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Constrained Bayesian Method for Testing Equi-Correlation Coefficient

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Abstract: The problem of testing the equi-correlation coefficient of a standard symmetric multivariate normal distribution is considered. Constrained Bayesian and classical Bayes methods, using the maximum likelihood estimation and Stein's approach, are examined. For the investigation of the obtained theoretical results and choosing the best among them, different practical examples are analyzed. The simulation results showed that the constrained Bayesian method (CBM) using Stein's approach has the advantage of making decisions with higher reliability for testing hypotheses concerning the equi-correlation coefficient than the Bayes method. Also, the use of this approach with the probability distribution of linear combinations of chi-square random variables gives better results compared to that of using the integrated probability distributions in terms of providing both the necessary precisions as well as convenience of implementation in practice. Recommendations towards the use of the proposed methods for solving practical problems are given.

Keywords: symmetric multivariate normal distribution; hypothesis; constrained Bayesian method; Bayes method; Stein's approach

MSC: 62F15; 62F03

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1. Introduction

Symmetric multivariate normal distribution (SMND) is widely used in many applications of different spheres of human activities such as psychology, education, genetics, and so on [1]. It is also used extensively in statistical inference procedures, e.g., in analysis of variance (ANOVA) for the modeling of the error part [2]. A random vector has an SMND if its components have equal means, equal variances, and equal correlation coefficients between the pairs of the components. The last is called the equi-correlation coefficient [1,3–6]. A vector has a standard symmetric multivariate normal distribution (SSMND) when the components have zero mean and unit variances. The consideration of SSMND instead of SMND does not reduce generality when the means and variances are known. On the other hand, SSMND is interesting from several theoretical aspects [1] (p. 2): it is an invariant model which belongs to a curved exponential family [6,7], allows us to find a simple estimator of the correlation coefficient [8,9], and is amenable to the derivation of a small-sample optimal test for the correlation coefficient [1].

Extensive research on inference for the correlation coefficient in SMND exists, starting from the early 1940s [10] up to recent years [11–14]. In the majority of the works, the problem of finding estimators of the correlation coefficient was considered (see, e.g., [14–18]), compared to the testing problem [1,3–5,10]. The likelihood ratio test (LRT) for the general case of testing $H_0 : \rho = \rho_0$ against $H_2 : \rho \neq \rho_0$ and the locally most powerful test (LMPT)

for testing $H_0 : \rho = 0$ against $H_1 : \rho > (<)0$ are presented in [1]. The LMPT is based on the best (minimum variance) natural unbiased estimator (BNUE) of ρ. The beta-optimal test and the power envelope for testing $H_0 : \rho = 0$ vs. $H_1 : \rho > 0$ is considered in [5]. The relative performances of these tests are compared with a locally best test proposed in [1]. The analogous test of [1] for the symmetric multivariate normal distribution is constructed in [10] where it is shown that the proposed test is uniformly most powerful (invariant) even in the presence of a nuisance parameter, σ^2.

In the present work, we for the first time consider the CBM for testing hypotheses $H_0 : \rho = \rho_0$ vs. $H_2 : \rho \neq \rho_0$ concerning the equi-correlation coefficient using the exact and asymptotical probability density functions (pdfs) of test statistics obtained in [1,18,19]. It allows us to make a decision with the restricted criteria of optimality on the desired levels, i.e., permits us to specify both the significance level and the power of the test, which is not achieved by the testing rules given in [1,3–5,10]. Along with the theoretical results, the results of the simulation are presented for the demonstration of the validity of the obtained results and the investigation of their properties.

The layout of the rest of this paper is as follows: A statement of the problem is given in Section 2. Approaches for handling directional hypotheses tests concerning the equi-correlation coefficient based on both the maximum ratio test and on Stein's approach using CBM methodology are developed in Section 3. The general statement of the CBM of testing hypotheses is offered in Section 4, and the application of the CBM's two possible statements to testing formulated hypotheses is given in Sections 5 and 6. Statistical computations under concrete examples are presented in Section 7 followed by a discussion and brief conclusions included in Sections 8 and 9.

2. The Problem Under Consideration

Consider first the problem of the testing of hypotheses concerning a specified value of the correlation coefficient of SMND.

Let the k-dimensional random vector X follow the k-variate normal distribution with zero mean vector and a correlation matrix W of the following structure:

$$W = (1-\rho)\cdot I_{k\times k} + \rho\cdot J_{k\times k}, \tag{1}$$

where $I_{k\times k}$ is an identity matrix and $J_{k\times k}$ is a matrix of ones.

It is known [1] that

$$W^{-1} = (c_{ij})_{k\times k},$$

where

$$c_{ii} = \{1+(k-2)\rho\}/\{(1-\rho)[1+(k-1)\rho]\},$$
$$c_{ij} = -\rho/\{(1-\rho)[1+(k-1)\rho]\}, i \neq j,$$

and the density function of X for non-singular W is the following:

$$p(X;\rho) = \frac{1}{(2\pi)^{k/2}|W|^{1/2}} \exp\left\{-\frac{1}{2}\left[\frac{\left(\sum_{i=1}^{k} x_i^2\right)}{(1-\rho)} + \frac{\left(\sum_{i=1}^{k} x_i\right)^2(-\rho)}{(1+(k-1)\rho)(1-\rho)}\right]\right\}, \tag{2}$$

$-\infty < x_i < +\infty, i = 1,\ldots,k; -1/(k-1) < \rho < 1$.

The support of ρ, $(-1/(k-1), 1)$ assures the non-singularity of the correlation matrix W [4].

Let us introduce the Helmert orthogonal transformation $Y_1, Y_2, \ldots$ on the uncorrelated multivariate normal vectors sequence $X_1, X_2, \ldots$, i.e., $Y_i = H\cdot X_i$, $i = 1, 2, \ldots$, and H is the Helmert matrix given as [2]

$$H = \begin{bmatrix} \frac{1}{\sqrt{k}} & \frac{1}{\sqrt{k}} & \frac{1}{\sqrt{k}} & & \dots & \frac{1}{\sqrt{k}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & & \dots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{1}{\sqrt{k(k-1)}} & \dots & \frac{1}{\sqrt{k(k-1)}} & & -\frac{(k-1)}{\sqrt{k(k-1)}} & \end{bmatrix}. \tag{3}$$

Then, each of $Y_1, Y_2, \dots$ are i.i.d. k-dimensional random vectors with zero means and a diagonal correlation matrix

$$V = HWH^T = diag(v_1, \dots, v_k), \tag{4}$$

where $v_1, \dots, v_k$ are the eigenvalues of the correlation matrix W, i.e., $Y_i \sim N_k(0, V)$ [4]. It is known that [6]

$$v_1 = 1 + (k-1)\rho \text{ and } v_2 = \dots = v_k = 1 - \rho. \tag{5}$$

Let $Y_1, \dots, Y_n$ be i.i.d. observations, i.e., a random sample from $N_k(0, V)$, and introduce variables

$$V_{1n} = \sum_{i=1}^{n} y_{i1}^2 \text{ and } V_{2n} = \sum_{i=1}^{n} \sum_{j=2}^{k} y_{ij}^2, \tag{6}$$

where $Y_i = (y_{i1}, y_{i2}, \dots, y_{ik})$, $i = 1, \dots, n$.

The maximum likelihood estimation (MLE) of the correlation coefficient ρ is [2,19]

$$\hat{\rho}_n = \frac{V_{1n} - V_{2n}(k-1)^{-1}}{V_{1n} + V_{2n}}. \tag{7}$$

The problem we want to solve can be formulated as follows: to test

$$H_0 : \rho = \rho_0, \text{ vs. } H_1 : \rho < (>)\rho_0, \tag{8a}$$

or

$$H_0 : \rho = \rho_0 \text{ vs. } H_2 : \rho \neq \rho_0, \left(2.8^2\right) \tag{8b}$$

on the basis of the sample $Y_1, \dots, Y_n$. Here, Y_i is a k-dimensional random vector with independent components each of which has zero mean and variances determined by (5).

The pdf of Y_i is

$$p(Y_i|\rho) = (2\pi)^{-\frac{k}{2}} |V|^{-\frac{1}{2}} \cdot \exp\left\{ -\frac{1}{2} Y_i \cdot V^{-1} \cdot Y_i^T \right\}, \; i = 1, \dots, n. \tag{9}$$

The joint pdf of the sample $Y_1, \dots, Y_n$ has the following form [1]:

$$\begin{aligned} p(Y_1, \dots, Y_n|\rho) &= (2\pi)^{-kn/2} \cdot |V|^{-\frac{n}{2}} \cdot \exp\left\{ -\frac{1}{2} \sum_{i=1}^{n} Y_i \cdot V^{-1} \cdot Y_i^T \right\} = \\ &= (2\pi)^{-kn/2} (1 + (k-1)\rho)^{-\frac{n}{2}} (1-\rho)^{-\frac{n(k-1)}{2}} \cdot \exp\left\{ -\frac{1}{2} \left(\frac{V_{1n}}{1 + (k-1)\rho} + \frac{V_{2n}}{1-\rho} \right) \right\}, \end{aligned} \tag{10}$$

where V_{1n} and V_{2n} are defined by (6).

3. Testing (2.8) Hypotheses

Let us transform hypotheses (8) into directional ones, i.e., instead of (8), consider the following hypotheses:

$$H_0 : \rho = \rho_0 \text{ vs. } H_- : -1/(k-1) < \rho < \rho_0 \text{ or } H_+ : \rho_0 < \rho < 1, \tag{11}$$

$\rho, \rho_0 \in (-1/(k-1), 1)$.

3.1. The Test Using Maximum Ratio Estimation of the Parameter

Consider testing the hypothesis

$$H_0 : \rho = \rho_0 \text{ vs. } H_A : \rho = \hat{\rho}_n,\ \rho_0 \neq \hat{\rho}_n, \tag{12}$$

where $\hat{\rho}_n$ is defined by (7).

We use the sample $Y^m = (Y_{n+1}, \ldots, Y_{n+m})$ and the pdf (10) with the parameters ρ_0 and $\hat{\rho}_n$ under the hypotheses H_0 and H_A, respectively, i.e., pdfs at null and alternative hypotheses are determined by (10) using the values of ρ_0 and $\hat{\rho}_n$ for null and alternative specifications, respectively.

The algorithm for making decisions is the following. Let us designate $Y_1, \ldots, Y_n, Y_{n+1}, \ldots, Y_{n+m}$ i.i.d. random vectors obtained by the Helmert orthogonal transformation of the sample $X_1, \ldots, X_n, X_{n+1}, \ldots, X_{n+m}$, i.e., $Y_i = H \cdot X_i$, $i = 1, 2, \ldots, n+m$. On the basis of the first half of the vectors $Y_1, \ldots, Y_n$, we compute the MLE of the parameter ρ by (7), and on the basis of the second half of the sample $Y_{n+1}, \ldots, Y_{n+m}$, we test the hypotheses (12). The pdfs used for testing are the following: $p(Y_{n+1}, \ldots, Y_{n+m}|H_0) = p(Y^m|\rho_0)$ and $p(Y_{n+1}, \ldots, Y_{n+m}|H_A) = p(Y^m|\hat{\rho}_A)$ determined by (10). When testing (12), the decision is made in favor of the alternative hypothesis, i.e., we accept H_- if $\hat{\rho}_n < \rho_0$, otherwise H_+.

3.2. Stein's Approach

Stein's method [20,21] integrates the density over ρ using special measures to obtain the density of the maximum invariant statistic, which can then be used to analyze the problems, for example, to find the uniformly most powerful invariant test [22].

The integrated conditional pdfs of the sample $Y^m = (Y_{n+1}, \ldots, Y_{n+m})$ are necessary in this case for the hypotheses under consideration, i.e., the following pdfs:

$$H_0 : p(Y^m|H_0) = (2\pi)^{-\frac{mk}{2}} (1 + (k-1)\rho_0)^{-\frac{m}{2}} (1-\rho_0)^{-\frac{m(k-1)}{2}} \cdot \exp\left\{ -\frac{1}{2} \left(\frac{V_{1m}}{1 + (k-1)\rho_0} + \frac{V_{2m}}{1-\rho_0} \right) \right\}, \tag{13}$$

$$H_- : p(Y^m|H_-) = \int_{-1/(k-1)}^{\rho_0} p(Y^m|\rho) \cdot \gamma_-(\rho) d\rho, \tag{14}$$

$$H_+ : p(Y^m|H_+) = \int_{\rho_0}^{1} p(Y^m|\rho) \cdot \gamma_+(\rho) d\rho, \tag{15}$$

where $p(Y^m|\rho)$ is defined by (10) for the sample $Y^m = (Y_{n+1}, \ldots, Y_{n+m})$; $\gamma_-(\rho)$ and $\gamma_+(\rho)$ are the densities of the parameter ρ at the alternative specifications H_- and H_+, respectively.

Because at alternative hypotheses H_- and H_+ the value of the parameter ρ belongs to the intervals $(-1/(k-1), \rho_0)$ and $(\rho_0, 1)$, respectively, and no information is available about the preference of some values from these intervals, we use the uniform distributions of ρ in the appropriate intervals, i.e., (14) and (15) pdfs transform in the following forms:

$$\begin{aligned} H_- : p(Y^m|H_-) &= (2\pi)^{-mk/2} \cdot \frac{1}{\rho_0 + (k-1)^{-1}} \cdot \\ \cdot \int_{-1/(k-1)}^{\rho_0} (1 + (k-1)\rho)^{-\frac{m}{2}} (1-\rho)^{-\frac{m(k-1)}{2}} &\cdot \exp\left\{ -\frac{1}{2} \left(\frac{V_{1m}}{1 + (k-1)\rho} + \frac{V_{2m}}{1-\rho} \right) \right\} d\rho, \end{aligned} \tag{16}$$

$$\begin{aligned} H_+ : p(Y^m|H_+) &= (2\pi)^{-mk/2} \cdot \frac{1}{1-\rho_0} \cdot \\ \cdot \int_{\rho_0}^{1} (1 + (k-1)\rho)^{-\frac{m}{2}} (1-\rho)^{-\frac{m(k-1)}{2}} &\cdot \exp\left\{ -\frac{1}{2} \left(\frac{V_{1m}}{1 + (k-1)\rho} + \frac{V_{2m}}{1-\rho} \right) \right\} d\rho, \end{aligned} \tag{17}$$

where V_{1m} and V_{2m} are computed by (6) for the sample $Y^m = (Y_{n+1}, \ldots, Y_{n+m})$.

For testing hypotheses (11) and (12), using pdfs (10) and (13), (16), and (17), respectively, let us use the CBM—a brief introduction to it is given below [22,23].

4. Constrained Bayesian Method of Testing Hypotheses

Let $x^T = (x_1, \ldots, x_n)$ be generated from $p(x;\theta)$, and the problem of interest is to test $H_i : \theta_i \in \Theta_i$, $i = 1,2,\ldots,S$, where $\Theta_i \subset R^m$, $i = 1,2,\ldots,S$, are disjoint subsets with $\cup\Theta_i = R^m$. The number of tested hypotheses is S. Let the prior on θ be denoted by $\sum_{i=1}^{S} \pi(\theta|H_i)p(H_i)$, where for each $i = 1,2,\ldots,S$, $p(H_i)$ is the a priori probability of hypothesis H_i, and $\pi(\theta|H_i)$ is a prior density with support Θ_i; $p(x|H_i)$ denotes the marginal density of x given H_i, i.e., $p(x|H_i) = \int_{\Theta_i} p(x|\theta)\pi(\theta|H_i)d\theta$, and $D = \{d\}$ is the set of solutions, where $d = \{d_1, \ldots, d_S\}$, it being so that

$$d_i = \begin{cases} 1, \ if \ hypotheis \ H_i \ is \ accepted, \\ 0, \ otherwise, \end{cases}$$

$\delta(x) = \{\delta_1(x), \delta_2(x), \ldots, \delta_S(x)\}$ is the decision function that associates each observation vector x with a certain decision:

$$x \overset{\delta(x)}{\to} d \in D;$$

(notation: depending upon the choice of x, there is a possibility that $\delta_j(x) = 1$ for more than one j or $\delta_j(x) = 0$ for all $j = 1, \ldots, S$).

Γ_j is the region of acceptance of the hypothesis H_j, i.e., $\Gamma_j = \{x : \delta_j(x) = 1\}$. It is obvious that $\delta(x)$ is completely determined by the Γ_j regions, i.e., $\delta(x) = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_S\}$.

Let us introduce the loss function $L(H_i, \delta(x))$ which determines the value of the loss in the case where the sample has a probability distribution corresponding to hypothesis H_i, but the decision $\delta(x)$ is made wrongly. In the general case, the loss function $L(H_i, \delta(x))$ consists of two components:

$$L(H_i, \delta(x)) = \sum_{j=1}^{S} L_1(H_i, \delta_j(x) = 1) + \sum_{j=1}^{S} L_2(H_i, \delta_j(x) = 0), \tag{18}$$

where $L_1(H_i, \delta_j(x) = 1)$ is the loss for the incorrect acceptance of H_i when H_j is true and $L_2(H_i, \delta_j(x) = 0)$ is the incorrect rejection of H_i in favor of H_j.

It is possible to formulate nine different statements of the CBM depending on what type of restriction is desired, determined by the aim of the specific problem to be solved [22,23].

For concreteness, let us introduce one of the possible statements, namely Task 1, as an example, for a demonstration of the specificity of the CBM. In this case, we have to minimize the averaged loss of the incorrectly accepted hypotheses

$$r_\delta = \min_{\{\Gamma_j\}} \left\{ \sum_{i=1}^{S} p(H_i) \sum_{j=1}^{S} \int_{\Gamma_j} L_1(H_i, \delta_j(x) = 1) p(x|H_i) dx \right\}, \tag{19}$$

subject to the averaged loss of the incorrectly rejected hypotheses

$$\begin{gathered} \sum_{i=1}^{S} p(H_i) \sum_{j=1}^{S} \int_{R^n - \Gamma_j} L_2(H_i, \delta_j(x) = 0) p(x|H_i) dx = \\ = \sum_{i=1}^{S} p(H_i) \sum_{j=1}^{S} \int_{R^n} L_2(H_i, \delta_j(x) = 0) p(x|H_i) dx - \end{gathered}$$

$$- \sum_{i=1}^{S} p(H_i) \sum_{j=1}^{S} \int_{\Gamma_j} L_2(H_i, \delta_j(x) = 0) p(x|H_i) dx \le r_1, \tag{20}$$

where r_1 is some real number determining the level of the averaged loss of the incorrectly rejected hypotheses.

By solving problem (19) and (20), we have

$$\Gamma_j = \left\{ x : \sum_{i=1}^{S} L_1(H_i, \delta_j(x) = 1) p(H_i) p(x|H_i) < \lambda \sum_{i=1}^{S} L_2(H_i, \delta_j(x) = 0) p(H_i) p(x|H_i) \right\},$$

$$j = 1, \ldots, S, \tag{21}$$

where the Lagrange multiplier λ ($\lambda > 0$) is defined so that the equality holds in (20).

The acceptance regions (21) of the hypotheses differ from the classical cases where the observation space is divided into two complementary sub-spaces for acceptance and rejection regions. Here, the observation space contains the regions where decisions can be made as well as the regions where they cannot be made. This also gives us the opportunity to develop the corresponding sequential tests [22–24].

5. CBM for Testing Hypotheses in (3.1)

Let us consider the CBM for testing the hypotheses (11) with a stepwise loss function.

One of possible statements of the CBM, namely, the so-called Task 7, has the following form [22–24]:

$$G_\delta = \max_{\{\Gamma_-, \Gamma_0, \Gamma_+\}} \{K_0 \cdot [p(H_-) \cdot P(Y^m \in \Gamma_- | H_-) + p(H_0) \cdot P(Y^m \in \Gamma_0 | H_0) + \\ + p(H_+) \cdot P(Y^m \in \Gamma_+ | H_+)]\}, \tag{22}$$

subject to

$$\begin{aligned} &K_1 \cdot [p(H_0) \cdot P(Y^m \in \Gamma_- | H_0) + p(H_+) \cdot P(Y^m \in \Gamma_- | H_+)] \le r_7^-, \\ &K_1 \cdot [p(H_-) \cdot P(Y^m \in \Gamma_0 | H_-) + p(H_+) \cdot P(Y^m \in \Gamma_0 | H_+)] \le r_7^0, \\ &K_1 \cdot [p(H_-) \cdot P(Y^m \in \Gamma_+ | H_-) + p(H_0) \cdot P(Y^m \in \Gamma_+ | H_0)] \le r_7^+, \end{aligned} \tag{23}$$

where $p(H_0)$, $p(H_-)$, and $p(H_+)$ are a priori probabilities of the appropriate hypotheses, $P(Y^m \in \Gamma_i | H_j)$, $i, j \in (-, 0, +)$, is the probability of the acceptance of the H_i hypothesis when the H_j hypothesis is true, on the basis of Y^m, and K_1 and K_0 define the loss functions of incorrectly accepting and incorrectly rejecting the hypotheses given by

$$L_1(H_i, \delta_j(Y^m) = 1) = \begin{cases} 0 \; at \; i = j, \\ K_1 \; at \; i \ne j; \end{cases} \quad \text{and } L_2(H_i, \delta_j(Y^m) = 0) = \begin{cases} K_0 \; at \; i = j, \\ 0 \; at \; i \ne j; \end{cases} \tag{24}$$

$\delta(Y^m) = \{\delta_-(Y^m), \delta_0(Y^m), \delta_+(Y^m)\}$ is a decision function, $\Gamma_j = \{Y^m : \delta_j(Y^m) = 1\}$, $j \in (-, 0, +)$, is the acceptance region of the hypothesis H_j, and r_7^-, r_7^0, and r_7^+ are the specified levels of the averaged losses of the incorrect acceptances of the hypotheses H_-, H_0, and H_+, respectively.

The solution of the problem (22) and (23) using undetermined Lagrange multipliers yields the acceptance regions for the hypotheses as follows:

$$\begin{aligned} \Gamma_- &= \left\{ Y^m : K_1 \cdot (p(H_0 | Y^m) + p(H_+ | Y^m)) < \frac{1}{\lambda_7^-} \cdot K_0 \cdot p(H_- | Y^m) \right\}, \\ \Gamma_0 &= \left\{ Y^m : K_1 \cdot (p(H_- | Y^m) + p(H_+ | Y^m)) < \frac{1}{\lambda_7^0} \cdot K_0 \cdot p(H_0 | Y^m) \right\}, \\ \Gamma_+ &= \left\{ Y^m : K_1 \cdot (p(H_- | Y^m) + p(H_0 | Y^m)) < \frac{1}{\lambda_7^+} \cdot K_0 \cdot p(H_+ | Y^m) \right\}, \end{aligned} \tag{25}$$

where the Lagrange multipliers λ_7^-, λ_7^0, and λ_7^+ are determined so that the equalities hold in the conditions (23).

In accordance with the theorems proven in [24], when the circumstances $\frac{\lambda_7^- + \lambda_7^+}{K_1 \cdot P_{\min}} = q$, $\frac{\lambda_7^- + \lambda_7^0 + \lambda_7^+}{K_1 \cdot P_{\min}} = q$ ($0 < q < 1$) are satisfied, where $P_{\min} = \min\{p(H_-), p(H_0), p(H_+)\}$, the conditions $mdFDR = SERR_{III} \le q$ or $mdFDR + FAR \le q$ are fulfilled. Here, the following criteria of optimality of arriving at the decisions are used: the mixed directional false

discovery rate ($mdFDR$), the summary type III error rate ($SERR_{III}$), and the false acceptance rate (FAR). They are determined by the following formulae:

$$mdFDR = P(Y^m \in \Gamma_-|H_+) + P(Y^m \in \Gamma_-|H_0) + P(Y^m \in \Gamma_+|H_-) + P(Y^m \in \Gamma_+|H_0), \quad (26)$$

$$FAR = P(Y^m \in \Gamma_0|H_-) + P(Y^m \in \Gamma_0|H_+). \quad (27)$$

Another possible statement of the CBM, namely, the so-called Task 2, has the following form [22–24]:

$$\begin{aligned} R_\delta = \min_{\{\Gamma_-,\Gamma_0,\Gamma_+\}} \{ & p(H_-)\cdot K_1\cdot[P(Y^m \in \Gamma_0|H_-) + P(Y^m \in \Gamma_+|H_-)] + \\ & + p(H_0)\cdot K_1\cdot[P(Y^m \in \Gamma_-|H_0) + P(Y^m \in \Gamma_+|H_0)] + \\ & + p(H_+)\cdot K_1\cdot[P(Y^m \in \Gamma_-|H_+) + P(Y^m \in \Gamma_0|H_+)]\}, \end{aligned} \quad (28)$$

subject to

$$P(Y^m \in \Gamma_-|H_-) \geq 1 - \frac{r_2^-}{p(H_-)\cdot K_0},\ P(Y^m \in \Gamma_0|H_0) \geq 1 - \frac{r_2^0}{p(H_0)\cdot K_0},$$

$$P(Y^m \in \Gamma_+|H_+) \geq 1 - \frac{r_2^+}{p(H_+)\cdot K_0}, \quad (29)$$

where r_2^-, r_2^0, and r_2^+ are specification levels in the considered statement.

The solution of the problem of (28) and (29) by the Lagrange method yields the following acceptance regions for the hypotheses:

$$\begin{aligned} \Gamma_- &= \{Y^m : K_1\cdot(p(H_0|Y^m) + p(H_+|Y^m)) < \lambda_2^-\cdot K_0\cdot p(H_-|Y^m)\}, \\ \Gamma_0 &= \left\{Y^m : K_1\cdot(p(H_-|Y^m) + p(H_+|Y^m)) < \lambda_2^0\cdot K_0\cdot p(H_0|Y^m)\right\}, \\ \Gamma_+ &= \{Y^m : K_1\cdot(p(H_-|Y^m) + p(H_0|Y^m)) < \lambda_2^+\cdot K_0\cdot p(H_+|Y^m)\}, \end{aligned} \quad (30)$$

where the Lagrange multipliers λ_2^-, λ_2^0, and λ_2^+ are determined so that the equalities hold in the conditions (29).

Theorems are proven in [24] in accordance with which (30) decision regions ensure specified error rates of type I and type II, determined by the following ratios:

$$\alpha \leq \frac{r_2^0}{K_0\cdot p(H_0)},\ \beta \leq \frac{r_2^-}{K_0\cdot p(H_-)} + \frac{r_2^+}{K_0\cdot p(H_+)}. \quad (31)$$

They also ensure the specification of the averaged loss of incorrectly accepted hypotheses R_δ and the averaged loss of incorrectly rejected hypotheses α_δ by the following ratios:

$$R_\delta \leq \frac{K_1}{K_0}\cdot\left[r_2^- + r_2^0 + r_2^+\right],\ \alpha_\delta \leq r_2^- + r_2^0 + r_2^+, \quad (32)$$

where

$$\begin{aligned} \alpha_\delta = {} & p(H_-)\cdot K_0\cdot(1 - P(Y^m \in \Gamma_-|H_-)) + p(H_0)\cdot K_0\cdot(1 - P(Y^m \in \Gamma_0|H_0)) + \\ & + p(H_+)\cdot K_0\cdot(1 - P(Y^m \in \Gamma_+|H_+)). \end{aligned}$$

6. Evolution of CBM 2 for Testing (3.1) Hypotheses

6.1. Using the Maximum Ratio Estimation

Like (30), the hypotheses acceptance regions, testing (12) hypotheses using the CBM 2, have the following forms:

$$\Gamma_0 = \left\{ Y^m : K_1 \cdot p(H_A) \cdot P(Y^m|H_A) < \lambda_2^0 \cdot K_0 \cdot p(H_0) \cdot P(Y^m|H_0) \right\},$$

$$\Gamma_A = \left\{ Y^m : K_1 \cdot p(H_0) \cdot P(Y^m|H_0) < \lambda_2^A \cdot K_0 \cdot p(H_A) \cdot P(Y^m|H_A) \right\}, \tag{33}$$

where the Lagrange multipliers λ_2^0 and λ_2^A are determined from the following conditions [23]:

$$K_0 \cdot p(H_0) \cdot (1 - P(Y^m \in \Gamma_0|H_0)) = r_2^0,$$
$$K_0 \cdot p(H_A) \cdot (1 - P(Y^m \in \Gamma_A|H_A)) = r_2^A, \tag{34}$$

where r_2^0 and r_2^A are the specified levels under H_0 and H_A, respectively.

For the conditional pdfs of Y^m, we have (13) under H_0 and the following under H_A:

$$H_A : P(Y^m|H_A) = (2\pi)^{-\frac{mk}{2}} (1 + (k-1)\hat{\rho}_n)^{-\frac{m}{2}} (1 - \hat{\rho}_n)^{-\frac{m(k-1)}{2}} \cdot \exp\left\{ -\frac{1}{2} \left(\frac{V_{1m}}{1 + (k-1)\hat{\rho}_n} + \frac{V_{2m}}{1 - \hat{\rho}_n} \right) \right\}. \tag{35}$$

Let us determine the Lagrange multipliers λ_2^0 and λ_2^A in (33) of (34). The acceptance regions (33) of the hypotheses under the conditional densities (13) and (35), after simple transformations, take the forms

$$\Gamma_0 = \left\{ Y^m : \left(d_1^0 \cdot V_{1m} + d_2^0 \cdot V_{2m} \right) < 2\left(g_2^0 - g_1^0 \right) \right\}, \tag{36}$$

where

$$d_1^0 = \frac{1}{1 + (k-1)\rho_0} - \frac{1}{1 + (k-1)\hat{\rho}_n},$$
$$d_2^0 = \frac{1}{1 - \rho_0} - \frac{1}{1 - \hat{\rho}_n},$$
$$g_1^0 = \frac{m}{2} \cdot \ln\left(\frac{1 + (k-1)\rho_0}{1 + (k-1)\hat{\rho}_n} \right) + \frac{m(k-1)}{2} \cdot \ln\left(\frac{1 - \rho_0}{1 - \hat{\rho}_n} \right),$$
$$g_2^0 = \ln\left(\lambda_2^0 \cdot \frac{p(H_0) \cdot K_0}{p(H_A) \cdot K_1} \right). \tag{37}$$

$$\Gamma_A = \left\{ Y^m : \left(d_1^A \cdot V_{1m} + d_2^A \cdot V_{2m} \right) < 2\left(g_2^A - g_1^A \right) \right\}, \tag{38}$$

where

$$d_1^A = -d_1^0,\ d_2^A = -d_2^0,\ g_1^A = -g_1^0,\ g_2^A = \ln\left(\lambda_2^A \cdot \frac{p(H_A) \cdot K_0}{p(H_0) \cdot K_1} \right). \tag{39}$$

For the determination of the Lagrange multipliers λ_2^0 and λ_2^A from the conditions (34), the computation of the probabilities $P(Y^m \in \Gamma_0|H_0)$ and $P(Y^m \in \Gamma_A|H_A)$ is essential. For this purpose, knowledge of the pdfs of the random variables

$$\xi_0 = d_1^0 \cdot V_{1m} + d_2^0 \cdot V_{2m},$$
$$\xi_A = d_1^A \cdot V_{1m} + d_2^A \cdot V_{2m}, \tag{40}$$

is necessary.

The properties of V_{1m} and V_{2m} are given in [1,4]:

$$\text{(i)}\ V_{1m} \sim (1 + (k-1)\rho)\chi_m^2;$$
$$\text{(ii)}\ V_{2m} \sim (1 - \rho)\chi_{m(k-1)}^2; \tag{41}$$

(iii) V_{1m} and V_{2m}areindependent.

The distribution function of a linear combination of chi-square random variables is considered in many works [24–27]. Below, we use the results of [26], in accordance with which the distribution function $F^l(z,\rho)$ of ξ_l, $l \in (0, A)$, from (40) is

$$F^l(z,\rho) = b_2^l \sum_{j=0}^{\infty} a_j^l \int_0^z f_j^l(y,\rho)dy, \tag{42}$$

where the formulae for the determination of the coefficients b_2^l, a_j^l, $j = 0, 1, \ldots$, and $f_j^l(y,\rho)$ density are given in Appendix A.

Using (42), conditions (34) will be written as follows:

$$F^0(2\cdot(g_2^0 - g_1^0), \rho_0) = 1 - \frac{r_2^0}{p(H_0)\cdot K_0},$$

$$F^A(2\cdot(g_2^A - g_1^A), \hat{\rho}_n) = 1 - \frac{r_2^A}{p(H_A)\cdot K_0}, \tag{43}$$

We solve the above with respect to the Lagrange multipliers λ_2^0 and λ_2^A. Then, using the sample Y^m, we can make the decision depending on which of the conditions (33) is satisfied.

Note 1. Because of the specificity on the acceptance regions of the hypotheses, in the CBM, they can be intersected or their union may not lead to the entire observation space. Thus, the situation can arise such that none of the (12) hypotheses are accepted on the basis of the sample $Y^m = (Y_{n+1}, \ldots, Y_{n+m})$. In this case, we enhance the sequential approach, i.e., we increase the sample size by one and test hypotheses (12). If a unique decision is not made on the basis of the sample $Y^{m+1} = (Y_{n+1}, \ldots, Y_{n+m}, Y_{n+m+1})$, we again increase its size by one, i.e., we obtain the sample $Y^{m+2} = (Y_{n+1}, \ldots, Y_{n+m}, Y_{n+m+1}, Y_{n+m+2})$ and so on until a unique decision is made.

Note 2. The decision-making procedure can be purely sequential when we start testing (12) hypotheses for $m = 1$. Otherwise, it is combined: after the original testing approach, if on the basis of m observations a simple decision is not made, we adopt the sequential approach and proceed as above until a decision is made.

Notes 1 and 2 concern the Stein's approach too, which is described below.

6.2. Using the Stein's Approach

We generate the sample vectors $Y_{i,l} \sim N_k(0, V)$, $i = 1, \ldots, n$, $l \in (-, 0, +)$, where $N_k(0, V)$ is a k-variate normal distribution. As a result, we have $Y_{i,l} = (y_{i1}^l, \ldots, y_{ik}^l)$, $i = 1, \ldots, n$, where the components of the vectors $Y_{i,l}$ are independent and are given by the ratios

$$y_{i1}^l \sim N(0, 1 + (k-1)\rho_l), \; y_{ij}^l \sim N(0, 1 - \rho_l), \; j = 2, \ldots, k. \tag{44}$$

Here, $N(0,\theta)$ is a one-dimensional normal distribution with zero mean and a variance equal to θ. Depending on which of equation from (5.8) we solve, ρ_l takes values

$$\rho_l \in (-1/(k-1), \rho_0), \; \rho_l = \rho_0, \;\; \text{or } \rho_l \in (\rho_0, 1), \tag{45}$$

under H_-, H_0, or H_+ hypotheses, respectively.

Let us suppose that for the vectors $Y_{i,l}$, $i = 1, \ldots, n$, condition $Y_{i,l} \in \Gamma_l$ is fulfilled n_l times; then, we have

$$P(Y_l \in \Gamma_l | H_l) \approx \frac{n_l}{n}, \tag{46}$$

where $Y_l = (y_1^l, \ldots, y_k^l)$ and Γ_l, $l \in (-, 0, +)$, are defined by (30).

By changing λ_2^l in Γ_l, we achieve the fulfilment of the condition

$$\left| P(Y_l \in \Gamma_l | H_l) - \left(1 - \frac{r_2^l}{p(H_l) \cdot K_0} \right) \right| \leq \varepsilon, \tag{47}$$

where ε is the desired accuracy of the solution of the Equation (29).

For the next generated value $Y_{n+1,l}$ and already determined λ_2^l, $l \in (-,0,+)$, we test the conditions (30) using distribution densities

$$p(Y_{n+1,l}|H_t) = (2\pi)^{-\frac{k}{2}} \cdot (1+(k-1)\rho_t)^{-\frac{1}{2}} \cdot (1-\rho_t)^{-\frac{k-1}{2}} \cdot \exp\left\{ \frac{1}{2} \left(\frac{(y_{n+1,1}^l)^2}{(1+(k-1)\rho_t)} + \sum_{j=2}^{k} \frac{(y_{n+1,j}^l)^2}{(1-\rho_t)} \right) \right\},$$
$$l, t \in (-,0,+).$$

That particular hypothesis is accepted which belongs to the acceptance region $Y_{n+1,l}$ corresponding to it. If the accepted hypothesis is H_l, a decision is correct; otherwise, the decision is incorrect.

If none of the hypotheses are accepted, a new random vector $Y_{n+2,l}$ is generated. Then, the conditions of (30) are tested using (10) (in which $n = 2$) until one of the tested hypotheses is not accepted.

Note 3. The fact that $\hat{\rho}_n$ is asymptotically normally distributed when $n \to \infty$ is shown in [19], i.e.,

$$\sqrt{n}(\hat{\rho}_n - \rho) \to N(0, \sigma_\rho^2), \text{ where } \sigma_\rho^2 = \frac{2(1-\rho)^2(1+(k-1)\rho)^2}{k(k-1)}. \tag{48}$$

On the basis of this fact, for a quite big n, we can test the directional hypotheses (11) using the CBM for the following conditional distributions, instead of (13), (16), and (17):

$$H_0 : p(z|H_0) = N(z; (\hat{\rho}_n - \rho_0), \sigma_{\rho_0}^2), \tag{49}$$

$$H_- : p(z|H_-) = \frac{1}{\rho_0 + 1} \int_{-1}^{\rho_0} N(z; (\hat{\rho}_n - \rho), \sigma_\rho^2) d\rho, \tag{50}$$

$$H_+ : p(z|H_+) = \frac{1}{1 - \rho_0} \int_{\rho_0}^{1} N(z; (\hat{\rho}_n - \rho), \sigma_\rho^2) d\rho. \tag{51}$$

For testing hypotheses (12) using the maximum ratio test, we have to use the following densities:

$$H_0 : p(z|H_0) = N(z; (\hat{\rho}_n - \rho_0), \sigma_{\rho_0}^2),$$
$$H_A : p(z|H_A) = N(z; 0, \sigma_{\hat{\rho}_n}^2). \tag{52}$$

Here, $N(z; a, \sigma^2)$ is a normal distribution with a mathematical expectation a and variance σ^2.

7. Computation Results

The CBM 2 for both the maximum ratio test and for Stein's approach with the uniform distribution of the densities of the parameter ρ at the alternative specifications H_- and H_+ was obtained through MATLAB R2021b codes, both for making computations and the investigation of the theoretical results, which are given above. The CBM with the maximum likelihood estimation using both densities (13) and (35) or densities (52) gives very unreliable results due to the informational closeness of the null and alternative hypotheses. Therefore, the computational results for only the Stein's approach for both the cases when the densities are given by formulae (13), (16), and (17) as well as when they are given by formulae (49), (50), and (51) are given below. The Bayes method when the distribution densities are given by the formulae (49), (50), and (51) gives very unreliable

results. Therefore, its results for only the distribution densities of (13), (16), and (17) are given below.

For the implementation of the CBM 2 using the Stein's approach for testing the hypotheses of (11) for the distributions (13), (16), and (17), the following algorithm was used:

1. Estimation of the parameter ρ is computed by (7) using the sample $Y_1, \ldots, Y_n$.
2. The values of the Lagrange multipliers λ_2^0, λ_2^-, and λ_2^+ are determined by solving equations (29) using densities (13), (16), and (17); necessary integrals are computed by the Monte Carlo method.
3. The sample $Y^m = (Y_{n+1}, \ldots, Y_{n+m})$ with the size m distributed by one of the distributions (13), (16), and (17), depending on which hypothesis we test, is used for making a decision using (30) acceptance regions for the hypotheses.

The same algorithm was used for testing (11) hypotheses in the second case where the densities (13), (16), and (17) were changed by the densities (49), (50), and (51).

Example: The results of testing (11) hypotheses using the CBM 2 (acceptance regions for the hypotheses are (30) where Lagrange multipliers are defined from (29)) or Bayes rules (hypotheses acceptance regions are (30) where Lagrange multipliers are equal to 1) for making a decision are given in Appendices B and C. The following notations are used there: AN—averaged number of observations necessary for making a decision; k—the size of the observation vector; n—the number of observations for computing the ρ parameter's estimation; ρ_0—the value of the correlation coefficient at hypothesis H_0; λ_2^0, λ_2^-, and λ_2^+—Lagrange multipliers under hypotheses H_0, H_-, and H_+, respectively; r_2^0, r_2^-, and r_2^+—restriction levels in (29) determining probabilities of correct decisions at H_0, H_-, and H_+, respectively; m—the number of generated random variables for computing probabilities of acceptance of hypothesis H_i under hypothesis H_j, i.e., for computing probabilities $P(x \in \Gamma_i | H_j)$, $i, j \in (-, 0, +)$; and Averaged—averaged values of the appropriate variables obtained at different computations (runs of the program) for one scenario.

In all computations presented in Appendices B and C, the following values were used: $k = 5$; $m = 5000$; a priori probabilities of hypotheses $p(H_0)$ =1/3, $p(H_-) = 1/3$, and $p(H_A) = 1/3$; $r_2^0 = r_2^- = r_2^+ = 0.01(6)$; and the accuracy of the solution of Equation (29) is equal to 0.001.

8. Discussion

The results of the computation showed the following characteristics of the considered problem.

When using the (13), (16), and (17) distribution laws in the CBM 2, the following was observed:

1. Distribution laws (13), (16), and (17) are very close to each other for hypotheses (11), as can be seen from the computed values of the divergences between hypotheses and from their graphical representations given in Figure A1 of Appendix D.
2. The minimum divergence between testing hypotheses that can be discriminated with given reliability decreases with increasing the size of the random vector. This fact is seen in Table of Appendix E and in the graphs of Figure A1.
3. The absolute value of the difference between correlation coefficients corresponding to the null and alternative hypotheses equal to 0.05, i.e., $|\rho_0 - \rho_A| = 0.05$, is sufficient in the CBM for making the correct decision with a high reliability for $k = 15$.
4. For the case $k = 5$, which is computed in Appendix B, $|\rho_0 - \rho_A|$ must be no less than 0.18 or 0.20 in the CBM for different values of ρ_0 for testing hypotheses with given reliabilities (equal to 0.05). The Bayes method does not guarantee the reliability of a decision equal to 0.05 for some divergences (see Appendix B).
5. The number of observations necessary for making a decision is equal to 51.
6. The number of observations for computing the ρ parameter's estimation is equal to 50.
7. The Bayes method uses fifty observations for computing the estimation of ρ and only one additional observation for making a decision. In general, it gives worse results than the CBM 2 (see Appendix B).

8. Conditions (31) and (32) are fulfilled for the CBM 2. This is violated for the Bayes method for some values of $|\rho_0 - \rho_A|$.

For the practical use of the presented method when the alternative values of the correlation coefficient are not known for testing hypotheses, we use Lagrange multipliers computed for a divergence equal to 0.18 between correlation coefficients. In this case, the necessary reliability of the decisions made when the divergence between the correlation coefficients of null and alternative hypotheses is greater than 0.18 is guaranteed. This is due to the fact that with the increase in the informational distance between the hypotheses, the errors of the decisions made decrease [28,29].

The probabilities of correct decisions for the different values of ρ_0 and for different divergences between correlations of null and alternative hypotheses when the Lagrange multipliers correspond to 0.18—to the minimal value of the divergence—are given in Appendix E.

The following was found using the (49), (50), and (51) distribution densities in the CBM 2:

1. It is seen from the (49), (50), and (51) formulae that the distribution densities depend on ρ_0 and $\hat{\rho}_n$ (the last is ML estimator of ρ_0). Therefore, the computed values of λ_2^-, λ_2^0, and λ_2^+ are the same for all possible alternative hypotheses H_- and H_+ when given ρ_0, and correct decisions are made with identical reliability in all cases.
2. Based on what has been stated above, in practical applications, we compute the Lagrange multipliers for given ρ_0 and for the minimal distance between the (11) testing hypotheses (for example, equal to 0.05) and use them to test the H_0 hypothesis versus any possible H_- and H_+ hypotheses.
3. The number of observations necessary for making a decision is equal to 26 in the considered case.
4. The number of observations for computing the ρ parameter's estimation, i.e., for $\hat{\rho}_n$, is equal to 15 in the considered case.
5. Conditions (31) and (32) are fulfilled.

Despite that the divergences between distribution densities (49), (50), and (51) are also very close to each other for the (11) hypotheses, the CBM 2 using Stein's approach for these distributions gives better results than the use of distribution laws (13), (16), and (17). The superiority is in the number of used observations, providing necessary reliabilities, and also in the convenience of implementation in practical situations. The basic superiority of this case is that for testing the (11) hypotheses the knowledge of ρ_A is not necessary. Only the knowledge of ρ_0 and the Lagrange multipliers computed for the values $|\rho_0 - \rho_A| = 0.05$ are required, i.e., in its practical use, for any possible values of ρ_0 and ρ-s, corresponding to alternative hypotheses H_- and H_+ such that $|\rho_0 - \rho_A| = 0.05$, the Lagrange multipliers are computed and they are used in real testing processes when the exact values of ρ_A are not known but $|\rho_0 - \rho_A| \geq 0.05$. Theorems confirming the convergence of the testing algorithms for both the (13), (16), and (17) and (49), (50), and (51) distribution densities can be found in [24,29]. Also, as was mentioned above, the results in Table A1 clearly demonstrate that conditions (31) and (32) are fulfilled for both cases.

Because the Bayes method yields very bad results when $|\rho_0 - \rho_A| < 0.18$, for this case, the computation results of the Bayes method are not provided when the distributions are (49), (50), and (51).

Note 4. Because the noted peculiarities remain in force for other possible values of ρ_0, the computation results for only $\rho_0 = 0$ are given in Appendix C to conserve space.

It is worth separately noting the fact that the method proposed in this paper can achieve better results than other methods known to us for the considered problem [1,5,10]. For example, comparing the results given in Appendix C of this paper with the results of the beta-optimal methods given in Table 5 of [5] clearly shows the superiority of the proposed method. On the other hand, in [5] (see p. 87 "Summary") it is mentioned that the beta-optimal methods outperform SenGupta's test, but those are motivated differently.

9. Conclusions

Constrained Bayesian and classical Bayes methods, using the maximum likelihood estimation and Stein's approach, for testing the equi-correlation coefficient of a standard symmetric multivariate normal distribution are developed and investigated. For the investigation of the obtained theoretical results and choosing the best among them, different practical examples are computed using computer codes developed in MATLAB R2021b. The simulation results showed that the CBM using Stein's approach gives opportunities to make decisions with higher reliability for testing the (11) hypotheses than the Bayes method. Also, the CBM 2 using Stein's approach with distribution densities (49), (50), and (51) gives better results in comparison with the (13), (16), and (17) densities in terms of the number of observations needed and providing the necessary reliabilities, as well the convenience of their implementations in practice. Recommendations and guidelines for the use of the proposed methods for solving practical problems are given.

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Appendix A. An Algorithm for Computation of (42) Distribution Function

The algorithm is developed on the basis of the results of [25].

Let us consider random variable

$$\xi_l = q_1^l \cdot V_{1m} + q_2^l \cdot V_{2m},\ l \in (0, A), \tag{A1}$$

where

$$q_1^l = d_1^l \cdot (1 + (k-1)\rho) \text{ and } q_2^l = d_2^l \cdot (1-\rho), \tag{A2}$$

where d_1^l and d_2^l are determined by (37) and (39).

Then, the probability distribution function of ξ_l is [25]

$$F^l(z, \rho) = b_2^l \sum_{j=0}^{\infty} a_j^l \int_0^z f_j^l(y, \rho) dy, \tag{A3}$$

where

$$m_1 = \frac{m}{2},\ m_2 = \frac{m(k-1)}{2},\ s = m_1 + m_2 = \frac{mk}{2},$$

$$(m)_0 = 1,\ (m)_j = m(m+1)\ldots(m+j-1),$$

$$b_2^l = \left(\frac{d_1^l \cdot (1 + (k-1)\rho)}{d_2^l \cdot (1-\rho)} \right)^{\frac{m(k-1)}{2}},$$

$$a_j^l = (m_2)_j \cdot \left(1 - q_1^l / q_2^l\right)^j / j!,$$

$$f_j^l(y, \rho) = \left(y^{s+j-1} \cdot \exp\left(-\frac{y}{2q_1^l} \right) \right) / \left(\left(2q_1^l\right)^{s+j} \cdot \Gamma(s+j) \right).$$

Appendix B. Stein's and Bayes Methods Using Distributions (13), (16), and (17)

(A). Hypothesis H_0 is true. The samples for making decisions are generated by (13) with the size $m = 5000$.

ρ_0 n =50	CBM					Bayes Method		
	AN	$P(x \in \Gamma_-\|H_0)$	$P(x \in \Gamma_0\|H_0)$	$P(x \in \Gamma_+\|H_0)$	$\lambda_2^0/\lambda_2^-/\lambda_2^+$	$P(x \in \Gamma_-\|H_0)$	$P(x \in \Gamma_0\|H_0)$	$P(x \in \Gamma_+\|H_0)$
−0.1	(1) 51.0490	0	0.9976	0.0024		0.0008	0.9850	0.0142
	(2) 51.0569	0	0.9974	0.0014		0.0001	0.9838	0.0152
$\|\rho_0 - \rho_A\|$=	(3) 51.0552	0	0.9978	0.0008	0.3418/	0.0004	0.9864	0.0132
0.145	(4) 51.0520	0	0.9966	0.0009	2.1034×10^{-14}/	0.0008	0.9848	0.0144
	(5) 51.0514	0	0.9982	0.0004	0.2441	0.0001	0.9828	0.0162
Averaged	51.0529	0	0.9975	0.0012		0.00044	0.9846	0.01464
0	(1) 51.0052	0	0.9582	0.0418		0.0026	0.9782	0.0192
	(2) 51.0098	0	0.9622	0.0378		0.0022	0.9778	0.0200
$\|\rho_0 - \rho_A\|$=	(3) 51.0092	0	0.9590	0.0410	0.5310/	0.0022	0.9786	0.0192
0.18	(4) 51.0082	0	0.9554	0.0446	0.0057/	0.0018	0.9798	0.0184
	(5) 51.0083	0	0.9568	0.0432	1.8768	0.0012	0.9792	0.0196
Averaged	51.00814	0	0.95832	0.04168		0.002	0.97872	0.01928
0.1								
	(1) 51	0.0056	0.9534	0.0410		0.0052	0.9730	0.0218
$\|\rho_0 - \rho_A\|$=	(2) 51.0053	0.0063	0.9548	0.0389	0.6836/	0.0034	0.9728	0.0238
0.18	(3) 51.0035	0.0042	0.9522	0.0436	2.9297/	0.0042	0.9740	0.0218
	(4) 51.0027	0.0036	0.9562	0.0402	2.5635	0.0040	0.9746	0.0214
	(5) 51	0.0074	0.9550	0.0376		0.0044	0.9740	0.0216
Averaged	51.0023	0.00542	0.95432	0.04026		0.00424	0.97368	0.02208
0.3								
	(1) 51.0185	0.0091	0.9724	0.0185		0.0052	0.9722	0.0226
$\|\rho_0 - \rho_A\|$=	(2) 51.0225	0.0050	0.9877	0.0073	0.8240/	0.0050	0.9718	0.0232
0.2	(3) 51.0211	0.0073	0.9864	0.0063	7.8125/	0.0054	0.9698	0.0248
	(4) 51.0202	0.0077	0.9878	0.0045	0.6773	0.0074	0.9656	0.0270
	(5) 51.0197	0.0256	0.9712	0.0032		0.0068	0.9676	0.0256
Averaged	51.0204	0.01094	0.9811	0.00796		0.00596	0.9694	0.02464
0.5	(1) 51.0269	0.0055	0.9742	0		0.0066	0.9692	0.0242
	(2) 51.0338	0.0036	0.9759	1.746×10^{-7}		0.0040	0.9690	0.0270
$\|\rho_0 - \rho_A\|$=	(3) 51.0327	0.0030	0.9698	$1.2001 \times$	0.9277/	0.0048	0.9666	0.0286
0.2	(4) 51.0317	0.0020	0.9710	10^{-11}	3.9063/	0.0062	0.9704	0.0234
	(5) 51.0314	0.0016	0.9742	1.5429×10^{-4}	0.0134	0.0060	0.9680	0.0260
Averaged	51.0313	0.00314	0.97302	4.1067×10^{-5}		0.00552	0.96864	0.02584
0.7	(1) 51.0368	0.0012	0.9988	0		0.0024	0.9728	0.0248
	(2) 51.0401	0.0008	0.9992	0	0.8789/	0.0038	0.9738	0.0224
$\|\rho_0 - \rho_A\|$=	(3) 51.0396	0.0002	0.9998	0	0.2365/	0.0026	0.9714	0.0260
0.18	(4) 51.0380	0.0003	0.9997	0	$8.7041 \times$	0.0044	0.9712	0.0244
	(5) 51.0368	0.0002	0.9998	0	10^{-10}	0.0040	0.9686	0.0274
Averaged	51.03826	0.00054	0.99946	0		0.00344	0.97156	0.025

(B). Hypothesis H_- is true. The samples for making decisions are generated by (10) with $\rho \in (-1/(k-1), \rho_0)$ for the size $m = 5000$.

ρ_0 $n = 50$	CBM					Bayes Method		
	AN	$P(x \in \Gamma_-\|H_-)$	$P(x \in \Gamma_0\|H_-)$	$P(x \in \Gamma_+\|H_-)$	$\lambda_2^0/\lambda_2^-/\lambda_2^+$	$P(x \in \Gamma_-\|H_-)$	$P(x \in \Gamma_0\|H_-)$	$P(x \in \Gamma_+\|H_-)$
-0.1	(1) 51.0502	1	0	0		1	0	0
	(2) 51.0583	1	0	0		1	0	0
$\|\rho_0 - \rho_A\|=$	(3) 51.0567	1	0	0	0.3422/	1	0	0
0.145	(4) 51.0549	1	0	0	2.1684×10^{-14}/	1	0	0
	(5) 51.0549	1	0	0	0.2756	1	0	0
Averaged	51.055	1	0	0		1	0	0
0	(1) 51.0560	1	0	0		1	0	0
	(2) 51.0613	1	0	0		0.9996	0.0004	0
$\|\rho_0 - \rho_A\|=$	(3) 51.0606	1	0	0	0.5127/	1	0	0
0.18	(4) 51.0589	1	0	0	0.0054/	0.9998	0.0002	0
	(5) 51.0538	1	0	0	2.0142	1	0	0
Averaged	51.058	1	0	0		0.99988	0.00012	0
0.1	(1) 51	0.9549	0.0451	0		0.8868	0.1132	0
	(2) 51	0.9555	0.0445	0		0.8910	0.1090	0
$\|\rho_0 - \rho_A\|=$	(3) 51.0053	0.9544	0.0456	0	0.6592/	0.8986	0.1014	0
0.18	(4) 51.0035	0.9541	0.0459	0	2.9297/	0.8900	0.1100	0
	(5) 51.0027	0.9544	0.0456	0	2.5635	0.8922	0.1078	0
Averaged	51.0023	0.95466	0.04534	0		0.89172	0.10828	0
0.3	(1) 51	0.9566	0.0372	0		0.8018	0.1982	0
	(2) 51	0.9568	0.0378	0		0.8122	0.1878	0
$\|\rho_0 - \rho_A\|=$	(3) 51	0.9533	0.0402	0	0.8423/	0.8080	0.1920	0
0.2	(4) 51.0059	0.9533	0.0467	0	7.8164/	0.8068	0.1932	0
	(5) 51.0039	0.9536	0.0464	0	0.6580	0.8098	0.1902	0
Averaged	51.00196	0.95472	0.04166	0		0.80772	0.19228	0
0.5	(1) 51	0.9585	0.0415	0		0.8978	0.1022	0
	(2) 51.0054	0.9568	0.0432	0		0.8988	0.1012	0
$\|\rho_0 - \rho_A\|=$	(3) 51.0036	0.9565	0.0435	0	0.9766/	0.8998	0.1002	0
0.2	(4) 51.0027	0.9561	0.0439	0	4.0893/	0.8924	0.1076	0
	(5) 51.0022	0.9564	0.0436	0	0.0153	0.8934	0.1066	0
Averaged	51.00278	0.95686	0.03484	0		0.89644	0.10356	0
0.7	(1) 51.0206	0.9884	0.0116	0		0.9898	0.0102	0
	(2) 51.0271	0.9891	0.0109	0		0.9860	0.0140	0
$\|\rho_0 - \rho_A\|=$	(3) 51.0256	0.9894	0.0106	0	0.8286/	0.9854	0.0146	0
0.18	(4) 51.0260	0.9896	0.0104	0	0.2651/	0.9876	0.0124	0
	(5) 51.0249	0.9894	0.0106	0	8.5265×10^{-10}	0.9872	0.0128	0
Averaged	51.02484	0.98918	0.01082	0		0.9872	0.0128	0

(C). Hypothesis H_+ is true. The samples for making decisions are generated by (10) with $\rho \in (\rho_0, 1)$ for the size $m = 5000$.

ρ_0 n =50		CBM				Bayes Method		
	AN	$P(x \in \Gamma_-\|H_+)$	$P(x \in \Gamma_0\|H_+)$	$P(x \in \Gamma_+\|H_+)$	$\lambda_2^0/\lambda_2^-/\lambda_2^+$	$P(x \in \Gamma_-\|H_+)$	$P(x \in \Gamma_0\|H_+)$	$P(x \in \Gamma_+\|H_+)$
−0.1	(1) 51.0414	0	0.0060	0.9940	0.3296/	0	0.0202	0.9798
	(2) 51.0408	0	0.0058	0.9942	2.4293 ×	0	0.0158	0.9842
$\|\rho_0 - \rho_A\|$=	(3) 51.0452	0	0.0050	0.9950	10^{-14}/	0	0.0180	0.9820
0.145	(4) 51.0434	0	0.0050	0.9950	0.2632	0	0.0204	0.9796
	(5) 51.0420	0	0.0072	0.9928		0	0.0196	0.9804
Averaged	51.04256	0	0.0058	0.9942		0	0.0188	0.9812
0	(1) 51.0038	0	0.0446	0.9554		0	0.0970	0.9030
	(2) 51.0089	0	0.0469	0.9531	0.4883/	0	0.0888	0.9112
$\|\rho_0 - \rho_A\|$=	(3) 51.0069	0	0.0473	0.9527	0.0057/	0	0.0914	0.9086
0.18	(4) 51.0061	0	0.0467	0.9533	1.9531	0	0.0912	0.9088
	(5) 51	0	0.0457	0.9543		0	0.0874	0.9126
Averaged	51.00514	0	0.04624	0.95376		0	0.09116	0.90884
0.1	(1) 51.0004	0	0.0474	0.9526		0	0.0858	0.9142
	(2) 51.0053	0	0.0497	0.9503	0.6348/	0	0.0870	0.9130
	(3) 51.0037	0	0.0442	0.9558	0.7334/	0	0.0826	0.9174
$\|\rho_0 - \rho_A\|$= 0.2	(4) 51.0030	0	0.0480	0.9520	1.5869	0	0.0734	0.9266
	(5) 51.0025	0	0.0488	0.9512		0	0.0836	0.9164
Averaged	51.00298	0	0.04762	0.95238		0	0.08248	0.91752
0.3	(1) 51.0336	0	0.0304	0.9696		0	0.0404	0.9596
	(2) 51.0336	0	0.0284	0.9716	0.8446/	0	0.0376	0.9624
	(3) 51.0393	0	0.0275	0.9725	7.0801/	0	0.0348	0.9652
$\|\rho_0 - \rho_A\|$= 0.2	(4) 51.0372	0	0.0289	0.9711	0.6084	0	0.0370	0.9630
	(5) 51.0384	0	0.0294	0.9706		0	0.0382	0.9618
Averaged	51.03642	0	0.02892	0.97108		0	0.0376	0.9624
0.5	(1) 51.0734	0	0.0002	0.9998		0	0	1
	(2) 51.0845	0	0.0006	0.9996	0.8667/	0	0.0002	0.9998
	(3) 51.0790	0	0.0004	0.9996	4.5166/	0	0	1
$\|\rho_0 - \rho_A\|$=	(4) 51.0789	0	0.0003	0.9997	0.0143	0	0.0004	0.9996
0.18	(5) 51.0781	0	0.0004	0.9996		0	0.0002	0.9998
Averaged	51.07878	0	0.00038	0.99966		0	0.00016	0.99984
0.7	(1) 51.0840	0	0	1		0	0	1
	(2) 51.0911	0	0	1	0.7813/	0	0	1
	(3) 51.0863	0	0	1	0.2518/	0	0	1
$\|\rho_0 - \rho_A\|$=	(4) 51.0860	0	0	1	7.9581 ×	0	0	1
0.18	(5) 51.0851	0	0	1	10^{-10}	0	0	1
Averaged	51.0865	0	0	1		0	0	1

Appendix C. Stein's Method Using Distributions (49), (50), and (51)

(A). Hypothesis H_0 is true. The samples for making decisions are generated by $N(z;(\hat{\rho}_n-\rho_0),\sigma^2_{\rho_0})$(see (49)) with the size $m=5000$.

$\rho_0=0$ n =15	CBM				
	AN	$P(x\in\Gamma_-\|H_0)$	$P(x\in\Gamma_0\|H_0)$	$P(x\in\Gamma_+\|H_0)$	$\lambda_2^0/\lambda_2^-/\lambda_2^+$
$\|\rho_0-\rho_+\|$= 0.05 nk =25	(1) 26.0506	0.0071	0.9810	0	
	(2) 26.0635	0.0020	0.9780	0	
	(3) 26.0563	0.0017	0.9770	0	
	(4) 26.0542	0.0010	0.9750	0	
	(5) 26.0540	0.0016	0.9710	0	
Averaged	6.05572	0.00268	0.9764	0	
$\|\rho_0-\rho_+\|$= 0.10 nk =25	(1) 26.0618	0.0036	0.9730	0	
	(2) 26.0569	0.0031	0.9680	0	
	(3) 26.0538	0.0010	0.9820	0	
	(4) 26.0516	0.0010	0.9780	0	
	(5) 26.0512	0.0012	0.9750	0	
Averaged	26.05506	0.00198	0.9752	0	
$\|\rho_0-\rho_+\|$= 0.15 nk =25	(1) 26.0653	0.0031	0.9780	0	
	(2) 26.0592	0.0014	0.9779	0	
	(3) 26.0580	0.0015	0.9740	0	7.3590087890625/
	(4) 26.0596	0.0004	0.9790	0	2.298583984375/
	(5) 26.0587	0.0010	0.9760	0	0.58624267578125
Averaged	26.06016	0.00148	0.97698	0	
$\|\rho_0-\rho_+\|$= 0.20 nk =25	(1) 26.0565	0.0007	0.9800	0	
	(2) 26.0562	0.0005	0.9830	0	
	(3) 26.0567	0.0006	0.9760	0	
	(4) 26.0557	0.0004	0.9780	0	
	(5) 26.0556	0.0004	0.9780	0	
Averaged	26.05614	0.00052	0.9790	0	
$\|\rho_0-\rho_+\|$= 0.249 nk =25	(1) 26.0564	0.0007	0.9750	0	
	(2) 26.0556	0.0002	0.9780	0	
	(3) 26.0558	0.0007	0.9860	0	
	(4) 26.0559	0.0006	0.9680	0	
	(5) 26.0557	0.0003	0.9730	0	
Averaged	26.05588	0.0005	0.9760	0	

nk—the number of observations distributed in accordance with $p(z|H_0)=N(z;(\hat{\rho}_n-\rho_0),\sigma^2_{\rho_0})$, the arithmetic mean of which is used for making a decision. n—the number of observations used for the computation of ML estimator $\hat{\rho}_n$.

(B). Hypothesis H_- is true. The samples for making decisions are generated by $N(z;(\hat{\rho}_n-\rho),\sigma_\rho^2)$(see (50)) with $\rho\in(-1/(k-1),\rho_0)$ for the size $m=5000$.

$\rho_0=0$ $n=15$	CBM				
	AN	$P(x\in\Gamma_-\|H_-)$	$P(x\in\Gamma_0\|H_-)$	$P(x\in\Gamma_+\|H_-)$	$\lambda_2^0/\lambda_2^-/\lambda_2^+$
$\|\rho_0-\rho_+\|=0.05$ $nk=25$	(1) 26.0530	0.9960	0.0040	0	
	(2) 26.0570	0.9985	0.0040	0	
	(3) 26.0497	0.9957	0.0050	0	
	(4) 26.0475	0.9960	0.0030	0	
	(5) 26.0476	0.9960	0.0040	0	
Averaged	26.05096	0.99644	0.0040	0	
$\|\rho_0-\rho_+\|=0.10$ $nk=25$	(1) 26.0397	0.9967	4.003×10^{-6}	0	7.3590087890625/ 2.298583984375/ 0.58624267578125
	(2) 26.0340	0.9971	4.003×10^{-6}	0	
	(3) 26.0298	0.9975	4.003×10^{-6}	0	
	(4) 26.0445	1	0	0	
	(5) 26.0560	1	0	0	
Averaged	26.0408	0.99826	2.4018×10^{-6}	0	
$\|\rho_0-\rho_+\|=0.15$ $nk=25$	(1) 26.0297	1	0	0	
	(2) 26.0223	1	0	0	
	(3) 26.0178	1	0	0	
	(4) 26.0148	1	0	0	
	(5) 26.0127	1	0	0	
Averaged	26.01946	1	0	0	

nk—the number of observations distributed in accordance with $p(z|H_-)=N(z;(\hat{\rho}_n-\rho),\sigma_\rho^2)$, $\rho\in(-1/(k-1),\rho_0)$, the arithmetic mean of which is used for making a decision.
n—the number of observations used for the computation of ML estimator $\hat{\rho}_n$.

(C). Hypothesis H_+ is true. The samples for making decisions are generated by $N(z;(\hat{\rho}_n-\rho_0),\sigma_{\rho_0}^2)$ (see (51)) with $\rho\in(\rho_0,1)$ for the size $m=5000$.

ρ_0 $n=15$	CBM				
	AN	$P(x\in\Gamma_-\|H_+)$	$P(x\in\Gamma_0\|H_+)$	$P(x\in\Gamma_+\|H_+)$	$\lambda_2^0/\lambda_2^-/\lambda_2^+$
$\|\rho_0-\rho_+\|=0.05$ $nk=25$	(1) 26.0126	2.0298×10^{-5}	0.0390	0.9945	
	(2) 26.0126	2.8654×10^{-5}	0.0450	0.9881	
	(3) 26.0125	4.0069×10^{-5}	0.0470	0.9789	
	(4) 26.0124	5.5761×10^{-5}	0.0351	0.9741	
	(5) 26.0123	7.7157×10^{-5}	0.0410	0.9686	
Averaged	26.01248	4.43878×10^{-5}	0.04142	0.98084	7.3590087890625/ 2.298583984375/ 0.58624267578125
$\|\rho_0-\rho_+\|=0.10$ $nk=25$	(1) 26.0274	0	4.6000×10^{-8}	0.9516	
	(2) 26.0209	0	4.6000×10^{-8}	0.9631	
	(3) 26.0168	0	4.6000×10^{-8}	0.9704	
	(4) 26.0140	0	4.6000×10^{-8}	0.9753	
	(5) 26.0119	0	4.6000×10^{-8}	0.9789	
Averaged	26.0182	0	4.6000×10^{-8}	0.96786	

$\lvert\rho_0 - \rho_+\rvert =$ 0.15 nk =25	(1) 26.0367	0	1.4000×10^{-5}	0.9930	7.3590087890625/ 2.298583984375/ 0.58624267578125
	(2) 26.0244	0	1.4000×10^{-5}	0.9953	
	(3) 26.0183	0	1.4000×10^{-5}	0.9965	
	(4) 26.0147	0	1.4000×10^{-5}	0.9972	
	(5) 26.0122	0	1.4000×10^{-5}	0.9977	
Averaged	6.02126	0	1.4000×10^{-5}	0.99594	
$\lvert\rho_0 - \rho_+\rvert =$ 0.20 nk =25	(1) 26.0440	0	0	1	
	(2) 26.0293	0	0	1	
	(3) 26.0220	0	0	1	
	(4) 26.0176	0	0	1	
	(5) 26.0147	0	0	1	
Averaged	6.02552	0	0	1	

nk—the number of observations distributed in accordance with $p(z|H_+) = N(z; (\hat{\rho}_n - \rho), \sigma_\rho^2)$, $\rho \in (\rho_0, 1)$, the arithmetic mean of which is used for making a decision.
n—the number of observations used for the computation of ML estimator $\hat{\rho}_n$.

Appendix D. The Kullback–Leibler Divergence Between the Distributions Corresponding to the Basic and Alternative Hypotheses

The formula for computing the Kullback–Leibler divergence [29] between the distributions corresponding to the basic and alternative hypotheses (11) has the following form:

$$J(H_0, H_A) = \frac{1}{2}\left(tr\left(V_A^{-1} \cdot V_0\right) - k + \ln\left(\frac{\det V_A}{\det V_0}\right)\right).$$

Table A1. The Kullback–Leibler divergence between the considered distributions.

Absolute Value of the Difference between Correlation Coefficients	Divergence between Hypotheses					
	k—Dimension of the Random Vector					
	2	**3**	**4**	**5**	**10**	**15**
0	−0.500000	−2	−2	−2	−2	−2
0.05	−0.499998	−1.998452	−1.999800	−1.999868	−1.978545	−1.913498
0.10	−0.499975	−1.996207	−1.999897	−1.9981428	−1.884038	−1.508130
0.15	−0.499870	−1.994991	−1.999923	−1.991384	−1.647123	−0.361850
0.20	−0.499578	−1.995222	−1.998287	−1.974225	−1.140647	2.5478240
0.25	−0.498936	−1.996664	−1.992357	−1.938667	−0.118422	9.863595
0.30	−0.497705	−1.998624	−1.978181	−1.872546	1.921221	28.907530
0.35	−0.495539	−1.999952	−1.949830	−1.756739	6.057835	81.612806
0.40	−0.491939	−1.998882	−1.898169	−1.560040	14.7615462	239.804571
0.45	−0.486178	−1.992709	−1.808639	−1.229397	34.099183	764.261963
0.50	−0.477174	−1.977174	−1.657081	−0.670322	80.263477	2723.863943
0.55	−0.463281	−1.945327	−1.401508	0.294955	200.982274	1.118287×10^4
0.60	−0.441894	−1.885285	−0.964754	2.023792	554.973471	5.477538×10^4
0.65	−0.408723	−1.775568	−0.194875	5.296226	1755.615892	3.355081×10^5
0.70	−0.356280	−1.574484	1.234836	12.003869	6677.652705	2.750979×10^6
0.75	−0.270482	−1.193147	4.113706	27.420558	32759.954823	3.3554421×10^7
0.80	−0.121937	−0.415705	10.684338	69.403428	2.325059×10^5	7.2660899×10^8
0.85	0.160835	1.394187	29.065156	218.913453	2.955932×10^6	3.8925993×10^{10}
0.90	0.801213	6.830008	103.504803	1080.614380	1.086956×10^8	1.086957×10^{13}
0.95	2.964254	36.955242	827.124043	16,658.959510	5.333333×10^{10}	1.706667×10^{17}

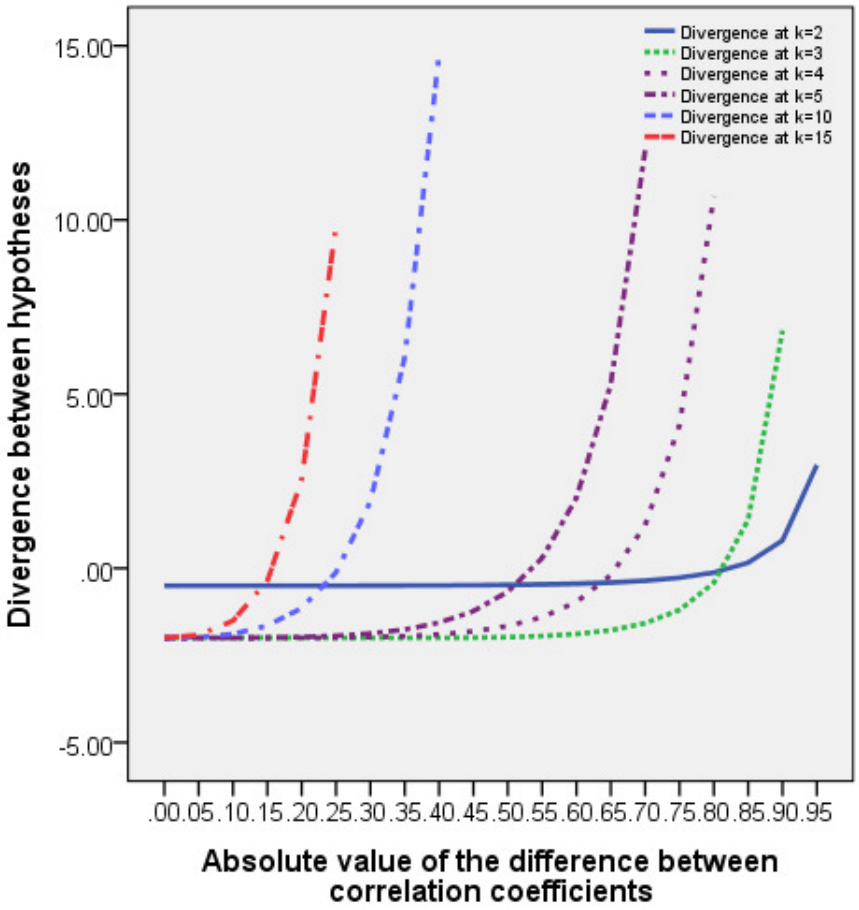


Figure A1. Dependence of the divergence between hypotheses on the difference between correlation coefficients, i.e., on $|\rho_0 - \rho_A|$, for different k-dimensions of the random vector.

Appendix E. The Probabilities of Correct Decisions for the Different Values of ρ_0 and for the Different Divergences Between Correlations of Basic and Alternative Hypotheses When the Lagrange Multipliers Correspond to the Minimal Value of the Divergence Equal to 0.20

Divergence between ρ_0 and ρ_+	Probabilities of correct decisions for $\rho_0 = 0/0.3/0.5$		Divergence between ρ_0 and ρ_-	Probabilities of correct decisions for $\rho_0 = 0/-0.1$	
	$P(x \in \Gamma_0 \mid H_0)$	$P(x \in \Gamma_+ \mid H_+)$		$P(x \in \Gamma_- \mid H_-)$	$P(x \in \Gamma_0 \mid H_0)$
0.20	0.9814/0.9724/0.9708	0.9732/0.9769/0.9998	0.145	0.9670/1	0.9776/1
0.25	0.9810/0.9760/0.9694	0.9842/0.9794/0.9999	0.146	0.9681/1	0.9754/1
0.30	0.9774/0.9774/0.9716	0.9892/0.9815/0.99993	0.147	0.9701/1	0.9792/1
0.35	0.9776/0.9762/0.9708	0.9919/0.9832/0.99995	0.148	0.9712/1	0.9794/1
0.40	0.9804/0.9766/0.9658	0.9935/0.9846/0.99996	0.149	0.9712/1	0.9752/1
0.45	0.9780/0.9760/0.9684	0.9946/0.9858/0.99997			
0.50	0.9788/0.9760/0.9684	0.9953/0.9868/0.99997			

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