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Monotonicities of Quasi-Normed Orlicz Spaces

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Abstract: In this paper, we introduce a new Orlicz function, namely a *b*-Orlicz function, which is not necessarily convex. The Orlicz spaces L^{Φ} generated by the *b*-Orlicz function Φ equipped with a Luxemburg quasi-norm contain both classical spaces $L^p(p \geq 1)$ and $L^p(0 < p < 1)$. The Orlicz spaces L^{Φ} are quasi-Banach spaces. Some basic properties in quasi-normed Orlicz spaces are discussed, and the criteria that a quasi-normed Orlicz space is strictly monotonic and lower (upper) locally uniformly monotonic are given.

Keywords: *b*-Orlicz function; Orlicz spaces; quasi-norm; strict monotonicity; lower local uniform monotonicity; upper local uniform monotonicity

MSC: 46A16; 46E30

1. Introduction

The study of quasi-Banach spaces did not start until in the late 1950s and early 1960s, being studied by Klee, Peck, Rolewicz, Waelbroeck and Zelazko. In 1969, the subject greatly promoted this theme by the paper of Duren, Romberg and Shields. The possibility of using quasi-Banach spaces was proved in classical function theory, and many important problems related to the Hahn–Banach theorem were emphasized. A crucial breakthrough was that Roberts, who proved that the Krein–Milman theorem failed in the general quasi-Banach space, developed a powerful new technology in 1976 [1,2].

In 1984, Davis, Garling and Tomczak-Jagermann introduced the concepts of complex convexity moduli and uniform PL-convexity of complex quasi-Banach spaces in [3], and it was shown that $L^p(X)(0 < p < \infty)$ is uniformly PL-convex under the assumption that the continuously quasi-normed space X is uniformly PL-convex. The monotonicity and convexity properties in quasi-Banach lattices were studied by Han Ju Lee in [4] by establishing the relationship between uniform monotonicity, uniform $\mathbb{C}$-convexity, H- and PL-convexity. In [5], examples of separable quasi-Banach spaces were constructed by Albiac and Kalton, which are Lipschitz isomorphic but not linearly isomorphic. Therefore, these showed that the Lipschitz structure of a separable quasi-Banach space does not determine its linear structure. In recent years, the nonlinear structure and geometric properties of quasi-Banach spaces have been studied in more detail (see [6–9]). In this paper, we introduce a new class of Orlicz functions, which remove the restriction of convexity and add a new condition. The Orlicz spaces generated by the new Orlicz function Φ contain both classical spaces $L^p(p \geq 1)$ (see [10]) and $L^p(0 < p < 1)$. Because the generating function has no convexity, the study of geometric properties of quasi-normed Orlicz spaces is more complicated than that of classical Orlicz spaces generated by convex functions. The unit ball of a non-convex Orlicz space is not necessarily convex, since a non-convex Orlicz function could lead to difficulties in applying established mathematical techniques and theories in this context. In ordering to study some basic properties, we first introduce some ideas and terminology.

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Citation: Ji, D.; Cui, Y. Monotonicities of Quasi-Normed Orlicz Spaces. *Axioms* **2024**, *13*, 696. https://doi.org/10.3390/axioms13100696

Academic Editor: Silvestru Sever Dragomir

Received: 23 August 2024
Revised: 22 September 2024
Accepted: 30 September 2024
Published: 7 October 2024

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2. Preliminaries

Let X be a vector space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A non-negative mapping defined on X is called a quasi-norm if it satisfies the following properties:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{F}$ and all $x \in X$;
(iii) There exists $K \geq 1$ such that $\|x_1 + x_2\| \leq K(\|x_1\| + \|x_2\|)$ for all $x_1, x_2 \in X$.

We say that X is a quasi-normed space if X equipped with the quasi-norm.

The Aoki–Rolewicz theorem (see [11,12]) shows that if $0 < p \leq 1$ is given, then there exists a constant B such that

$$\left\| \sum_{k=1}^{n} x_k \right\| \leq B \left(\sum_{k=1}^{n} \|x_k\|^p \right)^{1/p}, \tag{1}$$

for any $\{x_k\}_{k=1}^n$ in X. Consequently, we can find an equivalent p-subadditive quasi-norm $|||\cdot|||$, i.e.,

$$|||x_1 + x_2||| \leq (|||x_1|||^p + |||x_2|||^p)^{1/p},$$

for any $x_1, x_2 \in X$. X is said to p-normable if (1) holds. We will say that X is p-normed if the quasi-norm is p-subadditive on X. The complete quasi-normed space X with respect to the quasi-norm topology is called a quasi-Banach space. In general, we will assume that a quasi-Banach space is p-normed for some $p > 0$. For background information on quasi-Banach spaces, we refer the reader to [13] or [14].

Definition 1 (see [10]). *A function $\Phi : \mathbb{R} \to [0, +\infty)$ is called an Orlicz function if it has the following properties:*

(i) *Φ is even, continuous, convex and $\Phi(0) = 0$;*
(ii) *$\Phi(u) > 0$ for all $u > 0$.*

It is clear that $\Phi(u) = |u|^p$, where $1 < p < +\infty$, is an Orlicz function. In this paper, we will introduce a new class of Orlicz functions, as follows:

Definition 2. *A function $\Phi : \mathbb{R} \to [0, +\infty)$ is called a b-Orlicz function if it has the following properties:*

(i) *Φ is even, continuous and $\Phi(0) = 0$;*
(ii) *Φ is nondecreasing on $[0, +\infty)$ and $\Phi(u) > 0$ for all $u > 0$;*
(iii) *There exists $b > 1$ such that $\Phi(\frac{u}{b}) \leq \frac{1}{2}\Phi(u)$ for all $u \in \mathbb{R}$.*

Example 1. *Let $\Phi(u) = |u|^p$, where $0 < p < 1$, then Φ is a b-Orlicz function. In fact, we can take $b = 2^{\frac{1}{p}}$ such that $\Phi(\frac{u}{b}) \leq \frac{1}{2}\Phi(u)$ for all $u \in \mathbb{R}$.*

The following example shows that there exists a function which satisfies parts (i) and (ii) of Definition 2 but does not satisfy part (iii) of Definition 2.

Example 2. *Let*

$$\Phi(u) = \begin{cases} |u|, & |u| \in [0, b_1), \\ \frac{a_{n+1} - a_n}{b_{n+1} - b_n}(|u| - b_n) + a_n, & |u| \in [b_n, b_{n+1}), n = 1, 2, \ldots. \end{cases}$$

where $b_1 = 2$, $b_{n+1} = b_n^2$, $a_1 = 2$, $a_{n+1} = 2(a_n - \frac{1}{3})$, $n = 1, 2, \ldots$.

Now let us verify that, for any $b > 1$, there exists $u_0 > 0$ such that $\Phi(\frac{u_0}{b}) > \frac{1}{2}\Phi(u_0)$. In fact, for any $b > 1$, we can find b_{n_0} such that $b \leq b_{n_0}$. If we take $u_0 = bb_{n_0}$, we get

$$\frac{1}{2}\Phi(u_0) = \frac{1}{2}\Phi(bb_{n_0}) \leq \frac{1}{2}\Phi(b_{n_0}^2) = \frac{1}{2}\Phi(b_{n_0+1}) = \frac{1}{2}a_{n_0+1}$$
$$= a_{n_0} - \frac{1}{3} < a_{n_0} = \Phi(b_{n_0}) = \Phi(\frac{u_0}{b}).$$

Definition 3. *Let (G, Σ, μ) be a finite non-atomic measure space, i.e., (G, Σ, μ) is a measure space for which $\mu(G) < +\infty$, and if $A \in \Sigma$ with $\mu(A) > 0$, then, for any $a \in (0, \mu(A))$, there exists a subset $A_0 \subset A$ such that $\mu(A_0) = a$. Moreover, let L^0 be the space of all (equivalence) classes of Σ-measurable real-valued functions defined on G. Any b-Orlicz function Φ determines a mapping $I_\Phi : L^0 \to [0, +\infty]$, so we define the modular $I_\Phi(x) = \int_G \Phi(x(t))dt$ and the Orlicz space $L^\Phi = \{x \in L^0 : I_\Phi(\lambda x) < +\infty$ for some $\lambda > 0\}$.*

Lemma 1 (Levi Theorem). *Let $\{x_n\}_{n=1}^\infty$ be a sequence of measurable functions defined on G. If $0 \leq x_n(t) \leq x_{n+1}(t)$ for all $t \in G$ and $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n(t) = x(t)$ for almost all $t \in G$, then*

$$\lim_{n\to\infty} \int_G x_n(t)dt = \int_G x(t)dt$$

For any $x \in L^\Phi$, let

$$\|x\|_\Phi = \inf\{\lambda > 0 : I_\Phi(\frac{x}{\lambda}) \leq 1\}.$$

We can easily get that $\|\cdot\|_\Phi$ is a quasi-norm on L^Φ, called the Luxemburg quasi-norm. In fact, we only prove that $\|\cdot\|_\Phi$ satisfies part (iii) of the quasi-norm definition. By the definition of $\|\cdot\|_\Phi$, there exists $\lambda_n \downarrow \|x\|_\Phi$ such that $I_\Phi\left(\frac{x}{\lambda_n}\right) \leq 1$, and Lemma 1 ensures that

$$I_\Phi\left(\frac{x}{\|x\|_\Phi}\right) \leq 1,$$

for any $x \in L^\Phi \setminus \{0\}$. Then, for any $x, y \in L^\Phi \setminus \{0\}$, we obtain

$$\begin{aligned} I_\Phi\left(\frac{x+y}{b(\|x\|_\Phi + \|y\|_\Phi)}\right) &\leq \frac{1}{2} I_\Phi\left(\frac{x+y}{\|x\|_\Phi + \|y\|_\Phi}\right) \\ &= \frac{1}{2} I_\Phi\left(\frac{\|x\|_\Phi}{\|x\|_\Phi + \|y\|_\Phi}\frac{x}{\|x\|_\Phi} + \frac{\|y\|_\Phi}{\|x\|_\Phi + \|y\|_\Phi}\frac{y}{\|y\|_\Phi}\right) \\ &\leq \frac{1}{2} I_\Phi\left(\max\{\frac{x}{\|x\|_\Phi}, \frac{y}{\|y\|_\Phi}\}\right) \\ &\leq \frac{1}{2}\left(I_\Phi(\frac{x}{\|x\|_\Phi}) + I_\Phi(\frac{y}{\|y\|_\Phi})\right) \leq 1. \end{aligned}$$

Hence,

$$\|x + y\|_\Phi \leq b(\|x\|_\Phi + \|y\|_\Phi).$$

Definition 4 (see [10]). *We say that Φ satisfies the Δ_2-condition ($\Phi \in \Delta_2$ in short) if there exist constants $K > 2$ and $u_0 > 0$ such that*

$$\Phi(2u) \leq K\Phi(u),$$

whenever $u \geq u_0$.

Definition 5 (see [10]). *We say that Φ satisfies the Δ_l-condition ($\Phi \in \Delta_l$ in short) if, for any $l > 1$ and $u_0 > 0$, there exists a constant $K > 0$ such that*

$$\Phi(lu) \leq K\Phi(u),$$

whenever $u \geq u_0$.

It is easy to prove the following results in the same way as for the convex Orlicz function Φ (see [10]).

Lemma 2. *$\Phi \in \Delta_2$ if and only if $\Phi \in \Delta_l$.*

Definition 6 (see [10]). *We say that Φ satisfies the $\Delta^s_{1+\varepsilon}$-condition ($\Phi \in \Delta^s_{1+\varepsilon}$ in short) if, for any $\varepsilon > 0$ and $u_0 > 0$, there exists $\delta \in (0,1)$ such that*

$$\Phi((1+\delta)u) \leq (1+\varepsilon)\Phi(u),$$

whenever $u \geq u_0$.

Definition 7. *A point u_0 is said to be a strictly monotone point of Φ if, for any $u > u_0$, we have $\Phi(u) > \Phi(u_0)$.*

Definition 8 (see [8]). *A quasi-normed space $(X, \|\cdot\|)$ is called a quasi-normed Köthe space if it is a linear subspace of L^0 satisfying the following conditions:*

(i) If $x \in L^0$, $y \in X$ and $|x| \leq |y|$ a.e., then $x \in X$ and $\|x\| \leq \|y\|$;
(ii) There exists a strictly positive $x \in X$ (called a weak unit).

Clearly, each quasi-normed Köthe space is a quasi-Banach lattice.

Definition 9 (see [8]). *A quasi-Banach lattice $(X, \|\cdot\|)$ is said to be strictly monotone ($X \in (SM)$ for short) if, for any $x, y \in X$ such that $0 \leq y \leq x$, we have $\|y\| < \|x\|$ whenever $y \neq x$ (or equivalently $\|x - y\| < \|x\|$ whenever $y \neq 0$).*

Definition 10 (see [8]). *A quasi-Banach lattice $(X, \|\cdot\|)$ is said to be lower locally uniformly monotone ($X \in (LLUM)$ for short) if for any $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ in X such that $0 \leq x_n \leq x$ for all $n \in \mathbb{N}$ and $\|x_n\| \to \|x\|$ as $n \to \infty$, the condition $\|x - x_n\| \to 0$ as $n \to \infty$ holds.*

Definition 11 (see [8]). *A quasi-Banach lattice $(X, \|\cdot\|)$ is said to be upper locally uniformly monotone ($X \in (ULUM)$ for short) if for any $x \in X_+$ (the positive cone in X) and $\{x_n\}_{n=1}^{\infty}$ in X_+ such that $x \leq x_n$ for all $n \in \mathbb{N}$ and $\|x_n\| \to \|x\|$ as $n \to \infty$, the condition $\|x_n - x\| \to 0$ as $n \to \infty$ holds.*

3. Main Results

Lemma 3. *For any $\{x_n\}_{n=1}^{\infty} \subset L^{\Phi}$ and $x \in L^{\Phi}$, the following statements are true:*

(i) If $\lim\limits_{n\to\infty} \|x_n\|_{\Phi} = 0$, then $\lim\limits_{n\to\infty} I_{\Phi}(x_n) = 0$;
(ii) If $\|x\|_{\Phi} \leq 1$, then $I_{\Phi}(x) \leq 1$.

Proof. (i) By $\|x_n\|_{\Phi} \to 0$ as $n \to \infty$, we get that for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{2^{n_0}} < \varepsilon$ and $\|x_n\|_{\Phi} < \frac{1}{b^{n_0}}$, whenever $n > n_0$. Hence,

$$I_{\Phi}(x_n) = I_{\Phi}\left(\|x_n\|_{\Phi} \frac{x_n}{\|x_n\|_{\Phi}}\right) \leq I_{\Phi}\left(\frac{1}{b^{n_0}} \frac{x_n}{\|x_n\|_{\Phi}}\right) \leq \frac{1}{2^{n_0}} I_{\Phi}\left(\frac{x_n}{\|x_n\|_{\Phi}}\right) < \varepsilon,$$

whenever $n > n_0$. This shows that $I_{\Phi}(x_n) \to 0$ as $n \to \infty$.

(ii) $I_{\Phi}(x) = I_{\Phi}\left(\|x\|_{\Phi} \frac{x}{\|x\|_{\Phi}}\right) \leq I_{\Phi}\left(\frac{x}{\|x\|_{\Phi}}\right) \leq 1$. □

Theorem 1. *$(L^{\Phi}, \|\cdot\|_{\Phi})$ is a quasi-Banach space.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ in L^{Φ} be a Cauchy sequence.

(1) We first prove that $\{x_n(t)\}_{n=1}^{\infty}$ is a Cauchy sequence with respect to the measure. Assume that $\{x_n(t)\}_{n=1}^{\infty}$ is not a Cauchy sequence with respect to the measure, then there exist $\varepsilon_0 > 0$, $\delta_0 > 0$ and two subsequences $\{x_{m_i}\}$, $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\mu(\{t \in G : |x_{m_i}(t) - x_{n_i}(t)| \geq \varepsilon_0\}) \geq \delta_0.$$

By step (1) of Lemma 3 and $\|x_{m_i} - x_{n_i}\|_\Phi \to 0$ as $i \to \infty$, we have $I_\Phi(x_{m_i} - x_{n_i}) \to 0$ as $i \to \infty$. On the other hand,

$$\begin{aligned} I_\Phi(x_{m_i} - x_{n_i}) &= \int_G \Phi(x_{m_i}(t) - x_{n_i}(t))dt \\ &\geq \int_{\{t\in G:|x_{m_i}(t)-x_{n_i}(t)|\geq\varepsilon_0\}} \Phi(x_{m_i}(t) - x_{n_i}(t))dt \\ &\geq \Phi(\varepsilon_0)\delta_0 > 0, \end{aligned}$$

which is a contradiction.

(2) Since L^0 is complete in measure convergent, there exists $x \in L^0$ such that $x_n(t) \xrightarrow{\mu} x(t)$. By the Riesz theorem, there exists $\{x_{n_i}\} \subset \{x_n\}$ such that

$$\lim_{i\to\infty} x_{n_i}(t) = x(t), \quad \text{a. e.} \quad t \in G.$$

(3) We will prove that $x \in L^\Phi$. $\{x_{n_i}\}$ is a Cauchy sequence in quasi-norm $\|\cdot\|_\Phi$, which implies that for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $I_\Phi(x_{n_i} - x_{n_j}) < \varepsilon$ as $i, j > n_0$, i.e.,

$$\int_G \Phi(x_{n_i}(t) - x_{n_j}(t))dt < \varepsilon.$$

Let $j \to \infty$. Thus, with the Fatou Lemma, we have

$$\int_G \Phi(x_{n_i}(t) - x(t))dt \leq \liminf_{j\to\infty} \int_G \Phi(x_{n_i}(t) - x_{n_j}(t))dt \leq \varepsilon.$$

Hence, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $I_\Phi(x_{n_i} - x) < \varepsilon$ as $i > n_0$, that is,

$$\lim_{i\to\infty} I_\Phi(x_{m_i} - x) = 0.$$

Thus, $I_\Phi(x_{m_i} - x) < +\infty$. As Φ is increasing, we have

$$I_\Phi(\frac{x}{2}) = I_\Phi\left(\frac{x - x_{n_i}}{2} + \frac{x_{n_i}}{2}\right) \leq I_\Phi(x - x_{n_i}) + I_\Phi(x_{n_i}) < +\infty$$

and the inequality holds.

(4) We will prove that $\lim_{i\to\infty} \|x_{n_i} - x\|_\Phi = 0$. As $\{x_{n_i}\}$ is a Cauchy sequence in quasi-norm $\|\cdot\|_\Phi$, we know that for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\|x_{n_i} - x_{n_j}\|_\Phi \leq \varepsilon$ for all $i, j > n_0$, that is,

$$\left\|\frac{x_{n_i} - x_{n_j}}{\varepsilon}\right\|_\Phi \leq 1.$$

Then,

$$I_\Phi\left(\frac{x_{n_i} - x_{n_j}}{\varepsilon}\right) = \int_G \Phi\left(\frac{x_{n_i}(t) - x_{n_j}(t)}{\varepsilon}\right)dt \leq 1.$$

Let $j \to \infty$. Thus, with the Fatou Lemma, we have

$$\begin{aligned} I_\Phi\left(\frac{x_{n_i} - x}{\varepsilon}\right) &= \int_G \Phi\left(\frac{x_{n_i}(t) - x(t)}{\varepsilon}\right)dt \\ &\leq \liminf_{j\to\infty} \int_G \Phi\left(\frac{x_{n_i}(t) - x_{n_j}(t)}{\varepsilon}\right)dt \leq 1. \end{aligned}$$

Hence, $\|x_{n_i} - x\|_\Phi \leq \varepsilon$, whenever $n \geq n_0$. That is, $\lim_{i\to\infty} \|x_{n_i} - x\|_\Phi = 0$.

(5) $\lim_{n\to\infty} \|x_n - x\|_\Phi = 0$. Using

$$\begin{aligned}\|x_n - x\|_\Phi &= \|x_n - x_{n_i} + x_{n_i} - x\|_\Phi \\ &\le b\|x_n - x_{n_i}\|_\Phi + b\|x_{n_i} - x\|_\Phi\end{aligned}$$

we can easily get $\lim_{n\to\infty} \|x_n - x\|_\Phi = 0$. Consequently, we obtain $(L^\Phi, \|\cdot\|_\Phi)$, which is a quasi-Banach space. □

Theorem 2. *$I_\Phi\left(\frac{x}{\|x\|_\Phi}\right) = 1$ if and only if $\Phi \in \Delta_2$.*

Proof. Sufficiency. If $\Phi \in \Delta_2$, then $I_\Phi\left(\frac{x}{\lambda}\right)$ is a nonincreasing continuous function of λ on $(0, +\infty)$. By the definition of $\|\cdot\|_\Phi$, we obtain $I_\Phi\left(\frac{x}{\|x\|_\Phi}\right) = 1$.

Necessity. If $\Phi \notin \Delta_2$, by Lemma 2, there exists $u_k \uparrow +\infty$ such that $\frac{1}{\Phi(u_1)} \le \mu(G_0)$ and

$$\Phi\left(\left(1 + \frac{1}{k}\right)u_k\right) > 2^k \Phi(u_k),$$

where $G_0 \in \Sigma$ with $\mu(G_0) > 0$ are given previously. Select a sequence $\{G_k\}$ of disjoint subsets of G_0 such that

$$\mu(G_k) = \frac{1}{2^k \Phi(u_k)},\ k = 1, 2, \ldots.$$

Moreover, define

$$x_n(t) = \sum_{k=n+1}^{\infty} u_k \chi_{G_k}(t),\ n = 0, 1, 2, \ldots$$

where $\chi_E(t)$ is the characteristic function of set E. Then, for each $n \ge 0$, we have

$$I_\Phi(x_n) = \int_G \Phi(x_n(t))dt = \sum_{k=n+1}^{\infty} \Phi(u_k)\mu(G_k) = \frac{1}{2^n} < 1.$$

However, for any $\lambda \in (0, 1)$, let $n_0 \in \mathbb{N}$, which satisfies $1 + \frac{1}{n_0} \le \frac{1}{\lambda}$. Then, for all $n \ge n_0$, we have

$$\begin{aligned}I_\Phi\left(\frac{x_n}{\lambda}\right) &> \sum_{k=n+1}^{\infty} \Phi\left(\left(1 + \frac{1}{k}\right)u_k\right)\mu(G_k) \\ &> \sum_{k=n+1}^{\infty} 2^k \Phi(u_k)\mu(G_k) = \sum_{k=n+1}^{\infty} 1 = +\infty.\end{aligned}$$

Hence, $\|x_n\|_\Phi = 1$, which is a contradiction. □

Theorem 3. *$\Phi \in \Delta_2$ if and only if $\lim_{n\to\infty} I_\Phi(x_n) = 0$ implies $\lim_{n\to\infty} \|x_n\|_\Phi = 0$.*

Proof. Necessity. Since $\Phi \in \Delta_2$, for any $\varepsilon > 0$ and $u_0 > 0$, there exists $K > 0$ such that

$$\Phi\left(\frac{u}{\varepsilon}\right) \le K\Phi(u)\ (u \ge u_0).$$

Take a small enough $u_0 > 0$ such that $\Phi\left(\frac{u_0}{\varepsilon}\right)\mu(G) < \frac{1}{2}$. By $\lim_{n\to\infty} \int_G \Phi(x_n(t))dt = 0$, there exists $n_0 \in \mathbb{N}$ such that $\int_G \Phi(x_n(t))dt \le \frac{1}{2K}$ $(n \ge n_0)$. Thus, we get that

$$\begin{aligned} I_\Phi\left(\frac{x_n}{\varepsilon}\right) &= \int_G \Phi\left(\frac{x_n(t)}{\varepsilon}\right)dt \\ &= \int_{\{t\in G:|x_n(t)|<u_0\}} \Phi\left(\frac{x_n(t)}{\varepsilon}\right)dt + \int_{\{t\in G:|x_n(t)|\geq u_0\}} \Phi\left(\frac{x_n(t)}{\varepsilon}\right)dt \\ &\leq \Phi\left(\frac{u_0}{\varepsilon}\right)\mu(G) + K\int_G \Phi(x_n(t))dt \\ &= \frac{1}{2}+\frac{1}{2} = 1 \ (n \geq n_0). \end{aligned}$$

Hence, $\|x_n\|_\Phi \leq \varepsilon$ $(n \geq n_0)$, that is, $\lim_{n\to\infty} \|x_n\|_\Phi = 0$.

Sufficiency. It can be obtained by the same argument as in Theorem 3. □

Theorem 4. *Suppose* $\Phi \in \Delta_2$. *If* $x_n(t) \xrightarrow{\mu} x(t)$ *and* $I_\Phi(x_n) \to I_\Phi(x)$, *then*

$$\|x_n - x\|_\Phi \to 0.$$

Proof. By the absolute continuity of the integral, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_e \Phi(x(t))dt < \frac{\varepsilon}{8},$$

where $\mu(e) < \delta$. Since $x_n(t) \xrightarrow{\mu} x(t)$ and according to the Egorov theorem for the mentioned above $\delta > 0$, we obtain $e_0 \subset G$ such that $x_n(t)$ uniformly converges to $x(t)$ on $G \setminus e_0$. Thus, there exists $n_1 \in \mathbb{N}$ such that

$$|x_n(t) - x(t)| < 2\Phi^{-1}\left(\frac{\varepsilon}{2\mu(G)}\right),$$

whenever $n \geq n_1$. Consequently,

$$\int_{G\setminus e_0} \Phi\left(\frac{x_n(t)-x(t)}{2}\right)dt < \Phi\left(\Phi^{-1}\left(\frac{\varepsilon}{2\mu(G)}\right)\right)\mu(G) = \frac{\varepsilon}{2}.$$

By the assumption that $x_n(t)$ uniformly converges to $x(t)$ on $G \setminus e_0$, we have

$$I_\Phi(x_n\chi_{G\setminus e_0}) \to I_\Phi(x\chi_{G\setminus e_0}).$$

By $I_\Phi(x_n) \to I_\Phi(x)$, we obtain the following:

$$I_\Phi(x_n\chi_{e_0}) \to I_\Phi(x\chi_{e_0}).$$

Then, there exists $n_2 \in \mathbb{N}$ such that

$$\int_{e_0} \Phi(x_n(t))dt < \int_{e_0} \Phi(x(t))dt + \frac{\varepsilon}{4},$$

whenever $n \geq n_2$.

$$\begin{aligned} I_\Phi\left(\frac{x_n - x}{2}\right) &= \int_G \Phi\left(\frac{x_n(t)-x(t)}{2}\right)dt \\ &= \int_{G\setminus e_0} \Phi\left(\frac{x_n(t)-x(t)}{2}\right)dt + \int_{e_0} \Phi\left(\frac{x_n(t)-x(t)}{2}\right)dt \\ &\leq \int_{G\setminus e_0} \Phi\left(\frac{x_n(t)-x(t)}{2}\right)dt + \int_{e_0} \Phi\left(\frac{|x_n(t)|+|x(t)|}{2}\right)dt \\ &\leq \int_{G\setminus e_0} \Phi\left(\frac{x_n(t)-x(t)}{2}\right)dt + \int_{e_0} \Phi(|x_n(t)|)dt + \int_{e_0} \Phi(|x(t)|)dt. \end{aligned}$$

Hence, for any $\varepsilon > 0$, take $n_0 = \max\{n_1, n_2\}$; thus, we get

$$I_\Phi\left(\frac{x_n - x}{2}\right) < \frac{\varepsilon}{2} + 2\int_{e_0} \Phi(x(t))dt + \frac{\varepsilon}{4} < \varepsilon,$$

whenever $n \geq n_0$. Then, by $\Phi \in \Delta_2$, we get $\|x_n - x\|_\Phi \to 0$. □

If Φ is a constant on the interval $[a, b]$ and Φ is not a constant on either $[a - \delta, b]$ or $[a, b + \delta]$ for each $\delta > 0$, then we call $[a, b]$ a structural constant interval of Φ. Let $\{[a_n, b_n]\}_{n=1}^m$ be all structural constant intervals of Φ, where m is finite or infinite. The next result is different from the classical Orlicz spaces.

Theorem 5. *For any $x \in L_\Phi$, $I_\Phi(x) = 1$ implies $\|x\|_\Phi = 1$ if and only if the following conditions hold:*

(i) *$\{b_n\}_{n=1}^\infty$ is a bounded sequence;*

(ii) *for any $\{G_i\}_{i=1}^k$ such that $\bigcup_{i=1}^k G_i \subseteq G$ and $G_i \cap G_j = \emptyset$, $i \neq j$ and $i, j = 1, 2, \ldots, k$, either $\sum_{n=1}^k \Phi(a_n)\mu(G_n) < 1$ or $\sum_{n=1}^k \Phi(a_n)\mu(G_n) = 1$ implies $k = \infty$.*

Proof. Necessity. We first discuss the case $\limsup_{n\to\infty} b_n = \infty$, then there exists b_{n_0} and $G_0 \subset G$ such that $\Phi(b_{n_0})\mu(G_0) = 1$. Let

$$x_0(t) = a_{n_0}\chi_{G_0}.$$

Hence, $I_\Phi(x_0) = 1$. Let $\lambda_0 = \frac{a_{n_0}}{b_{n_0}}$; thus, we have

$$I_\Phi\left(\frac{x_0}{\lambda_0}\right) = \Phi(b_{n_0})\mu(G_0) = 1.$$

Furthermore, for any $0 < \lambda < \lambda_0$, we have

$$I_\Phi\left(\frac{x_0}{\lambda}\right) = \Phi\left(\frac{a_{n_0}}{\lambda}\right)\mu(G_{n_0}) > \Phi(b_{n_0})\mu(G_{n_0}) = 1.$$

Thus, $\|x_0\|_\Phi = \lambda_0 < 1$, which is a contradiction.

The case if there exists a segmentation $\{G_n\}_{n=1}^m$ of G, i.e., $G \subset \bigcup_{n=1}^m G_n$, and $G_i \cap G_j = \emptyset$, $i \neq j$ and $m < \infty$ satisfies $\sum_{n=1}^m \Phi(b_n)\mu(G_n) = 1$. Let

$$x_1(t) = \sum_{n=1}^m a_n\chi_{G_n}.$$

Take $\lambda_1 = \max\left\{\frac{a_n}{b_n} : n = 1, \ldots, m\right\}$, then $0 < \lambda_1 < 1$ and $\frac{a_n}{\lambda_1} \leq b_n$, $n = 1, \ldots, m$. Hence,

$$I_\Phi\left(\frac{x_1}{\lambda_1}\right) = \sum_{n=1}^m \Phi\left(\frac{a_n}{\lambda_1}\right)\mu(G_n) \leq \sum_{n=1}^m \Phi(b_n)\mu(G_n) = 1.$$

Thus, $\|x_1\|_\Phi \leq \lambda_1 < 1$, which is a contradiction.

Sufficiency. For any $x \in L_\Phi$ with $I_\Phi(x) = 1$, let

$$G_n = \{t \in G : a_n \leq |x(t)| \leq b_n\},$$

then $G_i \cap G_j = \emptyset$ $(i \neq j)$. If $\sum_{n=1}^m \Phi(b_n)\mu(G_n) < 1$, then $\mu(supp(x) \setminus \bigcup_{n=1}^m G_n) > 0$, where $supp(x) = \{t : x(t) \neq 0\}$. Since for any $\lambda \in (0, 1)$, we have

$$\begin{aligned} I_\Phi\left(\frac{x}{\lambda}\right) &= \int_{\bigcup_{n=1}^m G_n} \Phi\left(\frac{x(t)}{\lambda}\right)dt + \int_{supp(x)\setminus\bigcup_{n=1}^m G_n} \Phi\left(\frac{x(t)}{\lambda}\right)dt \\ &> \int_{\bigcup_{n=1}^m G_n} \Phi(x(t))dt + \int_{supp(x)\setminus\bigcup_{n=1}^m G_n} \Phi(x(t))dt = I_\Phi(x), \end{aligned}$$

and we obtain $I_\Phi(\frac{x}{\lambda}) > I_\Phi(x) = 1$, $\|x\|_\Phi \geq \lambda$, i.e., $\|x\|_\Phi = 1$.

Without a loss of generality, we suppose that $supp(x) = \bigcup_{n=1}^{m} G_n$, that is,

$$\sum_{n=1}^{m} \Phi(a_n)\mu(G_n) = 1,$$

it follows that $m = \infty$. It follows from $\{b_n\}_{n=1}^{\infty}$, which is a bounded sequence, that there exists $\{b_{n_i}\} \subset \{b_n\}$ such that $b_{n_i} - a_{n_i} \to 0$. By passing to a subsequence, it may be assumed that $b_n - a_n \to 0$. Thus, for any $\lambda \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{a_{n_0}}{\lambda} > b_{n_0}.$$

Then, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \Phi(\frac{a_n}{\lambda})\mu(G_n) &= \sum_{n\neq n_0} \Phi(\frac{a_n}{\lambda})\mu(G_n) + \Phi(\frac{a_{n_0}}{\lambda})\mu(G_{n_0}) \\ &> \sum_{n\neq n_0} \Phi(a_n)\mu(G_n) + \Phi(b_{n_0})\mu(G_{n_0}) \\ &= \sum_{n=1}^{\infty} \Phi(a_n)\mu(G_n) = 1,\end{aligned}$$

i.e., $I_\Phi(\frac{x}{\lambda}) > 1$, $\|x\|_\Phi \geq \lambda$. According to the arbitrariness of λ, we know that $\|x\|_\Phi \geq 1$, and since $I_\Phi(x) = 1$, we know that $\|x\|_\Phi \leq 1$. Hence, $\|x\|_\Phi = 1$. We complete the proof. □

Theorem 6. *$\|x_n\|_\Phi \to 1 \Rightarrow I_\Phi(x_n) \to 1$ if and only if $\Phi \in \Delta_{1+\varepsilon}^s$.*

Proof. Necessity. Assume $\Phi \notin \Delta_{1+\varepsilon}^s$. Then, there exist $\varepsilon_0 > 0$ and a sequence $u_n \uparrow +\infty$ such that

$$\Phi\left(\left(1+\frac{1}{n}\right)u_n\right) \geq (1+\varepsilon_0)\Phi(u_n)$$

for each $n \in \mathbb{N}$. Choose $G_n \subset G$ such that $\Phi(u_n)\mu(G_n) = \frac{1}{1+\varepsilon_0}$. Let $x_n = u_n\chi_{G_n}$, then

$$I_\Phi(x_n) = \int_G \Phi(x_n(t))dt = \Phi(u_n)\mu(G_n) = \frac{1}{1+\varepsilon_0}.$$

Hence, $\|x_n\|_\Phi \leq 1$. However,

$$\begin{aligned}I_\Phi\left(\frac{x_n}{\frac{1}{1+\frac{1}{n}}}\right) = I_\Phi\left(\left(1+\frac{1}{n}\right)x_n\right) &= \Phi\left(\left(1+\frac{1}{n}\right)u_n\right)\mu(G_n) \\ &\geq (1+\varepsilon_0)\Phi(u_n)\mu(G_n) = 1.\end{aligned}$$

Then, $\|x_n\|_\Phi \geq \frac{n}{n+1}$. Hence, we get $\|x_n\|_\Phi \to 1$, $I_\Phi(x_n) \nrightarrow 1$, which is a contradiction.

Sufficiency. Let $\Phi \in \Delta_{1+\varepsilon}^s$, which means that, for any $\varepsilon > 0$ and $u_0 > 0$, there exists $\delta \in (0,1)$ such that $\Phi((1+\delta)u) \leq (1+\varepsilon)\Phi(u)$ whenever $u \geq u_0$. We conclude

$I_\Phi\left(\frac{x_n}{\|x_n\|_\Phi}\right) = 1$. Suppose $\|x_n\|_\Phi \to 1$, then there exists $n_0 > 0$ such that $\|x_n\|_\Phi < 1+\delta$ whenever $n > n_0$ and

$$\begin{aligned} I_\Phi(x_n) &= I_\Phi\left(\|x_n\|_\Phi \frac{x_n}{\|x_n\|_\Phi}\right) \le I_\Phi\left((1+\delta)\frac{x_n}{\|x_n\|_\Phi}\right) = \int_G \Phi\left((1+\delta)\frac{x_n(t)}{\|x_n\|_\Phi}\right)dt \\ &= \int_{\{t\in G:\frac{|x_n(t)|}{\|x_n\|_\Phi}\le u_0\}} \Phi\left((1+\delta)\frac{x_n(t)}{\|x_n\|_\Phi}\right)dt \\ &+ \int_{\{t\in G:\frac{|x_n(t)|}{\|x_n\|_\Phi}> u_0\}} \Phi\left((1+\delta)\frac{x_n(t)}{\|x_n\|_\Phi}\right)dt \\ &< \Phi((1+\delta)u_0)\mu(G) + (1+\varepsilon)I_\Phi\left(\frac{x_n}{\|x_n\|_\Phi}\right) \\ &= \Phi((1+\delta)u_0)\mu(G) + 1 + \varepsilon. \end{aligned}$$

The condition $\|x_n\|_\Phi \to 1$, $\frac{1}{\|x_n\|_\Phi} \to 1$ implies that there exists $n_1 > 0$ such that $\|x_n\|_\Phi < 1+\delta$ as $n > n_1$ and

$$\begin{aligned} 1 &= I_\Phi\left(\frac{x_n}{\|x_n\|_\Phi}\right) = I_\Phi\left(\frac{1}{\|x_n\|_\Phi}x_n\right) \le I_\Phi((1+\delta)x_n) = \int_G \Phi((1+\delta)x_n(t))dt \\ &= \int_{\{t\in G:|x_n(t)|\le u_0\}} \Phi((1+\delta)x_n(t))dt + \int_{\{t\in G:|x_n(t)|>u_0\}} \Phi((1+\delta)x_n(t))dt \\ &< \Phi((1+\delta)u_0)\mu(G) + (1+\varepsilon)I_\Phi(x_n) \\ &= \Phi((1+\delta)u_0)\mu(G) + I_\Phi(x_n) + \varepsilon L, \end{aligned}$$

where $L = \sup_n\{I_\Phi(x_n)\}$.

Take a small enough $u_0 > 0$ such that $\Phi((1+\delta)u_0)\mu(G) < \varepsilon$; thus, we get

$$1 - (1+L)\varepsilon < I_\Phi(x_n) < 1 + 2\varepsilon,$$

when $n > \max\{n_0, n_1\}$, i.e., $\lim_{n\to\infty} I_\Phi(x_n) = 1$. □

The following example shows that although there exists an Orlicz function $\Phi \in \Delta_2$, $\Phi \notin \Delta^s_{1+\varepsilon}$ for any $\varepsilon > 0$.

Example 3. *Let*

$$\Phi(u) = \begin{cases} |u|, & |u| \in [0, b_1), \\ \frac{a_{n+1}-a_n}{b_{n+1}-b_n}(|u| - b_n) + a_n, & |u| \in [b_n, b_{n+1}), n = 1, 2, \ldots, \end{cases}$$

where $a_1 = 2$, $a_{2n} = 2a_{2n-1}$, $a_{2n+1} = a_{2n}$, $b_1 = 2$, $b_{2n} = \left(1+\frac{1}{n}\right)b_{2n-1}$, $b_{2n+1} = 2b_{2n}$, $n = 1, 2, \ldots$.

$\Phi \notin \Delta_{1+\delta}$. In fact, if we take $\varepsilon_0 = \frac{1}{2}$ and $u_n = b_{2n-1}$, $n = 1, 2, \ldots$, we have

$$\begin{aligned} \Phi(u_n) &= \Phi(b_{n-1}) = a_{2n-1}, \\ \Phi\left(\left(1+\frac{1}{n}\right)u_n\right) &= \Phi\left(\left(1+\frac{1}{n}\right)b_{2n-1}\right) = \Phi(b_{2n}) = a_{2n} = 2a_{2n-1} \\ &= 2\Phi(u_n) > (1+\varepsilon_0)\Phi(u_n). \end{aligned}$$

$\Phi \in \Delta_2$. In fact, if $u \in [b_{2n-1}, b_{2n})$, $n = 1, 2, \ldots$, then

$$\Phi(u) \ge \Phi(b_{2n-1}) \ge a_{2n-1}$$

and $b_{2n} < 2b_{2n-1} \leq 2u < 2b_{2n} = b_{2n+1}$, that is, $2u \in (b_{2n}, b_{2n+1})$. Thus

$$\Phi(2u) = \Phi(b_{2u}) = 2_a 2n = 2a_{2n-1}.$$

Hence, for any $K > 2$, we have $\Phi(2u) \leq K\Phi(u)$.

If $u \in [b_{2n}, b_{2n+1})$, $n = 1, 2, \ldots$, then

$$\Phi(u) \geq \Phi(b_{2n}) \geq a_{2n},$$

and $b_{2n+1} < 2b_{2n} \leq 2u < 2b_{2n+1} = b_{2n+3}$, that is, $2u \in (b_{2n+}, b_{2n+3})$. Thus

$$\Phi(2u) = \Phi(b_{2u+3})\Phi(b_{2u+2}) = a_{2n+2} = 2_a 2n + 1 = 2a_{2n}.$$

Hence, for any $K > 2$, we have $\Phi(2u) \leq K\Phi(u)$.

Theorem 7. *The following statements are equivalent:*

(i) L^Φ *is lower locally uniformly monotone;*

(ii) L^Φ *is strictly monotone;*

(iii) Φ *is strictly increasing on* $\mathbb{R}_+$ *and* Φ *satisfies the* Δ_2*-condition.*

Proof. $(i) \Rightarrow (ii)$ is obvious. Now, we will prove that $(ii) \Rightarrow (iii)$. If Φ is not strictly increasing on the interval $[0, +\infty)$, then there is an interval $[a, b]$ in $[0, +\infty)$ such that $\Phi(u) = C$ on $[a, b]$, where C is a constant. Take $G_0 \in \Sigma$, with $\mu(G_0) > 0$ and $0 < \Phi(b)\mu(G_0) \leq 1$. Let us take a sufficiently large strictly monotone point c of Φ such that

$$\Phi(b)\mu(G_0) + \Phi(c)\mu(G \backslash G_0) > 1.$$

By the fact that the measure μ is non-atomic, we have $G_1 \subset G \backslash G_0$ such that

$$\Phi(b)\mu(G_0) + \Phi(c)\mu(G_1) = 1.$$

Define

$$x = a\chi_{G_0} + c\chi_{G_1}, y = b\chi_{G_0} + c\chi_{G_1}.$$

Then, we can easily get that $x < y$ and $I_\Phi(x) = I_\Phi(y) = 1$. Since c is strictly a monotone point, thus $\|x\|_\Phi = \|y\|_\Phi = 1$, which is a contradiction.

Now, we will prove that $(iii) \Rightarrow (i)$. Assume that Φ is strictly increasing on $\mathbb{R}_+$ and Φ satisfies the Δ_2-condition. For any $0 \leq x_n \leq x$ and $\|x_n\|_\Phi \to \|x\|_\Phi$ as $n \to \infty$, we will prove that $\|x_n - x\|_\Phi \to 0$ as $n \to \infty$ holds.

First, we will prove that $I_\Phi\left(\frac{x}{\|x_n\|_\Phi}\right) \to 1$ as $n \to \infty$. If not, then $I_\Phi\left(\frac{x}{\|x_n\|_\Phi}\right) \geq 1 + \delta_1$, where δ_1 is a positive number. By the absolute continuity of the integral, for any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that $\int_e \Phi\left(\frac{x(t)}{\|x_n\|_\Phi}\right) dt < \varepsilon$, where $\mu(e) < \delta_0$. By the assumption that $\|x_n\|_\Phi \to \|x\|_\Phi$ as $n \to \infty$, we can assume that $\frac{\|x\|_\Phi}{\|x_n\|_\Phi} \leq 2$. Using $\Phi \in \Delta_2$, there exist $u_0 > 0$ and $K > 0$ such that $\Phi(2u) \leq K\Phi(u)$, whenever $u > u_0$, which together gives the previous condition, as follows:

$$\begin{aligned} I_\Phi\left(\frac{x}{\|x_n\|_\Phi}\right) &= I_\Phi\left(\frac{\|x\|_\Phi}{\|x_n\|_\Phi}\frac{x}{\|x\|_\Phi}\right) \leq I_\Phi\left(2\frac{x}{\|x\|_\Phi}\right) \\ &\leq \Phi(u_0)\mu(G) + KI_\Phi\left(\frac{x}{\|x\|_\Phi}\right) = \Phi(u_0)\mu(G) + K. \end{aligned}$$

Denote $\Phi(u_0)\mu(G) + K = M$; thus, we have

$$M \geq I_\Phi\left(\frac{x}{\|x_n\|_\Phi}\right) \geq \int_{\{t\in G:\ \frac{x(t)}{\|x_n\|_\Phi}\geq D\}} \Phi\left(\frac{x(t)}{\|x_n\|_\Phi}\right)dt$$
$$\geq \Phi(D)\mu(\{t \in G :\ \frac{x(t)}{\|x_n\|_\Phi} \geq D\}).$$

Thus, $\mu(\{t \in G : \frac{x(t)}{\|x_n\|_\Phi} \geq D\}) \leq \frac{M}{\Phi(D)}$. Since Φ is strictly increasing, we conclude that there exists $D > 0$ such that $\mu(\{t \in G : \frac{x(x)}{\|x_n\|_\Phi} \geq D\}) \leq \delta_0$. Let $G_n(D) = \{t \in G : \frac{x(x)}{\|x_n\|_\Phi} \geq D\}$; thus, we obtain

$$\int_{G_n(D)} \Phi\left(\frac{x(t)}{\|x_n\|_\Phi}\right)dt < \varepsilon.$$

Because Φ is continuous, then Φ is uniformly continuous in the interval $[0, D]$. For the above ε, there exists $\delta > 0$ such that

$$\Phi\left(\frac{x(t)}{\|x_n\|_\Phi}\right) - \Phi\left(\frac{x(t)}{\|x\|_\Phi}\right) < \varepsilon,$$

whenever $\frac{x(t)}{\|x_n\|_\Phi} - \frac{x(t)}{\|x\|_\Phi} < \delta$. We can easily understand that

$$\begin{aligned}
\delta_1 &\leq \int_G \Phi\left(\frac{x(t)}{\|x_n\|_\Phi}\right)dt - \int_G \Phi\left(\frac{x(t)}{\|x\|_\Phi}\right)dt \\
&= \int_{G\setminus G_n(D)} \Phi\left(\frac{x(t)}{\|x_n\|_\Phi}\right)dt + \int_{G_n(D)} \Phi\left(\frac{x(t)}{\|x_n\|_\Phi}\right)dt - \int_{G\setminus G_n(D)} \Phi\left(\frac{x(t)}{\|x\|_\Phi}\right)dt \\
&- \int_{G_n(D)} \Phi\left(\frac{x(t)}{\|x\|_\Phi}\right)dt \\
&\leq \int_{G\setminus G_n(D)} \left(\Phi\left(\frac{x(t)}{\|x_n\|_\Phi}\right) - \Phi\left(\frac{x(t)}{\|x\|_\Phi}\right)\right)dt + \int_{G_n(D)} \Phi\left(\frac{x(t)}{\|x_n\|_\Phi}\right)dt \\
&< \varepsilon\mu(G) + \varepsilon.
\end{aligned}$$

Due to $\varepsilon\mu(G) + \varepsilon < \frac{\delta_1}{2}$, we get a contradiction.

Now, we will prove that $\{x_n(t)\}_{n=1}^\infty$ converges to $x(t)$ in measure. Otherwise, there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that

$$\mu(\{t \in G : x(t) - x_n(t) \geq \varepsilon_0\}) \geq \delta_0.$$

By $\Phi \in \Delta_2$, there exists $D > 0$ such that

$$\mu(\{t \in G : x(t) - x_n(t) \geq \varepsilon_0, x(t) > D\}) \leq \frac{\delta_0}{3}.$$

Denote $A_n = \{t \in G : x(t) - x_n(t) \geq \varepsilon_0, x(t) \leq D\}$. By the assumption that Φ is strictly increasing and by the proof of Theorem 15 in [9], there exists $\delta' > 0$ such that

$$\Phi\left(\frac{x(t)}{\|x_n\|_\Phi}\right) \geq \Phi\left(\frac{x_n(t)}{\|x_n\|_\Phi}\right) + \delta'$$

for $t \in A_n$, so we have

$$\begin{aligned}\int_G \Phi\left(\frac{x(t)}{\|x_n\|_\Phi}\right)dt &= \int_{A_n} \Phi\left(\frac{x(t)}{\|x_n\|_\Phi}\right)dt + \int_{G\setminus A_n} \Phi\left(\frac{x(t)}{\|x_n\|_\Phi}\right)dt \\ &\geq \int_{A_n} \Phi\left(\frac{x_n(t)+\varepsilon_0}{\|x_n\|_\Phi}\right)dt + \int_{G\setminus A_n} \Phi\left(\frac{x_n(t)}{\|x_n\|_\Phi}\right)dt \\ &\geq \int_{A_n} \left(\Phi\left(\frac{x_n(t)}{\|x_n\|_\Phi}\right)+\delta'\right)dt + \int_{G\setminus A_n} \Phi\left(\frac{x_n(t)}{\|x_n\|_\Phi}\right)dt \\ &= \int_{A_n} \Phi\left(\frac{x_n(t)}{\|x_n\|_\Phi}\right)dt + \delta'\mu(A_n) + \int_{G\setminus A_n} \Phi\left(\frac{x_n(t)}{\|x_n\|_\Phi}\right)dt \\ &\geq 1+\delta'\delta_0,\end{aligned}$$

where $\delta' > 0$, which is a contradiction.

Now, we are going to show that $I_\Phi\left(\frac{x_n}{\|x_n\|_\Phi} - \frac{x}{\|x_n\|_\Phi}\right) \to 0$ as $n \to \infty$. Since $\{x_n(t)\}_{n=1}^\infty$ converges to $x(t)$ in measure and $\|x_n\|_\Phi \to \|x\|_\Phi$ as $n \to \infty$, we derive that $\left\{\frac{x_n(t)}{\|x_n\|_\Phi} - \frac{x(t)}{\|x_n\|_\Phi}\right\}_{n=1}^\infty$ converges to 0 in measure. The absolute continuity of the integral implies that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\int_e \Phi\left(\frac{x_n}{\|x_n\|_\Phi} - \frac{x}{\|x_n\|_\Phi}\right)dt < \varepsilon$, where $\mu(e) < \delta$, and the Egorov's theorem implies that for the above $\delta > 0$, there exists $e_0 \subset G$ such that $\left\{\frac{x_n(t)}{\|x_n\|_\Phi} - \frac{x(t)}{\|x_n\|_\Phi}\right\}_{n=1}^\infty$ uniformly converges to 0 on $G\setminus e_0$. Further, we obtain that

$$\int_{G\setminus e_0} \Phi\left(\frac{x_n(t)}{\|x_n\|_\Phi} - \frac{x(t)}{\|x_n\|_\Phi}\right)dt \to 0$$

as $n \to \infty$. Hence, for the above $\varepsilon > 0$, there exists $N \in \mathbb{N}_+$ such that

$$\int_{G\setminus e_0} \Phi\left(\frac{x_n(t)}{\|x_n\|_\Phi} - \frac{x(t)}{\|x_n\|_\Phi}\right)dt < \varepsilon,$$

whenever $n > N$. Therefore, we conclude that

$$\begin{aligned}\int_G \Phi\left(\frac{x_n(t)}{\|x_n\|_\Phi} - \frac{x(t)}{\|x_n\|_\Phi}\right)dt &= \int_{G\setminus e_0} \Phi\left(\frac{x_n(t)}{\|x_n\|_\Phi} - \frac{x(t)}{\|x_n\|_\Phi}\right)dt + \int_{e_0} \Phi\left(\frac{x_n(t)}{\|x_n\|_\Phi} - \frac{x(t)}{\|x_n\|_\Phi}\right)dt \\ &< \varepsilon + \varepsilon = 2\varepsilon,\end{aligned}$$

whenever $n > N$. By Theorem 3, we obtain that $\|x_n - x\|_\Phi \to 0$ as $n \to \infty$, which finishes the proof. □

Theorem 8. *L^Φ is upper locally uniformly monotone if and only if Φ is strictly an increasing continuous function on $\mathbb{R}_+$ and Φ satisfies the Δ_2-condition.*

Proof. We only need to prove the sufficiency. Let

$$0 \leq x(t) \leq x_n(t), \quad \|x\|_\Phi = 1 \quad \text{and} \quad \|x_n\|_\Phi \to 1.$$

By passing to a subsequence, it may be assumed further that $\|x_n\|_\Phi \geq 1/2$ for all $n \in \mathbb{N}$. Since $\Phi \in \triangle_2$, by the same argue of the proof of Theorem 7, we have

$$\lim_{n\to\infty} I_\Phi\left(\frac{x}{\|x_n\|_\Phi}\right) = 1.$$

Hence,

$$\lim_{n\to\infty}\left(I_\Phi\left(\frac{x_n}{\|x_n\|_\Phi}\right) - I_\Phi\left(\frac{x}{\|x_n\|_\Phi}\right)\right) = 0.$$

That is,

$$\lim_{n\to\infty}\int_G\left(\Phi\left(\frac{x_n(t)}{\|x_n\|_\Phi}\right)-\Phi\left(\frac{x(t)}{\|x_n\|_\Phi}\right)\right)dt=0.$$

Using

$$\Phi\left(\frac{x_n(t)}{\|x_n\|_\Phi}\right)-\Phi\left(\frac{x(t)}{\|x_n\|_\Phi}\right)\geq 0$$

for all $t\in G$, we have

$$\left\|\Phi\circ\frac{x_n}{\|x_n\|_\Phi}-\Phi\circ\frac{x}{\|x_n\|_\Phi}\right\|_{L^1(\Omega)}\to 0$$

as $n\to\infty$. Therefore, there exists a strictly increasing subsequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers, a sequence of positive numbers $\{\varepsilon_k\}_{k=1}^{\infty}\subset(0,1)$, and $y\in L_1(\Omega)^+$ such that

$$\lim_{k\to\infty}\varepsilon_k=0,$$

and

$$0\leq\Phi\circ\frac{x_n}{\|x_n\|_\Phi}-\Phi\circ\frac{x}{\|x_n\|_\Phi}\leq\varepsilon_k y$$

for all $k\in\mathbb{N}$ (see [15]). By the absolute continuity of the integral, we obtain that, for any $\varepsilon>0$, there exists $\delta>0$ such that

$$\int_e\Phi\left(\frac{x_n(t)}{\|x_n\|_\Phi}-\frac{x(t)}{\|x\|_\Phi}\right)dt<\varepsilon,$$

where $\mu(e)<\delta$. Egorov's theorem implies that for the above $\delta>0$, there exists $e_0\subset G$ such that

$$\left\{\frac{x_n(t)}{\|x_n\|_\Phi}-\frac{x(t)}{\|x\|_\Phi}\right\}_{n=1}^{\infty}$$

is uniformly convergent on $G\setminus e_0$; further, we obtain that

$$\int_{G\setminus e_0}\Phi\left(\frac{x_n(t)}{\|x_n\|_\Phi}-\frac{x(t)}{\|x\|_\Phi}\right)dt\to 0$$

as $n\to\infty$. Hence, for the above $\varepsilon>0$, there exists $N\in\mathbb{N}^+$ such that

$$\int_{G\setminus e_0}\Phi\left(\frac{x_n(t)}{\|x_n\|_\Phi}-\frac{x(t)}{\|x\|_\Phi}\right)dt<\varepsilon,$$

whenever $n>N$. Therefore, we conclude that

$$\begin{aligned}\int_G\Phi\left(\frac{x_n(t)}{\|x_n\|_\Phi}-\frac{x(t)}{\|x\|_\Phi}\right)dt&=\int_{G\setminus e_0}\Phi\left(\frac{x_n(t)}{\|x_n\|_\Phi}-\frac{x(t)}{\|x\|_\Phi}\right)dt+\int_{e_0}\Phi\left(\frac{x_n(t)}{\|x_n\|_\Phi}-\frac{x(t)}{\|x\|_\Phi}\right)dt\\&<\varepsilon+\varepsilon=2\varepsilon,\end{aligned}$$

whenever $n\geq N$. This implies that

$$\frac{x_{n_k}(t)}{\|x_{n_k}\|_\Phi}-\frac{x(t)}{\|x_{n_k}\|_\Phi}\to 0$$

in measure. Since Φ is a strictly monotonically increasing function on $\mathbb{R}^+$, it follows that

$$\Phi\left(\frac{x_{n_k}(t)}{\|x_{n_k}\|_\Phi}-\frac{x(t)}{\|x_{n_k}\|_\Phi}\right)\to 0$$

in measure. Without a loss of generality, we may assume that

$$\Phi\left(\frac{x_{n_k}(t)}{\|x_{n_k}\|_\Phi} - \frac{x(t)}{\|x_{n_k}\|_\Phi}\right) \to 0.$$

Hence,

$$\begin{aligned}\Phi\left(\frac{x_{n_k}(t)}{\|x_{n_k}\|_\Phi} - \frac{x(t)}{\|x_{n_k}\|_\Phi}\right) &\leq \Phi\left(\frac{x_{n_k}(t)}{\|x_{n_k}\|_\Phi}\right) \\ &\leq \Phi\left(\frac{x(t)}{\|x_{n_k}\|_\Phi}\right) + \varepsilon_k y(t) \\ &\leq \Phi(2x(t)) + y(t).\end{aligned}$$

By the Lebesgue dominated convergence theorem, we have

$$\lim_{k\to\infty} \int_G \Phi\left(\frac{x(t)}{\|x_{n_k}\|_\Phi} - \frac{x_{n_k}(t)}{\|x_{n_k}\|_\Phi}\right) dt = 0.$$

By $\Phi \in \Delta_2$, we have

$$\left\| \frac{x_{n_k} - x}{\|x_{n_k}\|_\Phi} \right\| \to 0.$$

as $k \to \infty$. Since $\|x_{n_k}\|_\Phi \to 1$ as $k \to \infty$, it follows that $\|x_{n_k} - x\|_\Phi \to 0$. By the twin sequences theorem, $\|x_n - x\|_\Phi \to 0$. as $n \to \infty$. This completes the proof. □

4. Conclusions

In this study, we defined a new class of Orlicz functions and successfully extended classical Orlicz spaces. We discussed some fundamental properties of quasi-normed Orlicz spaces and provided criteria for strict monotonicity and lower (upper) local uniform monotonicity within these spaces. Our in-depth exploration of the newly defined Orlicz spaces offers powerful tools for understanding broader function spaces, which are of great importance in classical function space theory and regarding their practical applications. Moving forward, we aim to explore a characterization for the uniform convexity of new Orlicz spaces. Our theoretical insights will provide valuable support for the field of non-convex optimization, contributing to its further development.

Author Contributions: All authors have made equal and substantial contributions to the composition of this study. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Harbin Science and Technology Plan Project grant number 2023ZCZJCG039.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: We extend our gratitude to the unidentified reviewers and the editors for their valuable feedback and recommendations, which significantly enhanced the quality of this manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

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