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Mikhlin-Type H^p Multiplier Theorem on the Heisenberg Group

Jinsen Xiao [1,†], Jianxun He [2,†] and Yingzhu Wu [1,*,†]

[1] Department of Mathematics, Guangdong University of Petrochemical Technology, Maoming 525000, China; jinsenxiao@gdupt.edu.cn
[2] Department of Basic Course Teaching, Software Engineering Institute of Guangzhou, Guangzhou 510990, China; hejianxun@gzhu.edu.cn
* Correspondence: wuyingzhu1978@gdupt.edu.cn
† These authors contributed equally to this work.

Abstract: The classical Fourier multiplier theorem by Mikhlin in Hardy spaces is extended to the Heisenberg group. The proof relies on the theories of atom and molecule functions and the property of special Hermite functions. The main result is a Mikhlin-type H^p multiplier theorem on the Heisenberg group. If an operator-valued function $M(\lambda)$ satisfies certain conditions, the right-multiplier operator T_M is bounded on the Hardy space $H^p(\mathbb{H}^n)$, which is defined in terms of maximal functions, and elements can be decomposed into atoms or molecules. The paper also discusses the relationship with other results and open problems.

Keywords: multiplier; Heisenberg group; Hardy space; Fourier transform

MSC: 43A85; 42B15; 26D10

1. Introduction

The classical L^p Mikhlin multiplier theorem on $\mathbb{R}^n$ states that, for $\alpha \in \mathbb{N}^n$ and the integer $s > \frac{n}{2}$, if a function $m(x)$ satisfies the following (see [1]),

$$\left| \left(\frac{\partial}{\partial x} \right)^{\alpha} m(x) \right| \leq c|x|^{-|\alpha|}, \ |\alpha| \leq s,$$

then the multiplier operator T_m can be defined in terms with Fourier transform by

$$\widehat{(T_m f)}(x) = \hat{f}(x) m(x), \ f \in \mathscr{S}(\mathbb{R}^n),$$

and can be extended to a bounded operator on $L^p(\mathbb{R}^n), 1 < p < \infty$. This result was extended by Taibleson and Weiss [2] to Hardy spaces $H^p(\mathbb{R}^n)$, $0 < p \leq 1$ for the integer $s > n(\frac{1}{p} - \frac{1}{2})$.

Since the 1980s, considerable attention has been paid to discovering different types of multiplier theorems on the stratified Lie groups. For the sublaplacian $\mathscr{L}$, it is known that if M is a bounded, Borel-measurable function on $(0, \infty)$, then the operator $M(\mathscr{L})$ can be defined by the spectral theorem according to the prescription

$$M(\mathscr{L}) = \int_0^{\infty} M(\lambda) dE(\lambda)$$

and $M(\mathscr{L})$ is bounded on L^2 (see [3]).

A natural direction of interest is to investigate the sufficient condition for the multiplier that ensures that the operator $M(\mathscr{L})$ is bounded on different spaces of distributions. The desired conditions are variants of the multiplier theorems of Hörmander-type [3–5], Marcinkiewicz-type [6–8] and Mikhlin-type [9]. The ideas in these works mainly focus on

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two important clarifications: the first, independently due to Mauceri and Meda [10] and to Christ [4], showed that the scale-invariant smoothness condition of order $s > Q/2$, where Q is the homogeneous dimension of the group, is sufficient on any stratified group; the second, according to Müller and Stein [11], proved that for the Heisenberg group $\mathbb{H}^n$, $s > d/2$, where $d = 2n + 1$ is the topological dimension, is enough. This result was extended to generalized Heisenberg groups by Hebisch [12], and to the free group $N_{3,2}$ and larger classes of two-step groups by Martini and Müller [13–15]. Essentially, these sufficient conditions all come down to the aspect of smoothness. The requirement of a certain order of smoothness, such as $s > Q/2$ or $s > d/2$, is the key determinant in ensuring the boundedness of the operator. Whether it is in the context of different group structures like stratified groups or the Heisenberg group and its extensions, the underlying factor that satisfies the desired boundedness property for the operator $M(\mathscr{L})$ is the smoothness condition.

All those works cited in the last paragraph are for the L^p multipliers. For the H^p case, Hulanicki and Stein [16] proved the Marcinkiewicz-type theorem on any stratified Lie group, as follow.

Theorem 1. *Suppose M is of class C^s on $(0, \infty)$ and*

$$\sup_{\lambda>0} |\lambda^j M^{(j)}(\lambda)| \leq C < \infty, \text{ for } 0 \leq j \leq s.$$

If r is a positive integer and $s > r + (3Q/2) + 2$, then $M(\mathscr{L})$ is bounded on H^p for $Q/(Q + r) < p < \infty$.

It follows from this theorem that $M(\mathscr{L})$ is bounded on H^p for $0 < p \leq 1$ when $s > Q(1/p + 1/2) + 2$. Michele and Mauceri [17] obtained an improvement for spectral multipliers of the sublaplacian as $s > Q(1/p - 1/2) + 1$. Their result was sharpened by Mauceri and Meda [10] for $s > Q(1/p - 1/2)$.

Note that the spectral multipliers of the sublaplacian $\mathscr{L}$, which were discussed in those papers we mentioned above, can be considered as a particular class of Fourier multipliers (essentially in the same way that radial Fourier multipliers on $\mathbb{R}^n$ are a particular class of Fourier multipliers). For the general Fourier multipliers, in the Heisenberg group, Lin [18] used the method of [17] to obtain the sufficiency of the Hörmander-type theorem with the smoothness condition of order $s \geq 4[Q(1/p - 1/2)/4 + 1]$, where $[\cdot]$ denotes the greatest integer function.

Our purpose in this paper is to prove the Mikhlin-type H^p multiplier theorem on the Heisenberg group $\mathbb{H}^n$. Let Δ_P be a difference-differential operator and $W^0_\alpha(\lambda)$ be a projection operator to a part of the main diagonal (both operators will be defined in the next section). Then, we have the following Theorem.

Theorem 2. *Let $0 < p \leq 1$ and let s be a positive integer satisfying $s > \frac{Q}{2}(\frac{1}{p} - \frac{1}{2})$. Suppose that the operator-valued function $M(\lambda)$ satisfies*

$$\|W^0_\alpha(\lambda)\Delta_P M(\lambda)\|^2 \leq c\big((2|\alpha| + n)|\lambda|\big)^{-d(P)}$$

for every monomial P with homogeneous degree $0 \leq d(P) \leq 2s$, and then the right-multiplier operator T_M defined by

$$\widehat{(T_M f)}(\lambda) = \hat{f}(\lambda)M(\lambda), \ f \in H^p \cap \mathscr{S}(\mathbb{H}^n)$$

is bounded on $H^p(\mathbb{H}^n)$.

This paper is organized as follows. In Section 2 we briefly summarize the harmonic analysis on the Heisenberg group needed in the sequel. Section 3 is devoted to the proof of our main result. In order to do this, some theorems and lemmas are stated in this Section.

We will adopt the convention that c denotes constants which may be different from one statement to another.

2. Preliminaries

We begin by recalling some notions from the articles [3,18] which have laid the foundation for harmonic analysis on the Heisenberg group. The $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$ is a Lie group structure on $\mathbb{C}^n \times \mathbb{R}$ with the multiplication law

$$(z,t) \circ (z',t') = (z+z', t+t'+2\mathrm{Im} z\bar{z'}),$$

where $z\bar{z'} = \sum_{j=1}^{n} z_j \bar{z'_j}$. For $(z,t) \in \mathbb{H}^n$, the dilation is defined by $\delta_\rho(z,t) = (\rho z, \rho^2 t), \rho > 0$, and its homogeneous norm is $|(z,t)| = (|z|^4 + |t|^2)^{1/4}$. The set $B_r(z_0,t_0) = \{(z,t) \in \mathbb{H}^n : |(z_0,t_0)^{-1} \circ (z,t)| < r\}$ is called the ball of radius r centered at (z_0,t_0), and its measure is given by $|B_r(z_0,t_0)| = cr^Q$, $Q = 2n+2$.

The Lie algebra $\mathcal{G}$ of $\mathbb{H}^n$, which admits a stratification by $\mathcal{G} = V_1 \oplus V_2$, is generated by the left-invariant vector fields

$$Z_j = \partial/\partial z_j + i\bar{z}_j \partial/\partial t, \bar{Z}_j = \partial/\partial \bar{z}_j - i z_j \partial/\partial t, 1 \le j \le n, \text{and } T = \partial/\partial t.$$

The horizontal layer is the first layer V_1 generated by $Z_j, \bar{Z}_j, 1 \le j \le n$, and the Heisenberg sublaplacian is defined by

$$\mathscr{L} = -\frac{1}{2} \sum_{j=1}^{n} (Z_j \bar{Z}_j + \bar{Z}_j Z_j).$$

We can consider that $X_j = Z_j$, $X_{n+j} = \bar{Z}_j, 1 \le j \le n$, and $X_{2n+1} = T$. For $I = (i_1, \cdots, i_{2n+1}) \in \mathbb{N}^{2n+1}$, we use the notation $X^I = X_1^{i_1} \cdots X_{2n+1}^{i_{2n+1}}$ to denote the left-invariant differential operator whose homogeneous degree is $d(I) = \sum_{k=1}^{2n} i_k + 2i_{2n+1}$.

A function P on $\mathbb{H}^n$ is called a polynomial if $P \circ \exp$ is a polynomial on $\mathcal{G}$. Every polynomial on $\mathbb{H}^n$ can be written uniquely as a finite sum

$$P = \sum_J a_J \eta^J, \ a_J \in \mathbb{C}, \ \eta^J = \eta_1^{j_1} \cdots \eta_{2n+1}^{j_{2n+1}},$$

where $\eta_j = \zeta_j \circ \log$, $\zeta_1, \cdots, \zeta_{2n+1}$ is the basis for $\mathcal{G}^*$ dual to the basis $X_1, \cdots, X_{2n+1}$ for $\mathcal{G}$. The monomial η^J is homogeneous for the degree $d(J) = \sum_{k=1}^{2n} j_k + 2j_{2n+1}$, and the homogeneous degree of P is given by $d(P) = \max\{d(J) : a_J \neq 0\}$. For $s \in \mathbb{N} = \{0,1,2,\cdots\}$, we denote by $\mathscr{P}_s$ the space of polynomials whose homogeneous degree is $\le s$. Let $x \in \mathbb{H}^n$ and $f \in C^{s+1}(\mathbb{H}^n)$, and the left Taylor polynomial of f at x of the homogeneous degree s will be the unique $P_s(f,x) \in \mathscr{P}_s$, such that $X^I P_s(f,x)(0) = X^I f(x)$ for $d(I) \le s$ (see [19]).

For $\lambda > 0$, let $\mathcal{H}_\lambda$ be the Bargmann space of all holomorphic functions F on $\mathbb{C}^n$ such that

$$\|F\|^2_{\mathcal{H}_\lambda} = \left(\frac{2\lambda}{\pi}\right)^n \int_{\mathbb{C}^n} |F(\xi)|^2 e^{-2\lambda|\xi|^2} d\xi d\bar{\xi} < \infty.$$

Then, $\mathcal{H}_\lambda$ is a Hilbert space with an orthogonal basis $\{E_\alpha^\lambda(\xi) = (\sqrt{2\lambda}\xi)^\alpha/\sqrt{\alpha!}, \alpha \in \mathbb{N}^n\}$. Let $\lambda \in \mathbb{R}^* = \mathbb{R}\backslash\{0\}$, and the Fock representation Π_λ of $\mathbb{H}^n$ acts on $\mathcal{H}_{|\lambda|}$ by

$$\Pi_\lambda(z,t)F(\xi) = \begin{cases} e^{i\lambda t + 2\lambda(\xi z - |z|^2/2)} F(\xi - \bar{z}), & \text{if } \lambda > 0, \\ e^{i\lambda t - 2\lambda(\xi \bar{z} - |z|^2/2)} F(\xi - z), & \text{if } \lambda < 0. \end{cases}$$

Let $\Phi_{\alpha,\beta}^\lambda(z,t) = \left(\Pi_\lambda(z,t)E_\alpha^\lambda, E_\beta^\lambda\right)_{\mathcal{H}_\lambda}$. Then, $\Phi_{\alpha,\beta}^\lambda(z) = \Phi_{\alpha,\beta}^\lambda(z,0)$ is called the special Hermite function (see [20,21]).

The group Fourier transform of a function $f \in \mathscr{S}(\mathbb{H}^n)$ is an operator-valued function defined by

$$\hat{f}(\lambda) = \int_{\mathbb{H}^n} f(z,t)\Pi_\lambda(z,t)dzd\bar{z}dt.$$

Let $d\mu(\lambda) = (2^{n-1}/\pi^{n+1})|\lambda|^n d\lambda$, and then one has the Plancherel formula

$$\|f\|_2^2 = \int_{\mathbb{R}^*} \|\hat{f}(\lambda)\|_{HS}^2 d\mu(\lambda),$$

where $\|\cdot\|_{HS}$ denotes the Hilbert–Schmidt norm.

Given a polynomial P in $z_j, \bar{z}_j, t$ on $\mathbb{H}^n$, we can introduce a differential operator Δ_P on the dual of $\mathbb{H}^n$ such that for $f \in \mathscr{S}(\mathbb{H}^n)$,

$$\Delta_P \hat{f}(\lambda) = \widehat{(Pf)}(\lambda). \tag{1}$$

Indeed, Δ_P can be defined for suitable operator-valued functions as follows. For $(\lambda, m, \alpha) \in \mathbb{R}^* \times \mathbb{Z}^n \times \mathbb{N}^n$, we use the notations

$$m_i^+ = \max(m_i, 0), m_i^- = -\min(m_i, 0),$$

$$m^+ = (m_1^+, \cdots, m_n^+), m^- = (m_1^-, \cdots, m_n^-),$$

and define the partial isometry $W_\alpha^m(\lambda)$ on $\mathcal{H}_{|\lambda|}$ by

$$W_\alpha^m(\lambda)E_\beta^\lambda = (-1)^{|m^+|}\delta_{\alpha+m^+,\beta}E_{\alpha+m^-}^\lambda, \text{ for } \lambda > 0,$$
$$W_\alpha^m(\lambda) = W_\alpha^m(-\lambda)^*, \text{ for } \lambda < 0,$$

where

$$\delta_{\alpha+m^+,\beta} = \begin{cases} 1 \text{ if } \alpha + m^+ = \beta, \\ 0 \text{ if } \alpha + m^+ \neq \beta. \end{cases}$$

Then, $\{W_\alpha^m(\lambda) : m \in \mathbb{Z}^n, \alpha \in \mathbb{N}^n\}$ forms an orthonormal basis for the Hilbert–Schmidt operators on $\mathcal{H}_{|\lambda|}$.

Let $f \in L^2(\mathbb{H}^n)$ and write it in the following form (see [22]):

$$f(z,t) = \sum_{m \in \mathbb{Z}^n} f_m(r_1, \cdots, r_n, t)e^{im\cdot\theta},$$

where $\theta = (\theta_1, \cdots, \theta_n)$ and $z = (r_1 e^{i\theta_1}, \cdots, r_n e^{i\theta_n})$. Then, the Fourier transform of f can be written as

$$\hat{f}(\lambda) = \sum_{m \in \mathbb{Z}^n, \alpha \in \mathbb{N}^n} \mathscr{R}_{f_m}(\lambda, m, \alpha)\, W_\alpha^m(\lambda), \tag{2}$$

where the Fourier coefficient $\mathscr{R}_{f_m}(\lambda, m, \alpha)$ is

$$\mathscr{R}_{f_m}(\lambda, m, \alpha) = \int_{\mathbb{H}^n} f_m(|z_1|, \cdots, |z_n|, t)e^{i\lambda t} \prod_{j=1}^n l_{\alpha_j}^{|m_j|}(2|\lambda||z_j|^2)dzd\bar{z}dt$$

with the Laguerre function $l_k^n(x)$ given by

$$l_k^s(x) = \left(\frac{k!}{(k+s)!}\right)^{1/2} x^{\frac{s}{2}} e^{-\frac{x}{2}} \sum_{j=1}^k \binom{k+s}{k-j} \frac{(-x)^j}{j!}.$$

From (1) and (2), the differential operator Δ_P can now be defined by

$$\Delta_P \mathscr{R}_f(\lambda, m, \alpha) = \mathscr{R}_{Pf}(\lambda, m, \alpha), \quad f \in \mathscr{S}(\mathbb{H}^n).$$

This equality, together with the property of the Laguerre function, helps us to derive an expression for Δ_P. Explicitly, it can be expressed through the combinations of $\{\Delta_{z_j}, \Delta_{\bar{z}_j}, \Delta_t\}$, where

$$\Delta_{z_j} = \begin{cases} -(2|\lambda|)^{-1/2}(\sqrt{\alpha_j+1}\tau_{j,1} - \sqrt{\alpha_j+|m_j|}\tau_{j,0})\sigma_{j,-1}, & \text{if } m_j > 0, \\ (2|\lambda|)^{-1/2}(\sqrt{\alpha_j+|m_j|+1}\tau_{j,0} - \sqrt{\alpha_j}\tau_{j,-1})\sigma_{j,-1}, & \text{if } m_j \le 0, \end{cases}$$

$$\Delta_{\bar{z}_j} = \begin{cases} (2|\lambda|)^{-1/2}(\sqrt{\alpha_j+|m_j|+1}\tau_{j,0} - \sqrt{\alpha_j}\tau_{j,-1})\sigma_{j,1}, & \text{if } m_j \ge 0, \\ -(2|\lambda|)^{-1/2}(\sqrt{\alpha_j+1}\tau_{j,1} - \sqrt{\alpha_j+|m_j|}\tau_{j,0})\sigma_{j,1}, & \text{if } m_j < 0, \end{cases}$$

$$\Delta_t = -i\frac{\partial}{\partial\lambda} - \frac{i}{2\lambda}\sum_{j=1}^{n}(\tau_{j,0} + \sqrt{\alpha_j(\alpha_j+|m_j|)}\tau_{j,-1} - \sqrt{(\alpha_j+1)(\alpha_j+|m_j|+1)}\tau_{j,1})$$

and

$$\sigma_{j,\iota}\mathscr{R}(\lambda, m, \alpha) = \mathscr{R}(\lambda, m + \iota e_j, \alpha),$$
$$\tau_{j,\iota}\mathscr{R}(\lambda, m, \alpha) = \mathscr{R}(\lambda, m, \alpha + \iota e_j).$$

Here, $\{e_j : 1 \le j \le n\}$ is the standard basis of $\mathbb{Z}^n$. For $I \in \mathbb{N}^{2n+1}$, we introduce a differential operator Δ_{P_I} defined by

$$\Delta_{P_I} = \prod_{j=1}^{n} \Delta_{z_j}^{i_j} \Delta_{\bar{z}_j}^{i_{n+j}} \Delta_t^{i_{2n+1}}. \tag{3}$$

Then, the operator Δ_P, defined above, can be expressed by

$$\Delta_P = \sum_I a_I \Delta_{P_I}, \quad a_I \in \mathbb{C}.$$

Moreover, this operator can be extended as formal difference-differential operators acting on operators which are of the form

$$M(\lambda) = \sum_{m\in\mathbb{Z}^n, \alpha\in\mathbb{N}^n} B(\lambda, m, \alpha) W_\alpha^m(\lambda).$$

A special case studied by Müller and Stein [11] is where

$$M(\lambda) = \sum_{\alpha\in\mathbb{N}^n} m(2|\alpha|+n) W_\alpha^0(\lambda),$$

which is defined in terms with the Heisenberg sublaplacian $\mathscr{L}$ whose group Fourier transform is $\hat{\mathscr{L}}(\lambda) = \sum_{\alpha\in\mathbb{N}^n}(2|\alpha|+n)W_\alpha^0(\lambda)$.

3. Proof of Theorem 2

In order to prove Theorem 2, we need the atomic and molecular decompositions of $H^p(\mathbb{H}^n)$. Recall that the Hardy space of Heisenberg group $H^p(\mathbb{H}^n)$ can be defined in terms of maximal functions. Following [16], a distribution f is said to be in $H^p(\mathbb{H}^n)$ if

$$\mathcal{M}_{\mathscr{L}} f(z,t) = \sup_{r>0} |e^{-r\mathscr{L}} f(z,t)|$$

belongs to $L^p(\mathbb{H}^n)$. The quasi-norm on H^p is

$$\|f\|_{H^p(\mathbb{H}^n)} = \|\mathcal{M}_{\mathscr{L}} f\|_{L^p(\mathbb{H}^n)}.$$

If $p > 1, H^p(\mathbb{H}^n) = L^p(\mathbb{H}^n)$. When $0 < p \leq 1$, the elements of $H^p(\mathbb{H}^n)$ can be decomposed into atoms or molecules.

Definition 1. *Let $0 < p \leq 1 \leq q \leq \infty$, $p \neq q$, $s \in \mathbb{N}$ and $s \geq [Q(1/p - 1)]$. A function a is called a $(p, q, s)-$atom with the center (z_0, t_0) if*

(i) $supp(a) \subset B_r(z_0, t_0)$;
(ii) $\|a\|_q \leq |B_r(z_0, t_0)|^{1/q - 1/p}$;
(iii) $\int_{\mathbb{H}^n} a(z,t)P(z,t)dzd\bar{z}dt = 0$ *for* $P \in \mathscr{P}_s$.

The following atomic decomposition theorem is due to Latter and Uchiyama [23].

Theorem 3. *Let $0 < p \leq 1$, and then $f \in H^p(\mathbb{H}^n)$ if and only if*

$$f = \sum_{j=1}^{\infty} \lambda_j a_j \text{ in the sense of distributions,}$$

where a_j's are $(p, q, s)-$atoms and $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$. Moreover,

$$\|f\|_{H^p}^p \approx \inf\{\Sigma_{j=1}^{\infty} |\lambda_j|^p : f = \Sigma_{j=1}^{\infty} \lambda_j a_j \text{ is an atomic decomposition of } f\}.$$

Definition 2. *Suppose that $0 < p \leq 1 \leq q \leq \infty$, $p \neq q$, $s \in \mathbb{N}$ and $s \geq [Q(1/p - 1)]$. Set $a = 1 - \frac{1}{p} + \epsilon$ and $b = 1 - \frac{1}{q} + \epsilon$, where $\epsilon > \max\{\frac{s}{Q}, \frac{1}{p} - 1\}$. A function $F \in L^q(\mathbb{H}^n)$ is called a $(p, q, s, \epsilon)-$molecule with the center (z_0, t_0) if*

(i) $F(z,t) \cdot |(z_0, t_0)^{-1} \circ (z,t)|^{Qb} \in L^q(\mathbb{H}^n))$;
(ii) *The molecular norm* $\mathfrak{N}(F) := \|F\|_q^{\frac{a}{b}} \cdot \|F|(z_0, t_0)^{-1} \circ (\cdot)|^{Qb}\|_q^{1 - \frac{a}{b}} < \infty$;
(iii) $\int_{\mathbb{H}^n} F(z,t)P(z,t)dzd\bar{z}dt = 0$ *for* $P \in \mathscr{P}_s$.

The following molecular decomposition theorem is due to Hemler [24].

Theorem 4. *Let $0 < p \leq 1$. Then, $f \in H^p(\mathbb{H}^n)$ if and only if*

$$f = \sum_{j=1}^{\infty} \lambda_j F_j \text{ in the sense of distributions,}$$

where F_j's are $(p, q, s, \epsilon)-$molecules with $\mathfrak{N}(F_j) \leq 1$ and $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$. Moreover,

$$\|f\|_{H^p}^p \approx \inf\{\Sigma_{j=1}^{\infty} |\lambda_j|^p : f = \Sigma_{j=1}^{\infty} \lambda_j F_j \text{ is a molecular decomposition of } f\}.$$

We state the following estimate for the remainder of the special Hermite function, which was proved in our earlier paper [25].

Lemma 1. *For any $x, y \in \mathbb{H}^n$, let $P_k(\Phi_{\alpha,\beta}^{\lambda}, x)$ be the left Taylor polynomial of $\Phi_{\alpha,\beta}^{\lambda}$ at x of homogeneous degree k. Then, we have*

$$\sum_{\beta \in \mathbb{N}^n} \left| \Phi_{\alpha,\beta}^{\lambda}(x \circ y) - P_k(\Phi_{\alpha,\beta}^{\lambda}, x)(y) \right|^2 \leq c|y|^{2k+2} \big((2|\alpha| + n)|\lambda|\big)^{k+1}.$$

Using this result, we can deduce the following crucial lemma.

Lemma 2. *Let P_I be defined in (3) and suppose a is a $(p,2,s)$-atom centered at origin. Then, for $d(P_I) \le s$, we have*

(i) $\|\Delta_{P_I}\hat{a}(\lambda)W_\alpha^0(\lambda)\|_{HS}^2 \le c\big((2|\alpha|+n)|\lambda|\big)^{s+1-d(P_I)}\|a\|_2^{2-\frac{2s+2+Q}{Q(1/p-1/2)}}$;

(ii) $\int_{\mathbb{R}^*}\|\Delta_{P_I}\hat{a}(\lambda)\|_{HS}^2 d\mu(\lambda) \le c\|a\|_2^{2-\frac{2d(P_I)}{Q(1/p-1/2)}}$.

Proof. Let

$$\hat{a}(\lambda,\alpha,\beta) = \int_{\mathbb{H}^n} a(z,t)\Phi_{\alpha,\beta}^\lambda(z,t)dzd\bar{z}dt.$$

Then, we have

$$\|\hat{a}(\lambda)W_\alpha^0(\lambda)\|_{HS}^2 = \sum_{\beta\in\mathbb{N}^n}\big|\hat{a}(\lambda,\alpha,\beta)\big|^2.$$

For the first estimate, we assume that $k\in\mathbb{N}$ satisfies $k+d(P_I)=s$ and let $P_k(\Phi_{\alpha,\beta}^\lambda,0)$ be the left Taylor polynomial of $\Phi_{\alpha,\beta}^\lambda$ at 0 of homogeneous degree k. According to Definition 1(ii), we can see that $\|a\|_2^{-1/[Q(1/p-1/2)]} \ge r$, which, together with Definition 1(iii) and Lemma 4, implies that

$$\begin{aligned}
&\|\Delta_{P_I}\hat{a}(\lambda)W_\alpha^0(\lambda)\|_{HS}^2\\
=\;& \sum_\beta\left|\int_{B_r} P_I(z,t)a(z,t)\Phi_{\alpha,\beta}^\lambda(z,t)dzd\bar{z}dt\right|^2\\
=\;& \sum_\beta\left|\int_{B_r} P_I(z,t)a(z,t)\Big(\Phi_{\alpha,\beta}^\lambda(z,t)-P_k(\Phi_{\alpha,\beta}^\lambda,0)(z,t)\Big)dzd\bar{z}dt\right|^2\\
\le\;& \int_{B_r}\left|P_I(z,t)a(z,t)\right|^2 dzd\bar{z}dt\sum_\beta\int_{B_r}\left|\Phi_{\alpha,\beta}^\lambda(z,t)-P_k(\Phi_{\alpha,\beta}^\lambda,0)(z,t)\right|^2 dzd\bar{z}dt\\
\le\;& c\|a\|^2 r^{2d(P_I)}\big((2|\alpha|+n)|\lambda|\big)^{k+1}\int_{B_r}|(z,t)|^{2(k+1)}dzd\bar{z}dt\\
\le\;& c\big((2|\alpha|+n)|\lambda|\big)^{s+1-d(P_I)}\|a\|_2^{2-\frac{2s+2+Q}{Q(1/p-1/2)}}.
\end{aligned}$$

For the second estimate, according to Definition 1(ii) and the Plancherel formula, we have

$$\begin{aligned}
\int_{\mathbb{R}^*}\|\Delta_{P_I}\hat{a}(\lambda)\|_{HS}^2 d\mu(\lambda) &= \int_{B_r}|P_I(z,t)a(z,t)|^2 dzd\bar{z}dt\\
&\le cr^{2d(P_I)}\|a\|_2^2\\
&\le c\|a\|_2^{2-\frac{2d(P_I)}{Q(1/p-1/2)}}.
\end{aligned}$$

This finishes the proof of this lemma. □

Now, we are ready to prove Theorem 2.

Proof of Theorem 2. Note that $2s-1 \ge [Q(\frac{1}{p}-1)]$, where s satisfies

$$s > \frac{Q}{2}\left(\frac{1}{p}-\frac{1}{2}\right) > \frac{Q}{2}\left(\frac{1}{p}-1\right).$$

As a consequence of the atomic and molecular characterizations of H^p, to show that T_M is bounded on H^p, it suffices to show that whenever a is a $(p,2,2s-1)$-atom, then T_Ma is a $(p,2,[Q(\frac{1}{p}-1)],\frac{2s}{Q}-\frac{1}{2})$-molecule and $\mathfrak{N}(T_Ma)<c$ (see [2,17]). Without a loss

of generality, assume that the atom a is centered at the origin, and thus we need to prove the following:

(i) $\mathfrak{N}(T_M a)^{\frac{2s}{Q}} = \|T_M a\|_2^{\frac{2s}{Q}-\frac{1}{p}+\frac{1}{2}} \|T_M a|(\cdot)|^{2s}\|_2^{\frac{1}{p}-\frac{1}{2}} < c$;

(ii) $\int_{\mathbb{H}^n} T_M a(z,t)P(z,t)dzd\bar{z}dt = 0,\ 0 \le d(P) \le [Q(\frac{1}{p}-1)]$.

To prove (i), according to the Plancherel formula and the condition of $M(\lambda)$, we obtain

$$\begin{aligned} \|T_M a\|_2^2 &= \int_{\mathbb{R}^*} \|\hat{a}(\lambda)M(\lambda)\|_{HS}^2 d\mu(\lambda) \\ &\le \sum_{\alpha} \int_{\mathbb{R}^*} \|\hat{a}(\lambda)W_\alpha^0(\lambda)\|_{HS}^2 \|W_\alpha^0(\lambda)M(\lambda)\|^2 d\mu(\lambda) \\ &\le c\|a\|_2^2, \end{aligned}$$

which implies that (i) can be obtained by the estimate

$$\|T_M a|(\cdot)|^{2s}\|_2^2 \le c\|a\|_2^{2-\frac{4s}{Q(1/p-1/2)}}. \tag{4}$$

Since $|(z,t)|^{2s} \le (|t|+|z|^2)^s$, then

$$\|T_M a|(\cdot)|^{2s}\|_2 \le c \sum_{d(P_I)=2s} \left(\int_{\mathbb{R}^*} \|\Delta_{P_I}(T_M a)\gamma(\lambda)\|_{HS}^2 d\mu(\lambda) \right)^{1/2},$$

where Δ_{P_I} is defined by (3). Thus, to establish (4), it is necessary to prove that

$$\int_{\mathbb{R}^*} \|\left(\Delta_{P_{I_1}}\hat{a}(\lambda)\right)\left(\Delta_{P_{I_2}}M(\lambda)\right)\|_{HS}^2 d\mu(\lambda) \le c\|a\|_2^{2-\frac{4s}{Q(1/p-1/2)}}, \tag{5}$$

where $\Delta_{P_{I_1}}$ and $\Delta_{P_{I_2}}$ satisfy $d(P_{I_1}) + d(P_{I_2}) = 2s$.

In order to obtain the Estimate (5), we first consider the case $d(P_{I_1}) = 2s$. Similar to the proof of Lemma 2(ii), we can easily obtain

$$\int_{\mathbb{R}^*} \|\left(\Delta_{P_{I_1}}\hat{a}(\lambda)\right)M(\lambda)\|_{HS}^2 d\mu(\lambda) \le c\int_{\mathbb{R}^*} \|\Delta_{P_{I_1}}\hat{a}(\lambda)\|_{HS}^2 d\mu(\lambda) \le c\|a\|_2^{2-\frac{4s}{Q(1/p-1/2)}}.$$

Next, for the case $0 \le d(P_{I_1}) < 2s$, note that $\sum_\alpha W_\alpha^0(\lambda) = \sum_\alpha W_\alpha^0(\lambda)W_\alpha^0(\lambda) = I(\lambda)$ is the identity operator, and then

$$\begin{aligned} &\int_{\mathbb{R}^*} \|\left(\Delta_{P_{I_1}}\hat{a}(\lambda)\right)\left(\Delta_{P_{I_2}}M(\lambda)\right)\|_{HS}^2 d\mu(\lambda) \\ \le\ & \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathbb{N}^n} \int_{2^k < (2|\alpha|+n)|\lambda| \le 2^{k+1}} \|\Delta_{P_{I_1}}\hat{a}(\lambda)W_\alpha^0(\lambda)\|_{HS}^2 \|W_\alpha^0(\lambda)\Delta_{P_{I_2}}M(\lambda)\|^2 d\mu(\lambda) \\ =\ & S_1 + S_2, \end{aligned}$$

where S_1 is the sum over $k \le k_0$ and k_0 is an integer satisfying

$$2^{k_0} < \|a\|_2^{2/(Q(1/p-1/2))} \le 2^{k_0+1}.$$

For convenience, set $\lambda(\alpha) = (2|\alpha|+n)|\lambda|$. Then, according to Lemma 2(i) and the condition of $M(\lambda)$, we can obtain

$$\begin{aligned} S_1 &\leq c\sum_{k\leq k_0}\sum_{\alpha\in\mathbb{N}^n}\int_{2^k<\lambda(\alpha)\leq 2^{k+1}} \|a\|_2^{2-\frac{4s+Q}{Q(1/p-1/2)}}\lambda(\alpha)^{2s-d(P_{I_1})}\lambda(\alpha)^{-d(P_{I_2})}|\lambda|^n d\lambda \\ &= c\|a\|_2^{2-\frac{4s+Q}{Q(1/p-1/2)}}\sum_{k\leq k_0}\sum_{\alpha\in\mathbb{N}^n}\frac{2^{k(n+1)}}{(2|\alpha|+n)^{n+1}} \\ &\leq c\|a\|_2^{2-\frac{4s+Q}{Q(1/p-1/2)}}2^{k_0Q/2} \\ &\leq c\|a\|_2^{2-\frac{4s}{Q(1/p-1/2)}}. \end{aligned}$$

For the estimate S_2, again using the condition of $M(\lambda)$, we have

$$\begin{aligned} S_2 &\leq c\sum_{k>k_0}\sum_{\alpha\in\mathbb{N}^n}\int_{2^k<\lambda(\alpha)\leq 2^{k+1}} \|\Delta_{P_{I_1}}\hat{a}(\lambda)W_\alpha^0(\lambda)\|_{HS}^2\lambda(\alpha)^{-d(P_{I_2})}d\mu(\lambda) \\ &\leq c\sum_{k>k_0}2^{-kd(P_{I_2})}\int_{\mathbb{R}^*}\|\Delta_{P_{I_1}}\hat{a}(\lambda)\|_{HS}^2 d\mu(\lambda) \\ &\leq c2^{-(k_0+1)d(P_{I_2})}\|P_{I_1}a\|_2^2 \\ &\leq c\|a\|_2^{-2d(P_{I_2})/(Q(1/p-1/2))}\|a\|_2^{2-2d(P_{I_1})/(Q(1/p-1/2))} \\ &\leq c\|a\|_2^{2-\frac{4s}{Q(1/p-1/2)}}, \end{aligned}$$

where in the fourth inequality we used Lemma 2(ii) and the fact that

$$2^{-(k_0+1)d(P_{I_2})} < \|a\|_2^{-2d(P_{I_2})/(Q(1/p-1/2))}.$$

This proves (5), and hence the inequality (i).

Now, we can continue to prove (ii). Suppose $d(P_{I_1})+d(P_{I_2}) \leq [Q(\frac{1}{p}-1)]$, and according to Lemma 2(i), we obtain

$$\begin{aligned} & \|\big(\Delta_{P_{I_1}}\hat{a}(\lambda)\big)W_0^0(\lambda)\big(\Delta_{P_{I_2}}M(\lambda)\big)\|_{HS}^2 \\ \leq\ & \|\Delta_{P_{I_1}}\hat{a}(\lambda)W_0^0(\lambda)\|_{HS}^2\|W_0^0(\lambda)\Delta_{P_{I_2}}M(\lambda)\|^2 \\ \leq\ & c\|a\|_2^{2-\frac{4s+Q}{Q(1/p-1/2)}}\big(n|\lambda|\big)^{2s-d(P_{I_1})-d(P_{I_2})} \\ & \to 0 \ \text{ as } \lambda\to 0. \end{aligned}$$

This means that

$$\Delta_P(T_Ma)\Upsilon(\lambda)W_0^0(\lambda)\to 0 \text{ as } \lambda\to 0,\ 0\leq d(P)\leq\big[Q\big(\frac{1}{p}-1\big)\big]$$

in the sense of weak convergence. Hence, we obtain

$$\int_{\mathbb{H}^n}P(z,t)T_Ma(z,t)dzd\bar{z}dt=0,\ \ 0\leq d(P)\leq\big[Q\big(\frac{1}{p}-1\big)\big].$$

This completes the proof. □

4. Discussion

Note that Theorem 2 remains true if the operator norm $\|\cdot\|$ is replaced by the Hilbert–Schmidt norm $\|\cdot\|_{HS}$ since we have the inequality $\|M\|\leq\|M\|_{HS}$. Hence, this result extends Theorem 2 of [26]. We also remark that in this theorem, the smoothness condition of order $s>\frac{Q}{2}(\frac{1}{p}-\frac{1}{2})$ with $Q=2n+2$ for the homogeneous degree of the Heisenberg

group is the same as that in [10]. However, whether the same result holds for the order $s > \frac{d}{2}(\frac{1}{p} - \frac{1}{2})$ with the topological dimension $d = 2n + 1$ is still an unsolved problem.

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