

## Article

# An RBF Method for Time Fractional Jump-Diffusion Option Pricing Model under Temporal Graded Meshes

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**Abstract:** This paper explores a numerical method for European and American option pricing under time fractional jump-diffusion model in Caputo scene. The pricing problem for European options is formulated using a time fractional partial integro-differential equation, whereas the pricing of American options is described by a linear complementarity problem. For European option, we present nonuniform discretization along time and the radial basis function (RBF) method for spatial discretization. The stability and convergence analysis of the discrete scheme are carried out in the case of European options. For American option, the operator splitting method is adopted which split linear complementary problem into two simple equations. The numerical results confirm the accuracy of the proposed method.

**Keywords:** Caputo fractional derivative; time fractional Black-Scholes model; jump-diffusion model; option pricing; RBF method; graded meshes

**MSC:** 35R09; 35R11; 65M12; 65M22



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## 1. Introduction

Option pricing refers to the process of calculating and determining the price of option contracts based on specific financial models and market conditions. Option pricing plays an important role in financial markets, not only helping investors manage risks and make investment decisions, but also promoting effective market operation and reasonable asset pricing. The famous option pricing model is Black-Scholes (BS) model, which was proposed by Black and Scholes [1] in 1973. The model is based on the assumption of a log normal distribution of stock prices, which considered the relationship between option prices and underlying asset prices, strike prices, risk-free interest rates, option expiration times, and underlying asset volatility. However, the strict assumptions of the BS model often do not hold true in the actual market. To overcome this problem, many improved models have been put forward, such as stochastic volatility models, jump-diffusion models, fractional BS models, etc. The improvement and research of stochastic models can be found in [2,3]. The paper mainly focuses on the jump-diffusion model and fractional BS model.

Merton [4] and Kou [5] considered jump-diffusion models, assuming that the underlying asset price follows a jump-diffusion with a normal distribution and the jump size has a double exponential distribution, respectively. Due to its ability to accurately reflect the volatility observed in real markets, the jump-diffusion model has been widely applied in financial markets. For the jump-diffusion models, numerical methods are usually used to solve the partial integro-differential equations (PIDEs) that the option prices satisfy. Toivanen [6] used finite difference (FD) discrete space differential operators on non-uniform grids and performed time discretization using implicit Rannacher schemes. For the calculation of integral terms, an recursive formula has been derived. Salmi and Toivanen [7] raised

a family of IMEX discretization schemes for PIDEs, Christara and Leung [8] demonstrated that the combination of quadratic spline collocation method and Picard iteration scheme can efficiently solve the pricing problem. Rad and Parand [9] proposed the local weak form meshless methods for jump-diffusion pricing models. Kwon and Lee [10] adopted the implicit FD method incorporating operator splitting method to solve the linear complementarity problems (LCPs) that American option satisfied. Saib et al. [11] and Chan and Hubbert [12] proposed the radial basis functions (RBF) method, and Haghi et al. [13] combined the RBF method with IMEX scheme for option pricing.

In 1997, Carpinterj and Mainardi [14] emphasized the practicality of fractional partial differential equations (FPDEs) in researching fractal geometry and dynamics, and then FPDEs are gradually integrated into financial theory. Wyss [15] and Chen et al. [16] proposed different time fractional BS equations (FBSEs) for option pricing. Subsequently, it is extended to more general situations, for example, the interest rate and volatility are functions of the underlying asset price and time [17,18]. Important issues that urgently need to be addressed is the solutions of time FBSEs. The analytical solutions obtained through integral transformation methods often involve the integral form of special functions, making them difficult to calculate [15]. However, methods such as homotopy perturbation method [19,20], homotopy analysis method [21,22], RPSM [23], and ADM [24] are widely used to solve FBSEs and get series solutions due to their independence from discretization and linearization.

Due to the dependence of analytical solutions on some special functions or convolutions of infinite series and integrals, it has become very important to study numerical methods for solving the option pricing problem controlled by FPDE. Song and Wang [25], and Koleva and Vulkov [26] proposed an implicit FD method for put option pricing and weighted FD under the Jumarie [27] model, respectively. Yang et al. [28] used the FD method to solve the time-space FBSEs. Zhang et al. [29] derived a numerical solution of barrier options in a time FBSEs. Cen et al. [30] transformed differential equations into integral form, discretized them on adaptive grids, and proved the convergence. Rahimkhani et al. [31] presented a new numerical method based on Hahn hybrid functions for solving of BS option pricing distributed order time FPDE. Maddouri [32] proposed a new  $\zeta$ -Caputo fractional BS model which was solved by numerical implicit scheme, as well as the stability and convergence were studied.

In order to improve the accuracy, compact scheme and high-order time discretization techniques are adopted. Staelen and Hendy [33] designed a compact spatial fourth-order difference scheme. Abdi et al. [34] fused the L1-2 format with the sixth and eighth order compact schemes to obtain two high-order numerical methods. References [35,36] also employed compact schemes and high-order time discretization methods. Another popular meshless method is RBF. Nikan et al. [37] solved the time FBSE using local RBF method instead of global RBF method. Delpasand and Hosseini [38] extended the global RBF method to approximate the option pricing models under two types of assets, using the L1 scheme based C-N method to discretize time variables. In addition, to address the issue of reduced convergence order observed under initial conditions with non-smoothness, a modified L1 scheme [39] and various RBFs [40,41] are adopted.

Mohapatra et al. [42] considered a time FBSE under jump-diffusion model. A graded mesh combined with the FD scheme were adopted to derive a completely discrete framework. Chen and Li [43] developed an IMEX compact FD scheme to solve the time fractional partial integro-differential equations (TFPIDEs). To our knowledge, there are few literature studies on the pricing of options under time fractional jump-diffusion models, especially for American options. In this work, the main objective is to investigate numerical methods for evaluating the option prices under a time fractional jump-diffusion model, including European and American options. For simplicity of the analysis, the European options pricing can be transformed into a TFPIDE with a Fredholm integral operator, which is solved by using a discrete scheme of non-uniform fractional discretization along time and spatial discretization using RBF method, as well as composite trapezoidal approximation for Fred-

holm operator, while American options can be described by LCP, which is decomposed into two simple equations using operator splitting method. Stability and convergence analysis are discussed, and the optimal convergence rate is shown for graded meshes.

The paper is organized as follows. Section 2 introduces some preliminary knowledge about fractional integrals and fractional derivatives. In Section 3, the pricing model of European and American options are presented under time fractional jump-diffusion model. Section 4 introduces the discrete scheme and operator splitting method of European and American options. Section 5 demonstrates the stability and convergence of the discretization method. Section 6 conducted numerical experiments and empirical analysis, and the experimental results verified our expectations. Section 7 is the conclusion.

## 2. Preliminaries

This section introduces some basic definitions and preliminary knowledge about fractional integrals and fractional derivatives, as well as some well-known properties [44].

**Definition 1.** Let  $v(x, t) \in \bar{\Omega}$  be continuous. The Riemann–Liouville fractional integral of the function  $v(x, t)$  is defined by:

$$\mathfrak{J}^\delta v(x, t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} v(x, s) ds,$$

where  $\delta \in \mathbb{R}^+$ , the set of all positive real numbers, be the order of the integral.

**Definition 2.** The Caputo fractional derivative of the function  $v(x, t)$  is defined by:

$$\mathfrak{D}_t^\delta v(x, t) = \mathfrak{J}^{n-\delta} \left( \frac{\partial^n v}{\partial t^n}(x, t) \right), \text{ for } (x, t) \in \Omega,$$

where  $\delta \in \mathbb{R}^+$  is the order of the derivative and  $n = \lceil \delta \rceil$ , the smallest integer which is greater than or equal to  $\delta$ .

Some properties of fractional integrals and derivatives are given as follows.

1. For any constant function  $f = c$  ( $c$  is a constant)  $\mathfrak{D}_t^\delta f = 0$ .
2. For any  $\gamma \in \mathbb{R}, n \in \mathbb{N}$ , we have

$$\mathfrak{D}_t^\delta t^\gamma = \begin{cases} 0, & n-1 < \delta < n, \gamma \leq n-1, \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\delta+1)} t^{\gamma-\delta}, & n-1 < \delta < n, \gamma > n-1. \end{cases}$$

3. For  $\delta \in (0, 1)$ , we have  $\mathfrak{D}_t^\delta \mathfrak{J}^\delta v(x, t) = v(x, t)$ , but  $\mathfrak{J}^\delta \mathfrak{D}_t^\delta v(x, t) = v(x, t) - v(x, 0^+)$ .
4. Fractional integrals and derivatives satisfy the linearity property:
  - (a)  $\mathfrak{J}^\delta \{c_1 u \pm c_2 v\} = c_1 \mathfrak{J}^\delta u \pm c_2 \mathfrak{J}^\delta v$ .
  - (b)  $\mathfrak{D}_t^\delta \{c_1 u \pm c_2 v\} = c_1 \mathfrak{D}_t^\delta u \pm c_2 \mathfrak{D}_t^\delta v$ . $c_1, c_2$  are some positive constants.

## 3. Option Pricing Model

Assume the stock price  $S_t$  is controlled by the following equation

$$\frac{dS_t}{S_t} = (r - q - \lambda\kappa)dt + \sigma dB_t + (Z - 1)dN_t,$$

where  $B_t$  and  $N_t$  denote the standard Brownian motion and the independent Poisson process with intensity  $\lambda > 0$ , respectively.  $Z - 1$  is an impulse function producing a jump

from  $S_t$  to  $S_t Z$ ,  $\kappa$  represents the expected value of jump size distribution under Merton model, and

$$\kappa = E(Z - 1) = e^{\mu + \frac{\gamma^2}{2}} - 1. \quad (1)$$

$f(z)$  is the density function of log-normal distribution. The density function  $f(z) \geq 0$  for all  $z$ , and meet  $\int_0^\infty f(z) dz = 1$ , which is defined by

$$f(z) = \frac{1}{\sqrt{2\pi}\gamma z} e^{-\frac{(\ln z - \mu)^2}{2\gamma^2}}.$$

In addition,  $\mu$  and  $\gamma$  signify the mean and variance of jumps, respectively, and parameters  $r, q$ , and  $\sigma$  represent risk-free interest rates, dividends, and volatility, respectively.

### 3.1. TFPIDE for European Option

Under the jump-diffusion model, European option price  $V(S, \tau)$  at time  $\tau$  and stock price  $S$  with strike price  $K$  and expiry date  $T$  satisfies the following TFPIDE

$$\begin{cases} \frac{\partial^\alpha V(S, \tau)}{\partial \tau^\alpha} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V(S, \tau)}{\partial S^2} + (r - q - \lambda \kappa) S \frac{\partial V(S, \tau)}{\partial S} \\ - (r + \lambda) V(S, \tau) + \lambda \int_0^\infty V(Sz, \tau) f(z) dz = 0, & 0 < \alpha < 1, \\ V(0, \tau) = \xi_1(\tau), \quad V(+\infty, \tau) = \xi_2(\tau), \\ V(S, T) = \zeta(S), \end{cases} \quad (2)$$

where  $\frac{\partial^\alpha V(S, \tau)}{\partial \tau^\alpha}$  is Caputo's fractional derivative which is given by

$$\frac{\partial^\alpha V(S, \tau)}{\partial \tau^\alpha} = \frac{1}{\Gamma(1 - \alpha)} \int_0^\tau (\tau - \eta)^{-\alpha} V(S, \eta) d\eta, \quad 0 < \alpha < 1.$$

For European call option, the conditions of boundary and payoff function are given as

$$\xi_1(\tau) = 0, \quad \xi_2(\tau) = \lim_{S \rightarrow \infty} S e^{-r(T-\tau)} - K e^{-r(T-\tau)}, \quad \zeta(S) = (S - K)^+,$$

and for European put option, the conditions of boundary and payoff function are given as

$$\xi_1(\tau) = K e^{-r(T-\tau)}, \quad \xi_2(\tau) = 0, \quad \zeta(S) = (K - S)^+.$$

Set  $S = K e^x$ ,  $\tau = T - t$  and  $z = e^{y-x}$ , the problem (2) is converted into a TFPIDE with a Fredholm integral operator. We have

$$\begin{cases} - \frac{\partial^\alpha V(K e^x, T - t)}{\partial t^\alpha} + \frac{1}{2} \sigma^2 \frac{\partial^2 V(K e^x, T - t)}{\partial x^2} + (r - \frac{1}{2} \sigma^2 - q - \lambda \kappa) \frac{\partial V(K e^x, T - t)}{\partial x} \\ - (r + \lambda) V(K e^x, T - t) + \lambda \int_{-\infty}^{+\infty} V(K e^y, T - t) f(e^{y-x}) e^{y-x} dz, & 0 < \alpha < 1, \\ V(K e^{-\infty}, T - t) = \xi_1(T - t), \quad V(K e^{+\infty}, T - t) = \xi_2(T - t), \\ V(K e^x, T) = \zeta(K e^x). \end{cases} \quad (3)$$

To apply numerical approximation, we limit the range of spatial variables to a bounded interval, considering the bounded domain  $\Omega = [-L, L] \times (0, T]$  instead of the domain  $\mathbf{R} \times (0, T]$ . Let  $U(x, t) = V(K e^x, T - t)$ ,  $Y_1 = \frac{1}{2} \sigma^2$ ,  $Y_2 = r - \frac{1}{2} \sigma^2 - q - \lambda \kappa$ , and  $Y_3 = r + \lambda$ , we have

$$\begin{cases} \frac{\partial^\alpha U(x, t)}{\partial t^\alpha} = \mathcal{T} U(x, t), & 0 < \alpha < 1, \\ U(-L, t) = \xi_1(t), \quad U(L, t) = \xi_2(t), \\ U(x, 0) = \zeta(x), \end{cases} \quad (4)$$

where

$$\begin{aligned}\mathcal{T}U(x, t) &= \mathcal{D}U(x, t) + \mathcal{I}U(x, t), \\ \mathcal{D}U(x, t) &= Y_1 \frac{\partial^2 U(x, t)}{\partial x^2} + Y_2 \frac{\partial U(x, t)}{\partial x} - Y_3 U(x, t),\end{aligned}$$

$$\mathcal{I}U(x, t) = \lambda \int_{-L}^L U(y, t) h(y - x) dy,$$

$$\text{and } h(y) = f(e^y) e^y = \frac{1}{\sqrt{2\pi}\gamma} e^{-\frac{(y-\mu)^2}{2\gamma^2}}.$$

### 3.2. LCP for American Option

Let  $V(S, \tau)$  be the American put option price with strike price  $K$  and expiry date  $T$  under the jump-diffusion model which satisfies the following LCP

$$\begin{cases} \frac{\partial^\alpha V(S, \tau)}{\partial \tau^\alpha} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V(S, \tau)}{\partial S^2} + (r - q - \lambda \kappa) S \frac{\partial V(S, \tau)}{\partial S} \\ - (r + \lambda) V(S, \tau) + \lambda \int_0^\infty V(Sz, \tau) f(z) dz \geq 0, & 0 < \alpha < 1, \\ V(S, \tau) \geq \zeta(S), \\ \left( \frac{\partial^\alpha V(S, \tau)}{\partial \tau^\alpha} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V(S, \tau)}{\partial S^2} + (r - q - \lambda \kappa) S \frac{\partial V(S, \tau)}{\partial S} \right. \\ \left. - (r + \lambda) V(S, \tau) + \lambda \int_0^\infty V(Sz, \tau) f(z) dz \right) (V(S, \tau) - \zeta(S)) = 0, \\ V(0, \tau) = \xi_1(\tau), \quad V(+\infty, \tau) = \xi_2(\tau), \\ V(S, T) = \zeta(S), \end{cases}$$

where the conditions of boundary and payoff function are given as

$$\xi_1(\tau) = K, \quad \xi_2(\tau) = 0, \quad \zeta(S) = (K - S)^+.$$

Similar to the handling of European options, by replacing the same variables, we obtain

$$\begin{cases} \frac{\partial^\alpha U(x, t)}{\partial t^\alpha} - \mathcal{T}U(x, t) \geq 0, & 0 < \alpha < 1, \\ U(x, t) \geq \zeta(x), \\ \left( \frac{\partial^\alpha U(x, t)}{\partial t^\alpha} - \mathcal{T}U(x, t) \right) (U(x, t) - \zeta(x)) = 0, \\ U(-L, t) = \xi_1(t), \quad U(L, t) = \xi_2(t), \\ U(x, 0) = \zeta(x). \end{cases} \quad (5)$$

## 4. The Discrete Problem

To construct the numerical solution of TFPIDE, we apply the local RBF method in spatial and graded meshes in time.

### 4.1. Nonuniform Fractional Discretization along Time

Due to the non-smooth of TFPIDE at  $S = K$ , it also leads to inaccuracy at  $S = K$  and  $t = 0$ , we consider a well-known graded temporal partition. Let

$$t_j = T \left( \frac{j}{N} \right)^\beta,$$

where  $j = 0, 1, \dots, N$ , and  $\beta \geq 1$  is the grading parameter that controls the density of the points. When  $\beta = 1$ , the points are equidistant. The step size is  $\Delta t_j = t_j - t_{j-1}$ ,  $j = 1, 2, \dots, N$ . At each nodes  $t = t_j$ ,  $j = 1, \dots, N$ , we have

$$\frac{\partial^\alpha U(x, t_j)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} (t_j - \eta)^{-\alpha} \frac{\partial U(x, \eta)}{\partial \eta} d\eta.$$

For the nonuniform nodes, the L1 discretization is used on graded meshes which obtains the following scheme

$$\begin{aligned} \frac{\partial^\alpha U(x, t_j)}{\partial t^\alpha} &\approx \partial_N^\alpha U(x, t_j) = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \frac{U^k - U^{k+1}}{\Delta t_{k+1}} \int_{t_k}^{t_{k+1}} (t_j - \eta)^{-\alpha} d\eta \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{j-1} \frac{U^k - U^{k+1}}{\Delta t_{k+1}} \left( (t_j - t_k)^{1-\alpha} - (t_j - t_{k+1})^{1-\alpha} \right) \\ &= \frac{c_{j,1}}{\Gamma(2-\alpha)} U^j - \frac{c_{j,j}}{\Gamma(2-\alpha)} U^0 + \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{j-1} (c_{j,k+1} - c_{j,k}) U^{j-k}, \end{aligned} \quad (6)$$

where  $c_{j,k} = \frac{(t_j - t_{j-k})^{1-\alpha} - (t_j - t_{j-k+1})^{1-\alpha}}{\Delta t_{j-k+1}}$ ,  $c_{j,1} = \Delta t_j^{-\alpha}$ . A short calculation using the mean value theorem revealed that

$$\frac{1-\alpha}{(t_j - t_{j-k})^\alpha} \leq c_{j,k} \leq \frac{1-\alpha}{(t_j - t_{j-k+1})^\alpha},$$

and

$$c_{j,k+1} \leq c_{j,k},$$

which can be found in [42,45].

#### 4.2. Spatial Discretization

In this part, we explain the spatial discretization of the local RBF method, which is considered as an extension of the classical FD method and is referred to as RBF-FD. Polynomial interpolation is used to calculate weights for the FD method, while RBF interpolation is calculated by fitting grid points and some adjacent points in the RBF-FD method. RBF-FD method is a local RBF method which can overcome the ill-conditioning of the linear system, while the global RBF method considers all grid points to determine interpolation coefficients, which can lead to ill-conditioning in dense linear systems.

Consider a finite set consisting of  $M$  scattered nodes  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_M$ , and a differential operator  $\mathfrak{D}$ . For a given node  $\mathbf{x}_1$ , the goal is to approximate  $\mathfrak{D}U(\mathbf{x}_1)$  as a linear combination of  $U$  at the  $M$  scattered nodes, so that

$$\mathfrak{D}U(\mathbf{x}_1) \approx \sum_{i=1}^M q_i U(\mathbf{x}_i).$$

A set of base function  $\phi_i(\mathbf{x})$ ,  $i = 1, 2, \dots, M$  are required for determining the RBF-FD weights  $q_i$ , and

$$\mathfrak{D}\phi_j(\mathbf{x}_1) \approx \sum_{i=1}^M q_i \phi_j(\mathbf{x}_i), \quad j = 1, 2, \dots, M.$$

By solving a linear simultaneous equations, weight coefficients are obtained. Next, we will adopt multiquadrics as RBFs

$$\phi_i(\mathbf{x}) = \sqrt{c^2 + \|\mathbf{x} - \mathbf{x}_i\|_2^2},$$

where  $c$  is the shape parameter, which controls the shape of RBF. The accuracy of RBF method highly depends upon the shape parameter  $c$  of the basis functions, which is responsible for the flatness of the functions.

Now, we demonstrate how to get accurate RBF-FD formulas in one-dimensional cases of our model. Define  $M + 1$  nodes in the interval  $[-L, L]$  such that  $x_i = -L + i\Delta x$ ,  $i = 0, 1, \dots, M$ , where  $\Delta x = \frac{2L}{M}$ . The boundary nodes are  $x_0 = -L$  and  $x_M = L$ . We mainly focus on internal nodes  $x = x_i$ ,  $i = 1, \dots, M - 1$ . Selecting multiquadric function

$$\phi(|x - y|) = \sqrt{c^2 + |x - y|^2},$$

as RBF, and using three equally spaced nodes, it holds that

$$\begin{aligned} \frac{\partial U(x_i, t)}{\partial x} &\approx \delta_x U_i = a_{i-1} U_{i-1} + a_i U_i + a_{i+1} U_{i+1}, \\ \frac{\partial^2 U(x_i, t)}{\partial x^2} &\approx \delta_x^2 U_i = b_{i-1} U_{i-1} + b_i U_i + b_{i+1} U_{i+1}. \end{aligned}$$

Assuming  $c \gg \Delta x$ , using Taylor expansion and omitting higher-order terms, as in [13,41], one yields

$$a_{i-1} = -a_{i+1} = -\frac{1}{2\Delta x} \left( 1 + \frac{(\Delta x)^2}{2c^2} \right), \quad a_i = 0,$$

$$b_{i-1} = b_{i+1} = \frac{1}{(\Delta x)^2} \left( 1 + \frac{(\Delta x)^2}{c^2} \right), \quad b_i = -\frac{2}{(\Delta x)^2} \left( 1 + \frac{(\Delta x)^2}{c^2} \right).$$

Let  $a = \frac{1}{2\Delta x} \left( 1 + \frac{(\Delta x)^2}{2c^2} \right)$ , and  $b = \frac{1}{(\Delta x)^2} \left( 1 + \frac{(\Delta x)^2}{c^2} \right)$ , hence

$$\begin{aligned} \frac{\partial U(x_i, t)}{\partial x} &\approx \delta_x U_i = -a U_{i-1} + a U_{i+1}, \\ \frac{\partial^2 U(x_i, t)}{\partial x^2} &\approx \delta_x^2 U_i = b U_{i-1} - 2b U_i + b U_{i+1}. \end{aligned} \quad (7)$$

#### 4.3. Discretization of the Integral Operator

Under the spatial and time discretization,

$$\lambda \int_{-L}^L U(y, t_j) h(y - x_i) dy = \lambda \int_{x_0}^{x_M} U(y, t_j) h(y - x_i) dy$$

holds. The integration is approached by the composite trapezoidal rule, and requires the use of the  $(j - 1)$ th level solution to approximate the  $j$ th level solution, therefore

$$\begin{aligned} \lambda \int_{x_0}^{x_M} U(y, t_j) h(y - x_i) dy &\approx \lambda \int_{x_0}^{x_M} U(y, t_{j-1}) h(y - x_i) dy \\ &= \lambda \sum_{k=0}^{M-1} \int_{x_k}^{x_{k+1}} U(y, t_{j-1}) h(y - x_i) dy \\ &= \frac{\lambda \Delta x}{2} \sum_{k=0}^{M-1} \left( U(x_k, t_{j-1}) h(x_k - x_i) + U(x_{k+1}, t_{j-1}) h(x_{k+1} - x_i) \right) \\ &= \frac{\lambda \Delta x}{2} \sum_{k=0}^M \omega_k U(x_k, t_{j-1}) h(x_k - x_i), \end{aligned} \quad (8)$$

where  $\omega_k = 1$  for  $k = 0, M$ , and  $\omega_k = 2$  for  $k = 1, \dots, M - 1$ .

According to Equations (6)–(8), we get a discretization scheme for European options on a non-uniform grid  $\{(x_i, t_j) : i = 0, 1, \dots, M; j = 0, 1, \dots, N\}$

$$\begin{cases} \partial_N^\alpha U(x_i, t_j) - Y_1 \delta_x^2 U(x_i, t_j) - Y_2 \delta_x U(x_i, t_j) + Y_3 U(x_i, t_j) = G(x_i, t_j) + R_{i,j}, \\ U(x_0, t_j) = \xi_1(t_j), \quad U(x_M, t_j) = \xi_2(t_j), \\ U(x_i, t_0) = \zeta(x_i), \end{cases} \quad (9)$$

where

$$G(x_i, t_j) = \frac{\lambda \Delta x}{2} \sum_{k=0}^{M-1} (U(x_k, t_{j-1})h(x_k - x_i) + U(x_{k+1}, t_{j-1})h(x_{k+1} - x_i)).$$

Neglecting  $R_{i,j}$  and denoting  $U_i^j$  as the approximate solution at each  $(x_i, t_j)$ , we have

$$\begin{cases} \partial_N^\alpha U_i^j - Y_1 \delta_x^2 U_i^j - Y_2 \delta_x U_i^j + Y_3 U_i^j = G(x_i, t_j), \\ U_0^j = \xi_1(t_j), \quad U_M^j = \xi_2(t_j), \\ U_i^0 = \zeta(x_i). \end{cases} \quad (10)$$

By organizing, it can be concluded that

$$\begin{cases} U_{i-1}^j(-bY_1 + aY_2) + U_i^j(\frac{c_{j,1}}{\Gamma(2-\alpha)} + 2bY_1 + Y_3) + U_{i+1}^j(-bY_1 - aY_2) = \mathcal{G}_i^{j-1}, \\ U_0^j = \xi_1(t_j), \quad U_M^j = \xi_2(t_j), \\ U_i^0 = \zeta(x_i), \end{cases} \quad (11)$$

where

$$\mathcal{G}_i^{j-1} = \begin{cases} \frac{c_{1,1}}{\Gamma(2-\alpha)} U_i^0 + G(x_i, t_1), \quad j = 1, \\ \frac{c_{j,j}}{\Gamma(2-\alpha)} U_i^0 - \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{j-1} (c_{j,k+1} - c_{j,k}) U_i^{j-k} + G(x_i, t_j), \quad j = 2, 3, \dots, N. \end{cases}$$

For American options, we adopt operator splitting method for solving the LCP. An auxiliary function  $\psi(x, t)$  is introduced for the LCP, and the LCP (5) can be transformed to

$$\begin{cases} \frac{\partial^\alpha U(x, t)}{\partial t^\alpha} - \mathcal{T}U(x, t) = \psi(x, t), \\ \psi(x, t) \geq 0, \quad U(x, t) \geq \zeta(x), \\ \psi(x, t)(U(x, t) - \zeta(x)) = 0. \end{cases} \quad (12)$$

Operator splitting method splits the Equation (12) into two subproblems by using intermediate solution  $\tilde{U}(x, t)$ . In the first step, solving  $\tilde{U}^j$  by the following equation

$$\begin{cases} \frac{c_{j,1}}{\Gamma(2-\alpha)} \tilde{U}^j - \frac{c_{j,j}}{\Gamma(2-\alpha)} U^0 + \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{j-1} (c_{j,k+1} - c_{j,k}) U^{j-k} = \mathcal{T}\tilde{U}^j + \psi^{j-1}, \\ \tilde{U}^j(-L) = K, \quad \tilde{U}^j(L) = 0, \end{cases} \quad (13)$$

and in the second step, we solve  $(\psi^j, U^j)$  by

$$\begin{cases} \frac{c_{j,1}}{\Gamma(2-\alpha)} (U^j - \tilde{U}^j) = \psi^j - \psi^{j-1}, \\ \psi^j \geq 0, \quad U^j \geq \zeta, \quad \psi^j(U^j - \zeta) = 0. \end{cases} \quad (14)$$

The solution of (14) is

$$(\psi^j, U^j) = \begin{cases} \left(0, \tilde{U}^j - \frac{\Gamma(2-\alpha)}{c_{j,1}} \psi^{j-1}\right), & \text{if } \frac{c_{j,1} \tilde{U}^j}{\Gamma(2-\alpha)} - \psi^{j-1} \geq \frac{c_{j,1} \zeta}{\Gamma(2-\alpha)}, \\ \left(\frac{c_{j,1}}{\Gamma(2-\alpha)} (\zeta - \tilde{U}^j) + \psi^{j-1}, \zeta\right), & \text{otherwise.} \end{cases} \quad (15)$$

For the first step, Equation (13) can be transformed to the following by Equations (6)–(8),

$$\begin{cases} \tilde{U}_{i-1}^j(-bY_1 + aY_2) + \tilde{U}_i^j\left(\frac{c_{j,1}}{\Gamma(2-\alpha)} + 2bY_1 + Y_3\right) + \tilde{U}_{i+1}^j(-bY_1 - aY_2) = \mathcal{G}_i^{j-1}, \\ \tilde{U}_0^j = K, \quad \tilde{U}_M^j = 0, \end{cases} \quad (16)$$

where

$$\mathcal{G}_i^{j-1} = \begin{cases} \frac{c_{1,1}}{\Gamma(2-\alpha)}U_i^0 + G(x_i, t_1) + \psi_i^0, & j = 1, \\ \frac{c_{j,j}}{\Gamma(2-\alpha)}U_i^0 - \frac{1}{\Gamma(2-\alpha)}\sum_{k=1}^{j-1}(c_{j,k+1} - c_{j,k})U_i^{j-k} + G(x_i, t_j) + \psi_i^{j-1}, & j = 2, 3, \dots, N. \end{cases}$$

## 5. Stability and Convergence Analysis

This section discusses the stability and convergence of the discrete scheme (11), and the stability multipliers  $\varphi_{jn}$  are taken into account to show the convergence results.

### 5.1. Stability Analysis

**Lemma 1.** The solution of Equation (11) satisfies

$$\|U^j\|_\infty \leq \Delta t_j^\alpha \left[ \Gamma(2-\alpha)\|G^j\|_\infty + c_{j,j}\|U^0\|_\infty + \sum_{k=0}^{j-1}(c_{j,k} - c_{j,k+1})\|U^{j-k}\|_\infty \right], \quad 1 \leq j \leq N.$$

**Proof.** By transferring items to Equation (11) and organizing them, it can be obtained that

$$\begin{cases} U_i^j\left(\frac{c_{j,1}}{\Gamma(2-\alpha)} + 2bY_1 + Y_3\right) = U_{i-1}^j(bY_1 - aY_2) + U_{i+1}^j(bY_1 + aY_2) + \mathcal{G}_i^{j-1}, \\ U_0^j = \xi_1(t_j), \quad U_M^j = \xi_2(t_j), \\ U_i^0 = \zeta(x_i). \end{cases} \quad (17)$$

Define

$$\|U^j\|_\infty = \max_{0 \leq i \leq M} |U_i^j|, \quad \|U\|_\infty = \max_{0 \leq j \leq N} \max_{0 \leq i \leq M} |U_i^j|.$$

For a fixed  $j \in 1, 2, \dots, N$ , there exists  $i^*$  such that  $|U_{i^*}^j| = \|U^j\|_\infty$  holds. Therefore, at each grid node  $(x_{i^*}, t_j)$ , there are

$$U_{i^*}^j\left(\frac{c_{j,1}}{\Gamma(2-\alpha)} + 2bY_1 + Y_3\right) = U_{i^*-1}^j(bY_1 - aY_2) + U_{i^*+1}^j(bY_1 + aY_2) + \mathcal{G}_{i^*}^{j-1},$$

where

$$\mathcal{G}_{i^*}^{j-1} = \frac{c_{j,j}}{\Gamma(2-\alpha)}U_{i^*}^0 + \frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{j-1}(c_{j,k} - c_{j,k+1})U_{i^*}^{j-k} + G_{i^*}^j.$$

Due to  $Y_3 \geq 0$  being established, and based on the selection of  $i^*$ , there are

$$\begin{aligned} \left(\frac{c_{j,1}}{\Gamma(2-\alpha)} + 2bY_1\right)\|U^j\|_\infty &\leq (bY_1 - aY_2)\|U^j\|_\infty + (bY_1 + aY_2)\|U^j\|_\infty + |\mathcal{G}_{i^*}^{j-1}| \\ &= 2bY_1\|U^j\|_\infty + |\mathcal{G}_{i^*}^{j-1}|, \end{aligned}$$

which can be inferred as

$$\begin{aligned}\|U^j\|_\infty &\leq \frac{\Gamma(2-\alpha)}{c_{j,1}}|\mathcal{G}_{i^*}^{j-1}| \\ &= \Delta t_j^\alpha |c_{j,j}U_{i^*}^0 + \sum_{k=0}^{j-1}(c_{j,k} - c_{j,k+1})U_{i^*}^{j-k} + \Gamma(2-\alpha)G_{i^*}^j| \\ &\leq \Delta t_j^\alpha \left[ \Gamma(2-\alpha)\|G^j\|_\infty + c_{j,j}\|U^0\|_\infty + \sum_{k=0}^{j-1}(c_{j,k} - c_{j,k+1})\|U^{j-k}\|_\infty \right].\end{aligned}$$

□

Define the real numbers  $\varphi_{jn}$  for  $j = 1, 2, \dots, N$  and  $n = 1, 2, \dots, j-1$  by

$$\varphi_{j,j} = 1, \quad \varphi_{j,n} = \sum_{k=0}^{j-n} (c_{j,k} - c_{j,k+1})\varphi_{j-k,n} \geq 0.$$

**Theorem 1.** For  $j = 1, 2, \dots, N$ , the solution of Equation (11) satisfies

$$\|U^j\|_\infty \leq \|U^0\|_\infty + \Delta t_j^\alpha \Gamma(2-\alpha) \sum_{n=1}^j \varphi_{j,n} \|G^n\|_\infty.$$

**Proof.** Using mathematical induction for proof. For  $j = 1$ , we have

$$\|U^1\|_\infty \leq \Delta t_1^\alpha c_{1,1} \|U^0\|_\infty + \Delta t_1^\alpha \Gamma(2-\alpha) \varphi_{1,1} \|G^1\|_\infty,$$

which can be obtained by Lemma 1.

Assuming for  $k = 1, 2, \dots, j-1$  and  $j = 2, \dots, N$ , Theorem 1 holds. Then for  $k = j$ , we get

$$\begin{aligned}\|U^j\|_\infty &\leq \Delta t_j^\alpha \left[ \Gamma(2-\alpha)\|G^j\|_\infty + c_{j,j}\|U^0\|_\infty + \sum_{k=0}^{j-1}(c_{j,k} - c_{j,k+1})\|U^{j-k}\|_\infty \right] \\ &= \Delta t_j^\alpha \left[ \Gamma(2-\alpha)\|G^j\|_\infty + c_{j,j}\|U^0\|_\infty \right. \\ &\quad \left. + \sum_{k=1}^{j-1}(c_{j,k} - c_{j,k+1}) \left( \|U^0\|_\infty + \Delta t_{j-k}^\alpha \Gamma(2-\alpha) \sum_{n=1}^{j-k} \varphi_{j-k,n} \|G^n\|_\infty \right) \right] \\ &= \Delta t_j^\alpha \left[ \Gamma(2-\alpha)\|G^j\|_\infty + c_{j,1}\|U^0\|_\infty \right. \\ &\quad \left. + \Gamma(2-\alpha) \sum_{k=1}^{j-1} \Delta t_{j-k}^\alpha (c_{j,k} - c_{j,k+1}) \left( \sum_{n=1}^{j-k} \varphi_{j-k,n} \|G^n\|_\infty \right) \right] \\ &= \Delta t_j^\alpha \left[ \Gamma(2-\alpha)\|G^j\|_\infty + c_{j,1}\|U^0\|_\infty \right. \\ &\quad \left. + \Gamma(2-\alpha) \sum_{n=1}^{j-1} \|G^n\|_\infty \left( \sum_{k=1}^{j-n} \Delta t_{j-k}^\alpha (c_{j,k} - c_{j,k+1}) \varphi_{j-k,n} \right) \right] \\ &= \|U^0\|_\infty + \Delta t_j^\alpha \Gamma(2-\alpha) \sum_{n=1}^j \|G^n\|_\infty \left( \sum_{k=1}^{j-n} \Delta t_{j-k}^\alpha (c_{j,k} - c_{j,k+1}) \varphi_{j-k,n} \right) \\ &= \|U^0\|_\infty + \Delta t_j^\alpha \Gamma(2-\alpha) \sum_{n=1}^j \varphi_{j,n} \|G^n\|_\infty.\end{aligned}$$

□

### 5.2. Convergence Analysis

To demonstrate the convergence, we first calculated the truncation error bound at points  $(x_i, t_j)$ . Then, using the stability results, the complete error bound of the proposed method was established.

Let  $\|\varepsilon_i^j\|_\infty = \|U(x_i, t_j) - U_i^j\|_\infty$ , where  $U(x_i, t_j)$  and  $U_i^j$  denote the exact solution and numerical solution at node  $(x_i, t_j)$ , respectively. Then, we get

$$\begin{cases} \partial_N^\alpha \varepsilon_i^j - Y_1 \delta_x^2 \varepsilon_i^j - Y_2 \delta_x \varepsilon_i^j + Y_3 \varepsilon_i^j = R_{i,j}, \\ \varepsilon_0^j = \varepsilon_M^j = 0, \\ \varepsilon_i^0 = 0. \end{cases} \quad (18)$$

According to the discretization on time, space and integration, the remaining term  $R_{i,j}$  consists of four parts. The first remaining term satisfies

$$\|\partial_t^\alpha U(x_i, t_j) - \partial_N^\alpha U(x_i, t_j)\|_\infty \leq C j^{-\min(\alpha\beta, 2-\alpha)}.$$

The second and third remaining terms satisfy

$$\left\| Y_1 \left( \frac{\partial^2 U(x_i, t_j)}{\partial x^2} - \delta_x^2 U(x_i, t_j) \right) \right\|_\infty \leq C(\Delta x)^2,$$

$$\left\| Y_2 \left( \frac{\partial U(x_i, t_j)}{\partial x} - \delta_x U(x_i, t_j) \right) \right\|_\infty \leq C(\Delta x)^2.$$

The remaining terms in part four are caused by the composite trapezoidal formula and the approximation of the  $(j-1)$ th layer solution instead of the  $j$ th layer solution. The latter will result in errors that depend on the grading parameters such that  $\max_j \Delta t_j = O(t_1)$ . According to [45], we have

$$\begin{aligned} |U(x_j, t_j) - U(x_j, t_{j-1})| &\leq \int_{t_{j-1}}^{t_j} \left| \frac{\partial U}{\partial \xi} \right| d\xi \leq C \int_{t_{j-1}}^{t_j} \xi^{\alpha-1} d\xi \\ &\leq C \Delta t_j t_{j-1}^{\alpha-1} \leq C \Delta t_j t_1^{\alpha-1} = C \left( \frac{\Delta t_j}{t_1} \right) t_1^\alpha \\ &\leq C(TN^{-\beta})^\alpha = CT^\alpha N^{-\alpha\beta} = CN^{-\alpha\beta}. \end{aligned}$$

Hence, the infinite norm of the remainder in the fourth part is less than or equal to  $C(N^{-\alpha\beta} + (\Delta x)^2)$ .

To obtain the convergence theorem for numerical formats, we first provide the following lemma.

**Lemma 2.** When parameter  $\delta \leq \alpha\beta$ , for  $n = 1, 2, \dots, N$ , one has

$$\Delta t_j^\alpha \sum_{n=1}^j n^{-\delta} \varphi_{j,n} \leq \frac{T^\alpha}{(1-\alpha)N^\delta}.$$

**Proof.** Use induction on  $j$ , the Lemma is proved [45].  $\square$

**Theorem 2.** The error bounds of the discretization scheme (11) satisfy

$$\|\varepsilon_i^j\|_\infty \leq CT^\alpha (N^{-\min(\alpha\beta, 2-\alpha)} + (\Delta x)^2).$$

**Proof.** According to Theorem 1, there is

$$\begin{aligned} \|\varepsilon_i^j\|_\infty &\leq \|\varepsilon_i^0\|_\infty + \Delta t_j^\alpha \Gamma(2-\alpha) \sum_{n=1}^j \varphi_{j,n} \|R_{i,n}\|_\infty \\ &\leq C\Gamma(2-\alpha)\Delta t_j^\alpha \left( \sum_{n=1}^j \varphi_{j,n} n^{-\min(\alpha\beta, 2-\alpha)} + \sum_{n=1}^j \varphi_{j,n} (\Delta x)^2 \right. \\ &\quad \left. + \sum_{n=1}^j \varphi_{j,n} (\Delta x)^2 + \sum_{n=1}^j \varphi_{j,n} (N^{-\alpha\beta} + (\Delta x)^2) \right). \end{aligned}$$

Let  $\delta = \min(\alpha\beta, 2-\alpha) \leq \alpha\beta$ , and according to Lemma 2,

$$\begin{aligned} \|\varepsilon_i^j\|_\infty &\leq \frac{C\Gamma(2-\alpha)T^\alpha}{1-\alpha} \left( N^{-\min(\alpha\beta, 2-\alpha)} + N^{-\alpha\beta} + (\Delta x)^2 \right) \\ &\leq CT^\alpha \left( N^{-\min(\alpha\beta, 2-\alpha)} + (\Delta x)^2 \right). \end{aligned}$$

Especially, if  $\alpha\beta \geq 2-\alpha$ , we have

$$\|\varepsilon_i^j\|_\infty \leq CT^\alpha \left( N^{-(2-\alpha)} + (\Delta x)^2 \right).$$

□

## 6. Numerical Experiments and Empirical Analysis

### 6.1. Numerical Experiments

We provide some numerical examples to illustrate the effectiveness of numerical methods. Theoretical analysis also holds true for the TFPIDE of the source term. For the completeness of the numerical experiment, we added source terms to the equation. All the results are calculated by using the mathematical software MATLAB R2023b. The algorithm is given as follows.

- Step 1. Give the initial parameters  $r, q, \sigma, \lambda, \mu, \gamma, \alpha, \beta, T, K, M, N$ .
- Step 2. Calculate  $c_{j,k}, a, b$  in Equations (6) and (7).
- Step 3. Calculate  $Y_1, Y_2, Y_3$ , and Coefficients in discrete scheme (11).
- Step 4. Compute the coefficient matrix of the linear equation system composed of the coefficients in (11).
- Step 5. Calculate the right-hand term of the linear equations system based on the expression of  $\mathcal{G}_i^{j-1}$ .
- Step 6. Solve systems of linear Equation (11).

**Example 1.** Consider the TFPIDE with nonhomogeneous boundary conditions

$$\begin{cases} \frac{\partial^\alpha U(x,t)}{\partial t^\alpha} = Y_1 \frac{\partial^2 U(x,t)}{\partial x^2} + Y_2 \frac{\partial U(x,t)}{\partial x} \\ \quad - Y_3 U(x,t) + \lambda \int_{-1}^1 U(y,t) h(y-x) dy + f(x,t), \quad 0 < \alpha < 1, \\ U(-1,t) = (1+t)^2 \exp(\frac{1}{2\gamma^2}), \quad U(1,t) = (1+t)^2 \exp(\frac{1}{2\gamma^2}), \\ U(x,0) = \exp(\frac{x^2}{2\gamma^2}). \end{cases} \quad (19)$$

The exact solution of (19) is

$$U(x,t) = (1+t)^2 \exp(\frac{x^2}{2\gamma^2}).$$

The parameters values are  $r = 0.05, q = 0, \sigma = 0.1, \lambda = 0.01, \mu = 0, \gamma = 1$ , and  $T = 1$ .

Figure 1 presents the exact solution and numerical solution with  $M = N = 32$  for  $\alpha = 0.3$  and  $\beta = (2 - \alpha)/(2\alpha)$ . Table 1 shows the comparison of solutions at some grid node, and the errors are also given. Figure 2 gives the maximum error for different  $\alpha$  and  $\beta$ , where  $0.2 \leq \alpha \leq 0.9$ . Here, let  $\beta = 1$ ,  $\beta = \frac{(2-\alpha)}{2\alpha}$ ,  $\beta = \frac{(2-\alpha)}{\alpha}$ , and  $\beta = \frac{2(2-\alpha)}{\alpha}$  in Figure 2a–d, respectively. The maximum error is defined by  $E_{M,N} = \max_{M,N} |U(x_i, t_j) - U_i^j|$ . From Figure 2, it can be seen that with the change of  $\alpha$ , the length of the interval for the maximum error change at  $\beta = \frac{2(2-\alpha)}{\alpha}$  and  $\beta = \beta = \frac{(2-\alpha)}{\alpha}$  are smaller than  $\beta = 1$  and  $\beta = \frac{(2-\alpha)}{2\alpha}$ . Numerical errors and convergence orders are given by Table 2 for different  $\alpha$ ,  $M$  and  $N$  with  $\beta = \frac{(2-\alpha)}{\alpha}$ . The numerical errors are computed by mean squared error. The convergence order is computed by

$$P_{M,N} = \log_2 \left( \frac{\text{Error}_{M,N}}{\text{Error}_{2M,2N}} \right).$$

The log-log plots of the errors and the error bounds are shown in Figure 3 for different values of  $\alpha$ . It is observed that the convergence rate is  $(2 - \alpha)$  when  $\beta = \frac{(2-\alpha)}{\alpha}$ , which is the optimal rate.

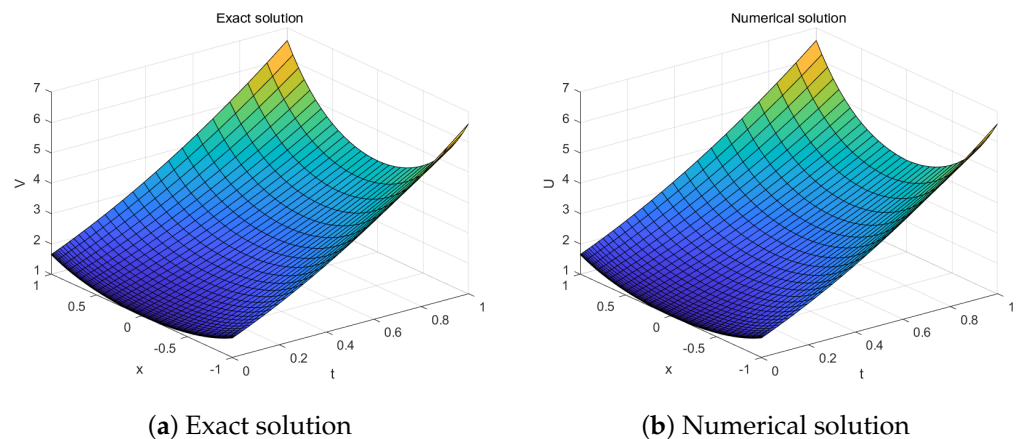


Figure 1. Exact solution and numerical solution with  $M = N = 32$ .

Table 1. Comparison of solutions at some grid node.

| $(x, t)$ | $(-0.75, 0.028)$ | $(-0.5, 0.028)$ | $(0.5, 0.2283)$ | $(0.75, 0.2283)$ | $(0.9375, 0.7566)$ |
|----------|------------------|-----------------|-----------------|------------------|--------------------|
| Exact    | 1.3333           | 1.1408          | 1.7168          | 2.0053           | 4.7979             |
| RBF      | 1.3321           | 1.1394          | 1.7097          | 1.9988           | 4.7885             |
| Error    | 0.0012           | 0.0013          | 0.0072          | 0.0065           | 0.0094             |

Table 2. Numerical error, convergence orders, and CPU times.

| $\alpha$ |         | $M = N = 8$           | $M = N = 16$          | $M = N = 32$          | $M = N = 64$          | $M = N = 128$         |
|----------|---------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|
| 0.7      | Error   | $2.70 \times 10^{-3}$ | $1.10 \times 10^{-3}$ | $4.51 \times 10^{-4}$ | $1.84 \times 10^{-4}$ | $7.75 \times 10^{-5}$ |
|          | Order   | 1.29                  | 1.28                  | 1.29                  | 1.29                  |                       |
|          | CPU (s) | 0.0146                | 0.0213                | 0.0281                | 0.0385                | 0.1064                |
| 0.5      | Error   | $4.2 \times 10^{-3}$  | $1.50 \times 10^{-3}$ | $5.15 \times 10^{-4}$ | $1.80 \times 10^{-4}$ | $6.43 \times 10^{-5}$ |
|          | Order   | 1.48                  | 1.53                  | 1.51                  | 1.49                  |                       |
|          | CPU (s) | 0.0161                | 0.0209                | 0.0291                | 0.0388                | 0.1013                |
| 0.3      | Error   | $5.6 \times 10^{-3}$  | $1.70 \times 10^{-3}$ | $5.18 \times 10^{-4}$ | $1.62 \times 10^{-4}$ | $4.94 \times 10^{-5}$ |
|          | Order   | 1.71                  | 1.71                  | 1.68                  | 1.71                  |                       |
|          | CPU (s) | 0.0152                | 0.0207                | 0.0270                | 0.0487                | 0.1057                |

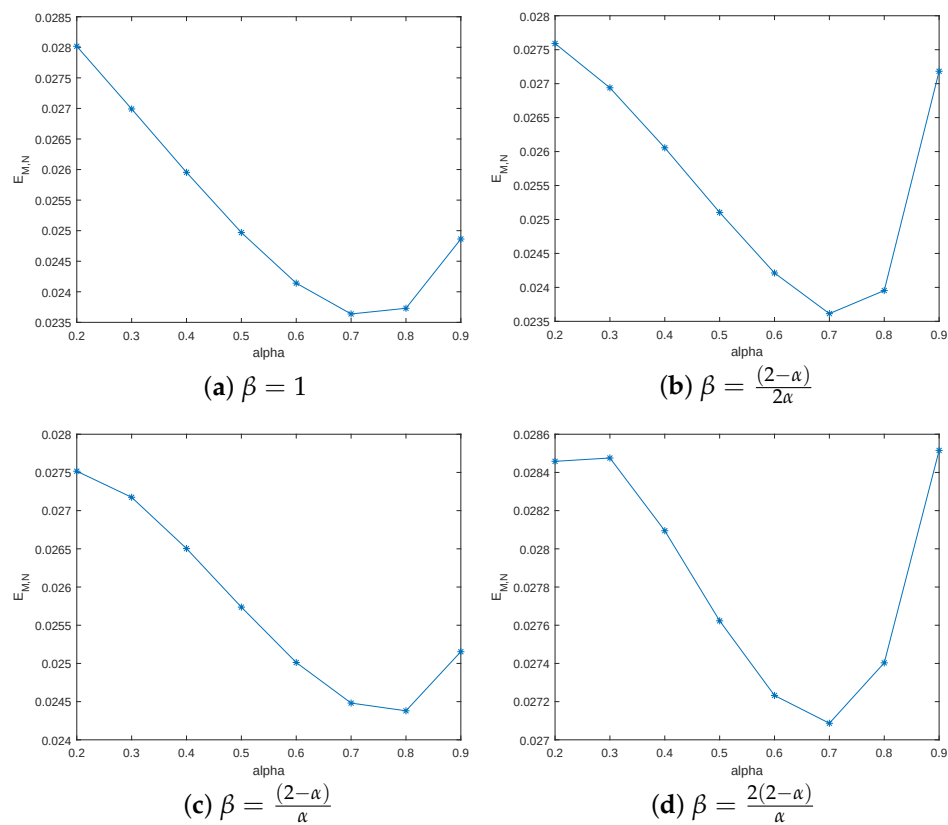


Figure 2. Error for different  $\alpha$  and  $\beta$ .

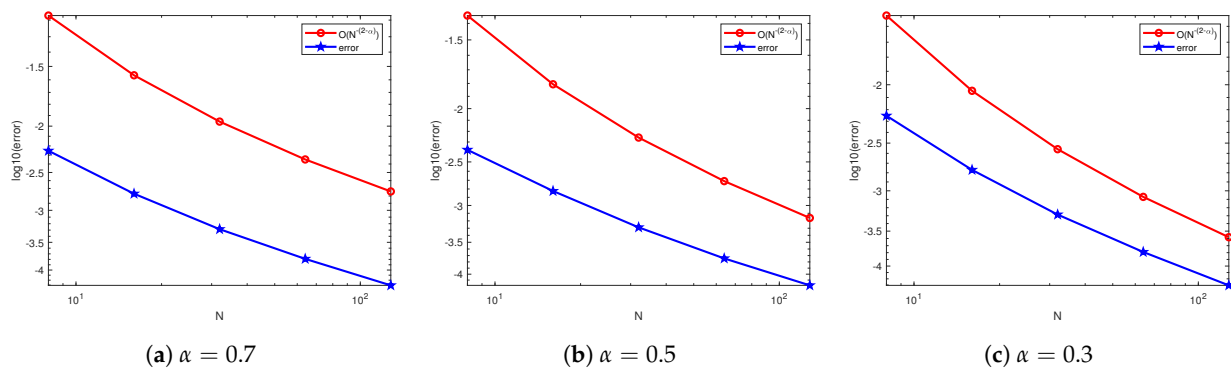


Figure 3.  $\log_{10}$  error against  $N$  for different  $\alpha$ .

**Example 2.** Consider the TFPIDE of European options with parameters  $r = 0.05, q = 0, \sigma = 0.15, \lambda = 0.1, \mu = -0.9, \gamma = 0.5, T = 0.5$  and  $K = 50$ . The grading parameter  $\beta = \frac{2-\alpha}{\alpha}$ , and  $\alpha = 0.3$ .

Figure 4 shows the European call and put option price using the proposed RBF method with  $M = N = 100$ . Due to the difficulty in obtaining option prices, we use the results obtained from the FD method as a benchmark to verify the effectiveness of our method. Figure 5 gives the European call and put option price using FD method. Figure 6a indicates that the error in calculating European call options using two methods ranges from 0 to 0.014, while Figure 6b indicates that the error in calculating European put options using both methods ranges from  $-6 \times 10^{-3}$  to  $4 \times 10^{-3}$ . Figures 7 and 8 display the numerical solution of the European call and put options for different  $0.1 \leq \alpha \leq 0.9$  and stock price. In Figure 7a, 7b and 7c, the curves from top to bottom correspond to 0.1, 0.2, 0.3, ..., 0.9, respectively. From these figures, it can be seen that for in-the-money, at-the-money, and out-of-the-money options, the option price decreases with the increase of  $\alpha$  for European call

options. This behavior is similar to the effect of fractional parameters on fluid temperature. By increasing the value of fractional parameter  $\alpha$ , temperature profile decreases for small time  $t$ , thermal conductivity decreases [46]. Similar results can be obtained from Figure 8b,c for European put options. But according to Figure 8a, it can be seen that the price increases with the increase of  $\alpha$  for European put in-the-money options.

**Example 3.** Consider the LCP of American put option with parameters  $r = 0.03, q = 0.01, \sigma = 0.1, \lambda = 0.1, \mu = -0.5, \gamma = 1, T = 1$  and  $K = 50$ . The grading parameter  $\beta = \frac{2-\alpha}{\alpha}$ , and  $\alpha = 0.3$ . Figure 9 shows the American put options for different time and stock price with  $M = N = 100$ . From Figure 10, it can be seen how time fractional derivative  $\alpha$  affects the prices of American options.

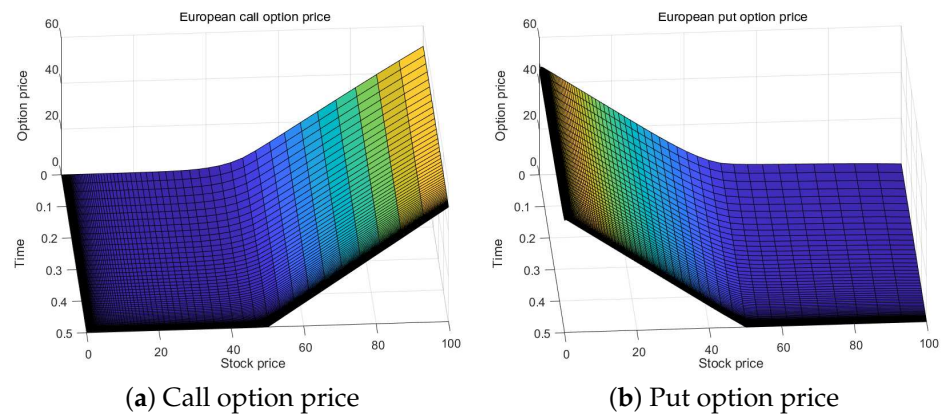


Figure 4. European call and put option price using RBF method with  $M = N = 100$ .

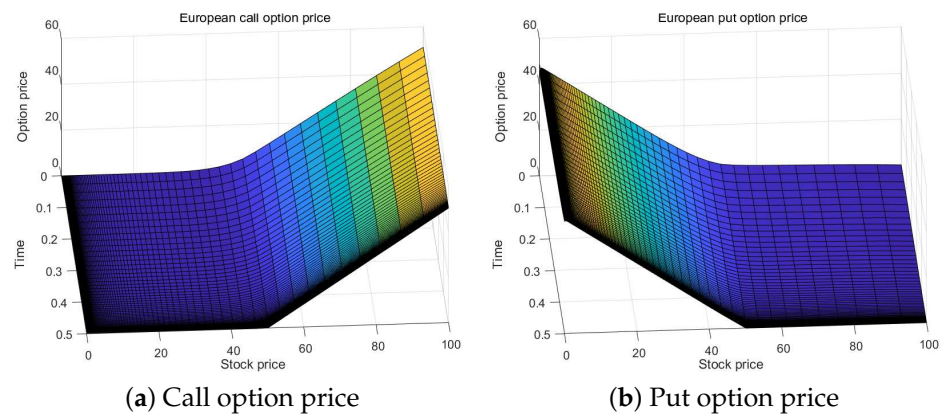


Figure 5. European call and put option price using FD method with  $M = N = 100$ .

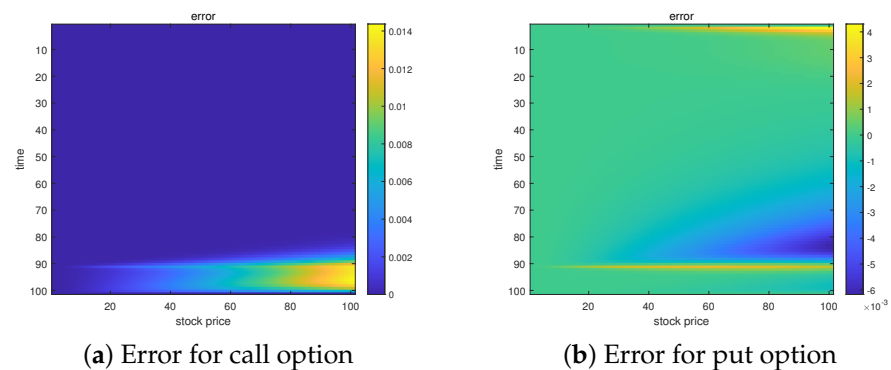


Figure 6. The error between RBF method and FD method for European call and put option.

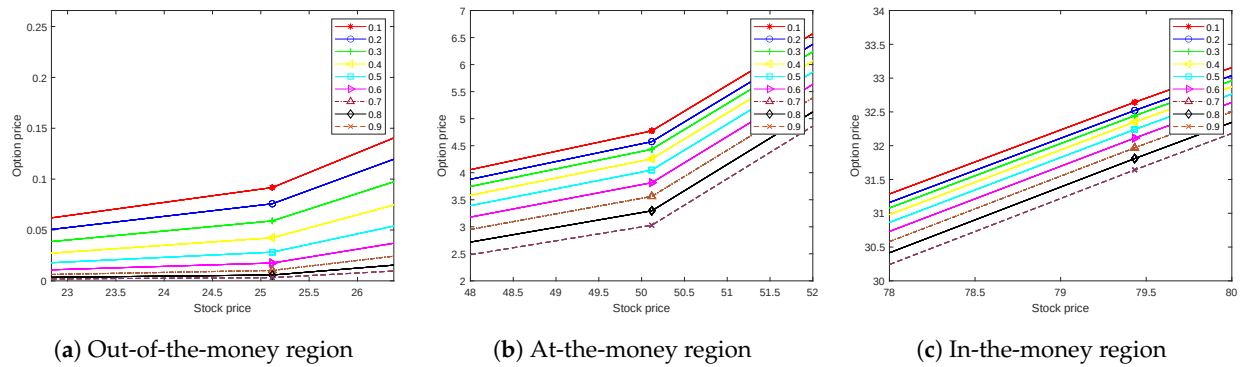


Figure 7. European call option prices for different  $\alpha$  and stock price.

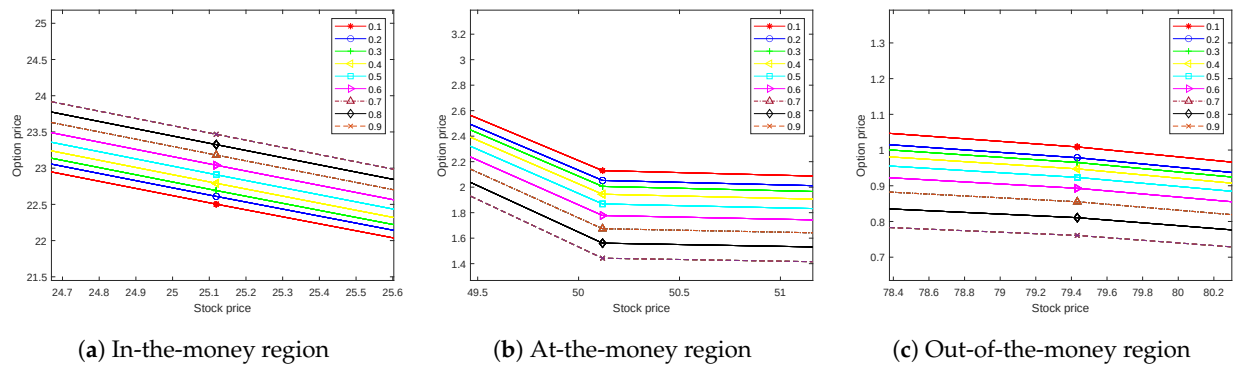


Figure 8. European put option prices for different  $\alpha$  and stock price.

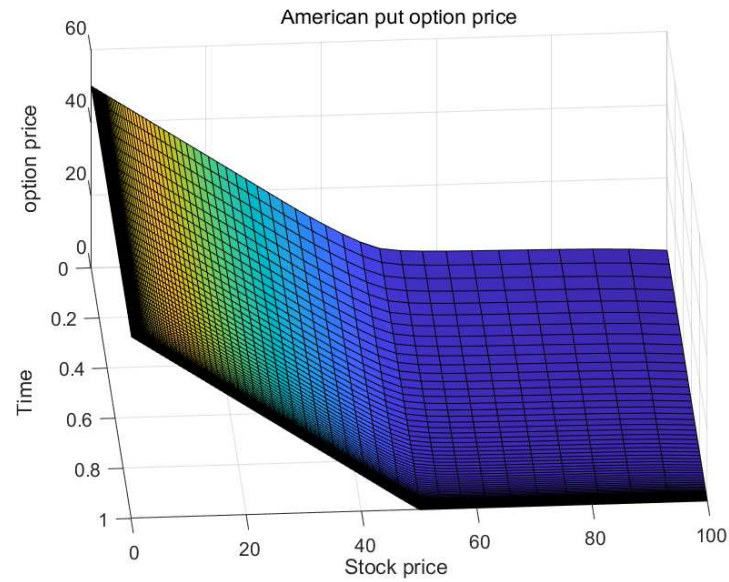
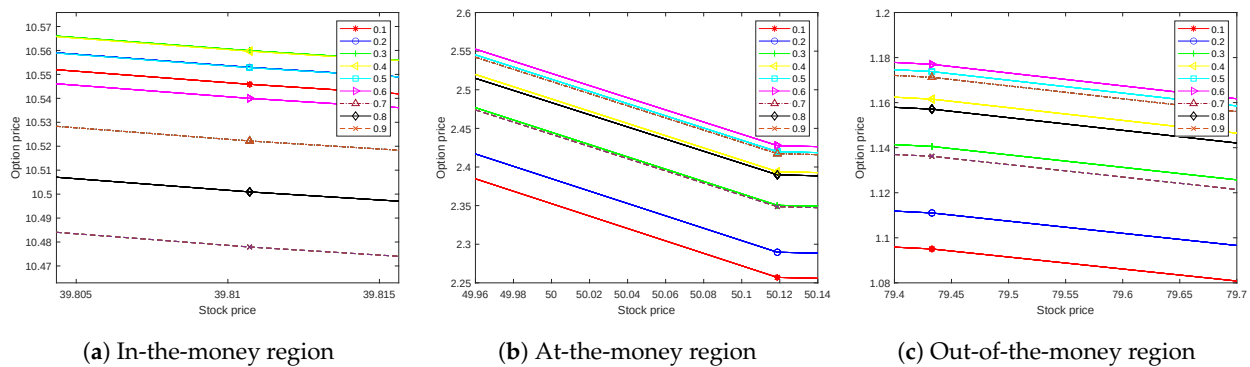


Figure 9. American put option price with  $M = N = 100$ .



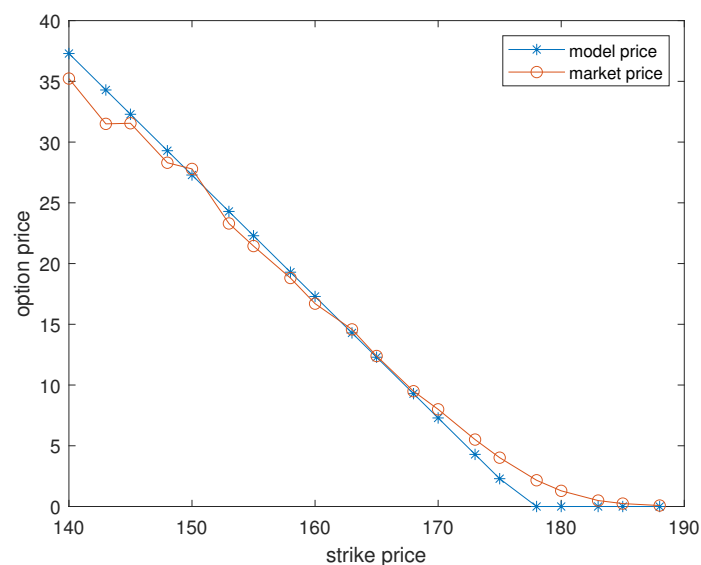
**Figure 10.** American option prices for different  $\alpha$  and stock price.

### 6.2. Empirical Analysis

We select Standard & Poor Depository Receipt (SPDR) option data from 8 November 2013 to 12 December 2013 for empirical analysis. Analyzing the historical closing prices of SPDR from 31 January 2011 to 8 November 2013, we give parameters as follows  $r = 0.0013, q = 0.00, \sigma = 0.1659, \lambda = 1.03, \mu = 0.0009, \gamma = 0.01, S_0 = 177.29, T = 1/12, \alpha = 0.3, M = N = 100$ . Figure 11 shows that our calculation results have a small error compared to real market data, while Table 3 shows that the error of the proposed method is smaller compared to the FD method, and the difference in CPU time occupied is also smaller, which indicate that we have indeed provided a truly effective method.

**Table 3.** Error and CPU time of RBF and FD method.

| K   | RBF                     |         | FD     |         |
|-----|-------------------------|---------|--------|---------|
|     | Error                   | CPU (s) | Error  | CPU (s) |
| 140 | 2.0599                  | 0.058   | 2.0617 | 0.058   |
| 150 | $1.3586 \times 10^{-5}$ | 0.057   | 0.0017 | 0.056   |
| 160 | 0.5899                  | 0.057   | 0.5917 | 0.057   |
| 170 | 0.7100                  | 0.058   | 0.7100 | 0.056   |



**Figure 11.** American put option price with  $M = N = 100$ .

## 7. Conclusions

In this paper, a new numerical method is put forward to solve the pricing problem of options in time fractional jump-diffusion models. The pricing problem of European option

is represented as a TFPIDE, while American option is expressed as a LCP. For European option, nonuniform discretization along time and RBF method for spatial discretization are adopted. For American options, the operator splitting method is used to decompose LCP into two sub problems that are easy to solve numerically, namely linear boundary value problem and algebraic system. Stability and convergence rate are given, and the convergence rate is  $(2 - \alpha)$  when  $\beta \geq \frac{2-\alpha}{\alpha}$ . The numerical results are consistent with the theoretical results and demonstrate the impact of  $\alpha$  on option prices. The method is also applicable to the Kou model and can be extended to pricing singular options.

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