axioms

MDPI

Article

Crossed Modules and Non-Abelian Extensions of Differential Leibniz Conformal Algebras

Hui Wu [1], Shuangjian Guo [2] and Xiaohui Zhang [1,*]

[1] School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China; huiwu@qfnu.edu.cn
[2] School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang 550025, China; 201301108@mail.gufe.edu.cn
* Correspondence: zhangxh2015@qfnu.edu.cn

Abstract: In this paper, we introduce two-term differential $Leib_\infty$-conformal algebras and give characterizations of some particular classes of such two-term differential $Leib_\infty$-conformal algebras. Furthermore, we discuss the classification of the non-Abelian extensions in terms of non-Abelian cohomology groups. Finally, we explore the inducibility of pairs of automorphisms and derive the analog Wells exact sequences under the circumstance of differential Leibniz conformal algebras.

Keywords: differential Leibniz conformal algebra; cohomology; crossed module; non-Abelian extension; Wells exact sequences

MSC: 17B10; 17B38; 17B56; 18G45

1. Introduction

Kac in [1] has proposed Lie conformal algebras, usually considered as an axiomatic description of the singular part of the operator product expansion of chiral fields in conformal field theory. The past few years have witnessed considerable scholarly attention to this algebraic structure in the past few years because they are closely related to vertex algebras [2]. Many more properties and structures of Lie conformal algebras have been developed; see [3–7] and references cited therein.

Leibniz conformal algebras were introduced in [8], which are closely related to field algebras [9] and vertex algebras. Later, the author further elaborated upon and elucidated the concept of a conformal representation of a Leibniz algebra in [10]. After that, Zhang introduced the cohomology of Leibniz conformal algebras in [11] and Wu articulated the notion of a Leibniz pseudoalgebra, which is a multivariable generalization of the concept of Leibniz conformal algebras in [12]. Recently, Feng and Chen studied $\mathcal{O}$-operators, also known as relative Rota–Baxter operators on Leibniz conformal algebras with respect to representations in [13]. Subsequently, the first author and Wang investigated some properties of relative Rota–Baxter operators on Leibniz conformal algebras with respect to representations and their connections with Leibniz dendriform conformal algebras in [14]. For further details on Leibniz conformal algebras, see [15,16]. Recently, the authors [17] introduced $Leib_\infty$-conformal algebras where the Leibniz conformal identity holds up to homotopy. Additionally, they presented equivalent descriptions of $Leib_\infty$-conformal algebras and identified certain characteristics of some particular classes of $Leib_\infty$-conformal algebras in terms of the cohomology of Leibniz conformal algebras and crossed modules of Leibniz conformal algebras as a generalization of [18]. This study would introduce two-term differential $Leib_\infty$-conformal algebra and generalize the common characteristics of some particular classes of such homotopy differential Leibniz conformal algebras, which constitute the major academic focus of this paper.

The extension problem has persisted and incurred scholarly dispute. Non-Abelian extensions were first developed in [19], which induces cohomology to the low dimensional

check for updates

Citation: Wu, H.; Guo, S.; Zhang, X. Crossed Modules and Non-Abelian Extensions of Differential Leibniz Conformal Algebras. *Axioms* **2024**, *13*, 685. https://doi.org/10.3390/axioms13100685

Academic Editor: Rutwig Campoamor-Stursberg

Received: 12 August 2024
Revised: 19 September 2024
Accepted: 30 September 2024
Published: 2 October 2024

Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

non-Abelian group. The authors examined non-Abelian extensions of Leibniz algebras in [20]. See [21] and references cited therein. Naturally, we look into non-Abelian extensions of a differential Leibniz conformal algebra by another differential Leibniz conformal algebras. Another interesting study linked to extensions of algebraic structures is given by the inducibility of a pair of automorphisms, which, after all, is intimately connected with extensions of algebras. Such a study was first initiated by Wells in extensions of abstract groups in [22]. Later, the authors investigated extending automorphism in [23]. In [24], the authors studied the inducibility of a pair of automorphisms about a non-Abelian extension of Lie algebras. The results of [20,24] have been extended to Rota–Baxter Leibniz algebras in [25]. Naturally, we study the inducibility of a pair of differential Leibniz conformal algebra automorphisms and characterize them by equivalent conditions. This forms the second research focus of this paper.

The paper is organized as follows. In Section 2, we recall some basic definitions of differential Leibniz conformal algebras. In Section 3, we introduce homotopy differential operators on two-term $Leib_\infty$-conformal algebras. A two-term $Leib_\infty$-conformal algebra equipped with a homotopy differential operators is called a two-term differential $Leib_\infty$-conformal algebra, and we give characterizations of some particular classes of such two-term differential $Leib_\infty$-conformal algebras. In Section 4, we introduce non-Abelian cohomology groups and classify the non-Abelian extensions in terms of non-Abelian cohomology groups. In Section 5, we explore the inducibility of pairs of automorphisms and derive the analog Wells exact sequences under the circumstance of differential Leibniz conformal algebras.

2. Preliminaries

Throughout the paper, all algebraic systems are supposed to be over a field $\mathbb{C}$. We denote by $\mathbb{Z}$ the set of all integers and $\mathbb{Z}_+$ the set of all nonnegative integers. We now recall some useful definitions in [8,11,26].

Definition 1. *A Leibniz conformal algebra is a $\mathbb{C}[\partial]$-module R endowed with a λ-bracket $[\cdot_\lambda \cdot]_R$, which defines a $\mathbb{C}$-bilinear map from $R \otimes R$ to $R[\lambda] = \mathbb{C}[\lambda] \otimes R$ such that the following axioms hold:*

$$[\partial x_\lambda y]_R = -\lambda [x_\lambda y]_R, [x_\lambda \partial y]_R = (\partial + \lambda)[x_\lambda y]_R, (\text{conformal sesquilinearity})$$
$$[x_\lambda [y_\mu z]_R]_R = [[x_\lambda y]_{R\lambda+\mu} z]_R + [y_\mu [x_\lambda z]_R]_R, (\text{Jacobi identity})$$

for any $x, y, z \in R$.

Definition 2. *A representation of a Leibniz conformal algebra R is a $\mathbb{C}[\partial]$-module R endowed with left and right λ-actions, which are two $\mathbb{C}$-linear maps*

$$\cdot_\lambda : R \otimes V \to V[\lambda], \quad {}_\lambda\cdot : V \otimes R \to V[\lambda]$$

that satisfy the following conditions:

$$\begin{aligned}
&(\partial x)_\lambda u = -\lambda x_\lambda u, \quad x_\lambda(\partial u) = (\partial + \lambda) x_\lambda u,\\
&(\partial u)_\lambda x = -\lambda u_\lambda x, \quad u_\lambda(\partial x) = (\partial + \lambda) u_\lambda x,\\
&u_\lambda [x_\mu y] = (u_\lambda x)_{\lambda+\mu} y + x_\mu (u_\lambda y),\\
&x_\mu (u_\lambda y) = (x_\mu u)_{\lambda+\mu} y + u_\lambda [x_\mu y]_R,\\
&x_\lambda (y_\mu u) = [x_\lambda y]_{R\lambda+\mu} u + y_\mu (x_\lambda u),
\end{aligned}$$

for any $x, y \in R$ and $u \in V$.

It follows that any Leibniz conformal algebra R is a representation of itself with

$$x \cdot_\lambda y = (L_x)_\lambda(y) = [x_\lambda y]_R \quad \text{and} \quad y \cdot_\lambda x = (R_x)_\lambda(y) = [y_\lambda x], \text{ for } x, y \in R.$$

Here, L_x and R_x denote the left and right λ-bracket on R by x, respectively. This is called the regular representation.

Let R be a Leibniz conformal algebra and V a representation of R. For $n \geq 1$, an n-λ-bracket on R with coefficients in V is a $\mathbb{C}$-linear map $f_{\lambda_1,\cdots,\lambda_{n-1}} : R^{\otimes n} \to V[\lambda_1, ..., \lambda_{n-1}]$ denoted by

$$x_1 \otimes \cdots \otimes x_n \mapsto f_{\lambda_1,\cdots,\lambda_{n-1}}(x_1, \cdots, x_n),$$

satisfying the following sesquilinearity conditions:

$$\begin{aligned} &f_{\lambda_1,\cdots,\lambda_{n-1}}(x_1, \cdots, \partial x_i, \cdots x_n) = -\lambda_i f_{\lambda_1,\cdots,\lambda_{n-1}}(x_1, \cdots, x_n), 1 \leq i < n, \\ &f_{\lambda_1,\cdots,\lambda_{n-1}}(x_1, \cdots, \partial x_n) = (\lambda_1 + \cdots + \lambda_{n-1} + \partial) f_{\lambda_1,\cdots,\lambda_{n-1}}(x_1, \cdots, x_n). \end{aligned}$$

Let $C^0_{\text{LeibC}} = V/\partial V$. For $n \geq 1$, let $C^n_{\text{LeibC}} = C^n_{\text{LeibC}}(R, V)$ be the space of all n-λ-brackets on R with coefficients in V. Define $C^*_{\text{LeibC}} = \oplus_{n\in\mathbb{N}} C^n_{\text{LeibC}}$ as the space of all poly λ-brackets.

For $n \geq 1$, $f \in C^n_{\text{LeibC}}$, define

$$\begin{aligned} &(\partial_{\text{LeibC}} f)_{\lambda_1,\cdots,\lambda_n}(x_1, \cdots, x_{n+1}) \\ = &\sum_{i=1}^{n} (-1)^{i+1} a_{i_{\lambda_i}} f_{\lambda_1,\cdots,\lambda_n}(x_1, \cdots, x_{n+1}) + (-1)^{n+1} f_{\lambda_1,\cdots,\lambda_{n-1}}(x_1, \cdots, x_n)_{\lambda_1+\cdots+\lambda_n} x_{n+1} \\ &+ \sum_{1\leq i<j\leq n+1} (-1)^i f_{\lambda_1,\cdots,\lambda_i+\lambda_j,\cdots,\lambda_n}(x_1, \cdots, [x_{i\lambda_i} x_j]_R, \cdots, x_{n+1}). \end{aligned}$$

The cohomology of this complex denoted by $H^*_{\text{LeibC}}(R, V)$ is called the cohomology of the Leibniz conformal algebra R with coefficients in a representation V.

Let R be a Leibniz conformal algebra. Recall that a $\mathbb{C}[\partial]$-linear map $d_R : R \to R$ is called a differential operator such that

$$d_R([x_\lambda y]) = [d_R(x)_\lambda y] + [x_\lambda d_R(y)] + \alpha[d_R(x)_\lambda d_R(y)], \quad \forall x, y \in R. \tag{1}$$

One denotes by $\text{Der}(R)$ the set of differential operators of the Leibniz conformal algebra R.

Definition 3. *A differential Leibniz conformal algebra is a Leibniz conformal algebra R with a differential operator $d_R \in \text{Der}(R)$. One denotes it by (R, d_R).*

Definition 4. *Given two differential Leibniz conformal algebras $(R, d_R), (Q, d_Q)$, a homomorphism of differential Leibniz conformal algebras from (R, d_R) to (Q, d_Q) is a Leibniz conformal algebra homomorphism $\varphi : R \to Q$ such that $\varphi \circ d_R = d_Q \circ \varphi$.*

Definition 5. *Let (R, d_R) be a differential Leibniz conformal algebra.*

(i) *A representation over the differential Leibniz conformal algebra (R, d_R) is a pair (V, d_V), where $d_V \in \text{Cend}(V)$, and V is a representation over the Leibniz conformal algebra R, such that for all $x \in R, u \in V$, the following equalities hold:*

$$\begin{aligned} &d_V(x \cdot_\lambda u) = d_R(x) \cdot_\lambda u + x \cdot_\lambda d_V(u) + \alpha d_R(x) \cdot_\lambda d_V(u), \\ &d_V(u \cdot_\lambda x) = u \cdot_\lambda d_R(x) + d_V(u) \cdot_\lambda x + \alpha d_V(u) \cdot_\lambda d_R(x). \end{aligned}$$

(ii) *Given two representations (U, d_U), (V, d_V) over (R, d_R), a conformal linear map $f : U \to V$ is called a homomorphism of representations, if $f \circ d_U = d_V \circ f$ and*

$$f(x \cdot_\lambda u) = x \cdot_\lambda f(u), \qquad f(u \cdot_\lambda x) = f(u) \cdot_\lambda x, \quad \forall x \in R, u \in V.$$

Define the set of n-cochains by

$$C^n_{\mathrm{DLeibC}}(R,V) := \begin{cases} C^n_{\mathrm{LeibC}}(R,V) \oplus C^{n-1}_{\mathrm{LeibC}}(R,V), & n \geq 2, \\ C^1_{\mathrm{LeibC}}(R,V) = \mathrm{Hom}(R,V), & n = 1, \\ C^0_{\mathrm{LeibC}}(R,V) = V, & n = 0. \end{cases}$$

For $n \geq 1$, we define a linear map $\delta : C^n_{\mathrm{LeibC}}(R,V) \to C^n_{\mathrm{LeibC}}(R,V)$ by

$$\begin{aligned} &\delta f_{\lambda_1,\cdots,\lambda_n}(x_1,\ldots,x_n): \\ &= \sum_{k=1}^{n} \alpha^{k-1} \sum_{1\leq i_1<\cdots<i_k\leq n} f_{\lambda_1,\cdots,\lambda_n}(x_1,\ldots,d_R(x_{i_1}),\ldots,d_R(x_{i_k}),\ldots,x_n) - d_V f_{\lambda_1,\cdots,\lambda_n}(x_1,\ldots,x_n), \end{aligned}$$

for any $f \in C^n_{\mathrm{LeibC}}(R,V)$ and

$$\delta v = -d_V(v), \quad \forall v \in C^0_{\mathrm{LeibC}}(R,V) = V/\partial V.$$

Define $\partial_{\mathrm{DLeibC}} : C^1_{\mathrm{LeibC}}(R,V) \to C^2_{\mathrm{LeibC}}(R,V)$ by

$$\partial_{\mathrm{DLeibC}}(f) = (\partial_{\mathrm{LieC}}(f), -\delta f), \qquad \forall f \in \mathrm{Hom}(R,V).$$

Then, for $n \geq 2$, we define $\partial_{\mathrm{DLeibC}} : C^n_{\mathrm{LeibC}}(R,V) \to C^{n+1}_{\mathrm{LeibC}}(R,V)$ by

$$\partial_{\mathrm{DLeibC}}(f_n, g_{n-1}) = (\partial_{\mathrm{LeibC}}(f_n), \partial_{\mathrm{LeibC}}(g_{n-1}) + (-1)^n \delta f_n),$$

for any $f_n \in C^n_{\mathrm{LeibC}}(R,V)$ and $g_{n-1} \in C^{n-1}_{\mathrm{LeibC}}(R,V)$. The cohomology of the cochain complex $(C^*_{\mathrm{DLeibC}}(R,V), \partial_{\mathrm{DLeibC}})$, denoted by $H^*_{\mathrm{DLeibC}}(R,V)$, is called the cohomology of the differential Leibniz conformal algebra (R, d_R) with coefficients in the representation (V, d_V).

3. Crossed Modules and Two-Term Differential $Leib_\infty$-Conformal Algebras

In this section, we introduce homotopy differential operators on two-term $Leib_\infty$-conformal algebras. A two-term $Leib_\infty$-conformal algebra equipped with a homotopy differential operator is called a two-term differential $Leib_\infty$-conformal algebra. We show that skeletal two-term differential $Leib_\infty$-conformal algebras correspond to the third cocycles of differential Leibniz conformal algebras. Next, we introduce crossed modules of differential Leibniz conformal algebras and show that crossed modules of differential Leibniz conformal algebras correspond to strict two-term differential $Leib_\infty$-conformal algebras.

Definition 6 ([17]). *A two-term $Leib_\infty$-conformal algebra is a triple $(R_1 \xrightarrow{\pi} R_0, \rho_2, \rho_3)$ consisting of a complex $R_1 \xrightarrow{\pi} R_0$ of $\mathbb{C}[\partial]$-modules equipped with*

- *a $\mathbb{C}$-linear conformal sesquilinear map $\rho_2 : R_i \otimes R_j \to R_{i+j}[\lambda]$, for $0 \leq i, j, i+j \leq 1$,*
- *a $\mathbb{C}$-linear conformal sesquilinear map $\rho_3 : R_0 \otimes R_0 \otimes R_0 \to R_1[\lambda,\mu]$*

that satisfy the following set of identities: for all $x, y, z, w \in R_0$ and $u, v \in R_1$,

$(Leib1)$ $(\rho_2)_\lambda(u,v)=0$,
$(Leib2)$ $\pi((\rho_2)_\lambda(x,u))=(\rho_2)_\lambda(x,\pi u)$,
$(Leib3)$ $\pi((\rho_2)_\lambda(u,x))=(\rho_2)_\lambda(\pi u,x)$,
$(Leib4)$ $(\rho_2)_\lambda(\pi u,v)=(\rho_2)_\lambda(u,\pi v)$,
$(Leib5)$ $\pi\Big((\rho_3)_{\lambda,\mu}(x,y,z)\Big)=(\rho_2)_\lambda(x,(\rho_2)_\mu(y,z))-(\rho_2)_{\lambda+\mu}((\rho_2)_\lambda(x,y),z)-(\rho_2)_\mu(y,(\rho_2)_\lambda(x,z))$,
$(Leib6)$ $(\rho_3)_{\lambda,\mu}(x,y,\pi v)=(\rho_2)_\lambda(x,(\rho_2)_\mu(y,v))-(\rho_2)_{\lambda+\mu}((\rho_2)_\lambda(x,y),v)-(\rho_2)_\mu(y,(\rho_2)_\lambda(x,v))$,
$(Leib7)$ $(\rho_3)_{\lambda,\mu}(x,\pi v,y)=(\rho_2)_\lambda(x,(\rho_2)_\mu(v,y))-(\rho_2)_{\lambda+\mu}((\rho_2)_\lambda(x,v),y)-(\rho_2)_\mu(v,(\rho_2)_\lambda(x,y))$,
$(Leib8)$ $(\rho_3)_{\lambda,\mu}(\pi v,x,y)=(\rho_2)_\lambda(v,(\rho_2)_\mu(x,y))-(\rho_2)_{\lambda+\mu}((\rho_2)_\lambda(v,x),y)-(\rho_2)_\mu(x,(\rho_2)_\lambda(v,y))$,
$(Leib9)$ $(\rho_2)_\lambda\Big(x,(\rho_3)_{\mu,\nu}(y,z,w)\Big)-(\rho_2)_\mu\Big(y,(\rho_3)_{\lambda,\nu}(x,z,w)\Big)+(\rho_2)_\nu\Big(z,(\rho_3)_{\lambda,\mu}(x,y,w)\Big)$
$+(\rho_2)_{\lambda+\mu+\nu}\Big((\rho_3)_{\lambda,\mu}(x,y,z),w\Big)-(\rho_3)_{\lambda+\mu,\nu}\Big((\rho_2)_\lambda(x,y),z,w\Big)-(\rho_3)_{\mu,\lambda+\nu}\Big(y,(\rho_2)_\lambda(x,z),w\Big)$
$-(\rho_3)_{\mu,\nu}\Big(y,z,(\rho_2)_\lambda(x,w)\Big)+(\rho_3)_{\lambda,\mu+\nu}\Big(x,(\rho_2)_\mu(y,z),w\Big)+(\rho_3)_{\lambda,\nu}\Big(x,z,(\rho_2)_\mu(y,w)\Big)$
$-(\rho_3)_{\lambda,\mu}\Big(x,y,(\rho_2)_\nu(z,w)\Big)=0.$

Definition 7. *Let $\mathcal{R}=(R_1\xrightarrow{\pi}R_0,\rho_2,\rho_3)$ be a two-term $Leib_\infty$-conformal algebra. A triple $\mathbf{d}=(d_0,d_1,d_2)$, where $d_0:R_0\to R_0$ and $d_1:R_1\to R_1$ are conformal linear maps and $d_2:\wedge^2R_0\to R_1[\lambda]$ is a conformal bilinear map, is called a homotopy differential operator on $\mathcal{R}$, if $\pi\circ d_1=d_0\circ\pi$, and for all $x,y,z\in R_0$ and $u\in R_1$,*

$(D1)$ $\pi((d_2)_\lambda(x,y))=d_0((\rho_2)_\lambda(x,y))-(\rho_2)_\lambda(d_0(x),y)-(\rho_2)_\lambda(x,d_0(y))-\alpha(\rho_2)_\lambda(d_0(x),d_0(y))$,
$(D2)$ $(d_2)_\lambda(x,\pi u)=d_1((\rho_2)_\lambda(x,u))-(\rho_2)_\lambda(d_0(x),u)-(\rho_2)_\lambda(x,d_1(u))-\alpha(\rho_2)_\lambda(d_0(x),d_1(u))$,
$(D3)$ $(d_2)_\lambda(\pi u,x)=d_1((\rho_2)_\lambda(u,x))-(\rho_2)_\lambda(u,d_0(x))-(\rho_2)_\lambda(d_1(u),x)-\alpha(\rho_2)_\lambda(d_1(u),d_0(x))$,
$(D4)$ $(\rho_3)_{\lambda,\mu}(d_0(x),y,z)+\alpha(\rho_3)_{\lambda,\mu}(x,d_0(y),z)+\alpha^2(\rho_3)_{\lambda,\mu}(x,y,d_0(z))-d_1(\rho_3)_{\lambda,\mu}(x,y,z)$
$=(\rho_2)_{\lambda+\mu}((d_2)_\lambda(x,y),z)-(\rho_2)_{\lambda+\nu}((d_2)_\lambda(x,z),y)-(\rho_2)_\lambda(x,(d_2)_\mu(y,z))+(d_2)_{\lambda+\mu}((\rho_2)_\lambda(x,y),z)$
$-(d_2)_{\lambda+\nu}((\rho_2)_\lambda(x,z),y)-(d_2)_\lambda(x,(\rho_2)_\mu(y,z))$.

A two-term differential $Leib_\infty$-conformal algebra is a two-term $Leib_\infty$-conformal algebra $\mathcal{R}=(R_1\xrightarrow{\pi}R_0,\rho_2,\rho_3)$ equipped with a homotopy differential operator $\mathbf{d}=(d_0,d_1,d_2)$. We denote a two-term differential $Leib_\infty$-conformal algebra by $(R_1\xrightarrow{\pi}R_0,\rho_2,\rho_3,d_0,d_1,d_2)$ or simply by $(\mathcal{R},\mathbf{d})$.

Definition 8. *Let $(\mathcal{R},\mathbf{d})$ be a two-term differential $Leib_\infty$-conformal algebra. It is said to be*

(i) Skeletal if $\pi=0$,
(ii) Strict if $\rho_3=0$ and $d_2=0$.

Theorem 1. *There is a one-to-one correspondence between skeletal two-term differential $Leib_\infty$-conformal algebras and triples of the form $(R_{\mathfrak{T}},V_{\mathfrak{S}},(f,\theta))$, where (R,d_R) is a differential Leibniz conformal algebra, (V,d_V) is a representation and $(f,\theta)\in C^3_{\mathrm{DLeibC}}(R,V)$ is a 3-cocycle.*

Proof. Let $(R_1\xrightarrow{\pi}R_0,\rho_2,\rho_3,d_0,d_1,d_2)$ be a skeletal two-term differential $Leib_\infty$-conformal algebra. Then, according to (Leib5) and (D1), we obtain (R_0,ρ_2) and operator d_0 is a differential Leibniz conformal algebra. On the other hand, by conditions (Leib6), (Leib7),

(Leib8), (D2) and (D3), we obtain that (R_1, d_1) is a representation of the differential Leibniz conformal algebra (R_0, d_0) with the left and right λ-actions

$$\begin{aligned} &\cdot_\lambda : R_0 \otimes R_1 \to R_1[\lambda], \quad x \cdot_\lambda u = (\rho_2)_\lambda(x, u), \\ &{}_\lambda\cdot : R_1 \otimes R_0 \to R_1[\lambda], \quad u \cdot_\lambda x = (\rho_2)_\lambda(u, x), \qquad \forall x \in R_0, u \in R_1. \end{aligned}$$

The conditions (Leib9) and (D4) are, respectively, equivalent to

$$\delta^3_{\mathrm{DLeibC}}(\rho_3, -d_2) = (\partial_{\mathrm{LeibC}}(\rho_3), -\partial_{\mathrm{LeibC}}(d_2) - \delta\rho_3) = 0.$$

Thus, $(\rho_3, -d_2) \in C^3_{\mathrm{DLeibC}}(R, V)$ is a 3-cocycle.

Conversely, given a triple $((R, d_R), (V, d_V), (\rho_3, d_2))$ as in the statement, define conformal bilinear maps ρ_2 by

$$l^2_\lambda(x, y) = [x_\lambda y]_R, \quad (\rho_2)_\lambda(x, u) = x \cdot_\lambda u, \quad (\rho_2)_\lambda(u, x) = u \cdot_\lambda x,$$

for $x, y \in R, u \in V$. Then, $(V \xrightarrow{0} R, \rho_2, f, d_R, d_V, -d_2)$ is a skeletal two-term differential $Leib_\infty$-conformal algebra. □

Next, we introduce crossed modules of differential Leibniz-conformal algebras and characterize strict two-term differential $Leib_\infty$-conformal algebras.

Definition 9. *A crossed module of differential Leibniz conformal algebras consists of* $((R_0, d_0), (R_1, d_1), \pi, \rho^L, \rho^R)$*, where* (R_0, d_0) *and* (R_1, d_1) *are differential Leibniz conformal algebras,* $\pi : (R_1, d_1) \to (R_0, d_0)$ *is a differential Leibniz conformal algebra homomorphism , and* $\rho^L : R_0 \otimes R_1 \to R_1[\lambda]$, $\rho^R : R_1 \otimes R_0 \to R_1[\lambda]$ *, and make* (R_1, d_1) *into a representation of the differential Leibniz conformal algebra* (R_0, d_0) *satisfying*

$$\begin{aligned} &(Ca) \qquad \pi(\rho^L(x)_\lambda(u)) = [x_\lambda \pi(u)]_{R_0}, \quad \pi(\rho^R(x)_\lambda(u)) = [\pi(u)_\lambda x]_{R_0}, \\ &(Cb) \qquad \rho^L(\pi(u))_\lambda(v) = [u_\lambda v]_{R_1}, \quad \rho^R(\pi(u))_\lambda(v) = [v_\lambda u]_{R_1}, \end{aligned}$$

for any $x \in R_0, u, v \in R_1$.

Proposition 1. *Let* $((R_0, d_0), (R_1, d_1), \pi, \rho^L, \rho^R)$ *be a crossed module of differential Leibniz conformal algebras. Then,* $(R_0 \oplus R_1, d_0 \oplus d_1)$ *is a differential Leibniz conformal algebra, where the bracket is*

$$[(x, u)_\lambda (y, v)] = ([x_\lambda y]_{R_0}, \rho^L(x)_\lambda v + \rho^R(y)_\lambda u + [u_\lambda v]_{R_1})$$

for any $x, y \in R_0, u, v \in R_1$.

Proof. Since R_0, R_1 are both Leibniz conformal algebras and (R_1, ρ^L, ρ^R) is a representation of R_0, then we have that $R_0 \oplus R_1$ is a Leibniz conformal algebra. Moreover, for any $(x, u), (y, v) \in R_0 \oplus R_1$, we have

$$\begin{aligned} &(d_0 \oplus d_1)[(x, u)_\lambda (y, v)] \\ &= (d_0 \oplus d_1)([x_\lambda y]_{R_0}, \rho^L(x)_\lambda v + \rho^R(y)_\lambda u + [u_\lambda v]_{R_1}) \\ &= (d_0([x_\lambda y]_{R_0}), d_1(\rho^L(x)_\lambda v) + d_1(\rho^R(y)_\lambda u) + d_1([u_\lambda v]_{R_1})) \\ &= ([d_0(x)_\lambda y]_{R_0} + [x_\lambda d_0(y)]_{R_0} + \alpha[d_0(x)_\lambda d_0(y)]_{R_0}, \rho^L(x)_\lambda d_1(v) + \rho^L(d_0(x))_\lambda v \\ &\quad + \alpha\rho^L(d_0(x))_\lambda d_1(v) + \rho^R(y)_\lambda d_1(u) + \rho^R(d_0(y))_\lambda u + \alpha\rho^R(d_0(y))_\lambda d_1(u) \\ &\quad + [d_1(u)_\lambda v]_{R_1} + [u_\lambda d_1(v)]_{R_1} + \alpha[d_1(u)_\lambda d_1(v)]_{R_1}) \\ &= [(d_0 \oplus d_1)(x, u)_\lambda (y, v)] + [(x, u)_\lambda (d_0 \oplus d_1)(y, v)] + \alpha[(d_0 \oplus d_1)(x, u)_\lambda (d_0 \oplus d_1)(y, v)]. \end{aligned}$$

This shows that the map $d_0 \oplus d_1 : R_0 \oplus R_1 \to R_0 \oplus R_1$ is a differential operator. And the proof is finished. □

Theorem 2. *There is a one-to-one correspondence between strict two-term differential $Leib_\infty$-conformal algebras and crossed modules of differential Leibniz conformal algebras.*

Proof. Let $(R_1 \xrightarrow{\pi} R_0, \rho_2, l^3 = 0, d_0, d_1, d_2 = 0)$ be a strict two-term differential $Leib_\infty$-conformal algebra. Then, according to (Leib5) and (D1), we obtain (R_0, ρ_2), and operator d_0 is a differential Leibniz conformal algebra. Next, we define $[\cdot_\lambda \cdot]_{R_1} : R_1 \otimes R_1 \to R_1[\lambda]$ by $[u_\lambda v]_{R_1} = (\rho_2)_\lambda(\pi u, v) = (\rho_2)_\lambda(u, \pi v)$, for any $u, v \in R_1$. By conditions (Leib6) and (D3), we obtain that (R_1, d_1) is a differential Leibniz conformal algebra. On the other hand, the condition (Leib2) implies that $\pi : (R_1, d_1) \to (R_0, d_0)$ is a differential Leibniz conformal algebra morphism. Finally, we define

$$\begin{aligned} &\rho^L : R_0 \to R_1 \to R_1[\lambda], \qquad \rho^L(x)_\lambda u = (\rho_2)_\lambda(x, u), \\ &\rho^R : R_1 \to R_0 \to R_1[\lambda], \qquad \rho^R(x)_\lambda u = (\rho_2)_\lambda(u, x), \qquad \forall \in R_0, u \in R_1. \end{aligned}$$

Then, we obtain that $((R_1, d_1), \rho^L, \rho^R)$ is a representation of the differential Leibniz conformal algebra (R_0, d_0); by the conditions (Leib9) and (D4), we also have

$$\begin{aligned} &\pi(\rho^L(x)_\lambda u) = \pi(\rho_2)_\lambda(x, u) = (\rho_2)_\lambda(x, \pi u), \qquad \pi(\rho^R(x)_\lambda u) = \pi(\rho_2)_\lambda(u, x) = (\rho_2)_\lambda(\pi u, x), \\ &\rho^L(\pi u)_\lambda v = \pi(\rho_2)_\lambda(\pi u, v) = [u_\lambda v]_{R_1}, \qquad \rho^R(\pi u)_\lambda v = \pi(\rho_2)_\lambda(v, \pi u) = [v_\lambda u]_{R_1}, \end{aligned}$$

for any $x \in R_0, u, v \in R_1$. Thus, $((R_0, d_0), (R_1, d_1), \pi, \rho^L, \rho^R)$ is a crossed module of differential Leibniz conformal algebras.

Conversely, let $((R_0, d_0), (R_1, d_1), \pi, \rho^L, \rho^R)$ be a crossed module of differential Leibniz conformal algebras. Define conformal bilinear maps $\rho_2 : R_i \times R_j \to R_{i+j}[\lambda], i + j \leq 1$ by

$$\begin{aligned} &(\rho_2)_\lambda(x, y) = [x_\lambda y]_{R_0}, \quad (\rho_2)_\lambda(x, u) = \rho^L(x)_\lambda u, \\ &(\rho_2)_\lambda(u, x) = \rho^R(x)_\lambda u, \quad (\rho_2)_\lambda(u, v) = 0, \end{aligned}$$

for $x, y \in R_0, u, v \in R_1$. Hence, $(R_1 \xrightarrow{\pi} R_0, \rho_2, \rho_3 = 0, d_0, d_1, d_2 = 0)$ is a strict two-term differential $Leib_\infty$-conformal algebra. □

Combining Proposition 1 and Theorem 2, we obtain the following result.

Proposition 2. *Let $(\mathcal{R}, \mathbf{d})$ be a strict two-term differential $Leib_\infty$-conformal algebra. Then, $(R_0 \oplus R_1, d_0 \oplus d_1)$ is a differential Leibniz conformal algebra, where the bracket is*

$$[(x, u)_\lambda (y, v)] = ((\rho_2)_\lambda(x, y), (\rho_2)_\lambda(x, v) + (\rho_2)_\lambda(u, y) + (\rho_2)_\lambda(u, v)),$$

for any $(x, u), (y, v) \in R_0 \oplus R_1$.

Example 1. *Let (R_0, d_0) be a differential Leibniz conformal algebra. Then, $((R_0, d_0), (R_0, d_0), \mathrm{id}, L_x, R_x)$ is a crossed module of differential Leibniz conformal algebras. Therefore, it follows that*

$$(R_0 \xrightarrow{\mathrm{id}} R_0, [\cdot, \cdot]_{R_0}, \rho_3 = 0, d_0, d_0, d_2 = 0)$$

is a strict two-term differential $Leib_\infty$-conformal algebra.

Example 2. *Let (R_0, d_0) and (R_1, d_1) be a differential Leibniz conformal algebras, let $f : (R_0, d_0) \to (R_1, d_1)$ be a differential Leibniz conformal algebra morphism and let $i : R_1 \to R_0$ be the inclusion map. Then, $(\mathrm{Ker} f, R_0, i, L_x, R_x)$ is a crossed module of differential Leibniz conformal algebras.*

4. Non-Abelian Extension of Differential Leibniz Conformal Algebras

In this section, we study non-Abelian extensions of a differential Leibniz conformal algebra by another differential Leibniz conformal algebra.

Definition 10. *Let (R, d_R) and (Q, d_Q) be two differential Leibniz conformal algebras. A non-Abelian extension of (R, d_R) by (Q, d_Q) is a differential Leibniz conformal algebra (E, d_E) equipped with a short exact sequence of differential Leibniz conformal algebras*

$$0 \to (Q, d_Q) \xrightarrow{i} (E, d_E) \xrightarrow{p} (R, d_R) \to 0. \tag{2}$$

Definition 11. *Let (E, d_E) and $(E', d_{E'})$ be two non-Abelian extensions of (R, d_R) by (Q, d_Q). They are said to be equivalent if there is a morphism $\tau : (E, d_E) \to (E', d_{E'})$ of differential Leibniz conformal algebras making the following diagram commutative:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (Q, d_Q) & \xrightarrow{i_1} & (E, d_E) & \xrightarrow{p_1} & (R, d_R) & \longrightarrow & 0 \\ & & \mathrm{id}\downarrow & & \tau\downarrow & & \mathrm{id}\downarrow & & \\ 0 & \longrightarrow & (Q, d_Q) & \xrightarrow{i_2} & (E', d_{E'}) & \xrightarrow{p_2} & (R, d_R) & \longrightarrow & 0. \end{array} \tag{3}$$

The set of all equivalence classes of non-Abelian extensions of (R, d_R) by (Q, d_Q) is denoted by $\mathrm{Ext}_{\mathrm{nab}}((R, d_R), (Q, d_Q))$.

Example 3. *Let $((R_0, d_0), (R_1, d_1), \pi, \rho^L, \rho^R)$ be a crossed module of differential Leibniz conformal algebras. Then, the exact sequence*

$$0 \to (R_1, d_1) \xrightarrow{i} (R_0 \oplus R_1, d_0 \oplus d_1) \xrightarrow{p} (R_0, d_0) \to 0,$$

is a non-Abelian extension of (R_0, d_{R_0}) by (R_1, d_{R_1}).

We denote the set of equivalence classes of non-Abelian 2-cocycles by $H^2_{\mathrm{nab}}((R, d_R), (Q, d_Q))$.

Let (E, d_E) be a non-Abelian extension of the differential Leibniz conformal algebra (R, d_R) by (Q, d_Q) as of (2). A section of p is a linear map $s : R \to E$ that satisfies $p \circ s = \mathrm{id}_R$. We define conformal maps $\omega : \wedge^2 R \to Q[\lambda], \cdot_\lambda : R \otimes Q \to Q[\lambda], _\lambda \cdot : Q \otimes R \to Q[\lambda]$ and $\Omega : R \to Q$ by

$$\begin{aligned} &\omega_\lambda(x, y) = [s(x)_\lambda s(y)]_E - s([x_\lambda y]_R), \\ &x \cdot_\lambda p = [s(x)_\lambda p]_E, \\ &p \cdot_\lambda x = [p_\lambda s(x)]_E, \\ &\Omega(x) := d_E(s(x)) - s(d_R(x)), \quad \forall x, y \in R, p \in Q. \end{aligned}$$

Further, we define $R \oplus Q$ by the bracket

$$[(x, p)_\lambda (y, q)] := ([x_\lambda y]_R, x \cdot_\lambda q + p \cdot_\lambda y + \omega_\lambda(x, y) + [p_\lambda q]_Q),$$

with the conformal linear map

$$d_\Omega(x, p) = (d_R(x), d_Q(p) + \Omega(x)).$$

Lemma 1. *With the above notations, $R \oplus Q$ is a Leibniz conformal algebra if and only if $\omega, \cdot_{\lambda,\lambda} \cdot$ satisfy the following conditions:*

$$[x_\lambda y]_R \cdot_{\lambda+\mu} p = x \cdot_\lambda (y \cdot_\mu p) - y \cdot_\mu (x \cdot_\lambda p) - [\omega_\lambda(x,y)_{\lambda+\mu} p]_Q, \tag{4}$$
$$p \cdot_\mu [x_\lambda y]_R = x \cdot_\lambda (p \cdot_\mu y) - (x \cdot_\lambda p) \cdot_{\lambda+\mu} y - [p_\mu \omega_\lambda(x,y)]_Q, \tag{5}$$
$$p \cdot_\mu [x_\lambda y]_R = (p \cdot_\mu x) \cdot_{\lambda+\mu} y + x \cdot_\lambda (p \cdot_\mu y) - [p_\mu \omega_\lambda(x,y)]_Q, \tag{6}$$
$$x \cdot_\lambda [p_\mu q]_Q = [(x \cdot_\lambda p)_{\lambda+\mu} q]_Q + [p_\mu (x \cdot_\lambda q)]_Q, \tag{7}$$
$$x \cdot_\lambda [p_\mu q]_Q = [p_\mu (x \cdot_\lambda q)]_Q - [(p \cdot_\mu x)_{\lambda+\mu} q]_Q, \tag{8}$$
$$[p_\lambda q]_Q \cdot_{\lambda+\mu} x = [p_\lambda (q \cdot_\mu x)]_Q - [q_\mu (p \cdot_\lambda x)]_Q, \tag{9}$$
$$x \cdot_\lambda \omega_\mu(y,z) - y \cdot_\mu \omega_\lambda(x,z) - \omega_\lambda(x,y) \cdot_\nu z = \omega_{\lambda+\mu}([x_\lambda y]_R, z) - \omega_\lambda(x, [y_\mu z]_R) + \omega_\mu(y, [x_\lambda z]_R). \tag{10}$$

Proof. For any $x, y, z \in R, p, q \in Q$, we have

$$\begin{aligned}
[\partial(x+p)_\lambda (y+q)] &= [(\partial x + \partial p)_\lambda (y+q)] \\
&= ([\partial x_\lambda y]_R, (\partial x \cdot_\lambda q) + (\partial p \cdot_\lambda y) + \omega_\lambda(\partial x, y) + [\partial p_\lambda q]_Q) \\
&= (-\lambda [x_\lambda y]_R, -\lambda (x \cdot_\lambda q) - \lambda (p \cdot_\lambda y) - \lambda \omega_\lambda(x,y) - \lambda [p_\lambda q]_Q) \\
&= -\lambda [(x+p)_\lambda (y+q)].
\end{aligned}$$

Similar, we have

$$[(x+p)_\lambda \partial(y+q)] = (\partial + \lambda)[(x+p)_\lambda (y+q)].$$

Further, assume that $R \oplus Q$ is a Leibniz conformal algebra. By

$$[x_\lambda [y_\mu p]] = [[x_\lambda y]_{R\lambda+\mu} p] + [y_\mu [x_\lambda p]],$$

we deduce that (4) holds. By

$$[x_\lambda [p_\mu y]] = [[x_\lambda p]_{\lambda+\mu} y] + [p_\mu [x_\lambda y]_R],$$

we deduce that (5) holds. Similar to deduce that (6) holds. By

$$[x_\lambda [p_\mu q]_Q] = [[x_\lambda p]_{\lambda+\mu} q] + [p_\mu [x_\lambda q]],$$

we deduce that (7) holds. Similarly, we deduce that (8)–(9) hold. By

$$[x_\lambda [y_\mu z]_R]_R = [[x_\lambda y]_{R\lambda+\mu} z]_R + [y_\mu [x_\lambda z]_R]_R,$$

we deduce that (10) holds.

Conversely, if (4)–(10) hold, it is straightforward to see that $R \oplus Q$ is a Leibniz conformal algebra. The proof is finished. □

Lemma 2. *The maps $\omega, \cdot_\lambda, \cdot_\lambda \cdot, \Omega$ defined above satisfy the following compatible conditions: for all $x, y \in R$ and $p \in Q$,*

$$\begin{aligned}
&d_Q \omega_\lambda(x,y) + \Omega([x_\lambda y]_R) = \Omega(x) \cdot_\lambda y + \omega_\lambda(d_R(x), y) + x \cdot_\lambda \Omega(y) + \omega_\lambda(x, d_R(y)) \\
&\quad + \alpha d_R(x) \cdot_\lambda \Omega(y) + \alpha \Omega(x) \cdot_\lambda d_R(y) + \alpha \omega_\lambda(d_R(x), d_R(y)) + \alpha [\Omega(x)_\lambda \Omega(y)]_E,
\end{aligned} \tag{11}$$
$$[\Omega(x)_\lambda p]_E + \alpha d_R(x) \cdot_\lambda d_Q(p) + \alpha [\Omega(x)_\lambda d_Q(p)]_E - d_Q(x \cdot_\lambda p) + d_R(x) \cdot_\lambda p + x \cdot_\lambda d_Q(p) = 0, \tag{12}$$
$$[p_\lambda \Omega(x)]_E + \alpha d_Q(p) \cdot_\lambda d_R(x) + \alpha [d_Q(p)_\lambda \Omega(x)]_E - d_Q(p \cdot_\lambda x) + d_Q(p) \cdot_\lambda x + p \cdot_\lambda d_R(x) = 0. \tag{13}$$

Proof. For any $x, y \in R$, we have

$$
\begin{aligned}
&\Omega(x)\cdot_\lambda y+\omega_\lambda(d_R(x),y)+x\cdot_\lambda\Omega(y)+\omega_\lambda(x,d_R(y))-d_Q\omega_\lambda(x,y)-\Omega([x_\lambda y]_R)\\
&\quad+\alpha d_R(x)\cdot_\lambda\Omega(y)+\alpha\Omega(x)\cdot_\lambda d_R(y)+\alpha\omega_\lambda(d_R(x),d_R(y))+\alpha[\Omega(x)_\lambda\Omega(y)]_E\\
&=\underbrace{[d_E(s(x))_\lambda s(y)]_E}_{2A}-\underbrace{[s(d_R(x))_\lambda s(y)]_E}_{1D}+\underbrace{[s(d_R(x))_\lambda s(y)]_E}_{1D}-\underbrace{s([d_R(x)_\lambda y]_R)}_{2B}+\underbrace{[s(x)_\lambda d_E(s(y))]_E}_{2A}\\
&\quad-\underbrace{[s(x)_\lambda s(d_R(y))]_E}_{1C}+\underbrace{[s(x)_\lambda s(d_R(y))]_E}_{1C}-\underbrace{s([x_\lambda d_R(y)]_R)}_{2B}-\underbrace{d_Q[s(x)_\lambda s(y)]_E}_{2A}+\underbrace{d_Q s([x_\lambda y]_R)}_{1B}
\end{aligned}
$$

$$
\begin{aligned}
&-\underbrace{d_E(s([x_\lambda y]_R))}_{1B}+\underbrace{s(d_R([x_\lambda y]_R))}_{2B}+\underbrace{\alpha[s(d_R(x))_\lambda d_E(s(y))]_E}_{1E}-\underbrace{\alpha[s(d_R(x))_\lambda s(d_R(y))]_E}_{1A}\\
&+\underbrace{\alpha[d_E(s(x))_\lambda s(d_R(y))]_E}_{1F}-\underbrace{\alpha[s(d_R(x))_\lambda s(d_R(y))]_E}_{1E}+\underbrace{\alpha[s(d_R(x))_\lambda s(d_R(y))]_E}_{1E}-\underbrace{\alpha s([d_R(x)_\lambda d_R(y)]_R)}_{2B}\\
&+\underbrace{\alpha[d_E(s(x))_\lambda d_E(s(y))]_E}_{2A}-\underbrace{\alpha[d_E(s(x))_\lambda s(d_R(y))]_E}_{1F}-\underbrace{\alpha[s(d_R(x))_\lambda d_E(s(y))]_E}_{1E}+\underbrace{\alpha[s(d_R(x))_\lambda s(d_R(y))]_E}_{1A}\\
&=0,
\end{aligned}
$$

and we deduce that (11) holds. Further, for any $x\in R$ and $p\in Q$, we have

$$
\begin{aligned}
&[\Omega(x)_\lambda p]_E+\alpha d_R(x)\cdot_\lambda d_Q(p)+\alpha[\Omega(x)_\lambda d_Q(p)]_E-d_Q(x\cdot_\lambda p)+d_R(x)\cdot_\lambda p+x\cdot_\lambda d_Q(p)\\
&=[d_E(s(x))_\lambda p]_E-\underbrace{[sd_R((x))_\lambda p]_E}_{A}+\underbrace{\alpha[s(d_R(x))_\lambda d_Q(p)]_E}_{B}+\alpha[d_E(s(x))_\lambda d_Q(p)]_E\underbrace{-\alpha[s(d_R(x))_\lambda d_Q(p)]_E}_{B}\\
&\quad-d_Q[s(x)_\lambda p]_E+\underbrace{[s(d_R(x))_\lambda p]_E}_{A}+[s(x)_\lambda d_Q(p)]_E\\
&=[d_E(s(x))_\lambda p]_E+[s(x)_\lambda d_Q(p)]_E+\alpha[d_E(s(x))_\lambda d_Q(p)]_E-d_Q[s(x)_\lambda p]_E\\
&=0.
\end{aligned}
$$

This means Equation (12) is satisfied. Similarly, one can check that Equation (13) holds. □

Definition 12.

(i) *Let (R,d_R) and (Q,d_Q) be two differential Leibniz conformal algebras. A non-Abelian 2-cocycle of (R,d_R) with values in (Q,d_Q) is a quadruple $(\omega,\cdot_\lambda,_\lambda\cdot,\Omega)$ of conformal linear maps $\omega:\wedge^2R\to Q[\lambda],\cdot_\lambda:R\otimes Q\to Q[\lambda],_\lambda\cdot:Q\otimes R\to Q[\lambda]$ and $\Omega:R\to Q$ satisfying the conditions (4)-(13).*

(ii) *Let $(\omega,\cdot_\lambda,_\lambda\cdot,\Omega)$ and $(\omega',\cdot'_\lambda,_\lambda\cdot',\Omega')$ be two non-Abelian 2-cocycles of (R,d_R) with values in (Q,d_Q). They are said to be equivalent if there exists a conformal linear map $\eta:R\to Q$ that satisfies*

$$\omega_\lambda(x,y)-\omega'_\lambda(x,y)=x\cdot'_\lambda\eta(y)+\eta(x)\cdot'_\lambda y-\eta[x_\lambda y]_R+[\eta(x)_\lambda\eta(y)]_Q, \tag{14}$$

$$x\cdot_\lambda p-x\cdot'_\lambda p=[\eta(x)_\lambda p]_E, \tag{15}$$

$$p\cdot_\lambda x-p\cdot'_\lambda x=[p_\lambda\eta(x)]_E, \tag{16}$$

$$\Omega(x)-\Omega'(x)=d_Q(\eta(x))-\eta(d_R(x)),\quad\forall x,y\in R,p\in Q. \tag{17}$$

We denote the set of equivalence classes of non-Abelian 2-cocycles by $H^2_{\mathrm{nab}}((R,d_R),(Q,d_Q))$.

With the above notations, we obtain the following result.

Theorem 3. *Let (R, d_R) and (Q, d_Q) be two differential Leibniz conformal algebras. Then, the set of equivalence classes of non-Abelian extensions of (R, d_R) by (Q, d_Q) is classified by $H^2_{\mathrm{nab}}((R, d_R), (Q, d_Q))$. In other words,*

$$\mathrm{Ext}_{\mathrm{nab}}((R, d_R), (Q, d_Q)) \cong H^2_{\mathrm{nab}}((R, d_R), (Q, d_Q)).$$

Proof. Let (E, d_E) and $(E', d_{E'})$ be two equivalent extensions of (R, d_R) by (Q, d_Q). If $s : R \to E$ is a section of the map p, then it is easy to observe that the map $s' := \eta \circ s$ is a section of the map p'. Let $(\omega', \cdot'_{\lambda,\lambda}\cdot', \Omega')$ be the non-Abelian 2-cocycle corresponding to the non-Abelian extension $(E', d_{E'})$ with section s', for any $x, y \in R, q \in Q$, we have

$$\begin{aligned}
\omega'_\lambda(x, y) &= [s'(x)_\lambda s'(y)]_E - s'[x_\lambda y]_R \\
&= [\eta \circ s(x)_\lambda \eta \circ s(y)]_E - \eta \circ s[x_\lambda y]_R \\
&= \eta([s(x)_\lambda s(y)]_E - s[x_\lambda y]_R) \\
&= \omega_\lambda(x, y).
\end{aligned}$$

Similarly, $x \cdot_\lambda q = x \cdot'_\lambda q, q \cdot_\lambda x = q \cdot'_\lambda x$ and $\Omega(x) = \Omega'(x)$. This shows that $(\omega', \cdot'_{\lambda,\lambda}\cdot', \Omega') = (\omega, \cdot_{\lambda,\lambda}\cdot, \Omega)$. Hence they give rise to the same element in $H^2_{\mathrm{nab}}((R, d_R), (Q, d_Q))$. Therefore, there is a well-defined map $\Pi : \mathrm{Ext}_{\mathrm{nab}}((R, d_R), (Q, d_Q)) \to H^2_{\mathrm{nab}}((R, d_R), (Q, d_Q))$.

Conversely, let $(\omega, \cdot_{\lambda,\lambda}\cdot, \Omega)$ be a non-Abelian 2-cocycle on (R, d_R) with values in (Q, d_Q). Define $E := R \oplus Q$ with the bracket

$$[(x, p)_\lambda (y, q)] := ([x_\lambda y]_R, x \cdot_\lambda q + p \cdot_\lambda y + \omega_\lambda(x, y) + [p_\lambda q]_Q).$$

and the conformal linear map

$$d^\Omega_E(x, p) = (d_R(x), d_Q(p) + \Omega(x)).$$

According to the conditions (4)-(10), it can be easily verified that E is a Leibniz conformal algebra. Moreover, we observe that

$$\begin{aligned}
&d^\Omega_E([(x, p)_\lambda (y, q)]) \\
&= d^\Omega_E([x_\lambda y]_R, x \cdot_\lambda q + p \cdot_\lambda y + \omega_\lambda(x, y) + [p_\lambda q]_Q) \\
&= (d_R([x_\lambda y]_R), d_Q(x \cdot_\lambda q) + d_Q(p \cdot_\lambda y) + d_Q(\omega_\lambda(x, y)) + d_Q([p_\lambda q]_Q) + \Omega([x_\lambda y]_R)) \\
&= ([d_R(x)_\lambda y]_R + [x_\lambda d_R(y)]_R + \alpha[d_R(x)_\lambda d_R(y)]_R, d_Q(x \cdot_\lambda q) + d_Q(p \cdot_\lambda y) + \Omega([x_\lambda y]_R) \\
&\quad + d_Q(\omega_\lambda(x, y)) + [d_Q(p)_\lambda q]_Q + [p_\lambda d_Q(q)]_Q + \alpha[d_Q(p)_\lambda d_Q(q)]_Q) \\
&= (\underbrace{[d_R(x)_\lambda y]_R}_{A} + \underbrace{[x_\lambda d_R(y)]_R}_{B} + \underbrace{\alpha[d_R(x)_\lambda d_R(y)]_R}_{C}, \underbrace{\Omega(x) \cdot_\lambda y}_{A} + \underbrace{\omega_\lambda(d_R(x), y)}_{A} + \underbrace{x \cdot_\lambda \Omega(y)}_{B} \\
&\quad + \underbrace{\omega_\lambda(x, d_R(y))}_{B} + \underbrace{\alpha d_R(x) \cdot_\lambda \Omega(y) + \alpha\Omega(x) \cdot_\lambda d_R(y) + \alpha\omega_\lambda(d_R(x), d_R(y)) + \alpha[\Omega(x)_\lambda \Omega(y)]_E}_{C} \\
&\quad + \underbrace{[\Omega(x)_\lambda q]_E}_{A} + \underbrace{\alpha d_R(x) \cdot_\lambda d_Q(q) + \alpha[\Omega(x)_\lambda d_Q(q)]_E}_{C} + \underbrace{d_R(x) \cdot_\lambda q}_{A} + \underbrace{x \cdot_\lambda d_Q(q)}_{B} + \underbrace{\alpha d_R(x) \cdot_\lambda d_Q(q)}_{C} \\
&\quad + \underbrace{[p_\lambda \Omega(y)]_E}_{B} + \underbrace{\alpha d_Q(p) \cdot_\lambda d_R(y) + \alpha[d_Q(p)_\lambda \Omega(y)]_E}_{C} + \underbrace{d_Q(p) \cdot_\lambda y}_{A} + p \cdot_\lambda d_R(y) + \underbrace{\alpha d_Q(p) \cdot_\lambda d_R(y)}_{C}
\end{aligned}$$

$$
\begin{aligned}
&+\underbrace{[d_Q(p)_\lambda q]_Q}_{A}+\underbrace{[p_\lambda d_Q(q)]_Q}_{B}+\underbrace{\alpha[d_Q(p)_\lambda d_Q(q)]_Q)}_{C}\\
&=\underbrace{[(d_R(x),d_Q(p)+\Omega(x))_\lambda(y,q)]}_{A}+\underbrace{[(x,p)_\lambda(d_R(y),d_Q(q)+\Omega(y))]}_{B}\\
&+\underbrace{\alpha[(d_R(x),d_Q(p)+\Omega(x))_\lambda(d_R(y),d_Q(q)+\Omega(y))]}_{C}\\
&=[d_E^\Omega(x,p)_\lambda(y,q)]+[(x,p)_\lambda d_E^\Omega(y,q)]+\alpha[d_E^\Omega(x,p)_\lambda d_E^\Omega(y,q)].
\end{aligned}
$$

This shows that d_E^Ω is a differential operator on the Leibniz conformal algebra E. In other words, (E, d_E^Ω) is a differential Leibniz conformal algebra. Further, it is easy to see that

$$0 \to (Q, d_Q) \xrightarrow{i} (E, d_E^\Omega) \xrightarrow{p} (R, d_R) \to 0$$

is a non-Abelian extension of the differential Leibniz conformal algebra (R, d_R) by (Q, d_Q).

Let $(\omega', \cdot'_{\lambda,\lambda}\cdot', \Omega')$ and $(\omega, \cdot_{\lambda,\lambda}\cdot, \Omega)$ be two equivalent 2-cocycles. Thus, there exists a conformal linear map $\eta : R \to Q$ such that the identities (14)–(17) hold. Let (E, d_E^Ω) be a differential Leibniz conformal algebra induced by the 2-cocycle $(\omega', \cdot'_{\lambda,\lambda}\cdot', \Omega')$. We define a map $\tau : R \oplus Q \to R \oplus Q$ by $\tau(x, p) = (x, p + \eta(x))$ for all $(x, p) \in R \oplus Q$. Then, we have

$$
\begin{aligned}
&\tau([(x,p)_\lambda(y,q)]_E)\\
&=\tau([x_\lambda y]_R, x\cdot_\lambda q + p\cdot_\lambda y + \omega_\lambda(x,y) + [p_\lambda q]_Q)\\
&=([x_\lambda y]_R, x\cdot_\lambda q + p\cdot_\lambda y + \omega_\lambda(x,y) + [p_\lambda q]_Q + \eta([x_\lambda y]_R))\\
&=([x_\lambda y]_R, x\cdot'_\lambda q + [\eta(x)_\lambda q]_E + p\cdot'_\lambda y + [p_\lambda \eta(y)]_E + \omega'_\lambda(x,y) + x\cdot'_\lambda \eta(y)\\
&+\eta(x)\cdot'_\lambda y - \eta[x_\lambda y]_R + [\eta(x)_\lambda \eta(y)]_Q + [p_\lambda q]_Q + \eta([x_\lambda y]_R)\\
&=([x_\lambda y]_R, x\cdot'_\lambda q + x\cdot'_\lambda \eta(y) + p\cdot'_\lambda y + \eta(x)\cdot'_\lambda y + \omega'_\lambda(x,y) + [(p+\eta(x))_\lambda(q+\eta(y))]_Q\\
&=[(x, p+\eta(x))_\lambda(y, q+\eta(y))]_{E'}\\
&=[\tau(x,p)_\lambda \tau(y,q)]_E.
\end{aligned}
$$

This is similar to checking that $\tau \circ d_E^\Omega = d_{E'}^{\Omega'} \circ \tau$. Hence, the map $\tau : (E, d_E^\Omega) \to (E', d_{E'}^{\Omega'})$ defines an equivalence between two non-Abelian extensions. Therefore, we obtain a well-defined map $\Gamma : H^2_{\mathrm{nab}}((R, d_R), (Q, d_Q)) \to \mathrm{Ext}_{\mathrm{nab}}((R, d_R), (Q, d_Q))$. Finally, it is straightforward to verify that the maps Π and Γ are inverse to each to each other. This completes the proof. □

5. Automorphisms of Differential Leibniz Conformal Algebras and the Wells Map

In this section, we study the inducibility of a pair of differential Leibniz conformal algebra automorphisms and characterize them by equivalent conditions.

Let (R, d_R) and (Q, d_Q) be two differential Leibniz conformal algebras, and let

$$0 \longrightarrow (Q, d_Q) \xrightarrow{i} (E, d_E) \xrightarrow{p} (R, d_R) \longrightarrow 0,$$

be a non-Abelian extension of (R, d_R) by (Q, d_Q). Let $\mathrm{Aut}_Q(E)$ be the set of all differential automorphisms $\Upsilon \in \mathrm{Aut}(E, d_E)$ that satisfy $\Upsilon|_Q \subset Q$. For any automorphism $\Upsilon \in \mathrm{Aut}_Q(E, d_E)$, then $\Upsilon|_Q \in \mathrm{Aut}(Q, d_Q)$. We define a conformal linear map $\bar{\Upsilon} : R \longrightarrow R$ by

$$\bar{\Upsilon}(x) = p\Upsilon s(x), \quad \forall\, x \in R.$$

Assume that s_1 and s_2 are two distinct sections of E, since $ps_1(x) - ps_2(x) = 0$, $s_1(x) - s_2(x) \in \mathrm{Ker}p \cong Q$, it follows that $\Upsilon(s_1(x) - s_2(x)) \in Q$. Thus, $p\Upsilon s_1(x) = p\Upsilon s_2(x)$, which indicates that $\bar{\Upsilon}$ is independent of the choice of a section.

For all $x, y \in R$, we have

$$\begin{aligned}\bar{\Upsilon}([x_\lambda y]_R) &= p\Upsilon(s[x_\lambda y]_R)\\ &= p\Upsilon([s(x)_\lambda s(y)]_E - \omega(x,y))\\ &= p\Upsilon([s(x)_\lambda s(y)]_E)\\ &= [p\Upsilon s(x)_\lambda p\Upsilon s(y)]_R\\ &= [\bar{\Upsilon}(x)_\lambda \bar{\Upsilon}(y)]_R.\end{aligned}$$

Further,

$$\begin{aligned}(d_R\overline{\Upsilon} - \overline{\Upsilon}d_R)(x) &= (d_R p\Upsilon s - p\Upsilon s d_R)(x)\\ &= (pd_E\Upsilon s - p\Upsilon s d_R)(x)\\ &= p\Upsilon(d_E s - s d_R)(x)\\ &= 0,\end{aligned}$$

which yields that $\bar{\Upsilon}$ is a homomorphism of differential Leibni- conformal algebras. It is easy to check that $\bar{\Upsilon}$ is bijective. Thus, $\bar{\Upsilon} \in \mathrm{Aut}(R, d_R)$. Then, we can define a group homomorphism

$$\Lambda : \mathrm{Aut}_Q(E, d_E) \longrightarrow \mathrm{Aut}(R, d_R) \times \mathrm{Aut}(Q, d_Q),\ \ \Lambda(\Upsilon) = (\bar{\Upsilon}, \Upsilon|_Q).$$

Definition 13. *A pair $(\Phi, \Psi) \in \mathrm{Aut}(R, d_R) \times \mathrm{Aut}(Q, d_Q)$ is said to be inducible if (Φ, Ψ) is an image of Λ.*

Below, we investigate when a pair (Φ, Ψ) is inducible.

Let $0 \longrightarrow (Q, d_Q) \xrightarrow{i} (E, d_E) \xrightarrow{p} (R, d_R) \longrightarrow 0$ be a non-Abelian extension of (R, d_R) by (Q, d_Q) and $(\omega, \cdot_\lambda, {}_\lambda\cdot, \Omega)$ be the corresponding non-Abelian 2-cocycle induced by a section s of E. Given any pair $(\Phi, \Psi) \in \mathrm{Aut}(R, d_R) \times \mathrm{Aut}(Q, d_Q)$. Define conformal maps $\omega^{\Phi,\Psi} : R \times R \longrightarrow Q[\lambda]$, $\cdot_\lambda^{\Phi,\Psi} : R \otimes Q \longrightarrow Q[\lambda]$, ${}_\lambda\cdot^{\Phi,\Psi} : Q \otimes R \longrightarrow Q[\lambda]$, $\Omega^{\Phi,\Psi} : R \longrightarrow Q$ respectively, by

$$\omega_\lambda^{\Phi,\Psi}(x,y) = \Psi\omega_\lambda(\Phi^{-1}(x), \Phi^{-1}(y)), \tag{18}$$

$$x \cdot_\lambda^{\Phi,\Psi} q = \Psi(\Phi^{-1}(x) \cdot_\lambda \Psi^{-1}(q)), \tag{19}$$

$$q \cdot_\lambda^{\Phi,\Psi} x = \Psi(\Psi^{-1}(q) \cdot_\lambda \Phi^{-1}(x)), \tag{20}$$

$$\Omega^{\Phi,\Psi}(x) = \Psi\Omega(\Phi^{-1}(x)), \tag{21}$$

for all $x, y \in R, q \in Q$.

Proposition 3. *With the above notations, $(\omega^{\Phi,\Psi}, \cdot_\lambda^{\Phi,\Psi}, {}_\lambda\cdot^{\Phi,\Psi}, \Omega^{\Phi,\Psi})$ is a non-Abelian 2-cocycle.*

Proof. Using (18)–(21), we obtain

$$\begin{aligned}&x \cdot_\lambda^{\Phi,\Psi} (y \cdot_\mu^{\Phi,\Psi} q) - y \cdot_\mu^{\Phi,\Psi} (x \cdot_\lambda^{\Phi,\Psi} q) - ([x_\lambda y]_R) \cdot_{\lambda+\mu}^{\Phi,\Psi} q\\ &= \Psi(\Phi^{-1}(x) \cdot_\lambda (\Phi^{-1}(y)_\mu \Psi^{-1}q)) - \Psi(\Phi^{-1}(y)_\mu(\Phi^{-1}(x)_\lambda \Psi^{-1}q)) - \Psi([\Phi^{-1}(x)_\lambda \Phi^{-1}(y)]_R \Psi^{-1}(q))\\ &= \Psi\big(\Phi^{-1}(x)_\lambda(\Phi^{-1}(y)_\mu \Psi^{-1}(q)) - \Phi^{-1}(y)_\mu(\Phi^{-1}(x)_\lambda \Psi^{-1}(q)) - [\Phi^{-1}(x)_\lambda \Phi^{-1}(y)]_R \Psi^{-1}(q)\big)\\ &= \Psi[\omega_\lambda(\Phi^{-1}(x), \Phi^{-1}(y))_{\lambda+\mu} \Psi^{-1}(q)]_Q\\ &= [\Psi\omega_\lambda \Phi^{-1}(x), \Phi^{-1}(y))_{\lambda+\mu} q]_Q\\ &= [\omega_\lambda^{\Phi,\Psi}(x,y)_{\lambda+\mu} q]_Q,\end{aligned}$$

which implies that (5) holds. Similarly, (6)-(13) hold. The proof is finished. □

Let $0 \longrightarrow (Q, d_Q) \xrightarrow{i} (E, d_E) \xrightarrow{p} (R, d_R) \longrightarrow 0$ be a non-Abelian extension of (R, d_R) by (Q, d_Q) . Suppose that $(\omega, \cdot_{\lambda,\lambda}\cdot, \Omega)$ is the corresponding non-Abelian 2-cocycle induced by a section s. Define a linear map $W : Aut(R, d_R) \times Aut(Q, d_Q) \to H^2_{\text{nab}}((R, d_R), (Q, d_Q))$ by

$$W(\Phi, \Psi) = [(\omega^{\Phi,\Psi}, \cdot_{\lambda}^{\Phi,\Psi}{}_{,\lambda}\cdot^{\Phi,\Psi}, \Omega^{\Phi,\Psi}) - (\omega, \cdot_{\lambda,\lambda}\cdot, \Omega)].$$

It is remarkable that the map W is not a group homomorphism in general. The map W is also said to be the Wells map.

Theorem 4. *Let* $0 \longrightarrow (Q, d_Q) \xrightarrow{i} (E, d_E) \xrightarrow{p} (R, d_R) \longrightarrow 0$ *be a non-Abelian extension of* (R, d_R) *by* (Q, d_Q) *and let* $(\omega, \cdot_{\lambda,\lambda}\cdot, \Omega)$ *be the corresponding non-Abelian 2-cocycle induced by a section* s*. A pair* $(\Phi, \Psi) \in \mathrm{Aut}(R, d_R) \times \mathrm{Aut}(Q, d_Q)$ *is inducible if and only if* $W(\Phi, \Psi) = 0$*.*

Proof. Suppose that $(\Phi, \Psi) \in \mathrm{Aut}(R, d_R) \times \mathrm{Aut}(Q, d_Q)$ is inducible; then, there is an automorphism $\Upsilon \in \mathrm{Aut}_Q(E, d_E)$ such that $\Upsilon|_Q = \Psi$ and $p\Upsilon s = \Phi$. For all $x \in R$, since s is a section of p, that is, $ps = \mathrm{id}$,

$$p(\Upsilon s\Phi^{-1} - s)(x) = x - x = 0,$$

which implies that $(\Upsilon s\Phi^{-1} - s)(x) \in \ker p \cong Q$. So we can define a conformal linear map $\eta : R \longrightarrow Q$ by

$$\eta(x) = (\Upsilon s\Phi^{-1} - s)(x), \ \ \forall\, x \in R.$$

For $x \in R, q \in Q$, we have

$$\begin{aligned}
&x \cdot_{\lambda}^{\Phi,\Psi} q - x \cdot_{\lambda} q \\
&= \Psi(\Phi^{-1}(x) \cdot_{\lambda} \Psi^{-1}(q)) - x \cdot_{\lambda} q \\
&= \Psi([s(\Phi^{-1}(x))_{\lambda}\Psi^{-1}(q)]_E) - [s(x)_{\lambda}q]_E \\
&= [\Upsilon s(\Phi^{-1}(x))_{\lambda}\Upsilon(\Psi^{-1}(q))]_E - [s(x)_{\lambda}q]_E \\
&= [\Upsilon s(\Phi^{-1}(x))_{\lambda}q]_E - [s(x)_{\lambda}q]_E \\
&= [\eta(x)_{\lambda}q]_Q.
\end{aligned}$$

Hence, we obtain (15). Similarly, by direct calculations, we observe that (14), (16), (17) hold. It follows from the above observation that the non-Abelian 2-cocycles $(\omega^{\Phi,\Psi}, \cdot_{\lambda}^{\Phi,\Psi}{}_{,\lambda}\cdot^{\Phi,\Psi}, \Omega^{\Phi,\Psi})$ and $(\omega, \cdot_{\lambda,\lambda}\cdot, \Omega)$ are equivalent via the conformal linear map $\eta : R \longrightarrow Q$. Hence, we have

$$W(\Phi, \Psi) = [(\omega^{\Phi,\Psi}, \cdot_{\lambda}^{\Phi,\Psi}{}_{,\lambda}\cdot^{\Phi,\Psi}, \Omega^{\Phi,\Psi}) - (\omega, \cdot_{\lambda,\lambda}\cdot, \Omega)] = 0.$$

Conversely, suppose that $(\Phi, \Psi) \in \mathrm{Aut}(R, d_R) \times \mathrm{Aut}(Q, d_Q)$, Since $W(\Phi, \Psi) = 0$, it follows that the non-Abelian 2-cocycles $(\omega^{\Phi,\Psi}, \cdot_{\lambda}^{\Phi,\Psi}{}_{,\lambda}\cdot^{\Phi,\Psi}, \Omega^{\Phi,\Psi})$ and $(\omega, \cdot_{\lambda,\lambda}\cdot, \Omega)$ are equivalent, there is a conformal linear map $\eta : R \longrightarrow Q$ satisfying (14)–(17). Due to s being a section of p, then for all $e \in E$ can be written as $e = q + s(x)$ for some $q \in Q, x \in R$. Define a conformal linear map $\Upsilon : E \longrightarrow E$ by

$$\Upsilon(e) = \Upsilon(q + s(x)) = \Psi(q) + \eta\Phi(x) + s\Phi(x).$$

If $\Upsilon(e) = 0$, then $s\Phi(x) = 0$ and $\Psi(q) + \eta\Phi(x) = 0$. In view of s and Φ being injective, we obtain $x = 0$; it follows that $q = 0$. Thus, $e = q + s(x) = 0$; that is, Υ is injective. For any $e = q + s(x) \in E$,

$$\Upsilon(\Psi^{-1}(q) - \Psi^{-1}\eta(x) + s\Phi^{-1}(x)) = q + s(x) = e,$$

which yields that Υ is surjective. In all, Υ is bijective.

Next, we show that Υ is a homomorphism of differential Leibniz conformal algebras. In fact, for all $e_i = q_i + s(x_i) \in E$ $(i = 1, 2)$,

$$
\begin{aligned}
&[\Upsilon(e_1)_\lambda \Upsilon(e_2)]_E \\
&= [(\Psi(q_1) + \eta\Phi(x_1) + s\Phi(x_1))_\lambda (\Psi(q_2) + \eta\Phi(x_2) + s\Phi(x_2))]_E \\
&= [\Psi(q_1)_\lambda \Psi(q_2)]_E + [\Psi(q_1)_\lambda \eta\Phi(x_2)]_E + [\Psi(q_1)_\lambda s\Phi(x_2)]_E + [\eta\Phi(x_1)_\lambda \Psi(q_2)]_E \\
&\quad + [\eta\Phi(x_1)_\lambda \eta\Phi(x_2)]_E + [\eta\Phi(x_1)_\lambda s\Phi(x_2)]_E + [s\Phi(x_1)_\lambda \Psi(q_2)]_E + [s\Phi(x_1)_\lambda \eta\Phi(x_2)]_E + [s\Phi(x_1)_\lambda s\Phi(x_2)]_E \\
&= [\Psi(q_1)_\lambda \Psi(q_2)]_E + \Psi((q_1) \cdot_\lambda x_2) \underbrace{-\Psi(q_1) \cdot_\lambda \Phi(x_2) + \Psi(q_1) \cdot_\lambda \Phi(x_2)}_{} + \Psi((x_1) \cdot_\lambda q_2) \\
&\quad \underbrace{-\Phi(x_1) \cdot_\lambda \Psi(q_2)}_{D} + \Psi(\omega_\lambda(x_1, x_2)) \underbrace{-\omega_\lambda(\Phi(x_1), \Phi(x_2))}_{B} \underbrace{-\Phi(x_1) \cdot_\lambda \eta\Phi(x_2)}_{A} \underbrace{-\eta\Phi(x_1) \cdot_\lambda \Phi(x_2)}_{C} \\
&\quad + \eta\Phi([(x_1)_\lambda x_2]_R) + \underbrace{\eta\Phi(x_1) \cdot_\lambda \Phi(x_2)}_{C} + \underbrace{\Phi(x_1) \cdot_\lambda \Psi(q_2)}_{D} + \underbrace{\Phi(x_1) \cdot_\lambda \eta\Phi(x_2)}_{A} + \underbrace{\omega_\lambda(\Phi(x_1), \Phi(x_2))}_{B} \\
&\quad + s[\Phi(x_1)_\lambda \Phi(x_2)]_R
\end{aligned}
$$

$$
\begin{aligned}
&= \Psi([q_{1\lambda} q_2]_E + (q_1) \cdot_\lambda x_2 + (x_1) \cdot_\lambda q_2 + \omega_\lambda(x_1, x_2)) + \eta\Phi([x_{1\lambda} x_2]_R) + s[\Phi(x_1)_\lambda \Phi(x_2)]_R \\
&= \Psi([q_{1\lambda} q_2]_E + [(q_1)_\lambda s(x_2)]_E + [s(x_1)_\lambda q_2]_E + \omega_\lambda(x_1, x_2)) + \eta\Phi([x_{1\lambda} x_2]_R) + s\Phi[x_{1\lambda} x_2]_R \\
&= \Upsilon([q_{1\lambda} q_2]_E + [q_{1\lambda} s(x_2)]_E + [s(x_1)_\lambda q_2]_E + [s(x_1)_\lambda s(x_2)]_R) \\
&= \Upsilon([(q_1 + s(x_1))_\lambda (q_2 + s(x_2))]_E) = \Upsilon([e_{1\lambda} e_2]_E).
\end{aligned}
$$

Similarly, one can check that $\Upsilon \circ d_E = d_E \circ \Upsilon$. This proves that Υ is an automorphism of differential Leibniz-conformal algebras. Thus, $\Upsilon \in \mathrm{Aut}_Q(E, d_E)$. Finally, we show that $\Upsilon|_Q = \Psi$ and $p\Upsilon s = \Phi$. In fact,

$$\Upsilon(q) = \Upsilon(q + s(0)) = \Psi(q), \ \ \forall\, q \in Q$$

and

$$(p\Upsilon s)(x) = p\Upsilon(0 + s(x)) = p(\chi(x) + s\Phi(x)) = ps\Phi(x) = \Phi(x), \ \ \forall\, x \in R.$$

Therefore, $\Upsilon|_Q = \Psi$ and $p\Upsilon s = \Phi$. Thus, $(\Phi, \Psi) \in \mathrm{Aut}(R, d_R) \times \mathrm{Aut}(Q, d_Q)$ is inducible. □

Theorem 5. *Let* $0 \longrightarrow (Q, d_Q) \xrightarrow{i} (E, d_E) \xrightarrow{p} (R, d_R) \longrightarrow 0$ *be a non-Abelian extension of* (R, d_R) *by* (Q, d_Q). *Then there is an exact sequence*

$$1 \longrightarrow \mathrm{Aut}_Q^{Q,R}(E, d_E) \xrightarrow{\iota} \mathrm{Aut}(E, d_E) \xrightarrow{\Lambda} \mathrm{Aut}(R, d_R) \times \mathrm{Aut}(Q_{\mathfrak{S}}) \xrightarrow{W} H^2_{\mathrm{nab}}((R, d_R), (Q, d_Q)),$$

where $\mathrm{Aut}_Q^{Q,R}(E, d_E) = \{\gamma \in \mathrm{Aut}(E, d_E) \mid \Lambda(\Upsilon) = (\mathrm{id}_R, \mathrm{id}_Q)\}$.

Proof. Obviously, $\mathrm{Ker}\Lambda = \mathrm{Im}\iota$ and ι is injective. By Theorem 6.3, one can easily check that $\mathrm{Ker}W = \mathrm{Im}\Lambda$. This completes the proof. □

More generally, if we define

$$
\begin{aligned}
\mathrm{Aut}_Q^Q(E, d_E) &= \{\Upsilon \in \mathrm{Aut}_Q(E, d_E) \mid \ \Upsilon|_Q = \mathrm{id}_Q)\}, \\
\mathrm{Aut}_Q^R(E, d_E) &= \{\gamma \in \mathrm{Aut}_Q(E, d_E) \mid \overline{\Upsilon} := p\Upsilon s = \mathrm{id}_R\},
\end{aligned}
$$

we obtain two morphisms of groups $\Lambda_R : \mathrm{Aut}_Q^Q(E, d_E) \to \mathrm{Aut}(R, d_R), \Upsilon \mapsto \overline{\Upsilon}$ and $\Lambda_Q : \mathrm{Aut}_Q^R(E, d_E) \to \mathrm{Aut}(Q, d_Q), \Upsilon \mapsto \Upsilon|_Q$. Define the maps $W_Q : \mathrm{Aut}(Q, d_Q) \to H^2_{\mathrm{nab}}((R, d_R), (Q, d_Q))$ and $W_R : \mathrm{Aut}(R, d_R) \to H^2_{\mathrm{nab}}((R, d_R), (Q, d_Q))$ by

$$W_R(\Phi) = [(\omega^{\Phi,\mathrm{id}}, \cdot_\lambda^{\Phi,\mathrm{id}}{}_{,\lambda}\, \cdot^{\Phi,\mathrm{id}}, \Omega^{\Phi,\mathrm{id}}) - (\omega, \cdot_{\lambda,\lambda}\, \cdot, \Omega)],$$
$$W_Q(\Psi) = [(\omega^{\mathrm{id},\Psi}, \cdot_\lambda^{\mathrm{id},\Psi}{}_{,\lambda}\, \cdot^{\mathrm{id},\Psi}, \Omega^{\mathrm{id},\Psi}) - (\omega, \cdot_{\lambda,\lambda}\, \cdot\Omega)].$$

Proposition 4. *Let* $0 \longrightarrow (Q, d_Q) \xrightarrow{i} (E, d_E) \xrightarrow{p} (R, d_R) \longrightarrow 0$ *be a non-Abelian extension of* (R, d_R) *by* (Q, d_Q)*. Then, there are two exact sequences of groups*

$$1 \longrightarrow \mathrm{Aut}_Q^{Q,R}(E, d_E) \xrightarrow{\iota} \mathrm{Aut}_Q^Q(E, d_E) \xrightarrow{\Lambda_R} \mathrm{Aut}(R, d_R) \xrightarrow{W_R} H^2_{\mathrm{nab}}((R, d_R), (Q, d_Q)),$$
$$1 \longrightarrow \mathrm{Aut}_Q^{Q,R}(E, d_E) \xrightarrow{\iota} \mathrm{Aut}_Q^R(E, d_E) \xrightarrow{\Lambda_Q} \mathrm{Aut}(Q, d_Q) \xrightarrow{W_Q} H^2_{\mathrm{nab}}((R, d_R), (Q, d_Q)).$$

Author Contributions: H.W. and S.G.: Writing—original draft (equal). X.Z.: Writing—review editing (equal). All authors have read and agreed to the published version of the manuscript.

Funding: The paper is supported the Guizhou Provincial Basic Research Program (Natural Science) (No. ZK[2021]006), the Natural Science Foundation of China (No. 12161013 and 12271292), and the Natural Science Foundation of Shandong Province (No. 2023MA008).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Kac, V. *Vertex Algebras for Beginners*; American Mathematical Soc.: Providence, RI, USA, 1998.
2. Hong, Y.; Li, F. Left-symmetric conformal algebras and vertex algebras. *J. Pure Appl. Algebra* **2015**, *219*, 3543–3567. [CrossRef]
3. Hong, Y.; Bai, C. On antisymmetric infinitesimal conformal bialgebras. *J. Algebra* **2021**, *586*, 325–356. [CrossRef]
4. Hong, Y.; Li, F. On left-symmetric conformal bialgebras. *J. Algebra Appl.* **2015**, *14*, 1450079. [CrossRef]
5. Liu, J.; Zhou, S.; Yuan, L. Conformal *r*-matrix-Nijenhuis structures, symplectic-Nijenhuis structures, and ON-structures. *J. Math. Phys.* **2022**, *63*, 101701. [CrossRef]
6. Xu, M.; Hong, Y.; Wu, Z. Finite irreducible conformal modules of rank two Lie conformal algebras. *J. Algebra Appl.* **2021**, *20*, 2150145. [CrossRef]
7. Yuan, L.; Liu, J. Twisting theory, relative Rota–Baxter type operators and L_∞-algebras on Lie conformal algebras. *J. Algebra* **2023**, *636*, 88–122. [CrossRef]
8. Bakalov, B.; Kac, V.; Voronov, A. Cohomology of conformal algebras. *Commun. Math. Phys.* **1999**, *200*, 561–598. [CrossRef]
9. Bakalov, B.; Kac, V. Field algebras. *Int. Math. Res. Not.* **2003**, *3*, 123–159. [CrossRef]
10. Kolesnikov, P. Conformal representations of Leibniz algebras. *Siberian Math. J.* **2008**, *49*, 429–435. [CrossRef]
11. Zhang, J. On the cohomology of Leibniz conformal algebras. *J. Math. Phys.* **2015**, *56*, 041703. [CrossRef]
12. Wu, Z. Leibniz *H*-pseudoalgebras. *J. Algebra* **2015**, *437*, 1–33. [CrossRef]
13. Feng, T.; Chen, L. Cohomology and deformations of $\mathcal{O}$-operators on Leibniz conformal algebras. 2023, *in press*.
14. Guo, S.; Wang, S. Twisted relative Rota–Baxter operators on Leibniz conformal algebras. *Commun. Algebra* **2024**, *52*, 3946–3959. [CrossRef]
15. Hong, Y.; Yuan, L. Unified products of Leibniz conformal algebras. *Commun. Algebra* **2021**, *49*, 2074–2090. [CrossRef]
16. Zhou, J.; Hong, Y. Quadratic Leibniz conformal algebras. *J. Algebra Appl.* **2019**, *18*, 1950195. [CrossRef]
17. Das, A.; Sahoo, A. Homotopification and categorification of Leibniz conformal algebras. *arXiv* **2023**, arXiv:2309.02116.
18. Khudaverdyan, D.; Poncin, N.; Qiu, J. On the infinity category of homotopy Leibniz algebras. *Theory Appl. Categ.* **2014**, *29*, 332–370.
19. Eilenberg, S.; Maclane, S. Cohomology theory in abstract groups. II. Group extensions with non-Abelian kernel. *Ann. Math.* **1947**, *48*, 326–341. [CrossRef]
20. Liu, J.; Sheng, Y.; Wang, Q. On non-Abelian extensions of Leibniz algebras. *Commun. Algebra* **2018**, *46*, 574–587. [CrossRef]

21. Das, A.; Sen, S. 2-term averaging L_∞-algebras and non-Abelian extensions of averaging Lie algebras. *J. Algebra* **2024**, *644*, 126–151. [CrossRef]
22. Wells, C. Automorphisms of group extensions. *Trans. Amer. Math. Soc.* **1971**, *155*, 189–194. [CrossRef]
23. Passi, S.; Singh, M.; Yadav, K. Automorphisms of Abelian group extensions. *J. Algebra* **2010**, *324*, 820–830. [CrossRef]
24. Bardakov, V.; Singh, M. Extensions and automorphisms of Lie algebras. *J. Algebra Appl.* **2017**, *16*, 1750162. [CrossRef]
25. Guo, Y.; Hou, B. Crossed modules and non-Abelian extensions of Rota–Baxter Leibniz algebras. *J. Geom. Phys.* **2023**, *191*, 104906. [CrossRef]
26. Guo, S.; Zhang, X. Cohomology of differential Leibniz conformal algebras and applications. 2024, *in press*.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.