

Article

A General Fixed Point Theorem for a Sequence of Multivalued Mappings in S -Metric Spaces

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Abstract: Over the years, the concept of metric space has been extended in several directions, and numerous common fixed point theorems for multivalued mappings in complete metric space have been demonstrated. In this paper, we prove a general fixed point theorem for a pair of multivalued mappings satisfying implicit relations, extending some results from the literature to S -metric spaces. As an application, we obtain new results for a sequence of multivalued mappings in S -metric spaces, generalizing some known results.

Keywords: fixed point; S -metric space; sequence of multivalued mappings; implicit relation

MSC: 54H25; 47H10



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1. Introduction

Due to its numerous applications, Fixed Point Theory is one of the most useful branches of Nonlinear Analysis, providing fruitful tools for the resolution of many problems arising from different fields of pure mathematics, applied sciences, engineering, and more.

The existence and uniqueness of a fixed point has many applications; for example, in the field of computer sciences [1].

Also, fixed point theory is important, for example, in geometry. It is worth mentioning some papers related to fixed points of automorphisms of the vector bundle moduli space [2], or fixed points of principal bundles over algebraic curves [3], for example.

Over the years, the concept of metric space has been extended in several directions by removing or modifying some axioms. As the loss or weakening of some of the metric axioms leads to the loss of some topological properties, obstacles have arisen in proving fixed point theorems, which have led researchers to develop new techniques in developing fixed point theories to solve more concrete applications.

In [4,5], Dhage introduced a new class of generalized metric spaces, named D -metric spaces.

In [6,7], Mustafa and Sims demonstrated that most of the claims concerning the fundamental topological structures in D -metric spaces are incorrect, and they introduced an appropriate notion of generalized metric space, named G -metric space.

In fact, Mustafa and Sims, as well as other authors, obtained numerous fixed point results in G -metric spaces for self mappings.

Avramescu [8], Markin [9], Nadler [10], and other authors proved, using the Hausdorff–Pompeiu metric, some fixed point theorems for multivalued mappings that generalize the Banach principle.

Some results for sequences of multivalued mappings are obtained in [11] and in the references therein.

In [12], the authors introduced a “generalization” of G -metric spaces, named S -metric spaces, and initiated the study of fixed points for mappings satisfying implicit relations in S -metric spaces.

Recently, in [13], the authors showed that the notion of an S -metric space is not a generalization of a G -metric, or vice versa. Therefore, these two notions are independent.

Other results on fixed points in S -metric spaces were obtained in [14–20], as well as other papers.

Several fixed point theorems and common fixed points theorems have been unified in [21], considering a general condition by an implicit relation.

This method is used in the study of fixed points in metric spaces, symmetric spaces, quasi-metric spaces, convex spaces, b -metric spaces, Hilbert spaces, fuzzy metric spaces, probabilistic metric spaces, partial metric spaces, dislocated metric spaces, G -metric spaces, and G_p -partial metric spaces.

Throughout the last years, in order to express different contractivity conditions in a unified way, the notion of simulation function on metric spaces has been introduced, and some fixed point results were obtained.

Since then, by slightly modifying this new notion, some existence and uniqueness of coincidence points of two nonlinear operators using this kind of control function were investigated, and it has been observed that several existing results can be concluded from these new results [22].

The study of fixed points for multivalued mappings and hybrid pairs of mappings satisfying the implicit relations is initiated in [23–25].

In this paper, we prove a general fixed point theorem for a pair of multivalued mappings, satisfying the implicit relations, extending some results from the literature to S -metric spaces. As an application, we obtain new results for a sequence of multivalued mappings in S -metric spaces, generalizing some known results.

2. Preliminaries

We consider a nonempty set X , and let (X, d) be a metric space. By $\mathcal{P}(X)$, we denote the set of all nonempty subsets of X , and by $\mathcal{P}_{Cl}(X) = \{Y : Y \in \mathcal{P}(X) \text{ and } Y \text{ is a closed set}\}$, we denote the set of all nonempty closed subsets of X .

A point $x \in X$ is called a fixed point of a mapping $T : X \rightarrow \mathcal{P}_{Cl}(X)$ if $x \in Tx$. The set of all fixed points of T is denoted by $\mathcal{F}(T)$.

Definition 1 ([12]). Let X be a nonempty set. and $S : X^3 \rightarrow \mathbb{R}_+$ be a function such that, for all $x, y, z, a \in X$:

(S_1) : $S(x, y, z) = 0$ if and only if $x = y = z$.

(S_2) : $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The function S is said to be an S -metric on X , and (X, S) is called an S -metric space.

Example 1 ([12]). Let $X = \mathbb{R}$, and

$$S(x, y, z) = |x - z| + |y - z|.$$

Then, S is an S -metric on X , named the usual S -metric of $\mathbb{R}$.

Lemma 1 ([12]). If S is an S -metric on a nonempty set X , then

$$S(x, x, y) = S(y, y, x) \text{ for all } x, y \in X.$$

Definition 2 ([12]). Let (X, S) be an S -metric space. For $r > 0$ and $x \in X$, the open ball with center x and radius r , denoted $B_S(x, r)$, is defined as being

$$B_S(x, r) = \{y \in X : S(x, x, y) < r\}.$$

Let τ be the collection of subsets $A \subset X$ with $x \in A$, if and only if there exists $r > 0$ such that $B_S(x, r) \subset A$. Then, τ is a topology on X , and A is said to be S -open. The complement of an S -open set is said to be S -closed.

Definition 3 ([12]). Let (X, S) be an S -metric space.

(a) A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$, denoted $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$, if

$$\lim_{n \rightarrow \infty} S(x_n, x_n, x) = 0.$$

(b) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if

$$\lim_{n \rightarrow \infty} S(x_n, x_n, x_m) = 0.$$

(c) The S -metric space (X, S) is complete if every Cauchy sequence is a convergent sequence.

Example 2 ([12]). Let (X, S) be as in Example 1. Then, (X, S) is complete.

Lemma 2 ([12]). Let (X, S) be an S -metric space. If $x_n \rightarrow x$ and $y_n \rightarrow y$, then

$$S(x_n, x_n, y_n) \rightarrow S(x, x, y).$$

Lemma 3 ([12]). If $\{x_n\}$ is a convergent sequence in an S -metric space (X, S) , then its limit is unique.

Lemma 4 ([26], Lemma 8). Let (X, S) be an S -metric space, and let A be a nonempty subset of X . A is said to be S -closed if and only if, for any sequence $\{x_n\}$ in A , such that $x_n \rightarrow x$ as $n \rightarrow \infty$, then $x \in A$.

Latif and Beg, in [27], proved the following common fixed point theorem for K -multivalued mappings.

Theorem 1 ([27], Theorem 4.1). Let M be a nonempty closed subset of a complete metric space X . Then, each closed-valued K -multivalued map $T : M \rightarrow 2^M$ has a fixed point.

In 2003, Sintămărian [28,29] extended Theorem 1 for pairs of multivalued mappings.

Theorem 2 ([28], Theorem 2). Let (X, d) be a complete metric space, and $T_1, T_2 : X \rightarrow \mathcal{P}_{Cl}(X)$ be two multivalued operators. We suppose that

(i) There exists $a_1 \in [0, \frac{1}{2})$ such that for each $x \in X$, $u_x \in T_1x$ and each $y \in X$, there exists $u_y \in T_2y$, such that

$$d(u_x, u_y) \leq a_1 [d(x, u_x) + d(y, u_y)];$$

(ii) There exists $a_2 \in [0, \frac{1}{2})$ such that, for each $x \in X$, $u_x \in T_2x$ and each $y \in X$, there exists $u_y \in T_1y$, such that

$$d(u_x, u_y) \leq a_2 [d(x, u_x) + d(y, u_y)].$$

Then,

$$\mathcal{F}(T_1) = \mathcal{F}(T_2) \in \mathcal{P}_{Cl}(X).$$

Theorem 3 ([29], Theorem 2.7). Let (X, d) be a complete metric space, and $T_1, T_2 : X \rightarrow \mathcal{P}_{Cl}(X)$ two multivalued operators. We suppose that

(i) There exists $a_{11}, a_{12}, \dots, a_{15} \geq 0$ with $a_{11} + a_{12} + a_{13} + 2a_{14} < 1$ such that, for each $x \in X$, any $u_x \in T_1x$, and for all $y \in X$, there exists $u_y \in T_2y$, such that

$$d(u_x, u_y) \leq a_{11}d(x, y) + a_{12}d(x, u_x) + a_{13}d(y, u_y) + a_{14}d(x, u_y) + a_{15}d(y, u_x);$$

- (ii) There exists $a_{21}, a_{22}, \dots, a_{25} \geq 0$ with $a_{21} + a_{22} + a_{23} + 2a_{24} < 1$ such that for all $x \in X$, any $u_x \in T_2x$, and for all $y \in X$, there exists $u_y \in T_1y$, such that

$$d(u_x, u_y) \leq a_{21}d(x, y) + a_{22}d(x, u_x) + a_{23}d(y, u_y) + a_{24}d(x, u_y) + a_{25}d(y, u_x).$$

Then,

$$\mathcal{F}(T_1) = \mathcal{F}(T_2) \in \mathcal{P}_{Cl}(X).$$

Theorem 4 ([29], Theorem 2.8). Let (X, d) be a complete metric space, and $T_1, T_2 : X \mapsto \mathcal{P}_{Cl}(X)$ be two multivalued operators. We suppose that

- (i) There exists $a_1 \in [0, 1)$, such that for all $x \in X$, any $u_x \in T_1x$, and for all $y \in X$, there exists $u_y \in T_2y$, such that

$$d(u_x, u_y) \leq a_1 \max \left\{ \begin{array}{l} d(x, y), d(x, u_x), d(y, u_y), \\ \frac{1}{2} [d(x, u_y) + d(y, u_x)] \end{array} \right\};$$

- (ii) There exists $a_2 \in [0, 1)$, such that for all $x \in X$, any $u_x \in T_2x$, and for all $y \in X$, there exists $u_y \in T_1y$, such that

$$d(u_x, u_y) \leq a_2 \max \left\{ \begin{array}{l} d(x, y), d(x, u_x), d(y, u_y), \\ \frac{1}{2} [d(x, u_y) + d(y, u_x)] \end{array} \right\}.$$

Then,

$$\mathcal{F}(T_1) = \mathcal{F}(T_2) \in \mathcal{P}_{Cl}(X).$$

Also, in [27], it is proved that for a sequence of multivalued mappings, the following theorem holds.

Theorem 5 ([27], Theorem 4.2). Let M be a nonempty closed subset of a complete metric space X , and $\{T_n\}$ a sequence of closed-valued maps of M into 2^M . Suppose that there exists a constant h with $0 \leq h < \frac{1}{2}$, such that

- (i) For any two maps T_i, T_j , and for any $x \in M$, $u_x \in T_i(x)$, there exists $u_y \in T_j(x)$ for all $y \in M$, with

$$d(u_x, u_y) \leq h[d(x, u_x) + d(y, u_y)].$$

Then, $\{T_n\}$ has a common fixed point.

Let $\mathcal{F}_6$ be the set of all continuous functions $F : \mathbb{R}_+^6 \mapsto \mathbb{R}$, satisfying the following conditions:

(F₁) : F is not increasing in variables t_3, t_4, t_5, t_6 ;

(F₂) : There exist $h \in [0, 1)$ and $g \geq 0$, such that for all $u, v, w \geq 0$,

$$F \left(\begin{array}{c} u, v, w + v, \\ w + u, u + v + w, w \end{array} \right) \leq 0$$

implies

$$u \leq hv + gw.$$

A generalization of Theorems 1–4 using the Latif–Beg method is obtained in [24].

Theorem 6 ([24]). Let (X, d) be a complete metric space, and $T_1, T_2 : X \mapsto \mathcal{P}_{Cl}(X)$ be two multivalued mappings, such that:

(a) For each $x \in X$ and $u_x \in T_1x$ and $y \in X$, there exists $u_y \in T_2y$, such that

$$\phi_1 \left(\begin{array}{c} d(u_x, u_y), d(x, y), d(x, u_x), \\ d(y, u_y), d(x, u_y), d(y, u_x) \end{array} \right) \leq 0;$$

(b) For each $x \in X$ and $u_x \in T_2x$ and $y \in X$, there exists $u_y \in T_1y$, such that

$$\phi_2 \left(\begin{array}{c} d(u_x, u_y), d(x, y), d(x, u_x) \\ , d(y, u_y), d(x, u_y), d(y, u_x) \end{array} \right) \leq 0,$$

where $\phi_1, \phi_2 \in \mathcal{F}_6$.

Then,

$$\mathcal{F}(T_1) = \mathcal{F}(T_2) \in \mathcal{P}_{Cl}(X).$$

3. Implicit Relations

Let $\mathcal{F}_{SM}$ be the set of all continuous functions $\phi : \mathbb{R}_+^6 \mapsto \mathbb{R}$, satisfying the following properties:

(ϕ_1) : ϕ is nonincreasing in variables t_3, t_4, t_5, t_6 .

(ϕ_2) : there exist $h \in [0, 1)$ and $g \geq 0$, such that for all $u, v, w \geq 0$,

$$\phi \left(\begin{array}{c} u, v, 2w + v, \\ 2w + u, 2u + 2v + w, w \end{array} \right) \leq 0$$

implies

$$u \leq hv + gw.$$

In the following examples, all of the functions ϕ are from $\mathcal{F}_{SM}$, and the proof of property (ϕ_1) is obvious.

Example 3.

$$\phi(t_1, \dots, t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6,$$

where $a, b, c, d, e \geq 0$ and $a + b + c + 4d < 1$.

For (ϕ_2), let $u, v, w \geq 0$ and

$$\begin{aligned} \phi \left(\begin{array}{c} u, v, 2w + v, \\ 2w + u, 2u + 2v + w, w \end{array} \right) &= u - av - b(2w + v) - c(2w + u) - \\ &\quad - d(2u + 2v + w) - ew \leq 0. \end{aligned}$$

Then,

$$u \leq \frac{a + b + 2d}{1 - (c + 2d)}v + \frac{2b + 2c + d + e}{1 - (c + 2d)}w.$$

Hence,

$$u \leq hv + gw,$$

where

$$0 \leq h = \frac{a + b + 2d}{1 - (c + 2d)} < 1$$

and

$$g = \frac{2b + 2c + d + e}{1 - (c + 2d)} \geq 0.$$

Example 4.

$$\phi(t_1, \dots, t_6) = t_1 - k \max \left\{ t_2, t_3, t_4, \frac{t_5 + t_6}{2} \right\},$$

where $k \in \left[0, \frac{1}{3}\right)$.

For (ϕ_2) , let $u, v, w \geq 0$, and

$$\phi\left(\begin{array}{c} u, v, 2w + v, \\ 2w + u, 2u + 2v + w, w \end{array}\right) = u - k \max\left\{\begin{array}{c} v, 2w + v, \\ 2w + u, \frac{2u + 2v + 2w}{2} \end{array}\right\} \leq 0.$$

If $u > \max\{v, w\}$, then $u(1 - 3k) \leq 0$; a contradiction.

Hence,

$$u \leq \max\{v, w\}$$

which implies

$$u \leq 3k \max\{v, w\} \leq 3k(v + w),$$

from where

$$u \leq hv + gw,$$

for

$$0 \leq h = 3k < 1$$

and

$$g = 3k \geq 0.$$

Example 5.

$$\phi(t_1, \dots, t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\},$$

where $k \in \left[0, \frac{1}{5}\right)$.

For (ϕ_2) , let $u, v, w \geq 0$ and

$$\phi\left(\begin{array}{c} u, v, 2w + v, \\ 2w + u, 2u + 2v + w, w \end{array}\right) = u - k \max\left\{\begin{array}{c} 2w + v, 2w + u, \\ 2u + 2v + w \end{array}\right\} \leq 0.$$

If $u > \max\{v, w\}$, then

$$u(1 - 5k) \leq 0,$$

a contradiction.

Hence,

$$u \leq \max\{v, w\}$$

which implies

$$u \leq 5k \max\{v, w\} \leq 5k(v + w).$$

Therefore,

$$u \leq hv + gw,$$

where

$$0 \leq h = 5k < 1$$

and

$$g = 5k \geq 0.$$

Similarly, it can be shown that the following functions are from $\mathcal{F}_{SM}$.

Example 6.

$$\phi(t_1, \dots, t_6) = t_1 - k \max\left\{t_2, \frac{t_3 + t_4}{2}, \frac{t_5 + t_6}{2}\right\},$$

where $k \in \left[0, \frac{1}{3}\right)$.

$$u(1 - 3k) \leq 0,$$

a contradiction.

$$u \leq 3k \max\{v, w\} \leq 3k(v + w),$$

from where

$$u \leq hv + gw,$$

for

$$0 \leq h = 3k < 1$$

and

$$g = 3k \geq 0.$$

Example 7.

$$\phi(t_1, \dots, t_6) = t_1 - at_2 - b \max\{t_3, t_4\} - c \min\{t_5, t_6\},$$

where $a, b, c \geq 0$ and $a + 2b < 1$.

Example 8.

$$\phi(t_1, \dots, t_6) = t_1 - at_2 - b(t_3 + t_4) - c(t_5 + t_6),$$

where $a, b, c \geq 0$ and $a + 2b + 4c < 1$.

Example 9.

$$\phi(t_1, \dots, t_6) = t_1^2 - k \max\{t_2^2, t_3t_4, t_5t_6\},$$

where $k \in \left[0, \frac{1}{9}\right)$.

Example 10.

$$\phi(t_1, \dots, t_6) = t_1 + \frac{t_1}{1 + t_5 + t_6} - k \max\{t_2, t_3, t_4\},$$

where $k \in \left[0, \frac{1}{2}\right)$.

4. Main Results

Theorem 7. Let (X, S) be a complete S -metric space, and $T_1, T_2 : X \mapsto \mathcal{P}_{Cl}(X)$ be two multivalued mappings, such that:

(a) For all $x, y \in X$, $u_x \in T_1x$, there exists $u_y \in T_2y$, such that

$$\phi_1 \left(\begin{array}{c} S(u_x, u_x, u_y), S(x, x, y), S(x, x, u_x), \\ S(y, y, u_y), S(x, x, u_y), S(y, y, u_x) \end{array} \right) \leq 0; \quad (1)$$

(b) For all $x, y \in X$, $u_x \in T_2x$, there exists $u_y \in T_1y$, such that

$$\phi_2 \left(\begin{array}{c} S(u_x, u_x, u_y), S(x, x, y), S(x, x, u_x), \\ S(y, y, u_y), S(x, x, u_y), S(y, y, u_x) \end{array} \right) \leq 0, \quad (2)$$

where $\phi_1, \phi_2 \in \mathcal{F}_{SM}$.

Then,

$$\mathcal{F}(T_1) = \mathcal{F}(T_2) \neq \emptyset$$

and

$$\mathcal{F}(T_1) = \mathcal{F}(T_2) \in \mathcal{P}_{Cl}(X).$$

Proof. Let x_0 be an arbitrary point of X , $x_1 \in T_1x_0$.

Then, by (a), there exists $x_2 \in T_2x_1$ such that

$$\phi_1 \left(\begin{array}{c} S(x_1, x_1, x_2), S(x_0, x_0, x_1), S(x_0, x_0, x_1), \\ S(x_1, x_1, x_2), S(x_0, x_0, x_2), 0 \end{array} \right) \leq 0.$$

By (S_2) and Lemma 1, we obtain

$$\begin{aligned} S(x_0, x_0, x_2) &= S(x_2, x_2, x_0) \leq 2S(x_2, x_2, x_1) + S(x_0, x_0, x_1) \\ &= 2S(x_1, x_1, x_2) + S(x_0, x_0, x_1), \end{aligned}$$

which implies

$$\phi_1 \left(\begin{array}{c} S(x_1, x_1, x_2), S(x_0, x_0, x_1), S(x_0, x_0, x_1), \\ S(x_1, x_1, x_2), 2S(x_1, x_1, x_2) + S(x_0, x_0, x_1), 0 \end{array} \right) \leq 0.$$

By (F_2) , we obtain

$$S(x_1, x_1, x_2) \leq hS(x_0, x_0, x_1).$$

By (b), there exists $u_3 \in T_1x_2$ such that

$$\phi_2 \left(\begin{array}{c} S(x_2, x_2, x_3), S(x_1, x_1, x_2), S(x_1, x_1, x_2), \\ S(x_2, x_2, x_3), S(x_1, x_1, x_3), 0 \end{array} \right) \leq 0.$$

By Lemma 1 and (S_2) , we obtain

$$\begin{aligned} S(x_1, x_1, x_3) &= S(x_3, x_3, x_1) \leq 2S(x_3, x_3, x_2) + S(x_1, x_1, x_2) \\ &= 2S(x_2, x_2, x_3) + S(x_1, x_1, x_2), \end{aligned}$$

which implies by (F_2)

$$S(x_2, x_2, x_3) \leq hS(x_1, x_1, x_2).$$

By induction, we obtain a sequence $\{x_n\}$ such that $x_0 \in X$, $x_{n-1} \in T_1x_{n-2}$, $x_n \in T_2x_{n-1}$, for $n = 2, 3, 4, \dots$ and

$$S(x_n, x_n, x_{n+1}) \leq hS(x_{n-1}, x_{n-1}, x_n) \leq \dots \leq h^n S(x_0, x_0, x_1).$$

We prove that $\{x_n\}$ is a Cauchy sequence in (X, S) .

For $n, m \in \mathbb{N}$, $m > n$, by (S_2) and Lemma 1, we necessarily have

$$\begin{aligned} S(x_n, x_n, x_m) &\leq 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1}) \\ &= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m) \\ &\vdots \\ &\leq 2(h^n + h^{n+1} + \dots + h^{m-1})S(x_0, x_0, x_1) \\ &\leq \frac{2h^n}{1-h} S(x_0, x_0, x_1). \end{aligned}$$

Letting n tend to infinity, we obtain

$$\lim_{m, n \rightarrow \infty} S(x_n, x_n, x_m) = 0.$$

Hence, $\{x_n\}$ is a Cauchy sequence in (X, S) .

Since (X, S) is S -complete, then $\{x_n\}$ is convergent to a point $x \in X$.

We prove that x is a fixed point of T_1 .

Since $x_n \in T_2x_{n-1}$, by (b), we have that there exists $u_n \in T_1x$, such that

$$\phi_2 \left(\begin{array}{c} S(x_n, x_n, x_{n-1}), S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), \\ S(x, x, u_n), S(x_{n-1}, x_{n-1}, u_n), S(x, x, x_n) \end{array} \right) \leq 0.$$

By Lemma 1 and (S_2) , we have

$$S(x_{n-1}, x_{n-1}, x_n) = S(x_n, x_n, x_{n-1}) \leq 2S(x_n, x_n, x) + S(x_{n-1}, x_{n-1}, x),$$

$$S(x, x, u_n) \leq 2S(x, x, x_n) + S(u_n, u_n, x),$$

and

$$S(u_n, u_n, x) \leq 2S(u_n, u_n, x_n) + S(x, x, x_n).$$

Hence,

$$S(x_{n-1}, x_{n-1}, u_n) \leq 2S(x_{n-1}, x_{n-1}, x) + 2S(u_n, u_n, x_n) + S(x, x, x_n)$$

which implies by (F_2) that

$$\phi_2 \left(\begin{array}{c} S(x_n, x_n, u_n), S(x_{n-1}, x_{n-1}, x_n), \\ 2S(x_n, x_n, x) + S(x_{n-1}, x_{n-1}, x), \\ 2S(x_n, x_n, x) + S(x_n, x_n, u_n), \\ 2S(x_n, x_n, u_n) + 2S(x_{n-1}, x_{n-1}, x_n) + S(x, x, x_n), \\ S(x, x, x_n) \end{array} \right) \leq 0.$$

By Lemma 3,

$$S(x, x, u_n) = S(x_n, x_n, x).$$

Hence, by (F_2) , we obtain

$$S(x_n, x_n, u_n) \leq hS(x_{n-1}, x_{n-1}, z) + gS(x_n, x_n, z).$$

On the other hand,

$$\begin{aligned} S(x, x, u_n) &\leq 2S(x, x, x_n) + S(u_n, u_n, x_n) \\ &\leq 2S(x_n, x_n, x) + hS(x_{n-1}, x_{n-1}, z) + gS(x_n, x_n, z). \end{aligned}$$

Letting n tend to infinity, we obtain

$$\lim_{n \rightarrow \infty} S(x, x, u_n) = \lim_{n \rightarrow \infty} S(u_n, u_n, x) = 0.$$

Hence,

$$\lim_{n \rightarrow \infty} u_n = x.$$

Since $u_n \in T_1x$ and T_1x is closed, then $x \in T_1x$ and x is a fixed point of T_1 . Therefore,

$$\mathcal{F}(T_1) \neq \emptyset.$$

Similarly, it is shown that x is a fixed point T_2 and

$$\mathcal{F}(T_2) \neq \emptyset.$$

Now, we prove that

$$\mathcal{F}(T_1) = \mathcal{F}(T_2).$$

Since $x \in T_1x$, let $u_x = x = y$. Then, there exists $u_y \in T_2y$ such that

$$\phi_1 \left(\begin{array}{c} S(x, x, u_y), 0, 0, \\ S(x, x, u_y), S(x, x, u_y), 0 \end{array} \right) \leq 0,$$

which implies

$$S(x, x, u_y) = 0.$$

Hence,

$$x = y = u_y \in T_2y = T_2x.$$

Thus, $x \in T_2x$ and $x \in \mathcal{F}(T_2)$.

Therefore,

$$\mathcal{F}(T_1) \subset \mathcal{F}(T_2).$$

Similarly, it is proved that

$$\mathcal{F}(T_2) \subset \mathcal{F}(T_1).$$

Hence,

$$\mathcal{F}(T_1) = \mathcal{F}(T_2) \neq \emptyset.$$

Now we have to prove that

$$\mathcal{F}(T_1) = \mathcal{F}(T_2) \in \mathcal{P}_{Cl}(X).$$

For this purpose, let $y_n \in \mathcal{F}(T_1) = \mathcal{F}(T_2)$, $y_n \rightarrow y$.

By (a), for $y_n \in T_1y_n$, we have that there exists $v_n \in T_2x$ such that

$$\phi_1 \left(\begin{array}{c} S(y_n, y_n, v_n), S(y_n, y_n, y), 0, \\ S(y, y, v_n), S(y_n, y_n, v_n), S(y_n, y_n, y) \end{array} \right) \leq 0.$$

By (F_1) and (S_2) we obtain

$$\phi_1 \left(\begin{array}{c} S(y_n, y_n, v_n), S(y_n, y_n, y), \\ 2S(y_n, y_n, y) + S(y_n, y_n, y), \\ 2S(y_n, y_n, y) + S(y_n, y_n, v_n), \\ 2S(y_n, y_n, v_n) + 2S(y_n, y_n, y) + S(y_n, y_n, y), \\ S(y_n, y_n, y) \end{array} \right) \leq 0,$$

which implies

$$S(y_n, y_n, v_n) \leq hS(y_n, y_n, y) + gS(y_n, y_n, y).$$

By (S_2) :

$$\begin{aligned} S(v_n, v_n, y) &\leq 2S(v_n, v_n, y_n) + S(y, y, y_n) \\ &\leq 2[hS(y_n, y_n, y) + gS(y_n, y_n, y)] + S(y_n, y_n, y). \end{aligned}$$

Letting n tend to infinity, we obtain

$$\lim_{n \rightarrow \infty} S(v_n, v_n, y) = \lim_{n \rightarrow \infty} S(y, y, v_n) = 0.$$

Hence,

$$\lim_{n \rightarrow \infty} v_n = y.$$

Since $v_n \in T_2y$ and T_2y is closed, then $y \in T_2y$ and $y \in \mathcal{F}(T_2)$.

By Lemma 4, $\mathcal{F}(T_2)$ is closed. Hence,

$$\mathcal{F}(T_1) = \mathcal{F}(T_2) \in \mathcal{P}_{Cl}(X)$$

and the proof is complete. $\square$

Theorem 8. Let (X, S) be a complete S -metric space, and $\{T_n\} \subset \mathcal{P}_{Cl}(X)$, $n \in \mathbb{N}$ be a sequence of multivalued mappings, such that:

(a) For all $x, y \in X$, $u_x \in T_i x$, there exists $u_y \in T_{i+1} y$, such that

$$\phi_i \left(\begin{array}{c} S(u_x, u_x, u_y), S(x, x, y), S(x, x, u_x), \\ S(y, y, u_y), S(x, x, u_y), S(y, y, u_x) \end{array} \right) \leq 0; \quad (3)$$

(b) For all $x, y \in X$, $u_x \in T_{i+1} x$, there exists $u_y \in T_i y$, such that

$$\phi_{i+1} \left(\begin{array}{c} S(u_x, u_x, u_y), S(x, x, y), S(x, x, u_x), \\ S(y, y, u_y), S(x, x, u_y), S(y, y, u_x) \end{array} \right) \leq 0, \quad (4)$$

where $\{\phi_i\}_{i \in \mathbb{N}} \in \mathcal{F}_{SM}$.

Then,

$$\mathcal{F}(T_1) = \mathcal{F}(T_2) = \dots = \mathcal{F}(T_n) \in \mathcal{P}_{Cl}(X), \quad n \in \mathbb{N}.$$

Proof. By Theorem 7 and $i = 1$, we have

$$\mathcal{F}(T_1) = \mathcal{F}(T_2) \in \mathcal{P}_{Cl}(X).$$

For $i = 2$ in Theorem 7, we obtain

$$\mathcal{F}(T_2) = \mathcal{F}(T_3) \in \mathcal{P}_{Cl}(X).$$

Continuing this process, we obtain

$$\mathcal{F}(T_j) = \mathcal{F}(T_{j+1}) \in \mathcal{P}_{Cl}(X), \quad j = \overline{1, n-1}.$$

□

Remark 1. (a) By Theorem 7 and Example 3, we obtain an result which extends Theorem 2 to S -metric spaces.

(b) By Theorem 8 and Example 3, we obtain an result which extends Theorem 3 to S -metric spaces.

(c) By Theorem 7 and Example 4, we obtain an result which extends Theorem 4 to S -metric spaces.

(d) By Theorem 7 and Example 5, we obtain an result which extends Theorem 5 to S -metric spaces.

(e) By Theorems 7 and 8 and Examples 6–10, we obtain new particular results.

5. Conclusions

In this paper, we proved a general fixed point theorem for a pair of multivalued mappings satisfying implicit relations, extending some well-known results to S -metric spaces.

As an application, we obtained new results for sequences of multivalued mappings in S -metric spaces, generalizing other results.

From our theorems and the other examples presented in the paper, we obtain new particular results.

As a future idea for study, we intend to extend these results to hybrid pairs of mappings.

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