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# Starlikeness, Convexity, Close-to-Convexity, and Quasi-Convexity for Functions with Fixed Initial Coefficients

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**Abstract:** In this paper, we employ the theory of differential subordination to establish a theorem that delineates certain sufficient conditions for starlikeness, convexity, close-to-convexity, and quasi-convexity in relation to functions with fixed initial coefficients. Furthermore, we introduce some results derived from these conditions. Building upon this framework, we derive an extension of Nunokawa's lemma for analytic functions with fixed initial coefficients.

**Keywords:** analytic functions; convex; close-to-convex; differential subordination; fixed initial coefficients; starlike, Nunokawa lemma; quasi-convex

**MSC:** primary 30C45; secondary 30C80

## 1. Introduction and Preliminaries

Following [1] by $\mathcal{H}$, we denote the set of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\},$$

where $n$ is a positive integer number, and $a \in \mathbb{C}$. Suppose that $n \in \mathbb{N}$, we consider the subclass $\mathcal{A}_n$ of $\mathcal{H}$ as follows:

$$\mathcal{A}_n = \{f \in \mathcal{H} : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots\}.$$

We set $\mathcal{A}_1 = \mathcal{A}$. Moreover, the subclass of $\mathcal{A}$ consisting of univalent functions in the open unit disk $\mathbb{U}$ is denoted by $\mathcal{S}$. We recall the following definitions from [2]. A function $f \in \mathcal{A}$ is called a starlike function of order $0 \leq \alpha < 1$, written $f \in \mathcal{S}^*(\alpha)$, if

$$\mathfrak{Re} \frac{z f'(z)}{f(z)} > \alpha \qquad (z \in \mathbb{U}).$$

Especially, we set $\mathcal{S}^*(0) \equiv \mathcal{S}^*$. Furthermore, a function $f \in \mathcal{A}$ is called convex of order $0 \leq \alpha < 1$, written $f \in \mathcal{K}(\alpha)$, if it satisfies

$$\mathfrak{Re}\left\{\frac{z f''(z)}{f'(z)} + 1\right\} > \alpha \qquad (z \in \mathbb{U}).$$

In particular, we put $\mathcal{K}(0) \equiv \mathcal{K}$. Let $f, g \in \mathcal{H}$. Then by the function $f$ subordinated to $g$, denoted by $f \prec g$, we mean that if there exists an analytic function in $\mathbb{U}$ such as $\omega$, with $\omega(0) = 0$ and $|\omega(z)| \leq |z| < 1$, such that $f(z) = g(\omega(z))$. Moreover, in the case that $g$ is univalent in $\mathbb{U}$, then $f \prec g$ if and only if $f(\mathbb{U}) \subset g(\mathbb{U})$ and $f(0) = g(0)$. Let also

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$0 \leq \alpha < 1$ and $0 \leq \beta < 1$; we, respectively, name the subclasses $\mathcal{C}(\alpha, \beta)$ and $\mathcal{C}^*(\alpha, \beta)$ of $\mathcal{A}$ close-to-convex and quasi-convex of order $\beta$ and type $\alpha$ in $\mathbb{U}$ and define

$$\mathcal{C}(\alpha, \beta) := \left\{ f \in \mathcal{A} : \mathfrak{Re}\left\{ \frac{zf'(z)}{g(z)} \right\} > \beta, \quad g \in \mathcal{S}^*(\alpha) \right\}$$

and

$$\mathcal{C}^*(\alpha, \beta) := \left\{ f \in \mathcal{A} : \mathfrak{Re}\left\{ \frac{[zf'(z)]'}{g'(z)} \right\} > \beta, \quad g \in \mathcal{K}(\alpha) \right\}.$$

The class $\mathcal{H}_\beta[a, n]$ consists of analytic functions in $\mathbb{U}$ with the fixed initial coefficients defined as follows:

$$\mathcal{H}_\beta[a, n] = \{ f \in \mathcal{H} : f(z) = a + \beta z^n + a_{n+1} z^{n+1} + \cdots \},$$

where $n$ is a positive integer number, $a \in \mathbb{C}$, and $\beta \in \mathbb{C}$ is a fixed number. Moreover,

$$\mathcal{A}_{n,b} = \{ f \in \mathcal{H} : f(z) = z + b z^{n+1} + a_{n+2} z^{n+2} + \dots \},$$

where $n$ is a positive integer number, and $b \in \mathbb{C}$ is a fixed number. We also set $\mathcal{A}_b = \mathcal{A}_{1,b}$. Moreover, let

$$\mathcal{S}^*{}_{n,\beta}(\alpha) := \left\{ f \in \mathcal{S}^*(\alpha) : f(z) = z + \beta z^{n+1} + \cdots \right\},$$

$$\mathcal{K}_{n,\beta}(\alpha) := \left\{ f \in \mathcal{K}^*(\alpha) : f(z) = z + \beta z^{n+1} + \cdots \right\},$$

$$\mathcal{C}_{n,\beta_1,\beta_2}(\alpha, \beta) := \left\{ f \in \mathcal{C}(\alpha, \beta) : f(z) = z + \beta_1 z^{n+1} + \cdots, \quad g \in \mathcal{S}^*_{n,\beta_2}(\alpha) \right\},$$

and

$$\mathcal{C}^*_{n,\beta_1,\beta_2}(\alpha, \beta) := \left\{ f \in \mathcal{C}^*(\alpha, \beta) : f(z) = z + \beta_1 z^{n+1} + \cdots, \quad g \in \mathcal{K}_{n,\beta_2}(\alpha) \right\},$$

where $0 \leq \alpha < 1$, $0 \leq \beta < 1$, and $\beta_1 \in \mathbb{C}$ and $\beta_2 \in \mathbb{C}$ are fixed (See [3]).

It is imperative to underscore the significance of the coefficients of analytic functions in the realm of geometric function theory. Specifically, constraints on the second coefficient of a univalent function yield well-established outcomes such as growth, distortion, and covering theorems (see [1]). Recently, the exploration of second-order differential subordination for analytic functions with fixed initial coefficients was undertaken by M. Ali et al. [3] who continue the research of S. S. Miller and P. T. Mocanu [1]. Subsequently, in the works of [4–6], notable results were derived through the application of first-order differential subordination for analytic functions with fixed initial coefficients.

Moreover, in [7], the inquiry into the radius of starlikeness for analytic functions with fixed second coefficients was addressed. Amani et al. [8] have contributed significant findings regarding functions with fixed initial coefficients. Furthermore, a multitude of authors have recently delved into various aspects of these functions (see [9]).

Inspired by the works of [10,11], we delineate the conditions for starlikeness and various notions related to convexity (close-to-convexity and quasi-convexity) concerning functions with fixed initial coefficients in this paper. Additionally, we present an extension of Nunokawa's lemma [12], tailored for functions with fixed initial coefficients.

Section 2 encompasses the derivation of novel conditions for starlikeness and various notions related to convexity (close-to-convexity and quasi-convexity) pertaining to functions with fixed initial coefficients, along with pertinent corollaries. Subsequently, in Section 3, we expound upon the extension of Nunokawa's lemma for the above-mentioned functions.

For proving the main results, we should express some basic definitions and results.

**Definition 1** (see [4]). *Let $Q$ denote the set of functions $q$ that are analytic and injective on $\overline{\mathbb{U}}\backslash E(q)$, where*

$$E(q) := \left\{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} q(z) = \infty\right\},$$

*such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(q)$.*

**Lemma 1** (see [3]). *Let $z_0 = r_0 e^{i\theta_0}$, $(r_0 < 1)$, and $g(z) = g_n z^n + g_{n+1} z^{n+1} + \ldots$ be continuous in $\overline{\mathbb{U}}_{r_0}$ and analytic in $\mathbb{U}_{r_0} \cup \{z_0\}$ with $g(z) \not\equiv 0$, and $n \geq 1$. If*

$$|g(z_0)| = \max_{|z|\leq r_0} |g(z)|,$$

*then*

$$\frac{z_0 g'(z_0)}{g(z_0)} = m \tag{1}$$

*and*

$$\mathfrak{Re}\left\{\frac{z_0 g''(z_0)}{g'(z_0)} + 1\right\} \geq m, \tag{2}$$

*where*

$$m \geq n + \frac{|g(z_0)| - |g_n| r_0^n}{|g(z_0)| + |g_n| r_0^n}. \tag{3}$$

**Lemma 2** (see [3]). *Let $q \in Q$ with $q(0) = a$ and $p \in \mathcal{H}_c[a, n]$ with $p(z) \not\equiv a$. If there is a point $z_0 \in \mathbb{U}$ such that $p(z_0) \in q(\partial\mathbb{U})$ and $p(\{z : |z| < |z_0|\}) \subset q(\mathbb{U})$, then*

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0) \tag{4}$$

*and*

$$\mathfrak{Re}\left\{1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right\} \geq m\mathfrak{Re}\left\{1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right\} \tag{5}$$

*where $q^{-1}(p(z_0)) = \zeta_0 = e^{i\theta_0}$ and*

$$m \geq n + \frac{|q'(0)| - |c||z_0|^n}{|q'(0)| + |c||z_0|^n} \tag{6}$$

## 2. Main Results

At the outset of this section, we introduce a fundamental theorem, as follows:

**Theorem 1.** *Let $f \in \mathcal{A}_{n,\beta_1}$, $g \in \mathcal{A}_{n,\beta_2}$, and $0 \leq \lambda \leq 1$. Moreover, define $\mathcal{F}(z)$ and $\mathcal{G}(z)$ as follows:*

$$\mathcal{F}(z) := zf'(z) + \lambda z^2 f''(z) \tag{7}$$

*and*

$$\mathcal{G}(z) := (1-\lambda)g(z) + \lambda z g'(z). \tag{8}$$

*(i) If $0 \leq (1+n\lambda)(\beta_2 - (1+n)\beta_1) \leq 1-\alpha$ and*

$$\mathfrak{Re}\left(\frac{z\mathcal{G}'(z)}{\mathcal{G}(z)}\right) - \mathfrak{Re}\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right) < M_1\left(\frac{1-\alpha}{2(1+\alpha)}\right), \tag{9}$$

*then*

$$\mathfrak{Re}\left(\frac{\mathcal{F}(z)}{\mathcal{G}(z)}\right) > \frac{1+\alpha}{2} \qquad (0 \leq \alpha < 1), \tag{10}$$

*where $M_1 = n + \frac{(1-\alpha)-(1+n\lambda)(\beta_2-(1+n)\beta_1)}{(1-\alpha)+(1+n\lambda)(\beta_2-(1+n)\beta_1)}$.*

*(ii)* *If* $0 \leq (1+n\lambda)((1+n)\beta_1 - \beta_2) \leq 2(1-\alpha)$ *and*

$$\mathfrak{Re}\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right) - \mathfrak{Re}\left(\frac{z\mathcal{G}'(z)}{\mathcal{G}(z)}\right) > \frac{\alpha}{2(\alpha-1)}M_2, \tag{11}$$

*then*

$$\mathfrak{Re}\left(\frac{\mathcal{F}(z)}{\mathcal{G}(z)}\right) > \alpha \qquad (0 < \alpha \leq \frac{1}{2}), \tag{12}$$

*where* $M_2 = n + \frac{2(1-\alpha)-(1+n\lambda)((1+n)\beta_1-\beta_2)}{2(1-\alpha)+(1+n\lambda)((1+n)\beta_1-\beta_2)}$.

*(iii)* *If* $0 \leq (1+n\lambda)((n+1)\beta_1 - \beta_2) \leq 2(1-\alpha)$ *and*

$$\mathfrak{Re}\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right) - \mathfrak{Re}\left(\frac{z\mathcal{G}'(z)}{\mathcal{G}(z)}\right) > \frac{(\alpha-1)}{2\alpha}M_2, \tag{13}$$

*then*

$$\mathfrak{Re}\left(\frac{\mathcal{F}(z)}{\mathcal{G}(z)}\right) > \alpha \qquad (\frac{1}{2} \leq \alpha < 1), \tag{14}$$

*where* $M_2$ *is given by (ii).*

**Proof.** For the proof of (i), set

$$f(z) = z + \beta_1 z^{n+1} + a_{n+2}z^{n+2} + \cdots$$

and

$$g(z) = z + \beta_2 z^{n+1} + b_{n+2}z^{n+2} + \cdots .$$

$$\mathcal{F}(z) = z + (1+n\lambda)(n+1)\beta_1 z^{n+1} + (1+(n+1)\lambda)(n+2)a_{n+2}z^{n+2} + \cdots$$

and

$$\mathcal{G}(z) = z + (1+n\lambda)(n+1)\beta_2 z^{n+1} + (1+(n+1)\lambda)(n+2)b_{n+2}z^{n+2} + \cdots .$$

Hence, one can show that

$$\frac{\mathcal{F}(z)}{\mathcal{G}(z)} = 1 + (1+n\lambda)((n+1)\beta_1 - \beta_2)z^n + \cdots ,$$

where $z \in \mathbb{U}$. Let us define

$$\frac{\mathcal{F}(z)}{\mathcal{G}(z)} = \frac{1+\alpha w(z)}{1+w(z)} \qquad (z \in \mathbb{U}, 0 \leq \alpha < 1). \tag{15}$$

Then, we can clearly verify that $w(z)$, $w(0) = 0$, and it is analytic in $\mathbb{U}$. Moreover,

$$w(z) = w_n z^n + w_{n+1}z^{n+1} + \cdots ,$$

where $w_n = \frac{1+n\lambda}{1-\alpha}(\beta_2 - (1+n)\beta_1)$. In view of (15) and computing, we obtain

$$z\left(\frac{\mathcal{F}'(z)}{\mathcal{F}(z)} - \frac{\mathcal{G}'(z)}{\mathcal{G}(z)}\right) = \frac{\alpha z w'(z)}{1+\alpha w(z)} - \frac{z w'(z)}{1+w(z)} \qquad (z \in \mathbb{U}). \tag{16}$$

Suppose that there exists $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$ and $|w(z)| < 1$, for $|z| < |z_0| = r_0$, where $z \in \mathbb{U}$. Hence, for $w(z_0) = e^{i\theta}$ and $0 \leq \theta < 2\pi$, Lemma 1 implies

$$\frac{z_0 w'(z_0)}{w(z_0)} = m \qquad (w(z_0) = e^{i\theta}, 0 \leq \theta < 2\pi), \tag{17}$$

where $m \geq n + \frac{(1-\alpha)-(1+n\lambda)(\beta_2-(1+n)\beta_1)}{(1-\alpha)+(1+n\lambda)(\beta_2-(1+n)\beta_1)}$. Consequently, (16) and (17) yield

$$\begin{aligned}\mathfrak{Re}\left\{z_0\left(\frac{\mathcal{F}'(z_0)}{\mathcal{F}(z_0)}-\frac{\mathcal{G}'(z_0)}{\mathcal{G}(z_0)}\right)\right\} &= \mathfrak{Re}\left\{\frac{\alpha z_0 w'(z_0)}{1+\alpha w(z_0)}-\frac{z_0 w'(z_0)}{1+w(z_0)}\right\}\\ &= m\mathfrak{Re}\left(\frac{\alpha e^{i\theta}}{1+\alpha e^{i\theta}}-\frac{e^{i\theta}}{1+e^{i\theta}}\right)\\ &\leq m\frac{\alpha-1}{2(1+\alpha)}.\end{aligned}$$

Thus, we have

$$\mathfrak{Re}\left\{z_0\left(\frac{\mathcal{G}'(z_0)}{\mathcal{G}(z_0)}-\frac{\mathcal{F}'(z_0)}{\mathcal{F}(z_0)}\right)\right\} \geq \left(\frac{1-\alpha}{2(1+\alpha)}\right)\left(n+\frac{(1-\alpha)-(1+n\lambda)(\beta_2-(1+n)\beta_1)}{(1-\alpha)+(1+n\lambda)(\beta_2-(1+n)\beta_1)}\right),$$

which contradicts assertion (9) from the hypothesis. Thus, the proof of (i) is complete. For proving (ii), we define

$$\frac{\mathcal{F}(z)}{\mathcal{G}(z)} = \alpha + (1-\alpha)p(z). \tag{18}$$

Note that $p$ is analytic in $\mathbb{U}$, $p \in \mathcal{H}_{\beta_3}[1,n]$ with $p(0)=1$, and $\beta_3 = \frac{1+n\lambda}{1-\alpha}[(1+n)\beta_1 - \beta_2]$. From (18), we have

$$z\left(\frac{\mathcal{F}'(z)}{\mathcal{F}(z)}-\frac{\mathcal{G}'(z)}{\mathcal{G}(z)}\right) = \frac{(1-\alpha)zp'(z)}{\alpha+(1-\alpha)p(z)} \qquad (z\in\mathbb{U}, 0<\alpha\leq\frac{1}{2}). \tag{19}$$

We now claim that $p(z) \prec q(z) = \frac{1+z}{1-z}$; otherwise, if $p \not\prec q$, then Lemma 2 implies that there exist $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that $p(z_0) = q(\zeta_0)$ and $z_0p'(z_0) = m\zeta_0q'(\zeta_0)$, where $m \geq n + \frac{2-\beta_3}{2+\beta_3}$. Then, by taking $\zeta_0 = e^{it}$, $(-\pi < t \leq \pi)$, we have $p(z_0) = ix$, where $x = \cot(\frac{t}{2}) \in \mathbb{R}$. Hence, by obtaining the inverse and derivative of $q$ and applying (19), we obtain

$$\begin{aligned}\mathfrak{Re}\left\{z_0\left(\frac{\mathcal{F}'(z_0)}{\mathcal{F}(z_0)}-\frac{\mathcal{G}'(z_0)}{\mathcal{G}(z_0)}\right)\right\} &= \mathfrak{Re}\left\{\frac{(1-\alpha)m\zeta_0q'(\zeta_0)}{\alpha+(1-\alpha)ix}\right\}\\ &= \mathfrak{Re}\left\{\frac{-m}{2}\frac{(1-\alpha)(1+x^2)}{\alpha+(1-\alpha)ix}\right\}\\ &= \frac{-m}{2}\frac{\alpha(1-\alpha)(1+x^2)}{\alpha^2+(1-\alpha)^2x^2}.\end{aligned}$$

Let $t = x^2$. Set

$$g(t) = \frac{-m\alpha(1-\alpha)(1+t)}{2(\alpha^2+(1-\alpha)^2t)} \qquad (t\in[0,\infty), 0<\alpha\leq\frac{1}{2}).$$

By computing, we can easily conclude that

$$g'(t) = \frac{-m\alpha(1-\alpha)(2\alpha-1)}{2(\alpha^2+(1-\alpha)^2t)^2} \geq 0 \qquad (t\in[0,\infty), 0<\alpha\leq\frac{1}{2}).$$

Then, we have

$$\frac{-m(1-\alpha)}{2\alpha} \leq g(t) \leq \frac{-m\alpha}{2(1-\alpha)} \qquad (t\in[0,\infty), 0<\alpha\leq\frac{1}{2}).$$

This implies that

$$\Re\mathfrak{e}\left\{z_0\left(\frac{\mathcal{F}'(z_0)}{\mathcal{F}(z_0)}-\frac{\mathcal{G}'(z_0)}{\mathcal{G}(z_0)}\right)\right\}\leq\frac{-m\alpha}{2(1-\alpha)}$$
$$\leq\frac{\alpha}{2(\alpha-1)}\left(n+\frac{2(1-\alpha)-(1+n\lambda)((1+n)\beta_1-\beta_2)}{2(1-\alpha)+(1+n\lambda)((1+n)\beta_1-\beta_2)}\right),$$

which contradicts assertion (11). Hence, $\Re\mathfrak{e}(p(z))>0$. So, (18) implies (12). The proof of (iii) is similar to (ii), and so we omit the analogous details of the proof. Thus, the proof is complete. □

**Remark 1.** *Letting $f\in\mathcal{A}_n$ and $g\in\mathcal{A}_n$ in Theorem 1 and using the corrections required in this theorem, one can extend and improve the main theorem in [10].*

Putting $\lambda=0$ in Theorem 1, we have $\mathcal{G}(z):=g(z)$ and $\mathcal{F}(z):=zf'(z)$, then we can obtain the following result:

**Corollary 1.** *Let $f\in\mathcal{A}_{n,\beta_1}$ and $g\in\mathcal{A}_{n,\beta_2}$.*

*(i) If $0\leq\beta_2-(1+n)\beta_1\leq1-\alpha$ and*

$$\Re\mathfrak{e}\left\{z\left(\frac{g'(z)}{g(z)}-\frac{f''(z)}{f'(z)}\right)\right\}<\frac{M_1(1-\alpha)+2(1+\alpha)}{2(1+\alpha)},$$

*then we have*

$$\Re\mathfrak{e}\left(\frac{zf'(z)}{g(z)}\right)>\frac{1+\alpha}{2}\qquad(0\leq\alpha<1),$$

*where $M_1=n+\frac{(1-\alpha)-(\beta_2-(1+n)\beta_1)}{(1-\alpha)+(\beta_2-(1+n)\beta_1)}$.*

*(ii) If $0\leq(1+n)\beta_1-\beta_2\leq2(1-\alpha)$ and*

$$\Re\mathfrak{e}\left\{z\left(\frac{f''(z)}{f'(z)}-\frac{g'(z)}{g(z)}\right)\right\}>\frac{\alpha M_2-2(\alpha-1)}{2(\alpha-1)},$$

*then we have*

$$\Re\mathfrak{e}\left(\frac{zf'(z)}{g(z)}\right)>\alpha\qquad(0<\alpha\leq\frac{1}{2}),$$

*where $M_2=n+\frac{2(1-\alpha)-((1+n)\beta_1-\beta_2)}{2(1-\alpha)+((1+n)\beta_1-\beta_2)}$.*

*(iii) If $0\leq(1+n)\beta_1-\beta_2\leq2(1-\alpha)$ and*

$$\Re\mathfrak{e}\left\{z\left(\frac{f''(z)}{f'(z)}-\frac{g'(z)}{g(z)}\right)\right\}>\frac{M_2(\alpha-1)-2\alpha}{2\alpha},$$

*then we have*

$$\Re\mathfrak{e}\left(\frac{zf'(z)}{g(z)}\right)>\alpha\qquad(\frac{1}{2}\leq\alpha<1),$$

*where $M_2$ is given by (ii).*

Putting $\lambda=1$ in Theorem 1 implies that $\mathcal{F}(z):=zf'(z)+z^2f''(z)$ and $\mathcal{G}(z):=zg'(z)$, then we can reach to the following:

**Corollary 2.** *Let $f\in\mathcal{A}_{n,\beta_1}$ and $g\in\mathcal{A}_{n,\beta_2}$.*

*(i) If $0\leq(1+n)(\beta_2-(1+n)\beta_1)\leq1-\alpha$ and*

$$\Re\mathfrak{e}\left\{z\left(\frac{g''(z)}{g'(z)}-\frac{2f''(z)+zf'''(z)}{f'(z)+zf''(z)}\right)\right\}<\frac{M_1(1-\alpha)}{2(1+\alpha)},$$

*then we have*

$$\Re\mathfrak{e}\left(\frac{zf'(z)+z^2f''(z)}{zg'(z)}\right)>\frac{1+\alpha}{2}\qquad(0\leq\alpha<1),$$

*where* $M_1=n+\frac{(1-\alpha)-(1+n)(\beta_2-(1+n)\beta_1)}{(1-\alpha)+(1+n)(\beta_2-(1+n)\beta_1)}$.

*(ii)* *If* $0\leq(1+n)((1+n)\beta_1-\beta_2)\leq 2(1-\alpha)$ *and*

$$\Re\mathfrak{e}\left\{z\left(\frac{2f''(z)+zf'''(z)}{f'(z)+zf''(z)}-\frac{g''(z)}{g'(z)}\right)\right\}>\frac{M_2\alpha}{2(\alpha-1)},$$

*then we have*

$$\Re\mathfrak{e}\left(\frac{f'(z)+zf''(z)}{g'(z)}\right)>\alpha\qquad(0<\alpha\leq\frac{1}{2}),$$

*where* $M_2=n+\frac{2(1-\alpha)-(1+n)((1+n)\beta_1-\beta_2)}{2(1-\alpha)+(1+n)((1+n)\beta_1-\beta_2)}$.

*(iii)* *If* $0\leq(1+n)((1+n)\beta_1-\beta_2)\leq 2(1-\alpha)$ *and*

$$\Re\mathfrak{e}\left\{z\left(\frac{2f''(z)+zf'''(z)}{f'(z)+zf''(z)}-\frac{g''(z)}{g'(z)}\right)\right\}>\frac{M_2(\alpha-1)}{2\alpha},$$

*then we have*

$$\Re\mathfrak{e}\left(\frac{f'(z)+zf''(z)}{g'(z)}\right)>\alpha\qquad(\frac{1}{2}\leq\alpha<1),$$

*where* $M_2$ *is given by (ii).*

Letting $f(z)=g(z)$ in Corollaries 1 and 2, the following results are obtained:

**Corollary 3.** *Let* $f\in\mathcal{A}_{n,\beta_1}$.

*(i)* *If* $\frac{\alpha-1}{n}\leq\beta_1\leq 0$ *and*

$$\Re\mathfrak{e}\left\{z\left(\frac{f'(z)}{f(z)}-\frac{f''(z)}{f'(z)}\right)\right\}<\frac{M_1(1-\alpha)+2(1+\alpha)}{2(1+\alpha)},$$

*then we have*

$$\Re\mathfrak{e}\left(\frac{zf'(z)}{f(z)}\right)>\frac{1+\alpha}{2}\qquad,i.e.,\qquad f\in\mathcal{S}^*_{n,\beta_1}\left(\frac{1+\alpha}{2}\right),\qquad(0\leq\alpha<1),$$

*where* $M_1=n+\frac{1-\alpha+n\beta_1}{1-\alpha-n\beta_1}$.

*(ii)* *If* $0\leq\beta_1\leq\frac{2(1-\alpha)}{n}$ *and*

$$\Re\mathfrak{e}\left\{z\left(\frac{f''(z)}{f'(z)}-\frac{f'(z)}{f(z)}\right)\right\}>\frac{\alpha M_2-2(\alpha-1)}{2(\alpha-1)},$$

*then we have*

$$\Re\mathfrak{e}\left(\frac{zf'(z)}{f(z)}\right)>\alpha\qquad,i.e.,\qquad f\in\mathcal{S}^*_{n,\beta_1}(\alpha),\qquad(0<\alpha\leq\frac{1}{2}),$$

*where* $M_2=n+\frac{2(1-\alpha)-n\beta_1}{2(1-\alpha)+n\beta_1}$.

*(iii)* *If* $0\leq\beta_1\leq\frac{2(1-\alpha)}{n}$ *and*

$$\Re\mathfrak{e}\left\{z\left(\frac{f''(z)}{f'(z)}-\frac{f'(z)}{f(z)}\right)\right\}>\frac{M_2(\alpha-1)-2\alpha}{2\alpha},$$

*then we have*

$$\mathfrak{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad ,i.e., \qquad f \in \mathcal{S}^*_{n,\beta_1}(\alpha), \qquad (\frac{1}{2} \leq \alpha < 1),$$

*where $M_2$ is given by (ii).*

**Corollary 4.** *Let $f \in \mathcal{A}_{n,\beta_1}$.*

*(i) If $\frac{\alpha-1}{n(n+1)} \leq \beta_1 \leq 0$ and*

$$\mathfrak{Re}\left\{z\left(\frac{f''(z)}{f'(z)} - \frac{2f''(z)+zf'''(z)}{f'(z)+zf''(z)}\right)\right\} < \frac{M_1(1-\alpha)}{2(1+\alpha)},$$

*then we have*

$$\mathfrak{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \frac{1+\alpha}{2} \qquad ,i.e., \qquad f \in \mathcal{K}_{n,\beta_1}(\frac{1+\alpha}{2}) \qquad (0 \leq \alpha < 1),$$

*where $M_1 = n + \frac{(1-\alpha)+n(n+1)\beta_1}{(1-\alpha)-n(1+n)\beta_1}$.*

*(ii) If $0 \leq \beta_1 \leq \frac{2(1-\alpha)}{n(n+1)}$ and*

$$\mathfrak{Re}\left\{z\left(\frac{2f''(z)+zf'''(z)}{f'(z)+zf''(z)} - \frac{f''(z)}{f'(z)}\right)\right\} > \frac{M_2\alpha}{2(\alpha-1)},$$

*then we have*

$$\mathfrak{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha \qquad ,i.e., \qquad f \in \mathcal{K}_{n,\beta_1}(\alpha) \qquad (0 < \alpha \leq \frac{1}{2}),$$

*where $M_2 = n + \frac{2(1-\alpha)-n(n+1)\beta_1}{2(1-\alpha)+n(n+1)\beta_1}$.*

*(iii) If $0 \leq \beta_1 \leq \frac{2(1-\alpha)}{n(n+1)}$ and*

$$\mathfrak{Re}\left\{z\left(\frac{2f''(z)+zf'''(z)}{f'(z)+zf''(z)} - \frac{f''(z)}{f'(z)}\right)\right\} > \frac{M_2(\alpha-1)}{2\alpha},$$

*then we have*

$$\mathfrak{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha \qquad ,i.e., \qquad f \in \mathcal{K}_{n,\beta_1}(\alpha) \qquad (\frac{1}{2} \leq \alpha < 1),$$

*where $M_2$ is presented by (ii).*

Letting $g(z) \in \mathcal{S}^*_{n,\beta_2}(\gamma)$, $0 \leq \gamma < 1$ in Corollary 1, we can gain the following interesting result:

**Corollary 5.** *Let $f \in \mathcal{A}_{n,\beta_1}$ and $g(z) \in \mathcal{S}^*_{n,\beta_2}(\gamma)$ with $0 \leq \gamma < 1$.*

*(i) If $0 \leq \beta_2 - (1+n)\beta_1 \leq 1-\alpha$ and*

$$\mathfrak{Re}\left\{z\left(\frac{g'(z)}{g(z)} - \frac{f''(z)}{f'(z)}\right)\right\} < \frac{M_1(1-\alpha)+2(1+\alpha)}{2(1+\alpha)},$$

*then $f \in \mathcal{C}_{n,\beta_1,\beta_2}(\gamma, \frac{1+\alpha}{2})$, where $0 \leq \alpha < 1$ and $M_1 = n + \frac{(1-\alpha)-(\beta_2-(1+n)\beta_1)}{(1-\alpha)+(\beta_2-(1+n)\beta_1)}$.*

*(ii)* *If* $0 \leq (1+n)\beta_1 - \beta_2 \leq 2(1-\alpha)$ *and*

$$\mathfrak{Re}\left\{z\left(\frac{f''(z)}{f'(z)} - \frac{g'(z)}{g(z)}\right)\right\} > \frac{\alpha M_2 - 2(\alpha-1)}{2(\alpha-1)},$$

*then* $f \in \mathcal{C}_{n,\beta_1,\beta_2}(\gamma,\alpha)$, *where* $0 < \alpha \leq \frac{1}{2}$ *and* $M_2 = n + \frac{2(1-\alpha)-((1+n)\beta_1-\beta_2)}{2(1-\alpha)+((1+n)\beta_1-\beta_2)}$.

*(iii)* *If* $0 \leq (1+n)\beta_1 - \beta_2 \leq 2(1-\alpha)$ *and*

$$\mathfrak{Re}\left\{z\left(\frac{f''(z)}{f'(z)} - \frac{g'(z)}{g(z)}\right)\right\} > \frac{M_2(\alpha-1) - 2\alpha}{2\alpha},$$

*then* $f \in \mathcal{C}_{n,\beta_1,\beta_2}(\gamma,\alpha)$, *where* $\frac{1}{2} \leq \alpha < 1$, *and* $M_2$ *is presented by (ii).*

Taking $g(z) \in \mathcal{K}_{n,\beta_2}(\gamma)$, $0 \leq \gamma < 1$ in Corollary 2, we have

**Corollary 6.** *Let* $f \in \mathcal{A}_{n,\beta_1}$ *and* $g(z) \in \mathcal{K}_{n,\beta_2}(\gamma)$ *with* $0 \leq \gamma < 1$.

*(i)* *If* $0 \leq (1+n)(\beta_2 - (1+n)\beta_1) \leq 1-\alpha$, *and* $0 \leq \gamma < 1$

$$\mathfrak{Re}\left\{z\left(\frac{g''(z)}{g'(z)} - \frac{2f''(z)+zf'''(z)}{f'(z)+zf''(z)}\right)\right\} < \frac{M_1(1-\alpha)}{2(1+\alpha)},$$

*then* $f \in \mathcal{C}^*_{n,\beta_1,\beta_2}(\gamma, \frac{1+\alpha}{2})$, *where* $0 \leq \alpha < 1$, *and* $M_1 = n + \frac{(1-\alpha)-(1+n)(\beta_2-(1+n)\beta_1)}{(1-\alpha)+(1+n)(\beta_2-(1+n)\beta_1)}$.

*(ii)* *If* $0 \leq (1+n)((1+n)\beta_1 - \beta_2) \leq 2(1-\alpha)$ *and*

$$\mathfrak{Re}\left\{z\left(\frac{2f''(z)+zf'''(z)}{f'(z)+zf''(z)} - \frac{g''(z)}{g'(z)}\right)\right\} > \frac{M_2\alpha}{2(\alpha-1)},$$

*then* $f \in \mathcal{C}^*_{n,\beta_1,\beta_2}(\gamma,\alpha)$, *where* $0 < \alpha \leq \frac{1}{2}$, *and* $M_2 = n + \frac{2(1-\alpha)-(1+n)((1+n)\beta_1-\beta_2)}{2(1-\alpha)+(1+n)((1+n)\beta_1-\beta_2)}$.

*(iii)* *If* $0 \leq (1+n)((1+n)\beta_1 - \beta_2) \leq 2(1-\alpha)$ *and*

$$\mathfrak{Re}\left\{z\left(\frac{2f''(z)+zf'''(z)}{f'(z)+zf''(z)} - \frac{g''(z)}{g'(z)}\right)\right\} > \frac{M_2(\alpha-1)}{2\alpha},$$

*then* $f \in \mathcal{C}^*_{n,\beta_1,\beta_2}(\gamma,\alpha)$, *where* $M_2$ *is presented by (ii).*

## 3. An Extension of Nunokawa's Lemma for Analytic Functions with Fixed Initial Coefficients

**Theorem 2.** *Let* $p \in \mathcal{H}_\beta[a,n]$, $0 \leq \alpha < \mathfrak{Re}a$ *and* $0 \leq \beta \leq 2(\mathfrak{Re}a - \alpha)$. *If there exists* $z_0 \in \mathbb{U}$ *such that* $\mathfrak{Re}(p(z)) > \alpha$, *for* $p(z_0) = \alpha + \beta_1 i$ *and* $|z| < |z_0|$ *with* $\beta_1 \neq 0$, *then*

$$\mathfrak{Re}\left(\frac{z_0 p'(z_0)}{p(z_0)}\right) \leq -\left(n + \frac{2(\mathfrak{Re}a-\alpha)-\beta}{2(\mathfrak{Re}a-\alpha)+\beta}\right)\left(\frac{\alpha|\alpha+\beta_1 i-a|^2}{2(\mathfrak{Re}a-\alpha)(\alpha^2+\beta_1^2)}\right) \leq 0, \tag{20}$$

*and*

$$\mathfrak{Im}\left(\frac{z_0 p'(z_0)}{p(z_0)}\right) \geq \left(n + \frac{2(\mathfrak{Re}a-\alpha)-\beta}{2(\mathfrak{Re}a-\alpha)+\beta}\right)\left(\frac{\beta_1|\alpha+\beta_1 i-a|^2}{2(\mathfrak{Re}a-\alpha)(\alpha^2+\beta_1^2)}\right) > 0, \tag{21}$$

*when* $\beta_1 > 0$, *and*

$$\mathfrak{Im}\left(\frac{z_0 p'(z_0)}{p(z_0)}\right) \leq \left(n + \frac{2(\mathfrak{Re}a-\alpha)-\beta}{2(\mathfrak{Re}a-\alpha)+\beta}\right)\left(\frac{\beta_1|\alpha+\beta_1 i-a|^2}{2(\mathfrak{Re}a-\alpha)(\alpha^2+\beta_1^2)}\right) < 0, \tag{22}$$

*when* $\beta_1 < 0$.

**Proof.** Define

$$q(z) = \frac{a - (2\alpha - \bar{a})z}{1 - z} \qquad (z \in \mathbb{U}),$$

where $\Re e\, a > \alpha$. It is easy to verify that $q \in Q$ with $E(q) = \{1\}$, $q(0) = p(0) = a$, and $q(\mathbb{U}) = \{w : \Re e\, w > \alpha\}$. Moreover, by computing, we obtain

$$q^{-1}(z) = \frac{z - a}{z - (2\alpha - \bar{a})} \quad , \quad q'(z) = \frac{2(\Re e\, a - \alpha)}{(1 - z)^2}. \tag{23}$$

Since $p(\mathbb{U}_{r_0}) \subset q(\mathbb{U})$ and $p(z_0) \in q(\partial\mathbb{U})$ with $|z_0| = r_0$, Lemma 2 implies that

$$p(z_0) = q(\zeta_0) \quad , \quad z_0 p'(z_0) = m\zeta_0 q'(\zeta_0), \tag{24}$$

where

$$\zeta_0 \in \partial\mathbb{U} \quad \textit{and} \quad m \geq n + \frac{2(\Re e\, a - \alpha) - \beta}{2(\Re e\, a + \alpha) + \beta}. \tag{25}$$

By combining (23) and (24), it can be observed that

$$\zeta_0 = \frac{p(z_0) - a}{p(z_0) - (2\alpha - \bar{a})}, \quad \zeta_0 q'(\zeta_0) = -\frac{|p(z_0) - a|^2}{2(\Re e\, a - \alpha)},$$

then have

$$\frac{z_0 p'(z_0)}{p(z_0)} = -\frac{m|\alpha + \beta_1 i - a|^2(\alpha - i\beta_1)}{2(\Re e\, a - \alpha)(\alpha^2 + \beta_1^2)} \tag{26}$$

So, (25) implies that

$$\Re e \frac{z_0 p'(z_0)}{p(z_0)} \leq -\left(n + \frac{2(\Re e\, a - \alpha) - \beta}{2(\Re e\, a - \alpha) + \beta}\right)\left(\frac{\alpha|\alpha + \beta_1 i - a|^2}{2(\Re e\, a - \alpha)(\alpha^2 + \beta_1^2)}\right) \leq 0,$$

$$\Im m \frac{z_0 p'(z_0)}{p(z_0)} \geq \left(n + \frac{2(\Re e\, a - \alpha) - \beta}{2(\Re e\, a - \alpha) + \beta}\right)\left(\frac{\beta_1|\alpha + \beta_1 i - a|^2}{2(\Re e\, a - \alpha)(\alpha^2 + \beta_1^2)}\right) > 0 \qquad (\beta_1 > 0),$$

and

$$\Im m \frac{z_0 p'(z_0)}{p(z_0)} \leq \left(n + \frac{2(\Re e\, a - \alpha) - \beta}{2(\Re e\, a - \alpha) + \beta}\right)\left(\frac{\beta_1|\alpha + \beta_1 i - a|^2}{2(\Re e\, a - \alpha)(\alpha^2 + \beta_1^2)}\right) < 0 \qquad (\beta_1 < 0).$$

Hence, the relations (20), (21), and (22) hold. Thus the proof is complete. □

**Remark 2.** *Putting $n = a = 1$ in Theorem 2, we find that Theorem 2 extends Theorem 1 in [11] for analytic functions with fixed initial coefficients.*

**Remark 3.** *Letting $\alpha = 0$ and $n = a = 1$ in Theorem 2, we find that Theorem 2 extends Corollary 1 in [12] for analytic functions with fixed initial coefficients.*

Letting $\alpha = 0$ in Theorem 2, we can obtain Corollary 2.4 in [13]:

**Corollary 7.** *Let $p \in \mathcal{H}_\beta[a, n]$ such that $0 \leq \beta \leq 2\Re e\, a$ and $\Re e\, a > 0$. If there exists $z_0 \in \mathbb{U}$ such that $|\arg(p(z))| < \frac{\pi}{2}$, for $|z| < |z_0|$ and $p(z_0) = \pm bi$ with $b > 0$, then $\Re e\left(\frac{z_0 p'(z_0)}{p(z_0)}\right) = 0$ and*

$$\Im m\left(\frac{z_0 p'(z_0)}{p(z_0)}\right) \geq \left(n + \frac{2\Re e\, p(0) - \beta}{2\Re e(p(0)) + \beta}\right)\left(\frac{b^2 - 2b\Im m(p(0)) + |p(0)|^2}{2b\Re e(p(0))}\right),$$

*when* $p(z_0) = bi$, *and*

$$\mathfrak{Im}\left(\frac{z_0 p'(z_0)}{p(z_0)}\right) \leq -\left(n + \frac{2\mathfrak{Re}\, p(0) - \beta}{2\mathfrak{Re}(p(0)) + \beta}\right)\left(\frac{b^2 + 2b\mathfrak{Im}(p(0)) + |p(0)|^2}{2b\mathfrak{Re}(p(0))}\right),$$

*when* $p(z_0) = -bi$.

**Remark 4.** *Putting* $n = 1$ *and* $\alpha = 0$ *in Theorem 2, we can conclude that Theorem 2 is the extended version of Theorem 2.1 in [14] for analytic functions with fixed initial coefficients.*

**Corollary 8.** *Let* $f \in \mathcal{A}_{n,\beta}$ *with* $\frac{f(z)}{z} \neq \alpha$ *in* $\mathbb{U}$. *Moreover, let* $\frac{1}{2} \leq \alpha < 1$ *and* $0 \leq \beta \leq 2(1-\alpha)$. *If*

$$\mathfrak{Re}\left(\frac{zf'(z) - f(z)}{f(z)}\right) \neq s \qquad (z \in \mathbb{U}),$$

*for all* $s \in \mathbb{R}$ *with* $s \leq \left(n + \frac{2(1-\alpha)-\beta}{2(1-\alpha)+\beta}\right)\left(\frac{\alpha-1}{2\alpha}\right)$, *then* $\mathfrak{Re}\frac{f(z)}{z} > \alpha$.

**Proof.** Define $p(z) = \frac{f(z)}{z}$. Therefore, $p \in \mathcal{H}_\beta[1, n]$. Let there exist $z_0 \in \mathbb{U}$ such that $\mathfrak{Re}\, p(z) > \alpha$ for $|z| < |z_0|$ and $p(z_0) = \alpha + i\beta_1$ with $\beta_1 \neq 0$. Now, utilizing Theorem 2 and the relation (26) in this Theorem, we have

$$\mathfrak{Re}\left(\frac{z_0 f'(z_0) - f(z_0)}{f(z_0)}\right) = \mathfrak{Re}\frac{z_0 p'(z_0)}{p(z_0)} = \frac{-m\alpha((\alpha-1)^2 + \beta_1^2)}{2(1-\alpha)(\alpha^2 + \beta_1^2)}, \tag{27}$$

where $m \geq n + \frac{2(1-\alpha)-\beta}{2(1-\alpha)+\beta}$. Define

$$f(t) = \frac{\alpha((\alpha-1)^2 + t)}{2(1-\alpha)(\alpha^2 + t)} \qquad (t = \beta_1^2 > 0).$$

By computing, we can readily deduce that

$$f'(t) = \frac{\alpha(2\alpha - 1)}{2(1-\alpha)(\alpha^2 + t)^2} \geq 0 \qquad (t > 0, \frac{1}{2} \leq \alpha < 1).$$

Thus

$$mf(t) \geq mf(0) = \frac{m(1-\alpha)}{2\alpha} \qquad (\frac{1}{2} \leq \alpha < 1),$$

where $m \geq n + \frac{2(1-\alpha)-\beta}{2(1-\alpha)+\beta}$. Thus (27) implies that

$$\mathfrak{Re}\left(\frac{z_0 f'(z_0) - f(z_0)}{f(z_0)}\right) \leq -\frac{m(1-\alpha)}{2\alpha} \leq -\left(n + \frac{2(1-\alpha)-\beta}{2(1-\alpha)+\beta}\right)\left(\frac{1-\alpha}{2\alpha}\right),$$

which contradicts the hypothesis. Hence, the proof is complete. □

Now, we consider an example for the above result.

**Example 1.** *Let us consider the function* $f$, *defined by*

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \qquad (z \in \mathbb{U}),$$

*for* $\frac{1}{2} \leq \alpha \leq 1$. *Then,* $f \in \mathcal{A}_{1,\beta}$ *and* $\frac{f(z)}{z} \neq \alpha$ *in* $\mathbb{U}$, *where* $\beta = 2(1-\alpha)$. *Since*

$$\frac{zf'(z)}{f(z)} = \frac{1 - (1-2\alpha)z}{1-z},$$

*we have*

$$\mathfrak{Re}\left(\frac{zf'(z)-f(z)}{f(z)}\right) > \alpha - 1.$$

*Therefore,*

$$\mathfrak{Re}\left(\frac{zf'(z)-f(z)}{f(z)}\right) \neq s$$

*for all* $s \leq \frac{\alpha-1}{2\alpha}$. *Thus,* $f$ *satisfies in*

$$\mathfrak{Re}\left\{\frac{f(z)}{z}\right\} = \mathfrak{Re}\left\{\frac{1}{(1-z)^{2(1-\alpha)}}\right\} > \alpha.$$

## 4. Conclusions

Our investigation focused on considering the analytic functions with fixed initial coefficients such that in Section 2, conditions for starlikeness, convexity, close-to-convexity, and quasi-convexity, along with the pertinent corollaries, are obtained. Furthermore, in Section 3, an extension of Nunokawa's lemma for analytic functions is presented and some new results are proven.

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