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Initial Coefficient Bounds for Bi-Close-to-Convex Classes of n -Fold-Symmetric Bi-Univalent Functions

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Abstract: In this article, the strong class of bi-close-to-convex functions of order α and β in n -fold symmetric bi-univalent functions, which is the subclass of σ , is introduced. The upper bound value for $|a_{n+1}|, |a_{2n+1}|$ for functions in these classes are obtained. Moreover, the Fekete–Szegő relation for our classes of functions are established.

Keywords: n -fold symmetric function; bi-univalent functions; univalent function; analytic function

MSC: 30C45; 30C50; 30C80



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1. Introduction and Definitions

Let $\mathcal{A}$ denote the family of functions g analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$g(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad z \in \mathbb{U}. \quad (1)$$

Let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ that are univalent in $\mathbb{U}$. Refer to [1] for further explanations on univalent functions. Let $\mathcal{S}^*(\beta)$, $\mathcal{C}(\beta)$, $0 \leq \beta < 1$ be the subclasses of $\mathcal{S}$ containing *starlike* and *convex* functions of order β . Their analytic representations are

$$\mathcal{S}^*(\beta) = \left\{ g : g \in \mathcal{S}, \Re \left(\frac{zg'(z)}{g(z)} \right) > \beta \quad (z \in \mathbb{U}) \right\},$$

and

$$\mathcal{C}(\beta) = \left\{ g : g \in \mathcal{S}, \Re \left(1 + \frac{zg''(z)}{g'(z)} \right) > \beta \quad (z \in \mathbb{U}) \right\},$$

respectively. The class $\mathcal{C}(0)$ contains all convex univalent functions and is denoted by $\mathcal{C}$. Various classes, such as $\mathcal{S}^*(\beta)$ and $\mathcal{C}(\beta)$, of starlike and convex functions have been investigated by many authors. These functions are commonly described by the value $zg'(z)/g(z)$ or $1 + zg''(z)/g'(z)$ lying in a starlike domain with respect to 1 in the right-half plane. Then, every function $g \in \mathcal{S}$ has an inverse g^{-1} of the form

$$g^{-1}(g(z)) = z, \quad z \in \mathbb{U}$$

and

$$g(g^{-1}(w)) = w, \quad |w| < r_0(g); r_0(g) \geq 1/4$$

where

$$g^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots \quad (2)$$

A function is said to be bi-univalent in $\mathbb{U}$ if both g and g^{-1} are univalent in $\mathbb{U}$. Let σ denote the class of bi-univalent function in $\mathbb{U}$ given by (2). The bound $|a_2| \leq 1.51$ is found from Lewin [2]. Brannan and Taha [3] computed the values $|a_2|$ and $|a_3|$ of the functions in the classes $\mathcal{S}_\sigma^*(\beta)$ and $\mathcal{C}_\sigma(\beta)$, motivated by Lewin [2] (refer [3]). Brannan and Clunie [4] found $|a_2| \leq \sqrt{2}$.

The coefficient computation problem is still unsolved: the bound for $|a_n|$ ($n \in \mathbb{N}$, $n \geq 3$) is unsolved ([5]).

Because of the work of Srivastava et al. [5], research on bi-univalent functions recently acquired more interest in this field. From this motivation, many researchers (see [5–12]) calculated various types of subclasses of the σ and nonsharp values on the first two coefficients in the Taylor–Maclaurin series. Really, the calculation of $|a_n|$ for $n \geq 3$ is still an open problem. On the other hand, there have been a few intriguing publications on bi-univalent functions with Faber polynomials under a gap series condition (see Jahangiri and Hamidi [13,14]).

Let $\mathcal{K}_\alpha$, $0 \leq \alpha \leq 1$ be the family of analytic functions $g(z)$ of the form (1). Then, there exists a convex function ϕ satisfying

$$\left| \arg \left(\frac{g'(z)}{\phi'(z)} \right) \right| < \frac{\alpha\pi}{2}.$$

Kaplan [15] and Reade [16] investigated these classes. Hence $\mathcal{K}_0 = \mathcal{C}$ and $\mathcal{K}_1$ are the families of *convex* univalent functions and *close-to-convex* functions, respectively. Moreover, $\mathcal{K}_{\alpha_1}$ is a proper subclass of $\mathcal{K}_{\alpha_2}$ if $\alpha_1 < \alpha_2$. The class of *close-to-convex functions of order β* , $0 \leq \beta \leq 1$ from the work of Reade [16] is as follows:

$$\Re \left(\frac{g'(z)}{\phi'(z)} \right) > \beta.$$

Because of Brannan and Taha [3] and Reade [16], this is evident to the class of strongly *bi-convex* and *bi-starlike* functions of order α (see the work of Brannan and Taha [3]).

Let $\mathcal{P}$ be the categories of analytic functions $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ such that $\Re(p(z)) > 0$ in $\mathbb{U}$. Srivastava et al. [17] have mentioned some illustrations of the class of n -fold symmetric bi-univalent functions and found the application on the coefficient estimates for $|a_{n+1}|$ and $|a_{2n+1}|$ to the new class of functions.

The function $h(z) = \sqrt[n]{g(z^n)}$ is univalent and maps the unit disk in $\mathbb{U}$ into a region with n -fold symmetry for each function in $\mathcal{S}$. Any function is called n -fold symmetric (see [18]) if it has the normalized form

$$h(z) = z + \sum_{k=1}^{\infty} a_{nk+1} z^{nk+1} \quad z \in \mathbb{U}, \quad (3)$$

and $\mathcal{S}_n$ represents the categories of n -fold symmetric univalent functions which are normalized by the above series expansion. In fact, the functions in class $\mathcal{S}$ are one-fold symmetric functions.

The n -fold symmetric bi-univalent functions are conceptualized similarly to n -fold symmetric univalent functions. For each integer n , each function in the class $h(z)$ in σ produces an n -fold symmetric bi-univalent function. The normalization of $h(z)$ is mentioned in (3), and h^{-1} is obtained from the following:

$$f(w) = w - a_{k+1}w^{k+1} + \left[(k+1)a_{k+1}^2 - a_{2k+1} \right] w^{2k+1} - \left[\frac{1}{2}(k+1)(3k+2)a_{k+1}^3 - (3k+2)a_{k+1}a_{2k+1} + a_{3k+1} \right] w^{3k+1} + \dots \quad (4)$$

where $h^{-1} = f$. Here, σ_n represents the the class of n -fold symmetric *bi-univalent* functions. The functions $\left(\frac{z^n}{1-z^n} \right)^{\frac{1}{n}}$, $\left(\frac{1}{2} \log \left(\frac{1+z^n}{1-z^n} \right) \right)^{\frac{1}{n}}$ and $(-\log(1-z^n))^{\frac{1}{n}}$ with the corresponding inverse functions $\left(\frac{w^n}{1+w^n} \right)^{\frac{1}{n}}$, $\left(\frac{e^{2w^n}-1}{e^{2w^n}+1} \right)^{\frac{1}{n}}$, and $\left(\frac{e^{w^n}-1}{e^{w^n}} \right)^{\frac{1}{n}}$ are a few examples of n -fold symmetric bi-univalent functions.

From Pommerenke [18], the n -fold symmetric functions in $\mathcal{P}$ take the following form:

$$p(z) = 1 + p_n z^n + p_{2n} z^{2n} + p_{3n} z^{3n} + \dots \quad (5)$$

Finding the initial coefficient bound for bi-close-to-convex classes of n -fold-symmetric bi-univalent functions of order α and β is the aim of this paper.

To obtain the main results, the following standard lemmas are used.

Lemma 1. If $p \in \mathcal{P}$, for each $k \geq 1$, then $|p_k| \leq 2$, and

$$\left| p_{2k} - \frac{p_k^2}{2} \right| \leq 2 - \frac{|p_k|^2}{2}.$$

for one-fold symmetric in Duren [1] and Ma and Minda [19].

Lemma 2. Let $\phi \in \mathcal{C}$ and $\lambda \in \mathbb{R}$. Then,

$$\left| c_{2k+1} - \lambda c_{k+1}^2 \right| \leq \begin{cases} 1 - \lambda & \text{for } \lambda < \frac{2}{3} \\ 1 & \text{for } \frac{2}{3} \leq \lambda \leq \frac{4}{3} \\ \lambda - 1 & \text{for } \lambda > \frac{4}{3} \end{cases}$$

for one-fold symmetry in Kanas [20].

2. The Bounds for Class $\mathcal{K}_{\sigma,n}[\alpha]$

In this section, the bound of the first two coefficients of the strong class of bi-close-to-convex functions of order α is derived. The work starts from the following well-known definitions.

Definition 1. Let $\mathcal{A}_{\sigma,n}(R)$ be the class of functions of the form (3) in $|z| < R$ such that its inverse function has an analytic continuation to $|z| < R$ (see (4)). Hence, the functions in the class $\mathcal{A}_{\sigma,n}(R)$ are called n -fold symmetric bi-analytic functions in $|z| < R$.

It will be easy to leave out the reference to the circular domain in Definition 1 when $R = 1$ and $n = 1$. Therefore, a *bi-analytic* function will lead to a *bi-analytic* on $\mathbb{U}$. It is shortened to $\mathcal{A}_{\sigma}(1) = \mathcal{A}_{\sigma}$. It should be noted that the class $\mathcal{A}_{\sigma}$ of bi-analytic functions is a legitimate subclass of $\mathcal{A}$.

Definition 2. Let $0 \leq \alpha \leq 1$. A function $h(z) \in \mathcal{A}_{\sigma,n}$, given by (3), with $h'(z) \neq 0$, is called a *strongly bi-close-to-convex function of order α* if there are bi-convex functions ϕ and ψ such that

$$\left| \arg \left(\frac{g'(z)}{\phi'(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{U}) \quad (6)$$

and

$$\left| \arg \left(\frac{f'(w)}{\psi'(w)} \right) \right| < \frac{\alpha\pi}{2} \quad (w \in \mathbb{U}). \quad (7)$$

Here, $f = h^{-1}$. Here $\mathcal{K}_{\sigma,n}[\alpha]$ represents the strong class of bi-close-to-convex n -fold symmetric functions.

The function ψ is the inverse of ϕ such that

$$\phi(z) = z + c_{n+1}z^{n+1} + c_{2n+1}z^{2n+1} + \dots \quad (8)$$

and

$$\psi(w) = w - c_{n+1}w^{n+1} + \left[(n+1)c_{n+1}^2 - c_{2n+1} \right] w^{2n+1} + \dots \quad (9)$$

Theorem 1. Let $0 \leq \alpha \leq 1$ and let $h(z)$ in $\mathcal{K}_{\sigma,n}[\alpha]$ from (3). Then,

$$|a_{n+1}| \leq \sqrt{1 + \frac{4(n+2)\alpha}{(2n+1)(n+1)}} \quad (10)$$

and

$$|a_{2n+1}| \leq 1 + \frac{2\alpha(n+2)}{2n+1}. \quad (11)$$

Proof. The (6) and (7) can be written in the form

$$\frac{g'(z)}{\phi'(z)} = [p(z)]^\alpha,$$

for some $p \in \mathcal{P}$. Thus,

$$g'(z) = [p(z)]^\alpha \phi'(z) \quad (12)$$

Similarly, there is a $q \in \mathcal{P}$ satisfying

$$\frac{f'(w)}{\psi'(w)} = [q(w)]^\alpha.$$

Therefore,

$$f'(w) = [q(w)]^\alpha \psi'(w). \quad (13)$$

Now $p(z), q(w)$ are expressed in the following form:

$$p(z) = 1 + p_n z^n + p_{2n} z^{2n} + p_{3n} z^{3n} + \dots \quad (14)$$

and

$$q(w) = 1 + q_n w^n + q_{2n} w^{2n} + q_{3n} w^{3n} + \dots \quad (15)$$

Then, from (12) and (13), one can obtain

$$(n+1)a_{n+1} = (n+1)c_{n+1} + \alpha p_n \quad (16)$$

$$(2n+1)a_{2n+1} = (2n+1)c_{2n+1} + (n+1)\alpha c_{n+1}p_n + \alpha p_{2n} + \frac{\alpha(\alpha-1)}{2}p_n^2 \quad (17)$$

$$-(n+1)a_{n+1} = \alpha q_n - (n+1)c_{n+1} \quad (18)$$

and

$$\begin{aligned} & (2n+1) \left((n+1)a_{n+1}^2 - a_{2n+1} \right) \\ &= (2n+1) \left((n+1)c_{n+1}^2 - c_{2n+1} \right) - (n+1)\alpha c_{n+1}q_n + \alpha q_{2n} + \frac{\alpha(\alpha-1)}{2}q_n^2. \end{aligned} \quad (19)$$

From (16) and (18), we obtain

$$p_n = -q_n. \quad (20)$$

By adding (17) and (19), we obtain

$$\begin{aligned} (n+1)(2n+1)a_{n+1}^2 &= (2n+1)(2n+1)c_{n+1}^2 \\ &+ (n+1)\alpha c_{n+1}(p_n - q_n) + \alpha \left(p_{2n} - \frac{p_n^2}{2} + q_{2n} - \frac{q_n^2}{2} \right) + \frac{\alpha^2}{2}(p_n^2 + q_n^2). \end{aligned} \quad (21)$$

An application of Lemma 1 leads to

$$\begin{aligned} (n+1)(2n+1)|a_{n+1}^2| &\leq (2n+1)(2n+1)|c_{n+1}^2| \\ &+ (n+1)\alpha|c_{n+1}||p_n - q_n| + \alpha \left(2 - \frac{1-\alpha}{2}|p_n^2| + 2 - \frac{1-\alpha}{2}|q_n^2| \right). \end{aligned}$$

Using $|c_k| \leq 1$, $|p_k| \leq 2$ and $|q_k| \leq 2$ for $k = 1, 2, \dots$, it is noted

$$(n+1)(2n+1)|a_{n+1}^2| \leq (n+1)(2n+1) + 4\alpha(n+1) + 4\alpha.$$

Therefore,

$$|a_{n+1}^2| \leq 1 + \frac{4(n+2)\alpha}{(2n+1)(n+1)}. \quad (22)$$

It produces the expected estimate of $|a_{n+1}|$.

For the bound (11), consider the relation (17).

$$(2n+1)a_{2n+1} = (2n+1)c_{2n+1} + \alpha(n+1)c_{n+1}p_n + \alpha p_{2n} + \frac{\alpha(\alpha-1)}{2}p_n^2.$$

The above equation reduces to

$$(2n+1)a_{2n+1} = (2n+1)c_{2n+1} + (n+1)\alpha c_{n+1}p_n + \alpha \left(p_{2n} - \frac{p_n^2}{2} \right) + \frac{\alpha^2}{2}p_n^2.$$

By applying Lemma 1 and $|c_k| \leq 1$,

$$(2n+1)|a_{2n+1}| \leq 2\alpha(n+1) + (2n+1) + 2\alpha.$$

The bound for $|a_{2n+1}|$ is obtained as declared in (11). Hence, Theorem 1 is proved. $\square$

$\mathcal{K}_\sigma[\alpha]$ can be $\mathcal{K}_\sigma[1] = \mathcal{K}_\sigma$, the class of *bi-close-to-convex* functions in the situation of one-fold symmetric functions for $\alpha = 1$

$\mathcal{K}_\sigma[\alpha]$ to $\mathcal{K}_\sigma[0] = \mathcal{C}_\sigma$, the class of *bi-convex* in the situation of one-fold symmetric functions for $\alpha = 0$ (see Brannan and Taha [3]).

The following corollaries are obtained from the theorem for one-fold symmetric functions when $\alpha = 0$ and $0 \leq \alpha \leq \frac{1}{2}$.

Corollary 1. If $g(z)$ is in $\mathcal{K}_\sigma[0] = \mathcal{C}_\sigma$, then $|a_2| \leq 1$.

Corollary 2. Let $g(z)$ represented in (1) be in $\mathcal{K}_\sigma[\alpha]$ whenever $0 \leq \alpha \leq \frac{1}{2}$. Then, $|a_2| \leq \sqrt{2}$.

Theorem 2. If $h(z)$ is in $\mathcal{K}_{\sigma,n}[\alpha]$, represented by (3) for $0 \leq \alpha \leq 1$, then

$$|a_{2n+1} - \lambda a_{n+1}^2| \leq \begin{cases} (1 - \lambda) + \frac{2\alpha((n+1)-2\lambda)}{(2n+1)} + \frac{((n+1)-2\lambda)\alpha M}{(2n+1)(n+1)} & \text{for } \lambda < 0, \\ (1 - \lambda) + \frac{2\alpha(n+1-2\lambda)}{(2n+1)} + \frac{\alpha M}{(2n+1)} & \text{for } 0 \leq \lambda < \frac{2}{3}, \\ 1 + \frac{2\alpha(n+1-2\lambda)}{(2n+1)} + \frac{\alpha M}{(2n+1)} & \text{for } \frac{2}{3} \leq \lambda < 1, \\ 1 + \frac{2\alpha(2\lambda-(n+1))}{(2n+1)} + \frac{\alpha M}{(2n+1)} & \text{for } 1 \leq \lambda \leq \frac{4}{3}, \\ (\lambda - 1) + \frac{2\alpha(2\lambda-(n+1))}{(2n+1)} + \frac{\alpha M}{(2n+1)} & \text{for } \frac{4}{3} < \lambda < 2, \\ (\lambda - 1) + \frac{2\alpha(2\lambda-(n+1))}{(2n+1)} + \frac{\alpha M(2\lambda-(n+1))}{(n+1)(2n+1)} & \text{for } \lambda \geq 2, \end{cases} \quad (23)$$

where $M \leq 2$.

Proof. From Equations (17) and (21), one can obtain

$$a_{2n+1} - \lambda a_{n+1}^2 = c_{2n+1} + \frac{(n+1)\alpha c_{n+1}p_n}{2n+1} + \frac{\alpha p_{2n}}{2n+1} + \frac{\alpha(\alpha-1)}{2(2n+1)}p_n^2$$

$$- \lambda \left[c_{n+1}^2 + \frac{\alpha c_{n+1}(p_n - q_n)}{2n+1} + \frac{\alpha(p_{2n} + q_{2n})}{(2n+1)(n+1)} + \frac{\alpha(\alpha-1)}{2(2n+1)(n+1)}(p_n^2 + q_n^2) \right].$$

By using the relations $q_n = -p_n$, $|c_k| \leq 1$, $|p_k| \leq 2$, and $|q_k| \leq 2$ for $k = 1, 2, \dots$, the above identity reduces to

$$|a_{2n+1} - \lambda a_{n+1}^2| \leq |c_{2n+1} - \lambda c_{n+1}^2| + \frac{2\alpha|n+1-2\lambda|}{2n+1}$$

$$+ \frac{\alpha|n+1-\lambda|}{(n+1)(2n+1)} \left[\left| p_{2n} - \frac{p_n^2}{2} \right| + \frac{\alpha|p_n^2|}{2} \right] + \frac{\alpha|\lambda|}{(n+1)(2n+1)} \left[\left| q_{2n} - \frac{q_n^2}{2} \right| + \frac{\alpha|q_n^2|}{2} \right].$$

The expressions $\left| p_{2n} - \frac{p_n^2}{2} \right| + \frac{\alpha|p_n^2|}{2}$ and $\left| q_{2n} - \frac{q_n^2}{2} \right| + \frac{\alpha|q_n^2|}{2}$ have the same bounds so that

$$|a_{2n+1} - \lambda a_{n+1}^2| \leq |c_{2n+1} - \lambda c_{n+1}^2| + \frac{2\alpha|n+1-2\lambda|}{2n+1} + \frac{\alpha(|n+1-\lambda| + |\lambda|)}{(n+1)(2n+1)} \left[\left| p_{2n} - \frac{p_n^2}{2} \right| + \frac{\alpha|p_n^2|}{2} \right].$$

Use of Lemma 1 gives

$$|a_{2n+1} - \lambda a_{n+1}^2| \leq |c_{2n+1} - \lambda c_{n+1}^2| + \frac{2\alpha|n+1-2\lambda|}{2n+1} + \frac{\alpha(|n+1-\lambda| + |\lambda|)}{(n+1)(2n+1)} \left[\left| 2 - \frac{p_n^2}{2} \right| + \frac{\alpha|p_n^2|}{2} \right].$$

That is,

$$|a_{2n+1} - \lambda a_{n+1}^2| \leq |c_{2n+1} - \lambda c_{n+1}^2| + \frac{2\alpha|n+1-2\lambda|}{2n+1} + \frac{\alpha(|n+1-\lambda| + |\lambda|)}{(n+1)(2n+1)} \left[2 - \frac{(1-\alpha)p_n^2}{2} \right].$$

Using $\left[2 - \frac{(1-\alpha)p_n^2}{2} \right] < 2$, the above equation reduces to

$$|a_{2n+1} - \lambda a_{n+1}^2| \leq |c_{2n+1} - \lambda c_{n+1}^2| + \frac{2\alpha|n+1-2\lambda|}{2n+1} + \frac{2\alpha(|n+1-\lambda| + |\lambda|)}{(n+1)(2n+1)}. \quad (24)$$

For the case $\lambda < 0$, by using Lemma 2, the above Equation (24) implies

$$|a_{2n+1} - \lambda a_{n+1}^2| \leq (1 - \lambda) + \frac{2\alpha((n+1) - 2\lambda)}{(2n+1)} + \frac{((n+1) - \lambda - \lambda)2\alpha}{(n+1)(2n+1)}$$

which brings the bound declared in (23) for $\lambda < 0$.

Again, for the case $0 \leq \lambda < \frac{2}{3}$, by using Lemma 2 in Equation (24), one can obtain

$$|a_{2n+1} - \lambda a_{n+1}^2| \leq (1 - \lambda) + \frac{2\alpha((n+1) - 2\lambda)}{(2n+1)} + \frac{((n+1) - \lambda + \lambda)2\alpha}{(n+1)(2n+1)}.$$

This gives the bound as asserted in (23) for $0 \leq \lambda < \frac{2}{3}$.

Now, for the case $\frac{2}{3} \leq \lambda < 1$, by using Lemma 2 in Equation (24), one can obtain

$$|a_{2n+1} - \lambda a_{n+1}^2| \leq 1 + \frac{2\alpha((n+1) - 2\lambda)}{(2n+1)} + \frac{((n+1) - \lambda + \lambda)2\alpha}{(n+1)(2n+1)}.$$

This gives the bound as asserted in (23) for $\frac{2}{3} \leq \lambda < 1$.

Now, for the case $1 \leq \lambda \leq \frac{4}{3}$, by using Lemma 2 in Equation (24), one can obtain

$$|a_{2n+1} - \lambda a_{n+1}^2| \leq 1 + \frac{2\alpha(2\lambda - (n+1))}{(2n+1)} + \frac{((n+1) - \lambda + \lambda)2\alpha}{(n+1)(2n+1)}.$$

This gives the bound declared in (23) for $1 \leq \lambda \leq \frac{4}{3}$.

Now, for the case $\frac{4}{3} < \lambda < 2$, by using Lemma 2 in Equation (24), one can obtain

$$|a_{2n+1} - \lambda a_{n+1}^2| \leq (\lambda - 1) + \frac{2\alpha(2\lambda - (n+1))}{(2n+1)} + \frac{((n+1) - \lambda + \lambda)2\alpha}{(n+1)(2n+1)}.$$

This gives the bound declared in (23) for $\frac{4}{3} < \lambda < 2$.

Last, for the case $\lambda \geq 2$, by using Lemma 2 in Equation (24), we obtain

$$|a_{2n+1} - \lambda a_{n+1}^2| \leq (\lambda - 1) + \frac{2\alpha(2\lambda - (n+1))}{(2n+1)} + \frac{(\lambda - (n+1) + \lambda)2\alpha}{(n+1)(2n+1)}$$

This gives the bound declared in (23) for $\lambda \geq 2$. $\square$

Theorem 2 reduces Theorem 2.2 of Sivasubramanian et al. [21] to the case of one-fold symmetry.

Corollary 3 ([21]). *If $g(z)$ is in the class $\mathcal{K}_\sigma[\alpha]$ given by (1) for $0 \leq \alpha \leq 1$, then*

$$|a_3 - \lambda a_2^2| \leq \begin{cases} (1 - \lambda) \left(1 + \frac{4}{3}\alpha + \frac{1}{3}\alpha M \right) & \text{for } \lambda < 0, \\ (1 - \lambda) \left(1 + \frac{4}{3}\alpha \right) + \frac{1}{3}\alpha M & \text{for } 0 \leq \lambda < \frac{2}{3}, \\ 1 + \frac{4}{3}\alpha(1 - \lambda) + \frac{1}{3}\alpha M & \text{for } \frac{2}{3} \leq \lambda < 1, \\ 1 + \frac{4}{3}\alpha(\lambda - 1) + \frac{1}{3}\alpha M & \text{for } 1 \leq \lambda \leq \frac{4}{3}, \\ (\lambda - 1) \left(1 + \frac{4}{3}\alpha \right) + \frac{1}{3}\alpha M & \text{for } \frac{4}{3} < \lambda < 2, \\ (\lambda - 1) \left(1 + \frac{4}{3}\alpha + \frac{1}{3}\alpha M \right) & \text{for } \lambda \geq 2 \end{cases}$$

where $M \leq 2$.

3. The Bounds for Class $\mathcal{K}_{\sigma,n}[\beta]$

Definition 3. Let $0 \leq \beta < 1$ and $h(z) \in \mathcal{A}_{\sigma,n}$, given by (3) such that $h'(z) \neq 0$ on $\mathbb{U}$. Then, $h(z)$ is said to be a bi-close-to-convex function of order β , if there exist bi-convex functions $\phi, \psi \in \mathcal{C}_\sigma$ such that

$$\operatorname{Re} \left(\frac{g'(z)}{\phi'(z)} \right) > \beta, \quad z \in \mathbb{U} \quad (25)$$

and

$$\operatorname{Re} \left(\frac{f'(w)}{\psi'(w)} \right) > \beta, \quad w \in \mathbb{U} \quad (26)$$

where $f = h^{-1}$. The term $\mathcal{K}_{\sigma,n}[\beta]$ represents the class of bi-close-to-convex functions of order β .

Theorem 3. If $h(z)$ is in $\mathcal{K}_{\sigma,n}[\beta]$ given by (3) for $0 \leq \alpha \leq 1$, then

$$|a_{n+1}| \leq \sqrt{1 + \frac{4(1 - \beta)(n + 2)}{(2n + 1)(n + 1)}}, \quad (27)$$

and

$$|a_{2n+1}| \leq 1 + \frac{2(1 - \beta)(n + 2)}{(2n + 1)}, \quad (28)$$

and

$$|a_{2n+1} - \lambda a_{n+1}^2| \leq \begin{cases} (1 - \lambda) + \frac{2(1-\beta)(n+1-2\lambda)}{2n+1} + \frac{(n+1-2\lambda)N}{(n+1)(2n+1)} & \text{for } \lambda < 0, \\ (1 - \lambda) + \frac{2(1-\beta)(n+1-2\lambda)}{2n+1} + \frac{N}{2n+1} & \text{for } 0 \leq \lambda < \frac{2}{3}, \\ 1 + \frac{2(1-\beta)(n+1-2\lambda)}{2n+1} + \frac{N}{2n+1} & \text{for } \frac{2}{3} \leq \lambda < 1, \\ 1 + \frac{2(1-\beta)(2\lambda-(n+1))}{2n+1} + \frac{N}{2n+1} & \text{for } 1 \leq \lambda \leq \frac{4}{3}, \\ (\lambda - 1) + \frac{2(1-\beta)(2\lambda-(n+1))}{2n+1} + \frac{N}{2n+1} & \text{for } \frac{4}{3} < \lambda < 2, \\ (\lambda - 1) + \frac{2(1-\beta)(2\lambda-(n+1))}{2n+1} + \frac{(2\lambda-(n+1))N}{(2n+1)(n+1)} & \text{for } \lambda \geq 2, \end{cases} \quad (29)$$

where

$$N \leq 2(1 - \beta).$$

Proof. The inequalities (25) and (26) imply

$$\frac{g'(z)}{\phi'(z)} = (1 - \beta)p(z) + \beta$$

and

$$\frac{f'(w)}{\psi'(w)} = (1 - \beta)q(w) + \beta$$

for some p and q from the class $\mathcal{P}$ involved with the expansion (14) and (15). Therefore,

$$g'(z) = \phi'(z) \left[(1 - \beta)p(z) + \beta \right], \quad f'(w) = \psi'(w) \left[(1 - \beta)q(w) + \beta \right]. \quad (30)$$

From the above two equations (30), it is easy to obtain

$$(n + 1)a_{n+1} = (1 - \beta)p_n + (n + 1)c_{n+1}, \quad (31)$$

$$(2n + 1)a_{2n+1} = (2n + 1)c_{2n+1} + (n + 1)(1 - \beta)c_{n+1}p_n + (1 - \beta)p_{2n}, \quad (32)$$

$$-(n + 1)a_{n+1} = -(n + 1)c_{n+1} + (1 - \beta)p_n, \quad (33)$$

$$\begin{aligned} & (2n + 1) \left((n + 1)a_{n+1}^2 - a_{2n+1} \right) \\ &= (2n + 1) \left((n + 1)c_{n+1}^2 - c_{2n+1} \right) - (n + 1)(1 - \beta)c_{n+1}q_n + (1 - \beta)q_{2n}. \end{aligned} \quad (34)$$

Equations (31) and (33) yield that $q_n = -p_n$. The addition of (32) and (34) implies that

$$\begin{aligned} & (2n + 1)(n + 1)a_{n+1}^2 = (2n + 1)(n + 1)c_{n+1}^2 \\ & + (n + 1)(1 - \beta)c_{n+1}(p_n - q_n) + (1 - \beta)(p_{2n} + q_{2n}). \end{aligned} \quad (35)$$

By the relation $q_n = -p_n$, $|c_k| \leq 1$ and Lemma 1, the equation reduces to

$$(n+1)(2n+1)|a_{n+1}^2| \leq (n+1)(2n+1) + 4(1-\beta)(n+1) + 4(1-\beta).$$

It brings out the desired bound for $|a_{n+1}|$, as declared in (27).

The same method as for $|a_{n+1}|$ is used in Equation (32) to obtain the bound (28). Now, by (32) and (35), one can obtain, for the real number λ ,

$$a_{2n+1} - \lambda a_{n+1}^2 = c_{2n+1} + \frac{(1-\beta)(n+1)c_{n+1}p_n}{(2n+1)} + \frac{(1-\beta)p_{2n}}{(2n+1)} \\ - \lambda \left[c_{n+1}^2 + \frac{(1-\beta)c_{n+1}(p_n - q_n)}{(2n+1)} + \frac{(1-\beta)(p_{2n} + q_{2n})}{(2n+1)(n+1)} \right].$$

Hence,

$$|a_{2n+1} - \lambda a_{n+1}^2| \leq |c_{2n+1} - \lambda c_{n+1}^2| \\ + \frac{2(1-\beta)|n+1-2\lambda|}{(2n+1)} + \frac{2(1-\beta)(|n+1-\lambda| + |\lambda|)}{(2n+1)(n+1)}. \quad (36)$$

When $\lambda < 0$, by using Lemma 2 in the above Equation (36), one can obtain

$$|a_{2n+1} - \lambda a_{n+1}^2| \leq 1 - \lambda + \frac{2(1-\beta)(n+1-2\lambda)}{(2n+1)} + \frac{2(1-\beta)[(n+1-\lambda) - \lambda]}{(2n+1)(n+1)}.$$

This brings out the bound as declared in (29) for $\lambda < 0$.

Also, for $0 \leq \lambda < \frac{2}{3}$, by using Lemma 2 in Equation (36), one can find

$$|a_{2n+1} - \lambda a_{n+1}^2| \leq 1 - \lambda + \frac{2(1-\beta)(n+1-2\lambda)}{(2n+1)} + \frac{2(1-\beta)[(n+1-\lambda) + \lambda]}{(2n+1)(n+1)}.$$

The bound is obtained as asserted in (29) for $0 \leq \lambda < \frac{2}{3}$.

Now, for the case $\frac{2}{3} \leq \lambda < 1$, by using Lemma 2 in Equation (36), one can obtain

$$|a_{2n+1} - \lambda a_{n+1}^2| \leq 1 + \frac{2(1-\beta)(n+1-2\lambda)}{(2n+1)} + \frac{2(1-\beta)[(n+1-\lambda) + \lambda]}{(2n+1)(n+1)}.$$

The bound is obtained as asserted in (29) for $\frac{2}{3} \leq \lambda < 1$.

Now, for the case $1 \leq \lambda \leq \frac{4}{3}$, by using Lemma 2 in Equation (36), one can obtain

$$|a_{2n+1} - \lambda a_{n+1}^2| \leq 1 + \frac{2(1-\beta)(2\lambda - (n+1))}{(2n+1)} + \frac{2(1-\beta)[(n+1-\lambda) + \lambda]}{(2n+1)(n+1)}.$$

The bound is obtained as asserted in (29) for $1 \leq \lambda \leq \frac{4}{3}$.

Now, for the case $\frac{4}{3} < \lambda < 2$, by using Lemma 2 in Equation (36), one can obtain

$$|a_{2n+1} - \lambda a_{n+1}^2| \leq \lambda - 1 + \frac{2(2\lambda - (n+1)(1-\beta))}{(2n+1)} + \frac{2[(n+1-\lambda) + \lambda](1-\beta)}{(n+1)(2n+1)}.$$

The bound is obtained as asserted in (29) for $\frac{4}{3} < \lambda < 2$.

Last, for the case $\lambda \geq 2$, by using Lemma 2 in Equation (36), one can obtain

$$\left| a_{2n+1} - \lambda a_{n+1}^2 \right| \leq \lambda - 1 + \frac{2(1-\beta)(2\lambda - (n+1))}{(2n+1)} + \frac{2(1-\beta)(\lambda - (n+1) + \lambda)}{(n+1)(2n+1)}.$$

The bound is obtained as asserted in (29) for $\lambda \geq 2$. $\square$

Theorem 3 reduces Theorem 3.1 of Sivasubramanian et al. [21] for one-fold symmetric functions.

Corollary 4 ([21]). *If $g(z)$ of the form (1) is in $\mathcal{K}_\sigma(\beta)$ and $0 \leq \beta < 1$, then*

$$|a_3 - \lambda a_2^2| \leq \begin{cases} (1-\lambda) \left(1 + \frac{4}{3}(1-\beta) + \frac{1}{3}N \right) & \text{for } \lambda < 0, \\ (1-\lambda) \left(1 + \frac{4}{3}(1-\beta) \right) + \frac{1}{3}N & \text{for } 0 \leq \lambda < \frac{2}{3}, \\ 1 + \frac{4}{3}(1-\lambda)(1-\beta) + \frac{1}{3}N & \text{for } \frac{2}{3} \leq \lambda < 1, \\ 1 + \frac{4}{3}(\lambda-1)(1-\beta) + \frac{1}{3}N & \text{for } 1 \leq \lambda \leq \frac{4}{3}, \\ (\lambda-1) \left(1 + \frac{4}{3}(1-\beta) \right) + \frac{1}{3}N & \text{for } \frac{4}{3} < \lambda < 2, \\ (\lambda-1) \left(1 + \frac{4}{3}(1-\beta) + \frac{1}{3}N \right) & \text{for } \lambda \geq 2, \end{cases}$$

For one-fold symmetric functions and $\frac{1}{2} \leq \beta < 1$, Theorem 3 reduces the following corollary:

Corollary 5. *If $g(z)$ given by (1) is in $\mathcal{K}_\sigma(\beta)$, then $|a_2| \leq \sqrt{2}$ for $\frac{1}{2} \leq \beta < 1$.*

4. Conclusions

This research article's motivation is to introduce various classes and n -fold symmetric bi-univalent functions. The upper bounds for the second and third Taylor–Maclaurin coefficients are obtained for functions in each subclass. Furthermore, some of the results' implications are discussed.

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