convergence of symmetrization processes. *Indiana Univ. Math. J.* **2022**, *71*, 785–817. [CrossRef]
9. Bianchi, G.; Gardner, R.J.; Gronchi, P. Full rotational symmetry from reflections or rotational symmetries in finitely many subspaces. *Indiana Univ. Math. J.* **2022**, *71*, 767–784. [CrossRef]
10. Klain, D.A. Steiner symmetrization using a finite set of directions. *Adv. Appl. Math.* **2012**, *48*, 340–353. [CrossRef]
11. Basit, B.; Hoehner, S.; Lángi, Z.; Ledford, J. Steiner symmetrization on the sphere. *arXiv* **2024**, arXiv:2406.10614.
12. Sarvas, J. Symmetrization of condensers in n-space. *Ann. Fenn. Math.* **1972**, *1*, 44. [CrossRef]
13. Solynin, A.Y. Continuous symmetrization via polarization. *St. Petersburg Math. J.* **2013**, *24*, 117–166. [CrossRef]
14. Figiel, T.; Lindenstrauss, J.; Milman, V. The dimension of almost spherical sections of convex bodies. *Acta Math.* **1977**, *139*, 53–94. [CrossRef]
15. Barchiesi, M.; Cagnetti, F.; Fusco, N. Stability of the Steiner symmetrization of convex sets. *J. Eur. Math. Soc. (JEMS)* **2013**, *15*, 1245–1278. [CrossRef]
16. Bianchi, G.; Gronchi, P. Steiner symmetrals and their distance from a ball. *Isr. J. Math.* **2003**, *135*, 181–192. [CrossRef]
17. 17 Blaschke, W. Affine Geometrie XIV: Eine Minimumaufgabe für Legendres Trägheitsellipsoid. *Ber. Verh. Sächs. Akad. Leipzig Math.-Phys. Kl* **1918**, *70*, 72–75.
18. Gruber, P.M. *Convex and Discrete Geometry*; Springer: Berlin, Germany, 2007.
19. Klartag, B. 5n Minkowski symmetrization suffice to arrive at an approximate Euclidean ball. *Ann. Math.* **2002**, *156*, 947–960. [CrossRef]
20. Bianchini, C.; Colesanti, A. A sharp Rogers and Shephard inequality for the p-difference body of planar convex bodies. *Proc. Am. Math. Soc.* **2008**, *136*, 2575–2582. [CrossRef]
21. Bianchi, G.; Klain, D.A.; Lutwak, E.; Yang, D.; Zhang, G. A countable set of directions is sufficient for Steiner symmetrization. *Adv. Appl. Math.* **2011**, *47*, 869–873. [CrossRef]
22. Lutwak, E. The Brunn–Minkowski–Firey theory, II: Affine and geominimal surface areas. *Adv. Math.* **1996**, *118*, 229–235. [CrossRef]
23. Mani-Levitska, P. Random Steiner symmetrizations. *Stud. Sci. Math. Hung.* **1986**, *21*, 373–378.
24. Lin, Y. Smoothness of the Steiner symmetrization. *Proc. Am. Math. Soc.* **2018**, *146*, 345–357. [CrossRef]
25. Ni, W.; Chen, Z. Hausdorff measure of the range of space-time anisotropic Gaussian random fields. *J. Theor. Probab.* **2021**, *34*, 264–282. [CrossRef]
26. Cagnetti, F.; Perugini, M.; Stöger, D. Rigidity for perimeter inequality under spherical symmetrisation. *Calc. Var. Partial. Differ. Equ.* **2020**, *59*, 139. [CrossRef]
27. Banuelos, R.; Latala, R.; Méndez-Hernández, P.J. A Brascamp-Lieb-Luttinger-type inequality and applications to symmetric stable processes. *Proc. Am. Math. Soc.* **2001**, *129*, 2997–3008. [CrossRef]
28. Burchard, A.; Chambers, G.R.; Dranovsi, A. Ergodic properties of folding maps on spheres. *Discret. Contin. Dyn. Syst.* **2017**, *37*, 1183–1200. [CrossRef]
29. Evans, L.C.; Gariepy, R.F. *Measure Theory and Fine Properties of Functions*, Revised ed.; CRC Press: Boca Raton, FL, USA, 2015.
30. Schneider, R. *The Brunn–Minkowski Theory, Encyclopedia Mathematical Applications*, expanded ed.; Cambridge University Press: Cambridge, UK, 2014; Volume 151.
31. Burchard, A.; Hajaiej, H. Rearrangement inequalities for functionals with monotone integrands. *J. Funct. Anal.* **2006**, *233*, 561–582. [CrossRef]
32. Burchard, A.; Schmuckenschläger, M. Comparison theorems for exit times. *Geom. Funct. Anal.* **2001**, *11*, 651–692. [CrossRef]
33. Cianchi, A.; Chlebík, M.; Fusco, N. The perimeter inequality under Steiner symmetrization: Cases of equality. *Ann. Math.* **2005**, *162*, 525–555.
34. Hack, T.; Pivovarov, P. Randomized Urysohn-type inequalities. *Mathematika* **2021**, *67*, 100–115. [CrossRef]
35. Basit, B.; Lángi, Z. Discrete isoperimetric problems in spaces of constant curvature. *Mathematicka* **2023**, *69*, 33–50. [CrossRef]
36. Leichtweiss, K. On Steiner's symmetrization of convex bodies in non-Euclidean geometry. *Results Math.* **2008**, *52*, 339–346. [CrossRef]

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