axioms

Review

Join Spaces and Lattices

Violeta Leoreanu-Fotea [1,*] and Sarka Hoskova-Mayerova [2]

1 Faculty of Mathematics, "Al.I. Cuza" University, Bd. Carol I, No. 11, 700506 Iaşi, Romania
2 Department of Mathematics and Physics, Faculty of Military Technology, University of Defence, 662 10 Brno, Czech Republic; sarka.mayerova@unob.cz
* Correspondence: violeta.fotea@uaic.ro

Abstract: Hypergroups represent a generalization of groups, introduced by Marty, that are rich in applications in several sectors of mathematics and in other fields. An important class of hypergroups called join spaces is presented in this paper, along with some connections to lattice theory, in particular, to modular and to distributive lattices. In particular, we study join spaces associated with chains through functions and we analyze when such join spaces are isomorphic. Moreover, a combinatorial problem is presented for a finite context, focusing on calculating the number of isomorphisms classes of join spaces.

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1. Introduction

The composition of two elements is an element in groups, while in an algebraic hypergroup, the composition of two elements is a nonempty subset. F. Marty observed that the elements of a factor group are subsets and this was the starting point for hypergroup theory, see [1].

He introduced the hypergroup concept in 1934 at the 8th Congress of the mathematicians from the Scandinavian countries. Over time, new results have also appeared interesting, but especially since the 1970s, this theory developed a lot in Europe, the United States, Asia, and Australia. Some sound names in this field such as Dresher, Ore, Koskas, and Krasner made contributions in the field of homomorphisms of hypergroups and in the theory of subhypergroups.

Hypergroups have applications in several sectors of mathematics and in other fields, see [2]. Complete parts were studied by Koskas, then by Corsini, Leoreanu, Davvaz, Vougiouklis, and Freni.

Fundamental equivalence relations are important in algebraic hyperstructures because they establish a natural connection between algebraic hyperstructures and classical algebraic structures. The relation β connects the class of hypergroups to the class of groups. More exactly, the quotient of a hypergroup has a group structure. Using relation β, Migliorato defines the notion of an n-complete hypergroup.

In [3], basic notions and results about algebraic hypergroups are presented, in particular about semihypergroups, hypergroups, subhypergroups, homomorphisms and isomorphisms, fundamental relations and the corresponding quotient structures, join spaces, canonical hypergroups, Rosenberg hypergroups, topological hypergroups, and also connections with hypergraphs and n-ary relations, while in [4] hyperstructures and their representations are studied.

Hyperlattices were introduced in 1994 by Mittas and Konstantinidou, see [5], and later on they were studied by many mathematicians, see, e.g., [6,7]. Connections between hypergroups and lattices or hyperlattices have been considered and analyzed by

Nakano [8] and Varlet [9], then by Comer [10] and later by Kehagias, Konstantinidou, and Serafimidis [11,12], Călugăreanu and Leoreanu [13], Tofan and Volf [14], and Njionou, Ngapeya and Leoreanu-Fotea [15].

2. Join Spaces and Connections with Lattices

In this section, we present the join space notion and we analyze some connections with lattice theory.

Let $H \neq \emptyset$. A function $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *hyperoperation*, where $\mathcal{P}^*(H)$ denotes the set of nonempty subsets of H.

If S, T are subsets of H, then $S \circ T = \bigcup_{s \in S; t \in T} s \circ t$.

The structure $(H, \circ)$ is a *hypergroup* if for all u, v, w of H we have

$$(u \circ v) \circ w = u \circ (v \circ w) \text{ and } u \circ H = H \circ u = H.$$

For all elements $a, b \in H$, denote $a/b = \{u \in H \mid a \in u \circ b\}$.

Definition 1. *A hypergroup $(H, \circ)$ is called a join space if it is commutative and for all a, b, c, d of H, we have*

$$a/b \cap c/d \neq \emptyset \Rightarrow a \circ d \cap b \circ c \neq \emptyset. \tag{1}$$

In other words, $a \in u \circ b, c \in u \circ d \Rightarrow a \circ d \cap b \circ c \neq \emptyset$.

The condition (1) is often called the *join space condition*.

Join spaces were defined by W. Prenowitz. He and J. Jantosciak applied them in both Euclidean and non-Euclidean geometry, see [16]. Using join spaces, descriptive, projective, and spherical geometry were subsequently rebuilt.

Join spaces can be also studied in connections with binary relations, fuzzy sets, or rough sets, see [2,17].

We present here some examples of join spaces:

Example 1. *Let H be a non-empty set. If R is an equivalence relation on it, then denote the equivalence class of $a \in H$ by $[a]$ and define the next hyperoperation on H:*

$$\forall a, b \in H,\ a \circ b = [a] \cup [b].$$

Then, $(H, \circ)$ is a join space.

For all $a, b, c \in H$ we have $a \circ H = H,\ a \circ (b \circ c) = (a \circ b) \circ c = [a] \cup [b] \cup [c]$, whence $(H, \circ)$ is a commutative hypergroup. Moreover, if $a, b, c, d \in H$ and there is $u \in H$ such that $a \in u \circ b,\ c \in u \circ d$ then $a \in [u] \cup [b],\ c \in [u] \cup [d]$.

If $a \in [b]$, then $a \in a \circ d \cap b \circ d$. Similarly, if $c \in [d]$, then $c \in a \circ d \cap b \circ c$.

If $a \notin [b]$ and $c \notin [d]$, then $a \in [u]$ and $c \in [u]$, whence $[a] = [u] = [c]$, hence $u \in a \circ d \cap b \circ c$. Therefore, $(H, \circ)$ is a join space.

Example 2. *Let $(G, \cdot)$ be a commutative group. For all $x \in G$, consider a nonempty set $A(x)$, such that if $x, y \in G,\ x \neq y$, then $A(x) \cap A(y) = \emptyset$.*

Set $H_G = \cup_{x \in G} A(x)$ and $f : K_G \to G,\ f(a) = x \Leftrightarrow a \in A(x)$.

For all $a, b \in H_G$ we define $a \,\square\, b = A(f(a)f(b))$. Then, $(H_G, \square)$ is a join space.

Indeed, it can be checked that $\forall a \in H_G,\ a \,\square\, H_G = H_G$ and $\forall a, b, c \in H_G,\ (a \,\square\, b) \,\square\, c = a \,\square\, (b \,\square\, c) = A(f(a)f(b)f(c))$.

Now, if $a, b, c, d, u \in H_G$ are such that $a \in u \,\square\, b,\ c \in u \,\square\, d$ then $a \in A(f(u)f(b))$ and $c \in A(f(u)f(d))$, whence $f(a) = f(u)f(b)$ and $f(c) = f(u)f(d)$.

Hence, $f(a)f(d) = f(c)f(b)$, so $a \,\square\, d \cap b \,\square\, c \neq \emptyset$. Thus, $(H_G, \square)$ is a join space.

Example 3. *Consider* $(H, \circ)$ *a hypergroup and* $(G, \cdot)$ *a commutative group. Consider a family* $\{A_i\}_{i \in G}$ *of nonempty sets, such that* $A_1 = H$ *and for* $i, j \in G,\ i \neq j,\ A_i \cap A_j = \emptyset$. *We define the next hyperoperation on* $T = \cup_{i \in G} A_i$:

$$\forall a, b \in H,\ a \,\Box\, b = a \circ b;\ \forall a \in A_i,\ b \in A_j, (i, j) \neq (1, 1), ij = k,\ a \,\Box\, b = A_k.$$

Then, $(T, \Box)$ *is a join space if and only if* $(H, \circ)$ *is a join space.*

Let $(H, \circ)$ be a join space.
We have $a \,\Box\, T = T$, since $(G, \cdot)$ is a group. The associativity law holds.
Moreover, if $a \in v \,\Box\, b,\ a \in A_i,\ b \in A_j$, then $v \in A_{ij^{-1}}$.
Similarly, if $c \in v \,\Box\, d,\ c \in A_k,\ d \in A_s$, then $v \in A_{ks^{-1}}$. Thus, $ij^{-1} = ks^{-1}$, whence $is = kj$.
If $(i, j, k, s) \neq (1, 1, 1, 1)$, then $a \,\Box\, d \cap c \,\Box\, b \neq \emptyset$.
If $(i, j, k, s) = (1, 1, 1, 1)$, then we use the fact $(H, \circ)$ is a join space.
Therefore, $(T, \Box)$ is a join space.
Conversely, suppose that $(T, \Box)$ is a join space. If $u \in H$ and $a \in u \circ b,\ c \in u \circ d$, then $a \in u \,\Box\, b,\ c \in u \,\Box\, d$, whence $a \,\Box\, d \cap c \,\Box\, b \neq \emptyset$, which means that $a \circ d \cap c \circ b \neq \emptyset$. Thus, $(H, \circ)$ is a join space.

The study of algebraic hypergroups and connections with lattices and ordered sets was initiated by J. Mittas and then by M. Konstantinidou and K. Serafimidis, Ch. Massouros, G. Massouros, and later by Ath. Kehagias. Connections between ordered sets, quasi-orders, and hypergroups were also studied by Chvalina.

In what follows, we present some connections with lattice theory, see [18]. Two important classes of lattices are characterized using hypergroups: distributive and modular lattices, see [9,10,19].

Connection 1. *In* [9], *J.Varlet provided the following characterization of distributive lattices:*
Consider the next hyperoperation on a lattice $\mathbf{L} = (L, \vee, \wedge)$*:*
$\forall a, b \in L$ *we set*

$$a \diamond b = [a \wedge b, a \vee b] = \{x \in L : a \wedge b \leq x \leq a \vee b\}.$$

Theorem 1. $\mathbf{L}$ *is a distributive lattice if and only if* $(L, \diamond)$ *is a join space.*

In [19], we considered and analyzed a family of hyperoperations $\{\diamond_{pq}\}_{p,q \in L}$ defined as follows.
Let $p, q \in L$ be arbitrary. For all $a, b \in L$ set

$$a \diamond_{pq} b = [a \wedge b \wedge p, a \vee b \vee q].$$

Theorem 2. *If the lattice* $\mathbf{L}$ *is distributive and* $p, q \in L$*, then* $(L, \diamond_{pq})$ *is a join space.*

We mention some important steps from the proof of this theorem:
First, check that

$$(a \diamond_{pq} b) \diamond_{pq} c = [a \wedge b \wedge c \wedge p, a \vee b \vee c \vee q].$$

In order to prove "$\supseteq$", we consider $t \in [a \wedge b \wedge c \wedge p, a \vee b \vee c \vee q]$ an arbitrary element and we set $s = (a \wedge b) \vee (a \wedge t) \vee (b \wedge t)$. Thus, $s \in a \diamond_{pq} b$ whence $t \in s \diamond_{pq} c$.
Now, we consider $a, b, c, d, u \in L$ that satisfy $a \in b \diamond_{pq} u$ and $c \in d \diamond_{pq} u$.
Set $z = (a \wedge d \wedge p) \vee (b \wedge c \wedge p)$. We obtain $z \in b \diamond_{pq} c$.
Hence, $z \in (a \diamond_{pq} d) \cap (b \diamond_{pq} c)$ and so $(L, \diamond_{pq})$ is a join space.

Connection 2. *The next example of a join space is useful to characterize modular lattices.*

Let $\mathbf{L} = (L, \vee, \wedge)$ *be a lattice. In* [8] *H. Nakano introduced the following hyperoperation on* L*:*

$$a \circ b = \{u \in H \mid a \vee b = a \vee u = b \vee u\}.$$

Later, S. Comer [10] showed that:

Theorem 3. $\mathbf{L}$ *is a modular lattice if and only if* $(L, \circ)$ *is a join space.*

Another interesting proof of the above theorem is given in [2]. Several properties of this join space were presented in [13].

In [19], a new family of hyperoperations determined by a lattice is analyzed. For all $p \in L$ set

$$a \circ_p b = \{u \in L \mid a \vee b \vee p = a \vee u \vee p = b \vee u \vee p\}.$$

Notice that $a \circ_p b \neq \emptyset$, since $a \vee b \in a \circ_p b$.

For $p \in L$ set $L_p = \{x \in L \mid x \geq p\}$ and denote by $\mathbf{L}_p$ the restriction of $\mathbf{L}$ to L_p.

Theorem 4. *Let* $p \in L$*. If* $\mathbf{L}_p$ *is modular, then* $(L, \circ_p)$ *is a join space.*

Similar results can be obtained by considering the hyperoperation:

$$a \Box_p b = \{u \in L \mid a \wedge b \wedge p = a \wedge u \wedge p = b \wedge u \wedge p\}.$$

The hyperproduct is not empty since $a \wedge b \in a \Box_p b$.

Connection 3. *Another connection between join spaces and lattices was highlighted by Tofan and Volf* [14]*, as follows:*

If $(L, \vee, \wedge)$ *is a lattice and* $f : H \to L$ *is a function, such that* $f(L)$ *is a distributive sublattice of* $(L, \vee, \wedge)$*, then define*

$$\forall a, b \in H,\ a \diamond_f b = \{c \in H \mid inf\,\{f(a), f(b)\} \leq f(c) \leq sup\{f(a), f(b)\}\}.$$

We obtain a commutative hypergroup $(H, \diamond_f)$*.*

Indeed, in the above conditions, the next equality is checked:

$$(a \diamond_f b) \diamond_f c = \{u \in H \mid \inf\,\{f(a), f(b), f(c)\} \leq f(u) \leq \sup\{f(a), f(b), f(b)\}\} =$$

$$= a \diamond_f (b \diamond_f c),\ \forall a, b, c \in H.$$

Moreover, we shall prove here the next result, as follows:

Theorem 5. *The next statements are equivalent:*

- $f(L)$ *is a distributive sublattice.*
- $\diamond_f$ *satisfies the join space condition.*

Proof. First, let us check that the join space condition is satisfied for a distributive sublattice $f(L)$. Let $u \in H$: $a \in u \diamond_f b, c \in u \diamond_f d$.

Then, we shall check that there is $v \in a \diamond_f d \cap b \diamond_f c$.

Since $\{\inf\{f(b), f(u)\} \leq f(a) \leq \sup\{f(b), f(u)\}$ and $\{\inf\{f(d), f(u)\} \leq f(c) \leq \sup\{f(d), f(u)\}$, according to the distributivity, it follows that

$$\inf\{f(a), f(d)\} \leq \inf\{\sup\{f(b), f(u)\}, f(d)\} \leq \sup\{\inf\{f(b), f(d)\}, f(c)\} \leq \sup\{f(b), f(c)\}.$$

From here we obtain $s = \sup\{\inf\{f(a), f(d)\}, \inf\{f(b), f(c)\}\} \leq \sup\{f(b), f(c)\}$.

Similarly, we have $s \leq \sup\{f(a), f(d)\}$.

Hence, $\sup\{\inf\{f(a), f(d)\}, \inf\{f(b), f(c)\}\} \leq \inf\{\sup\{f(a), f(d)\}, \sup\{f(b), f(c)\}$.

Therefore, $(a \diamond_f d) \cap (b \diamond_f c) \neq \emptyset$.

Now, let us note that the reciprocal statement also holds: if the join space condition is satisfied, then $f(L)$ is a distributive sublattice.

Indeed, if $f(L)$ is not distributive, then it will contain a sublattice

$$\{f(u), f(v), f(w), f(t), f(s)\} \text{ with } \sup\{f(u), f(w)\} = \sup\{f(v), f(w)\} = f(t),$$

$\inf\{f(u), f(w)\} = \inf\{f(v), f(w)\} = f(s)$ and $f(u) > f(v)$ of $f(u), f(v), f(w)$ are not comparable two by two.

In both situations, $w \in u/v \cap v/u$, since

$$\inf\{f(v), f(w)\} = f(s) \leq f(u) \leq \sup\{f(v), f(w)\} = f(t)$$

and

$$\inf\{f(u), f(w)\} = f(s) \leq f(v) \leq \sup\{f(u), f(w)\} = f(t).$$

Thus, $w \in u/v \cap v/u$, and $u \diamond_f u = \{u\} \neq \{v\} = v \diamond_f v$, a contradiction. Thus, the sublattice $f(L)$ is distributive. □

Canonical hypergroups are an important class of join spaces and were introduced by J. Mittas [20]. They are the additive structures of Krasner hyperrings and were used by R. Roth to obtain results in the finite group character theory, see [21]. McMullen and Price studied finite abelian hypergroups over splitting fields [22].

More recent studies of canonical hypergroups were conducted by C and G. Massouros (in connection with automata), P. Corsini (sd-hypergroups), and K. Serafimidis, M. Konstantinidou, and J. Mittas, while feebly canonical hypergroups were analyzed by P. Corsini and M. De Salvo.

Canonical hypergroups are exactly join spaces with a scalar identity e, which means that $\forall u,\ u \circ e = e \circ u = u$. Obviously, commutative groups are canonical hypergroups. Other examples of canonical hypergroups are given in [3].

More general structures were also considered, namely polygroups, also called quasi-canonical hypergroups, by Bonansinga, Corsini, and Ch. Massouros. Comer analyzed the applications of polygroups in the theory of graphs, relations, Boolean, and cylindrical algebras.

A particular type of polygroup, namely the hypergroup of bilateral classes, was investigated by Drbohlav, Harrison. and Comer. Polygroups satisfy the same conditions as canonical hypergroups, with the exception of commutativity.

In the next two sections, we associate join spaces with chains and we analyze when they are isomorphic. Moreover, a combinatorial problem is presented: we calculate how many isomorphism classes of join spaces are.

3. Join Spaces Associated with a Chain: The Finite Case

In what follows, we associate a join space structure with a chain, through a function. We then study under what conditions such join spaces, considered for different functions, are isomorphic, for the finite case.

Let H be finite and $f : H \to C$ where C is a chain. Consider the next hyperoperation on H:

$$\forall a, b \in H,\ a \diamond_f b = \{c \in H \mid f(a) \wedge f(b) \leq f(c) \leq f(a) \vee f(b)\}.$$

We have $f(a) \wedge f(b) = min\{f(a), f(b)\},\ f(a) \vee f(b) = max\{f(a), f(b)\}$.

According to Theorem 5 or by a direct check, we then utilize the following theorem.

Theorem 6. *The structure $(H, \diamond_f)$ is a join space.*

Set $|H| = n$. We define the next equivalence relation on H:

$$h \sim_f k \Leftrightarrow f(h) = f(k).$$

Denote $s = |H/_\sim|$ and order $H/_\sim$ as follows: $\bar{h} \leq \bar{k} \Leftrightarrow f(h) \leq f(k)$, for $h, k \in H$.
We denote $H/_\sim = \{\bar{h_1}, \bar{h_2}, \ldots, \bar{h_s}\}$ where $f(h_1) < f(h_2) < \ldots < f(h_s)$.

For all $i \in \{1, \ldots, s\}$ set $a_i = |f^{-1}(f(h_i))|$. We have $a_i \geq 1$, $\sum_{i=1}^{s} a_i = n$.
Denote $\lambda(f) = (a_1, a_2, \ldots, a_s)$ the ordered partition of n into s parts and $\tau(a_1, a_2, \ldots, a_s) = (b_1, b_2, \ldots, b_s)$ where $\forall i, b_i = a_{s-i+1}$.

Theorem 7. *If $f, g : H \to C$ are two maps, then $(H, \diamond_f) \cong (H, \diamond_g) \Leftrightarrow \lambda(f) = \lambda(g)$ or $\lambda(g) = \tau(\lambda(f))$.*

Proof. "$\Leftarrow$"
Suppose $\lambda(f) = \lambda(g) = (a_1, a_2, \ldots, a_s)$. Set $H = \cup_{j=1}^{s} A_j = \cup_{j=1}^{s} A'_j$, where for all $j \in \{1, 2, \ldots, s\}$, $A_j = f^{-1}\big(f(h_j)\big)$, $A'_j = g^{-1}\big(g(h_j)\big)$.
Set $A_j = \{h_{1,j}, h_{2,j}, \ldots, h_{a_j,j}\}$, $A'_j = \{h'_{1,j}, h'_{2,j}, \ldots, h'_{a_j,j}\}$.
We order H as follows:
$\forall j \in \{1, 2, \ldots, s\}$, $\forall k, k' \in \{1, 2, \ldots, a_j\}$, $h_{k,j} < h_{k',j} \Leftrightarrow k < k'$,
$\forall j, j' \in \{1, 2, \ldots, s\}, j \neq j'$, $\forall k \in \{1, 2, \ldots, a_j\}, \forall k' \in \{1, 2, \ldots, a'_j\}$, $h_{k,j} < h_{k',j'} \Leftrightarrow j < j'$.
For $i, j \in \{1, 2, \ldots, s\}, k \in \{1, 2, \ldots, a_i\}, l \in \{1, 2, \ldots, a_j\}$ we have

$$h_{k,i} \diamond_f h_{l,j} = \cup_{t \in [i \wedge j, i \vee j]} A_t,$$

$$h_{k,i} \diamond_g h_{l,j} = \cup_{t \in [i \wedge j, i \vee j]} A'_t.$$

Consider the map: $\phi : (H, \diamond_f) \to (H, \diamond_g)$, $\phi(h_{k,i}) = h'_{k,i}$.
We have

$$\phi(h_{k,i} \diamond_f h_{l,j}) = h'_{k,i} \diamond_g h'_{l,j} = \phi\big(h_{k,i}\big) \diamond_g \phi(h_{l,j}),$$

which means that $(H, \diamond_f) \cong (H, \diamond_g)$.
Suppose now that $\lambda(g) = \tau(\lambda(f))$.
We have $H = \cup_{j=1}^{s} A_j = \cup_{j'=1}^{s} A'_{j'}$ where $j' = \tau(j) = s - j + 1$.
Moreover, $A_j = \{h_{1,j}, h_{2,j}, \ldots, h_{a_j,j}\}$, $A'_{j'} = \{h'_{1,j'}, h'_{2,j'}, \ldots, h'_{a'_{j'},j'}\}$ and $A_j = f^{-1}\big(f(h_j)\big)$, $A'_{j'} = g^{-1}\big(g(h_{j'})\big)$ with $a'_{j'} = a_j$.
Consider the function: $\phi : (H, \diamond_f) \to (H, \diamond_g)$, $\phi(h_{k,j}) = h'_{k,j'}$. We obtain
$\phi(h_{k,i} \diamond_f h_{l,j}) = \phi(\cup_{i \wedge j \leq t \leq i \vee j} A_t) = \cup_{\tau(i \wedge j) \leq \tau(t) \leq \tau(i \vee j)} A'_{\tau(t)} = h'_{k,i'} \diamond_g h'_{l,j'} =$
$\phi(h_{k,i}) \diamond_g \phi(h_{l,j})$.

Therefore, $(H, \diamond_f) \cong (H, \diamond_g)$.
"$\Rightarrow$"
Let $p : H \to \{1, 2, \ldots, s\}$ be defined as follows: $p(h) = j$ such that $h \in A_j$. Similarly, $p' : H \to \{1, 2, \ldots, s'\}$ is defined, where $s' = |H/_{\sim_g}|$.
Denote the isomorphism by $\phi : (H, \diamond_f) \to (H, \diamond_g)$.
Denote the set $\{t \mid u \wedge v \leq t \leq u \vee v\}$ by $I(u, v)$. For all $h, k \in H$ we have

$$\phi(h \diamond_f k) = \phi(\cup_{j \in I\left(p(h), p(k)\right)} A_j) = \cup_{j \in I\left(p(h), p(k)\right)} \phi(A_j).$$

On the other hand, $\phi(h) \diamond_g \phi(k) = \cup_{j \in I\left(p'(\phi(h)), p'(\phi(k))\right)} A'_j$.
For every $j \in \{1, 2, \ldots, s\}$ and every $h \in A_j$ we have
$\phi(A_j) = \phi(h \diamond_f h) = \phi(h) \diamond_g \phi(h) = A'_{p'(\phi(h))}$.
Consider the function $\alpha : \{1, 2, \ldots, s\} \to \{1, 2, \ldots, s'\}, \alpha(p(h)) = p'(\phi(h))$. We obtain

$$\alpha : I\big(p(h), p(k)\big) \to I\Big(p'\big(\phi(h)\big), p'\big(\phi(k)\big)\Big).$$

α is injective:

Indeed, if $\alpha(j_1) = \alpha(j_2)$ then $\phi(A_{j_1}) = \phi(A_{j_2})$. Hence, for every $n \in \{1, 2, \ldots a_{j_1}\}$ there exists $t \in \{1, 2, \ldots a_{j_2}\}$ such that $\phi(h_{n,j_1}) = \phi(h_{t,j_2})$ whence $j_1 = j_2$, which means that α is injective.

α is surjective:

Indeed, for each $t \in I\big(\alpha(i), \alpha(j)\big)$, there is $u \in I(i, j)$ for which $\alpha(u) = t$ since $p'(\phi(A_i)) = \alpha(i)$.

Therefore, α is a bijective function from $I(i, j)$ to $I(\alpha(i), \alpha(j))$.

Particularly, $\alpha : I(1, s) \to I(\alpha(1), \alpha(s))$. We have

$$\phi(H) = H = \cup_{j \in \{1,\ \ldots,\ s\}} \phi(A_j) = \cup_{j \in \{1,\ \ldots,\ s\}} A'_{\alpha(j)} = \cup_{t \in I\big(\alpha(1), \alpha(s)\big)} A'_t.$$

So, $I\big(\alpha(1), \alpha(s)\big) = I(1, s')$, whence it follows that $\{\alpha(1), \alpha(s)\} = \{1, s'\}$. We have

$$s' = |H/\sim_g| = |\alpha(p(H))| = |p(H)| = s.$$

Hence, $\{\alpha(1), \alpha(s)\} = \{1, s\}$.

Moreover, for all $j \in \{1, \ldots, s\}$, we have $a_j = |A_j| = |\phi(A_j)| = |A'_{\alpha(j)}| = a'_{\alpha(j)}$.

From $\{\alpha(1), \alpha(s)\} = \{1, s\}$ it follows that

$$\begin{aligned} I(2, s-1) &= I(\alpha(1), \alpha(s)) - \{\alpha(1), \alpha(s)\} \\ &= \alpha(I(1, s)) - \{\alpha(1), \alpha(s)\} \\ &= I(\alpha(2), \alpha(s-1)), \end{aligned}$$

so $\{\alpha(2), \alpha(s-1)\} = \{2, s-1\}$.

Hence,

$$\forall k,\ \alpha(k) = \{k, s-k+1\}. \tag{2}$$

Denote by B the set of bijections of $I(1, s)$ defined to itself.

We show that $\forall k,\ \alpha(k) = k$ or $\forall k,\ \alpha(k) = s - k + 1$. Denote $s - k + 1$ by $\tau(k)$.

For $s \leq 3$, we have $B = \{id_{I(1,s)}, \tau\}$.

If $s > 3$ and we suppose that $\alpha(1) = 1, \alpha(2) = \tau(2) = s - 1$, then $\alpha(I(1, 2)) = I(1, s-1)$, so $2 = |\alpha(I(1, 2))| \neq |I(1, s-1)| \geq 3$, which is a contradiction.

Similarly, for $\alpha(1) = s, \alpha(2) = 2$, we obtain a contradiction.

Suppose now that there is $k \in I(1, s)$ such that

$$\alpha(I(1, k)) = id_{I(1,k)} \text{ and } \alpha(k+1) = \tau(k+1).$$

We obtain

$$\begin{aligned} k+1 &= |I(1, k+1)| \\ &= |\alpha(I(1, k+1))| \\ &= k + s - 2k, \end{aligned}$$

whence $s = 2k + 1$.

Therefore, $\alpha(k+1) = k+1$, which means that

$$\alpha(I(1, k+1)) = id_{I(1,k+1)}.$$

If $\alpha(I(1, k)) = \tau(I(1, k)),\ \alpha(k+1) = k+1$, then

$$k + 1 = |\alpha(I(1, k+1))| = k + s - 2k,$$

whence $s = 2k + 1$. Thus, $\tau(k+1) = k+1$, that is $\alpha(I(1, s+1)) = \tau(I(1, k+1))$.

Hence, $\lambda(f) = \lambda(g)$ or $\lambda(g) = \tau(\lambda(f))$. $\square$

Now we calculate how many isomorphism classes for join spaces can be constructed in this way.

Denote by $Q_f(n)$ the quotient set which contains classes of join spaces $(H, \diamond_f)$ associated with maps $f : H \to C$.

Theorem 8. *(i)* *If* $n = 2k + 1$ *then* $|Q_f(n)| = 2^{k-1}(2^k + 1)$.
(ii) *If* $n = 2k$ *then* $|Q_f(n)| = 2^{k-1}(2^{k-1} + 1)$.

Proof. Denote by $(o.p.)(n)$ the set of ordered partitions of n.

According to [23], we have $|(o.p.)(n)| = 2^{n-1}$.

Let us number the symmetrical ordered partitions of n, that is partitions $(a_1, a_2, \ldots, a_s)$ for which $\tau(a_1, a_2, \ldots, a_s) = (a_1, a_2, \ldots, a_s)$.

Set $(s.o.p.)(n)$ the set of all symmetrical ordered partitions of n.

We have the two cases:

Case 1.

If $n = 2k + 1$ and if $p \in (s.o.p.)(n)$, then $p = (2k + 1)$ or $p = (i_1, \ldots, i_s, 2t + 1, i_s, \ldots, i_1)$, where $t \in \{0, 1, \ldots, k - 1\}$ and $s \in \{k - t, k - t - 1, \ldots, 1\}$.

We have

$$2\sum_{j=1}^{s} i_j + 2t + 1 = 2k + 1 \text{ whence } \sum_{j=1}^{s} i_j = k - t.$$

According to [23], for all t we have $|(s.o.p.)(k - t)| = 2^{k-t-1}$. Hence,

$$|(s.o.p.)(n)| = \sum_{t=0}^{k-1} 2^{k-t-1} + 1 = 2^{(k-1)+1} = 2^k.$$

Case 2.

If $n = 2k$ and if $p \in (s.o.p.)(n)$, then $p = (2k)$ or $p = (i_1, \ldots, i_s, 2t, i_s, \ldots, i_1)$, where $t \in \{0, 1, \ldots, k - 1\}$ and $s \in \{k - t, k - t - 1, \ldots, 1\}$.

We obtain $\sum_{j=1}^{s} i_j = k - t$.

According to [23], for all t we have $|(s.o.p.)(k - t)| = 2^{k-t-1}$.

Hence,

$$|(s.o.p.)(n)| = \sum_{t=0}^{k-1} 2^{k-t-1} + 1 = 2^k.$$

Therefore, we can conclude:

- If $n = 2k + 1$ then $|Q_f(n)| = 2^k + 1/2(2^{n-1} - 2^k) = 2^{k-1}(2^k + 1)$;
- If $n = 2k$ then $|Q_f(n)| = 2^k + 1/2(2^{n-1} - 2^k) = 2^{k-1}(2^{k-1} + 1)$.

□

4. Join Spaces Associated with a Chain: The General Case

In this section, we consider an arbitrary set H and we analyze when the join spaces associated with a chain are isomorphic.

Let us present first the context:

Let $f : H \to C$ and consider the equivalence relation on H:

$$h \sim_f k \Leftrightarrow f(h) = f(k).$$

We order $H/_{\sim_f}$ as follows: $[h]_f \leq [k]_f \Leftrightarrow f(h) \leq f(k)$, for $h, k \in H$.

Denote $H/_{\sim_f} = \{[h_i]_f \mid i \in I\}$ and $[h_i]_f$ by H_i, for all $i \in I$.

We order I as follows:

$$i \leq j \Leftrightarrow \forall h \in H_i, \forall k \in H_j,\ f(h) \leq f(k).$$

Since C is a chain, it follows that $(I, \leq)$ is a chain, too.

Moreover, for all $i \in I$ denote $|H_i| = \alpha_i$.

If $g : H \to C$, then $H/_{\sim_g} = \{H'_{i'} \mid i' \in I'\}$. Similarly,

$$i' \leq j' \Leftrightarrow \forall h' \in H'_{i'}, \forall k' \in H'_{j'},\ g(h') \leq g(k')$$

and for all $i' \in I'$ denote $|H'_{i'}| = \alpha_{i'}$. We have that $(I', \leq)$ is a chain, too.

Theorem 9. *If $f, g : H \to C$ are two functions, then $(H, \diamond_f) \cong (H, \diamond_g)$ if and only if there exists a strictly monotonous bijection $\varphi : I \to I' :\ \forall i \in I, \alpha_i = \alpha'_{\varphi(i)}$.*

Proof. "$\Leftarrow$"

For all $i \in I,\ |H_i| = \alpha_i = \alpha'_{\varphi(i)} = |H'_{\varphi(i)}|$. Define

$\phi : (H, \diamond_f) \to (H, \diamond_g)$ as follows: $\forall i \in I, \forall h_i \in H_i, \phi(h_i) = h'_{\varphi(i)}$ where we choose $h'_{\varphi(i)} \in H'_{\varphi(i)}$. Now, $\forall i, j \in I$, $\forall h_i \in H_i, h_j \in H_j$ we have

$$\phi(h_i) \diamond_g \phi(h_j) = h'_{\varphi(i)} \diamond_g h'_{\varphi(j)} = \cup_{t \in [\varphi(i) \wedge \varphi(j), \varphi(i) \vee \varphi(j)]} H'_{t'},$$

$$\phi(h_i \diamond_f \phi h_j) = \cup_{k \in [i \wedge j, i \vee j} H'_{\varphi(k)]}.$$

If φ is strictly increasing, then $\varphi(i \wedge j) = \varphi(i) \wedge \varphi(j),\ \varphi(i \vee j) = \varphi(i) \vee \varphi(j)$.

Since φ is a bijective function, we have

$$\phi(h_i \diamond_f h_j) = \cup_{t' \in [\varphi(i) \wedge \varphi(j), \varphi(i) \vee \varphi(j)]} H'_{t'}.$$

Hence, $\phi(h_i \diamond_f h_j) = \phi(h_i) \diamond_g \phi(h_j)$.

If φ is strictly decreasing, then $\varphi(i \wedge j) = \varphi(i) \vee \varphi(j),\ \varphi(i \vee j) = \varphi(i) \wedge \varphi(j)$.

Since φ is a bijection, we have

$$\phi(h_i \diamond_f h_j) = \cup_{t' \in [\varphi(i) \wedge \varphi(j), \varphi(i) \vee \varphi(j)]} H'_{t'}.$$

Hence, $\phi(h_i \diamond_f h_j) = \phi(h_i) \diamond_g \phi(h_j)$.

Therefore, ϕ is an isomorphism.

"$\Rightarrow$"

Denote by ϕ the isomorphism from $(H, \diamond_f)$ to $(H, \diamond_g)$. Set $p : H \to I$ and $p' : H \to I'$, where for all $h_i \in H_i,\ h'_{i'} \in H'_{i'}$, we obtain $p(h_i) = i$ and $p'(h'_{i'}) = i'$.

For $h, k \in H_i$ we obtain $h \diamond_f h = h \diamond_f k = H_i$, whence

$$\phi(h) \diamond_g \phi(k) = \phi(h \diamond_f k) = \phi(h) \diamond_g \phi(k) = H'_{\varphi(i)}.$$

Hence, $p'(\phi(h)) = p'(\phi(k))$.

Define $\varphi : I \to I'$ by : $\varphi(i) = p'(\phi(x))$ where $p(x) = i$.

We check that φ is a bijective function.

Suppose that there are $i_1, i_2 \in I,\ i_1 \neq i_2$ such that $\varphi(i_1) = \varphi(i_2)$.

Thus, $\phi(H_{1_1}) = \phi(H_{i_2})$, which is a contradiction with $H_{i_1} \cap H_{i_2} = \emptyset$.

On the other hand, since $\phi(H) = H$ we obtain

$$H = \cup_{i' \in I'} H'_{i'} = \phi(\cup_{i \in I} H_i) = \cup_{i \in I} \phi(H_i) = \cup_{i \in I} H'_{\varphi(i)} = \cup_{i \in Im\varphi} H'_{\varphi(i)},$$

whence $I' = Im\varphi$. Hence, φ is bijective and $|I| = |I'|$.

Thus, $\forall i \in I,\ |H_{\varphi(i)}| = |\phi(H_i)| = |H_i|$.

Hence, $\forall i \in I, \alpha'_{\varphi(i)} = \alpha_i$. Let us prove now that φ is strictly monotonous.

If $i \leq j$ are elements of I, then we denote $[i, j] = \{t \in I | i \wedge j \leq t \leq i \vee j\}$ and for $i', j' \in I, i \leq j'$ then $[i', j'] = \{t' \in I' | i' \wedge j' \leq t' \leq i' \vee j'\}$.

If $i, j \in I, i \le j$ and $h \in H_i, k \in H_j$, then $\phi(h \diamond_f k) = \phi(h) \diamond_g \phi(k)$ whence

$$\phi(\cup_{t\in[i,j]} H_t) = \cup_{t\in[i\wedge j, i\vee j]} H'_{\varphi(t)}.$$

We obtain that

$$\forall i \le j,\ \varphi([i,j]) = [\varphi(i) \wedge \varphi(j), \varphi(i) \vee \varphi(j)]. \quad (3)$$

For $i \ne j$ we have $\varphi(i) < \varphi(j)$ or $\varphi(j) < \varphi(i)$.

If $\varphi(i) < \varphi(j)$, then φ is strictly increasing on I. Indeed, it follows from (3).

Similarly, if $\varphi(j) < \varphi(i)$, then φ is strictly decreasing on I.

Therefore, we obtain the thesis. □

Let us present some examples.

Example 4. *If $f, g : H \to [a,b]$, where $[a,b]$ is a real interval and $g(h) = a + b - f(h)$, then $[h]_f = [h]_g$, whence $I = I'$ and φ is the identity function. Hence, $(H, \diamond_f) \cong (H, \diamond_g)$.*

Example 5. *If $f, g : H \to \mathbb{R}$, where $\mathbb{R}$ is the real number set and $g(h) = -f(h)$, then $[h]_f = [h]_g$, and again $I = I'$, φ is the identity function and $(H, \diamond_f) \cong (H, \diamond_g)$.*

Example 6. *If $f, g : H \to C$ are such that $|H/\sim_f| = |H/\sim_g| = n$, $H/\sim_f = \{H_1, H_2, \ldots, H_n\}$, $H/\sim_g = \{H'_1, H'_2, \ldots, H'_n\}$ such that $|H_1| = |H'_n| = 1$, $|H_2| = |H'_{n-1}| = 2$, and so on.*

In general, $|H_i| = |H'_{n-i+1}| = i$ for all $i \in \{1, 2, \ldots, n\}$.

Then, $I = I' = \{1, 2, \ldots, n\}$ and $\varphi(i) = n - i + 1$ is a strictly decreasing function. Hence, $(H, \diamond_f) \cong (H, \diamond_g)$.

5. Conclusions

We study classes of isomorphism for join spaces associated with chains and a combinatorial problem is analyzed for the finite case, which is to determine the number of the classes of isomorphism.

As a future problem, we can study classes of isomorphism for join spaces associated with lattices. For two maps $f, g : H \to L$, where $(L, \vee, \wedge)$ is a lattice, we intend to determine when $(H, \diamond_f) \cong (H, \diamond_g)$, to consider the corresponding equivalence classes and examine the finite case.

Another study problem would be to determine hypergroups/join spaces associated with other classes of lattices, such as Boolean lattices and to obtain characterizations of these classes of lattices. In this way, some results of the lattice theory could be demonstrated with the help of the hypergroup theory. For example, in a modular lattice, the ideals are exactly the subhypergroups of the associated join space structure.

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