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The Bundle of Tensor Densities and Its Covariant Derivatives

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Abstract: We construct the smooth vector bundle of tensor densities of arbitrary weight in a coordinate-independent way. We prove the general existence of a globally smooth tensor density field, as well as the existence of a globally smooth metric density for a pseudo-Riemannian manifold, specifically. We study the coordinate description of a covariant derivative over densities, and define a natural extension of affine connections to densities. We provide an equivalent characterization, in the case of a pseudo-Riemannian manifold.

Keywords: densities; tensor densities; vector bundles; covariant derivatives; Riemannian metrics

MSC: 53A45; 53C05; 53C07; 53Z05

1. Introduction

If (x^i) and (y^i) represent the coordinates of two different charts of a smooth manifold $\mathcal{M}$, it is well-known that a global (r,s)-tensor field is well-defined when its coefficients $T^{b_1\dots b_s}_{a_1\dots a_r}(x^i)$ and $\tilde{T}^{b_1\dots b_s}_{a_1\dots a_r}(y^i)$ are related in the following manner:

$$T^{b_1\dots b_s}_{a_1\dots a_r}(x^i) = \tilde{T}^{j_1\dots j_s}_{i_1\dots i_r}\left(y^i(x^j)\right)\frac{\partial y^{i_1}}{\partial x^{a_1}}\cdot\ldots\cdot\frac{\partial y^{i_r}}{\partial x^{a_r}}\cdot\frac{\partial x^{b_1}}{\partial y^{j_1}}\cdot\ldots\cdot\frac{\partial x^{b_s}}{\partial y^{j_s}}.$$

(Throughout this paper all indices will run from 1 to n. We will use $a,b,c,\dots$ to indicate a free index, but $i,j,k,\dots$ to indicate summation indices that follow Einstein's convention.). In the context of general relativity, however, some natural objects arise which do not follow this law of transformation. The two main examples are the determinant of the metric of the spacetime, on the one hand, and the Levi-Civita symbol ε_{ijkl}, on the other hand. Indeed, their coordinate transformations include a factor that contains the determinant of the Jacobian of the coordinate change. For instance, for the determinant of the metric we have:

$$\sqrt{|\det\left(g_{ab}(x^i)\right)|} = \left|\det\left(\frac{dy^i}{dx^j}\right)\right|\sqrt{\left|\det\left(\tilde{g}_{ab}\left(y^i(x^j)\right)\right)\right|}.$$

Objects of the above type are usually called *densities*, and they also appear in the context of integration for non-orientable manifolds (see, for example, chapter 16 in [1]). If it is rather the absolute value of the determinant that appears with a power of some $\alpha\in\mathbb{R}$ in the coordinate transformation, we obtain a *density of weight* α. In this paper, we will generalize this one step further: we will consider a mixture of both tensor fields and densities of weight α. These objects are often called *tensor densities*. They have been studied since at least the first half of the previous century. In fact, they were already proposed by Einstein in [2] (with the name *Tensordichten*), while other early references such as [3] or [4] use the term *relative tensors*.

Our caveat to the more modern literature is that tensor densities are almost everywhere defined directly by their law of transformation under a change of coordinates (see, e.g., [5,6] or expression (3) in our paper). This is also often the case in the literature on general relativity (see, e.g., [7]). Quite surprisingly, notwithstanding tensor densities are very useful

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in contemporary articles (see, e.g., [8–11], to mention just a few), one can almost find no general treatment of them with the modern language of global analysis and differential geometry. For this reason, we will introduce the theory from scratch in this paper and verify all constructions at the required level of detail. The main purpose of this paper is to generalize results that seem to be only known in a local flavor to a modern global treatment. In this sense, it is a mostly a methodological paper with the aim of clarifying some details of the description. In addition to the local–global aspects, the paper also contains results on the characterization of the relevant covariant derivatives in this context. We will give a global meaning to the local constructions that are spread throughout the literature by stating properties that uniquely identify the covariant derivatives of interest.

In Section 2, we introduce a coordinate-free definition of an (r, s)-tensor density of arbitrary weight (both at the level of a vector space and of a manifold), and in Section 3 we show that tensor densities on a manifold can be given the structure of a smooth vector bundle (Proposition 3).

In Section 4, we study the existence of smooth tensor density fields. First, we show how to construct a globally smooth tensor density field for any smooth manifold (Proposition 4). Second, we consider the case of a pseudo-Riemannian manifold, because the applications of tensor densities are to be found mostly in general relativity (see, e.g., [2,7,12]). Still in Section 4, we construct a density of arbitrary weight that is, roughly speaking, given by the determinant of the metric, and we show that it is a globally smooth density field (Proposition 6).

Once we have established the vector bundle structure of tensor densities, we may consider linear connections on that vector bundle. As is the case for tensor fields, also when dealing with tensor densities, it is often useful to be able to covariantly differentiate them. The general properties of a covariant derivative for densities are considered in Section 5.

It is well-known that any pseudo-Riemannian metric determines a unique affine connection, the Levi-Civita connection. In the literature, one may often find, again only with a coordinate expression, an extension of the Levi-Civita connection to tensor densities. For example, this extension can be found in special cases in [3,5] (for densities), and in [12] (for $(2, 0)$-tensor densities). Despite only being defined by a coordinate expression, no motivation is given as to why it is the "natural" extension. Here, in Section 5, we provide two such defining characterizations of the extension to densities. First, for a general affine connection, we show (in Proposition 7) that the extension is the unique one that follows the rule of differentiation of powers. Next, for the special case of a Levi-Civita connection (Proposition 8), we prove that the extension is the unique covariant derivative that is metrical in an appropriate sense. We end the paper with a natural extension of an affine connection to the most general context: that of tensor densities (Proposition 9).

2. Tensor Densities with a Weight

Let $\mathcal{M}$ be a smooth manifold. Although older sources ([5–7]) take densities or tensor densities on a manifold to be a set of functions that transform with the determinant of the Jacobian matrix of the coordinate change on $\mathcal{M}$, there are also some modern sources like [1,13] that give an intrinsic approach (be it in the case of scalar densities only).

To start the paper, we will consider first a real vector space E of dimension n. In the next section, a tangent space $T_p\mathcal{M}$ will play the role of E. The following definition can be found in [13], albeit with a different phrasing:

Definition 1. *For $\alpha \in \mathbb{R}$, an α-density on E is a map:*

$$\rho : E \times \overset{n}{\cdots} \times E \longrightarrow \mathbb{R}$$

such that, for any linear map $\Psi : E \to E$, and any set of vectors $v_1, \ldots, v_n \in E$, we have:

$$\rho(\Psi v_1, \ldots, \Psi v_n) = |\det(\Psi)|^\alpha \cdot \rho(v_1, \ldots, v_n)$$

(If $\alpha = 0$, we take that $|x|^\alpha = 1$, even if $x = 0$).

We will denote the set of α-densities on E by $|\Lambda|^\alpha(E)$.

α is called *the weight of the density*. Inspired by this intrinsic approach to scalar densities we may now define tensor densities. The definition below is not to be found, as such, in the literature. We will show later that its transformation under a change of basis corresponds to that what is often called a tensor density in the literature.

Definition 2. *For $\alpha \in \mathbb{R}$, an (r,s)-tensor density of weight α on E is a map:*

$$A : E\times \overset{r}{\cdots} \times E \times E^* \overset{s}{\cdots} \times E^* \times E\times \overset{n}{\cdots} \times E \longrightarrow \mathbb{R}$$

such that:

- *A is tensorial in its first $r+s$ variables: given any $u_1,\dots,u_n \in E$, the map*

$$t : E\times \overset{r}{\cdots} \times E \times E^* \overset{s}{\cdots} \times E^* \longrightarrow \mathbb{R}$$
$$(v_1,\dots,v_r;\alpha^1,\dots\alpha^s) \mapsto t(v_1,\dots,v_r;\alpha^1,\dots\alpha^s) := A(v_1,\dots,v_r;\alpha^1,\dots\alpha^s;u_1,\dots,u_n)$$

  is an (r,s)-tensor on E.
- *A is density-like in its last n variables: given any $v_1,\dots,v_r \in E$, $\alpha^1,\dots\alpha^s \in E^*$, the map*

$$\rho : E\times \overset{n}{\cdots} \times E \longrightarrow \mathbb{R}$$
$$(u_1,\dots,u_n) \mapsto \rho(u_1,\dots,u_n) := A(v_1,\dots,v_r;\alpha^1,\dots\alpha^s;u_1,\dots,u_n)$$

  is an α-density on E.

We will denote the set of (r,s)-tensor densities of weight α on E by $T^r_s|\Lambda|^\alpha(E)$.

We can define the following sum and product by $\mathbb{R}$ on $T^r_s|\Lambda|^\alpha(E)$:

$$(c_1\cdot A_1 + c_2\cdot A_2)(v_i;\alpha^i;u_i) := c_1\cdot\big(A_1(v_i;\alpha^i;u_i)\big) + c_2\cdot\big(A_2(v_i;\alpha^i;u_i)\big), \qquad \forall c_1,c_2\in\mathbb{R}.$$

When the last $u_1,\dots u_n$ variables remain fixed, this corresponds to the usual sum and product for tensors, meaning that the tensorial character of the first $r+s$ variables is conserved. As for the last n variables, it is easy to see that:

$$\begin{aligned}(c_1\cdot\rho_1 + c_2\cdot\rho_2)(\Psi u_1,\dots,\Psi u_n) &= c_1\cdot\rho_1(\Psi u_1,\dots,\Psi u_n) + c_2\cdot\rho_2(\Psi u_1,\dots,\Psi u_n)\\ &= |\det(\Psi)|^\alpha\cdot\big(c_1\cdot\rho_1(u_1,\dots,u_n) + c_2\cdot\rho_2(u_1,\dots,u_n)\big)\\ &= |\det(\Psi)|^\alpha\cdot(c_1\cdot\rho_1 + c_2\cdot\rho_2)(u_1,\dots,u_n).\end{aligned}$$

Therefore, with this sum and product, any set of tensor densities is a real vector space, since $c_1\cdot A_1 + c_2\cdot A_2$ is again an element of $T^r_s|\Lambda|^\alpha(E)$. The identity element of this vector space is the map that sends any combination of elements to 0 (which is trivially a tensor density of any weight).

Notice that the set of $(0,0)$-tensor densities of weight α is exactly the set of α-densities: $T^0_0|\Lambda|^\alpha(E) = |\Lambda|^\alpha(E)$. We will first show that this vector space is one-dimensional. To prove that, we will use the following result, which is also the main reason for the notation "$|\Lambda|^\alpha(E)$":

Lemma 1. *For any $\omega \in \Lambda^n(E)$ (the set of n-covariant alternating tensors), and any $\alpha\in\mathbb{R}$, the map $|\omega|^\alpha$ defined by:*

$$|\omega|^\alpha : E\times \overset{n}{\cdots} \times E \longrightarrow \mathbb{R}$$
$$(v_1,\dots,v_n) \mapsto |\omega|^\alpha(v_1,\dots,v_n) := |\omega(v_1,\dots,v_n)|^\alpha$$

is an element of $|\Lambda|^\alpha(E)$.

Proof. It is sufficient to see that, for any $\omega \in \Lambda^n(E)$ and any linear map Ψ,

$$\omega(\Psi v_1, \ldots, \Psi v_n) = \det(\Psi) \cdot \omega(v_1, \ldots, v_n).$$

But, this is a known property whose proof can be found in Proposition 14.9 of [1]. □

Throughout this section $\{e_1, \ldots, e_n\}$ will be a basis for E, with dual basis $\{e^1, \ldots, e^n\}$. Then $\hat{\omega} := e^1 \wedge \cdots \wedge e^n$ is a non-zero element of $\Lambda^n(E)$, and thus $|\hat{\omega}|^\alpha$ is a non-zero element of $|\Lambda|^\alpha(E)$. This element has the property $|\hat{\omega}|^\alpha(e_1, \ldots, e_n) = |\hat{\omega}(e_1, \ldots, e_n)|^\alpha = |1|^\alpha = 1$.

If ρ is another element of $|\Lambda|^\alpha(E)$, we can define $\rho_0 := \rho(e_1, \ldots, e_n)$. Since any linear map on a finite dimensional vector space is uniquely defined by the image of a basis, for any $v_1, \ldots, v_n \in E$, there is a unique linear map $\Psi : E \to E$ such that $\Psi(e_a) = v_a$, for every $a = 1, \ldots, n$. Then, we have that:

$$\begin{aligned} \rho(v_1, \ldots, v_n) &= \rho(\Psi e_1, \ldots, \Psi e_n) = |\det(\Psi)|^\alpha \cdot \rho(e_1, \ldots, e_n) = |\det(\Psi)|^\alpha \rho_0 \\ &= |\det(\Psi)|^\alpha \rho_0 \cdot |\hat{\omega}|^\alpha(e_1, \ldots, e_n) = \rho_0 \cdot |\hat{\omega}|^\alpha(\Psi e_1, \ldots, \Psi e_n) \\ &= \rho_0 \cdot |\hat{\omega}|^\alpha(v_1, \ldots, v_n). \end{aligned}$$

This means that any $\rho \in |\Lambda|^\alpha(E)$ can be expressed as a multiple of $|\hat{\omega}|^\alpha$, so that $|\Lambda|^\alpha(E)$ is a one-dimensional vector space.

Notice that the value of $|\hat{\omega}|^\alpha$ is always ≥ 0, so that, depending on the sign of ρ_0, the values that a vector density will take will be either all ≥ 0 or all ≤ 0. For this reason, we can classify the set of vector densities of weight α as being either positive-valued, negative-valued, or zero.

In addition to being the most basic example of tensor densities, α-densities also convey the essential difference between tensors and tensor densities. To show this, let us first define an analogue of the "tensorial product", but for tensors and densities.

Given some $t \in T^r_s(E)$ and $\rho \in |\Lambda|^\alpha(E)$, we define $t \otimes \rho$ as the map:

$$\begin{gathered} t \otimes \rho : E \times \overset{r}{\cdots} \times E \times E^* \overset{s}{\cdots} \times E^* \times E \times \overset{n}{\cdots} \times E \longrightarrow \mathbb{R} \\ (v_1, \ldots, v_r; \alpha^1, \ldots \alpha^s; u_1, \ldots, u_n) \mapsto t(v_1, \ldots, v_r; \alpha^1, \ldots \alpha^s)\rho(u_1, \ldots, u_n). \end{gathered}$$

This is obviously an element of $T^r_s|\Lambda|^\alpha(E)$, so that we have a map $\otimes : T^r_s(E) \times |\Lambda|^\alpha(E) \to T^r_s|\Lambda|^\alpha(E)$.

It is easy to see that this product has properties similar to those of the tensorial product:

$$\begin{aligned} (t_1 + t_2) \otimes \rho &= t_1 \otimes \rho + t_2 \otimes \rho, \\ t \otimes (\rho_1 + \rho_2) &= t \otimes \rho_1 + t \otimes \rho_2, \\ \lambda(t \otimes \rho) = (\lambda t) \otimes \rho &= t \otimes (\lambda \rho) \qquad \forall \lambda \in \mathbb{R}. \end{aligned} \tag{1}$$

The following result says that $\otimes$ is in fact sufficient to construct the whole space of tensor densities as a product of tensors and densities:

Proposition 1. *For any element $A \in T^r_s|\Lambda|^\alpha(E)$, there exist $t \in T^r_s(E)$ and $\rho \in |\Lambda|^\alpha(E)$ such that $A = t \otimes \rho$.*

Proof. The map

$$\begin{gathered} t : E \times \overset{r}{\cdots} \times E \times E^* \times \overset{s}{\cdots} \times E^* \longrightarrow \mathbb{R} \\ (v_1, \ldots, v_r; \alpha^1, \ldots \alpha^s) \mapsto t(v_1, \ldots, v_r; \alpha^1, \ldots \alpha^s) := A(v_1, \ldots, v_r; \alpha^1, \ldots \alpha^s; e_1, \ldots, e_n) \end{gathered}$$

is a tensor by the definition of tensor density.

Take $v_1, \ldots, v_r \in E$, $\alpha^1, \ldots, \alpha^s \in E^*$ and $u_1, \ldots, u_n \in E$. Any linear map is completely characterized by the image of a basis. Therefore, we can consider the unique linear map Ψ that sends $e_a \mapsto u_a$. Then, using the density-like character of the last n variables of A, we have that:

$$A(v_i; \alpha^i; u_i) = A\big(v_i; \alpha^i; \Psi(e_i)\big) = A\big(v_i; \alpha^i; e_i\big) \cdot |\det(\Psi)|^\alpha = t(v_i; \alpha^i) \cdot |\det(\Psi)|^\alpha.$$

Since $|\hat{\omega}|^\alpha(e_1, \ldots, e_n) = 1$, and since $|\hat{\omega}|^\alpha$ is an α-density, we have:

$$|\det(\Psi)|^\alpha = |\hat{\omega}|^\alpha(e_i) \cdot |\det(\Psi)|^\alpha = |\hat{\omega}|^\alpha\big(\Psi(e_i)\big) = |\hat{\omega}|^\alpha(u_i).$$

In conclusion, $A(v_i; \alpha^i; u_i) = t(v_i; \alpha^i) \cdot |\hat{\omega}|^\alpha(u_i)$, which means that, indeed, $A = t \otimes |\hat{\omega}|^\alpha$. □

The above result allows us to construct a basis for $T^r_s|\Lambda|^\alpha(E)$, by starting from the well-known basis for $T^r_s(E)$:

$$\left\{e^{a_1} \otimes \cdots \otimes e^{a_r} \otimes e_{b_1} \otimes \cdots \otimes e_{b_s}\right\}_{a_i, b_j = 1, \ldots, n}.$$

For the rest of this section, we will denote each of these n^{rs} elements by $E^{a_1 \ldots a_r}_{b_1 \ldots b_s}$.

Every $A \in T^r_s|\Lambda|^\alpha(E)$ can be written as $t \otimes |\hat{\omega}|^\alpha$, for some unique t. Indeed, if $t \otimes |\hat{\omega}|^\alpha = s \otimes |\hat{\omega}|^\alpha$ for some other tensor $s \in T^r_s(E)$, then, for any $v_i \in E$, $\alpha^i \in E^*$, $t(v_i; \alpha^i) = (t \otimes |\hat{\omega}|^\alpha)(v_i; \alpha^i; e_i) = (s \otimes |\hat{\omega}|^\alpha)(v_i; \alpha^i; e_i) = s(v_i; \alpha^i)$, and therefore $t = s$.

As t is a tensor, it can also be uniquely written as $t = t^{j_1 \ldots j_s}_{i_1 \ldots i_r} E^{i_1 \ldots i_r}_{j_1 \ldots j_s}$. For this reason, any $A \in T^r_s|\Lambda|^\alpha(E)$ has a unique decomposition as

$$A = t \otimes |\hat{\omega}|^\alpha = \big(t^{j_1 \ldots j_s}_{i_1 \ldots i_r} E^{i_1 \ldots i_r}_{j_1 \ldots j_s}\big) \otimes |\hat{\omega}|^\alpha = t^{j_1 \ldots j_s}_{i_1 \ldots i_r} \big(E^{i_1 \ldots i_r}_{j_1 \ldots j_s} \otimes |\hat{\omega}|^\alpha\big).$$

In conclusion, we have the following result:

Proposition 2. *$T^r_s|\Lambda|^\alpha(E)$ is a real vector space of dimension n^{rs}.*

If $\{e_1, \ldots e_n\}$ is a basis of E, with dual basis $\{e^1, \ldots e^n\} \subset E^*$, we have a basis of $T^r_s|\Lambda|^\alpha(E)$ given by:

$$\left\{\big(e^{a_1} \otimes \cdots \otimes e^{a_r} \otimes e_{b_1} \otimes \cdots \otimes e_{b_s}\big) \otimes |e^1 \wedge \cdots \wedge e^n|^\alpha\right\}_{a_i, b_j = 1, \ldots, n}.$$

Given a basis of E, the components $A^{b_1 \ldots b_s}_{a_1 \ldots a_r}$ of any tensor density A in this basis can be found easily by evaluating with the corresponding elements:

$$A^{b_1 \ldots b_s}_{a_1 \ldots a_r} = A(e_{a_1}, \ldots, e_{a_r}; e^{b_1}, \ldots, e^{b_s}; e_1, \ldots, e_n).$$

Since we have only defined the product $\otimes$ between tensors and α-densities, we have written expressions of the form $(t \otimes s) \otimes \rho$ to indicate that the product between tensors has preference. From now on, we will write, directly, $t \otimes s \otimes \rho$, since there can be no possible confusion: the α-density is always the last element of the product.

3. The Vector Bundle of Tensor Densities

At each point, the tangent spaces $T_p\mathcal{M}$ are the fundamental vector spaces associated with smooth manifolds. We now study the collection of all tensor densities defined on each $T_p\mathcal{M}$, and give it the structure of a vector bundle.

Let $\mathcal{M}$ be a smooth manifold $\mathcal{M}$ of dimension n which is Hausdorff and second-countable (needed, later, for the existence of partitions of unity, see, e.g., [14]). We can

consider, at every point $p \in \mathcal{M}$, the vector space $T^r_s|\Lambda|^\alpha(T_p\mathcal{M})$. Then, we define the bundle of tensor densities of weight α in $\mathcal{M}$ as:

$$T^r_s|\Lambda|^\alpha\mathcal{M} := \bigsqcup_{p\in\mathcal{M}} T^r_s|\Lambda|^\alpha(T_p\mathcal{M}).$$

As usual, there is a projection $\pi : T^r_s|\Lambda|^\alpha\mathcal{M} \to \mathcal{M}$, that sends $A \in T^r_s|\Lambda|^\alpha(T_p\mathcal{M})$ to p.

Proposition 3. *$\pi : T^r_s|\Lambda|^\alpha\mathcal{M} \to \mathcal{M}$ is a smooth vector bundle over M, of dimension n^{rs}.*

Proof. In order to prove this proposition, we will rely on "the vector bundle chart lemma", Lemma 10.6 of [1]. If we can show that its conditions (i), (ii) and (iii) are satisfied, we may conclude that the set $T^r_s|\Lambda|^\alpha\mathcal{M}$ becomes the total manifold of a smooth vector bundle (in particular, the proof of the lemma also defines a unique topology for the set that differs from the disjoint union topology).

The open sets of the atlas of $\mathcal{M}$, written as $\{(\mathcal{U}_a, \varphi_a)\}$, give an open cover of $\mathcal{M}$ (condition (i) in Lemma 10.6 of [1]). We will denote the corresponding natural basis for $T_p\mathcal{M}$, at every $p \in \mathcal{U}_a$, as $\{\frac{\partial}{\partial x^i}\big|_p\}$.

Then, for any $A \in \pi^{-1}(\mathcal{U}_a)$, such that $\pi(A) = p$, we can write its expression in the corresponding basis for $T^r_s|\Lambda|^\alpha(T_p\mathcal{M})$ as:

$$A = A^{j_1\cdots j_s}_{i_1\ldots i_r} dx^{i_1}|_p \otimes \cdots \otimes dx^{i_r}|_p \otimes \frac{\partial}{\partial x^{j_1}}\bigg|_p \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}}\bigg|_p \otimes \left|dx^1|_p \wedge \cdots \wedge dx^n|_p\right|^\alpha.$$

To avoid cumbersome expressions, we will denote the elements of this basis as $|D^{a_1\ldots a_r}_{b_1\ldots b_s}x_p|^\alpha$. In particular, we will also write: $|Dx_p|^\alpha := \left|dx^1|_p \wedge \cdots \wedge dx^n|_p\right|^\alpha$.

For any chart $\mathcal{U}_a$, we can define the map:

$$\begin{aligned} \phi_a : \pi^{-1}(\mathcal{U}_a) &\longrightarrow \mathcal{U}_a \times \mathbb{R}^{n^{rs}} \\ A \mapsto \phi_a(A) &:= (\pi(A); A^{j_1\ldots j_s}_{i_1\ldots i_r}). \end{aligned}$$

If we restrict ϕ_a to some $\pi^{-1}(p)$, it basically sends any tensor density of $T^r_s|\Lambda|^\alpha(T_p\mathcal{M})$ to its n^{rs} coefficients. This is a linear isomorphism, so ϕ_a is also a bijection. With this, we have verified, in our case, condition (ii) of Lemma 10.6 of [1].

To check condition (iii) of Lemma 10.6 of [1], we must consider, for any two charts $(\mathcal{U}_a, \varphi_a)$ and $(\mathcal{U}_b, \varphi_b)$, the composition:

$$\phi_a \circ \phi_b^{-1} : (\mathcal{U}_a \cap \mathcal{U}_b) \times \mathbb{R}^{n^{rs}} \longrightarrow (\mathcal{U}_a \cap \mathcal{U}_b) \times \mathbb{R}^{n^{rs}}.$$

If we write the coordinate basis of $(\mathcal{U}_a, \varphi_a)$ and $(\mathcal{U}_b, \varphi_b)$ by $\{\frac{\partial}{\partial x^i}\}$ and $\{\frac{\partial}{\partial y^i}\}$, respectively, then its relation to the dual bases is

$$dy^c\left(\frac{\partial}{\partial x^d}\right) = \frac{\partial y^c}{\partial x^d}, \qquad \frac{\partial}{\partial y^c}\left(dx^d\right) = \frac{\partial x^d}{\partial y^c},$$

and so on, where $\left(\frac{\partial y^i}{\partial x^j}\right)$ is the Jacobian matrix of the coordinate change from x^i to y^i (i for the row, j for the column). In each point p, it can be thought of as the matrix representation of a linear map $\Psi : T_p\mathcal{M} \to T_p\mathcal{M}$, $\frac{\partial}{\partial y^c}\Big|_p \mapsto \frac{\partial}{\partial x^c}\Big|_p = \frac{\partial y^i}{\partial x^c}\Big|_{\varphi_a(p)} \frac{\partial}{\partial y^i}\Big|_p$, with determinant function $\Delta := \det\left(\frac{\partial y^i}{\partial x^j}\right)$.

For a fixed $(p, \tilde{A}^{j_1\ldots j_s}_{i_1\ldots i_r}) \in (\mathcal{U}_a \cap \mathcal{U}_b) \times \mathbb{R}^{n^{rs}}$, we will have that:

$$(\phi_b)^{-1}(p, \tilde{A}^{j_1\ldots j_s}_{i_1\ldots i_r}) = \tilde{A}^{j_1\ldots j_s}_{i_1\ldots i_r} \cdot |D^{a_1\ldots a_r}_{b_1\ldots b_s}y_p|^\alpha =: A.$$

To find the image of A by ϕ_a, we need to find the coefficients of this density in the basis given by $|D^{a_1\dots a_r}_{b_1\dots b_s} x_p|^\alpha$. We have seen that these can be computed as

$$A^{b_1\dots b_s}_{a_1\dots a_r} = A\left(\frac{\partial}{\partial x^{a_1}}\Big|_p, \dots, \frac{\partial}{\partial x^{a_r}}\Big|_p; dx^{b_1}|_p, \dots, dx^{b_s}|_p; \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right).$$

Performing this evaluation with the coefficients given by the y coordinates, we obtain:

$$A^{b_1\dots b_s}_{a_1\dots a_r} = \tilde{A}^{j_1\dots j_s}_{i_1\dots i_r} \frac{\partial y^{i_1}}{\partial x^{a_1}}\Big|_{\varphi_a(p)} \cdot \dots \cdot \frac{\partial y^{i_r}}{\partial x^{a_r}}\Big|_{\varphi_a(p)} \cdot \frac{\partial x^{b_1}}{\partial y^{j_1}}\Big|_{\varphi_b(p)} \cdot \dots \cdot \frac{\partial x^{b_s}}{\partial y^{j_s}}\Big|_{\varphi_b(p)} \cdot |Dy|^\alpha\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right).$$

In the above expression we have dropped the $|_p$ dependence, for the sake of readability. For the evaluation of the density part, we obtain:

$$\begin{aligned} |Dy_p|^\alpha\left(\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right) &= |Dy_p|^\alpha\left(\Psi\frac{\partial}{\partial y^1}\Big|_p, \dots, \Psi\frac{\partial}{\partial y^n}\Big|_p\right) \\ &= |\Delta(p)|^\alpha \cdot |Dy_p|^\alpha\left(\frac{\partial}{\partial y^1}\Big|_p, \dots, \frac{\partial}{\partial y^n}\Big|_p\right) = |\Delta(p)|^\alpha. \end{aligned}$$

Putting everything together, we obtain:

$$A^{b_1\dots b_s}_{a_1\dots a_r} = \tilde{A}^{j_1\dots j_s}_{i_1\dots i_r} \frac{\partial y^{i_1}}{\partial x^{a_1}}\Big|_{\varphi_a(p)} \cdot \dots \cdot \frac{\partial y^{i_r}}{\partial x^{a_r}}\Big|_{\varphi_a(p)} \cdot \frac{\partial x^{b_1}}{\partial y^{j_1}}\Big|_{\varphi_b(p)} \cdot \dots \cdot \frac{\partial x^{b_s}}{\partial y^{j_s}}\Big|_{\varphi_b(p)} \cdot |\Delta(p)|^\alpha. \tag{2}$$

In conclusion, we have that $\varphi^\alpha_a \circ (\varphi^\alpha_b)^{-1}(p, \tilde{A}^{j_1\dots j_s}_{i_1\dots i_r}) = \left(p, \tau_{ab}(p)(\tilde{A}^{j_1\dots j_s}_{i_1\dots i_r})\right)$, where this τ_{ab} will be a map $\tau_{ab} : \mathcal{U}_a \cap \mathcal{U}_b \to End(\mathbb{R}^{n^{rs}})$, that sends p to the linear map $\tilde{A}^{b_1\dots b_s}_{a_1\dots a_r} \mapsto A^{b_1\dots b_s}_{a_1\dots a_r}$.

To use the aforementioned Lemma 10.6 in [1], we still need to show that τ_{ab} is smooth, and that its image τ_{ab} is non-singular.

Since $\tau_{ab}(p)$ is a linear map, it is determined by its coefficients, and it will be smooth if these coefficients are smooth. Here, $\frac{\partial y^c}{\partial x^d}\big|_{\varphi_a}$ and $\frac{\partial x^c}{\partial y^d}\big|_{\varphi_b}$ are smooth, given that the coordinate functions x^i and y^i are smooth. $|\Delta(p)|^\alpha : \mathcal{U}_a \cap \mathcal{U}_b \to \mathbb{R}$ is also smooth, because the determinant of the Jacobian of the coordinate change is never zero, and because its coefficients are $\frac{\partial y^i}{\partial x^j}\big|_{\varphi(p)}$, which are smooth functions. It follows that τ_{ab} is indeed smooth.

We can interpret $\tau_{ab}(p)$ as the composition of two linear maps. One is multiplying by $|\Delta(p)|^\alpha \neq 0$, which is a linear isomorphism. The other is sending:

$$\tilde{A}^{b_1\dots b_s}_{a_1\dots a_r} \mapsto \tilde{A}^{j_1\dots j_s}_{i_1\dots i_r} \frac{\partial y^{i_1}}{\partial x^{a_1}}\Big|_{\varphi_a(p)} \cdot \dots \cdot \frac{\partial y^{i_r}}{\partial x^{a_r}}\Big|_{\varphi_a(p)} \cdot \frac{\partial x^{b_1}}{\partial y^{j_1}}\Big|_{\varphi_b(p)} \cdot \dots \cdot \frac{\partial x^{b_s}}{\partial y^{j_s}}\Big|_{\varphi_b(p)}.$$

This is exactly a change of basis for the vector space of tensors on $T_p\mathcal{M}$, so it is necessarily an isomorphism too. With this, we have verified the last condition (iii) in Lemma 10.6 of [1]. The conclusion is that $T^r_s|\Lambda|^\alpha\mathcal{M}$ has a unique topology and a smooth differentiable structure which makes it the total manifold of a vector bundle. □

We can now consider smooth sections of the bundle π, that is, smooth maps $T : \mathcal{M} \to T^r_s|\Lambda|^\alpha\mathcal{M}$ such that $\pi \circ T = Id_{\mathcal{M}}$. To keep the analogy with tensor fields or vector fields, we will call them (r,s)-*tensor density fields of weight* α, and denote their set as $\mathcal{E}^r_s(\alpha)(\mathcal{M})$, or just $\mathcal{E}^r_s(\alpha)$.

Now that we have established in the above proof the coordinate transformation laws for tensor densities in (2), we can comment on how they appear in the literature. The main focus of the literature on this topic is on the case where $r = s = 0$. Tensor densities of this

type are sometimes referred as *densities* ([13]), *scalar densities* ([7]), *odd scalar densities* ([6]), or *relative scalars* ([5]). In [8] they are called *projective densities* (of weight α). To be coherent with our chosen notation for tensor densities, we will call them here em α-density fields, and denote their set simply by $\mathcal{E}(\alpha)$. The case where the weight $\alpha = 1$ is clearly that of *volume elements*, necessary for the integration on non-orientable manifolds (see, e.g., Corollary 7–8 in [6]).

Given any coordinate chart $(\mathcal{U}, (x^i))$, we may consider the following local sections of $T^r_s|\Lambda|^\alpha\mathcal{M}$ over $\mathcal{U}$:

$$|D^{a_1\dots a_r}_{b_1\dots b_s}x| : \mathcal{U} \longrightarrow T^r_s|\Lambda|^\alpha\mathcal{M}$$
$$p \mapsto |D^{a_1\dots a_r}_{b_1\dots b_s}x_p|.$$

At every p, these represent a basis of $T^r_s|\Lambda|^\alpha(T_p\mathcal{M})$, so that they give a local frame over $\mathcal{U}$, and one can easily verify that it is actually smooth. Therefore, a map $T : \mathcal{M} \to T^r_s|\Lambda|^\alpha\mathcal{M}$, such that $\pi \circ T = Id_{\mathcal{M}}$, is a smooth section over $\mathcal{U}$ if and only if the components in this local frame are smooth (see Proposition 10.22 in [1]). Given that $\mathcal{M}$ is completely covered by coordinate charts, we can summarize this in the following Definition and Lemma:

Definition 3. *Let $T : \mathcal{M} \to T^r_s|\Lambda|^\alpha\mathcal{M}$ be a map such that $A(p) \in T^r_s|\Lambda|^\alpha(T_p\mathcal{M})$. Given a coordinate chart $(\mathcal{U}, (x^i))$ of $\mathcal{M}$, the coefficients of T in this chart are the n^{rs} maps:*

$$T^{b_1\dots b_s}_{a_1\dots a_r} : \mathcal{U} \to \mathbb{R}$$

such that $T(p) = T^{j_1\dots j_s}_{i_1\dots i_r}(p)|D^{i_1\dots i_r}_{j_1\dots j_s}x_p|^\alpha$.

Lemma 2. *T is a smooth tensor density field of weight α, if and only if, for every coordinate chart of $\mathcal{M}$, the coefficients of T are smooth.*

In view of the discussion in the proof of Proposition 3, we will have that, if $\tilde{T}^{j_1\dots j_s}_{i_1\dots i_r}$ are the coefficients of T in another chart with coordinates (y^i):

$$T^{b_1\dots b_s}_{a_1\dots a_r} = \tilde{T}^{j_1\dots j_s}_{i_1\dots i_r}\frac{\partial y^{i_1}}{\partial x^{a_1}}\cdot\ldots\cdot\frac{\partial y^{i_r}}{\partial x^{a_r}}\cdot\frac{\partial x^{b_1}}{\partial y^{j_1}}\cdot\ldots\cdot\frac{\partial x^{b_s}}{\partial y^{j_s}}\cdot\left|\det\left(\frac{\partial y^i}{\partial x^j}\right)\right|^\alpha. \tag{3}$$

This law of transformation under a change of coordinates is actually the definition by which tensor densities (or α-densities) are introduced in most sources [3–5,7].

We also have that:

$$|D^{a_1\dots a_r}_{b_1\dots b_s}y|^\alpha = \frac{\partial y^{a_1}}{\partial x^{i_1}}\cdot\ldots\cdot\frac{\partial y^{a_r}}{\partial x^{i_r}}\cdot\frac{\partial x^{j_1}}{\partial y^{b_1}}\cdot\ldots\cdot\frac{\partial x^{j_s}}{\partial y^{b_s}}\cdot\left|\det\left(\frac{\partial y^i}{\partial x^j}\right)\right|^\alpha\cdot|D^{i_1\dots i_r}_{j_1\dots j_s}x|^\alpha.$$

In the case of α-density fields, we obtain:

$$|Dy|^\alpha = \left|\det\left(\frac{\partial y^i}{\partial x^j}\right)\right|^\alpha\cdot|Dx|^\alpha. \tag{4}$$

In the vector space set-up, we had defined a product $\otimes : T^r_s(E)\times|\Lambda|^\alpha(E) \to T^r_s|\Lambda|^\alpha(E)$. We can extend this product easily to sections $\otimes : \mathfrak{T}^r_s(\mathcal{M})\times\mathcal{E}(\alpha)(\mathcal{M}) \to \mathcal{E}^r_s(\alpha)(\mathcal{M})$, by defining:

$$(T\otimes\delta)(p) := T(p)\otimes\delta(p), \qquad \forall\, T\in\mathfrak{T}^r_s(\mathcal{M}), \quad \delta\in\mathcal{E}(\alpha)(\mathcal{M}).$$

It will have the same algebraic properties as those in (1), mutatis mutandis. The only thing we need to verify is that the element obtained is actually a smooth section.

By definition, the coefficients of $T \otimes \delta$ will be given by:

$$(T \otimes \delta)^{b_1 \dots b_s}_{a_1 \dots a_r} = T\left(\frac{\partial}{\partial x^{a_1}}, \dots, \frac{\partial}{\partial x^{a_r}}; dx^{b_1}, \dots, dx^{b_s}\right) \cdot \delta\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right).$$

These terms are exactly the coefficients of the tensor field T and the α-density field δ in the corresponding chart, which are smooth because both T and δ are smooth sections. It follows that $T \otimes \delta$ is also smooth.

Finally, recall that, according to Lemma 1, every tensor density can be expressed as the product of a tensor and a density. This will not always be the case for tensor density fields, in principle. Indeed, the proof of the Lemma relied on the choice of a basis for the vector space. There is in general no "smooth" way of choosing a basis for every $T_p\mathcal{M}$, therefore we cannot conclude that every tensor density field can be decomposed as the product of a tensor field and a density field. Locally on a coordinate open $\mathcal{U}$, however, we can perform the decomposition, since any smooth local section T can be written as $T = T^{j_1 \dots j_s}_{i_1 \dots i_r} |D^{i_1 \dots i_r}_{j_1 \dots j_s} x|^\alpha$, which means that:

$$T = \left(T^{j_1 \dots j_s}_{i_1 \dots i_r} dx^{i_1} \otimes \dots \otimes dx^{i_r} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_s}}\right) \otimes \left|dx^1 \wedge \dots \wedge dx^n\right|^\alpha.$$

4. Existence of α-Density Fields

For each meaningful definition of new concept, it is important to show that it actually exists in a natural context. In this section we will first prove that a strictly positive density field of any weight can be constructed on any general manifold. Then, we consider the case of a (pseudo-) Riemannian manifold. In that case, it is well-known that a one-density field can be defined by means of the determinant of the metric.

Let us take as an open cover of $\mathcal{M}$ the coordinate open sets $\{\mathcal{U}_a\}_{a \in A}$ of the maximal atlas of $\mathcal{M}$, with a subordinate smooth partition of unity $\{\psi_a\}_{a \in A}$. For every p in some fixed $\mathcal{U}_a$, we have already defined the α-density in $T_p\mathcal{M}$ that is induced by the coordinate basis: we denote it by $|D(x_a)_p|^\alpha$. With this, we can now define the following map:

$$\begin{aligned} \varepsilon : \mathcal{M} &\longrightarrow |\Lambda|^\alpha \mathcal{M} \\ p \mapsto \varepsilon(p) &:= \sum_{a \in A} \psi_a(p) \cdot |D(x_a)_p|^\alpha. \end{aligned}$$

The above sum makes sense. Indeed, if we were to evaluate it on a small enough neighborhood of any $p \in \mathcal{M}$, only a finite number of $a \in A$ would have $\psi_a \neq 0$. Furthermore, as $\mathrm{supp}(\psi_a) \subset \mathcal{U}_a$, $\psi_a \neq 0$ means that $p \in \mathcal{U}_a$.

For some given $p \in \mathcal{M}$, let us fix now $a_0 \in A$ such that the aforementioned small enough neighborhood of p is contained in $\mathcal{U}_{a_0}$. Let us define $\tilde{A}$ as the finite subset of indices for which ψ_a can be non-zero. Then, we can compute the coefficient of ε in the chart $(\mathcal{U}_{a_0}, (x^i_{a_0}))$, for any $q \in \mathcal{U}_{a_0}$:

$$\varepsilon_0(q) := \varepsilon\left(\left.\frac{\partial}{\partial (x_{a_0})^1}\right|_q, \dots, \left.\frac{\partial}{\partial (x_{a_0})^n}\right|_q\right) = \sum_{a \in \tilde{A}} \psi_a(q) \cdot |D(x_a)_q|^\alpha \left(\left.\frac{\partial}{\partial (x_{a_0})^1}\right|_q, \dots, \left.\frac{\partial}{\partial (x_{a_0})^n}\right|_q\right).$$

Since for any $a \in \tilde{A}$, either $q \in \mathcal{U}_a$ or $\psi_a(q) = 0$, we can use expression (4) to write:

$$\psi_a(q) \cdot |D(x_a)_q|^\alpha = \psi_a(q) \cdot \left|\det\left(\frac{\partial (x_a)^i}{\partial (x_{a_0})^j}\right)\right|^\alpha |D(x_{a_0})_q|^\alpha.$$

With this, we obtain:

$$\begin{aligned}\varepsilon_0(q) &:= \sum_{a\in\tilde{A}} \psi_a(q)\cdot\left|\det\left(\frac{\partial(x_a)^i}{\partial(x_{a_0})^j}\right)\right|^\alpha |D(x_{a_0})_q|^\alpha\left(\left.\frac{\partial}{\partial(x_{a_0})^1}\right|_q,\ldots,\left.\frac{\partial}{\partial(x_{a_0})^n}\right|_q\right)\\ &= \sum_{a\in\tilde{A}} \psi_a(q)\cdot\left|\det\left(\frac{\partial(x_a)^i}{\partial(x_{a_0})^j}\right)\right|^\alpha.\end{aligned}$$

By the smoothness of ψ_a, its coordinate representation of $\tilde{\psi}_a := \psi_a \circ \varphi_{a_0}^{-1}$ is $\mathscr{C}^\infty$. In the previous section, we saw that also $|\Delta(q)|^\alpha$ is smooth. Therefore, $\tilde{\varepsilon}_0 := \varepsilon_0 \circ \varphi_{a_0}^{-1}$ will be $\mathscr{C}^\infty$. If we now apply Lemma 2, we can conclude that ε is smooth.

Finally, notice that, as $|d_p x^1 \wedge \cdots \wedge d_p x^n|^\alpha$ is a positive-valued vector density for any coordinate basis, and $\psi_a \geq 0$, with always some $\psi_a(p) > 0$ for any $x \in \mathcal{M}$, we have that $\varepsilon(p)$ is a positive-valued density for any $p \in \mathcal{M}$. $\varepsilon(p)$ cannot be identically 0 at any p, for if $\psi_a(p) > 0$, it is necessary that $\varepsilon(p)\left(\frac{\partial}{\partial(x_a)^1}\big|_p,\ldots,\frac{\partial}{\partial(x_a)^n}\big|_p\right) > 0$.

We can summarize this whole discussion in:

Proposition 4. *For any $\alpha \in \mathbb{R}$, there exists at least one smooth α-density field, $\varepsilon \in \mathcal{E}(\alpha)$, such that $\varepsilon(p) \in |\Lambda|^\alpha(T_p\mathcal{M})$ is positive-valued for any $p \in \mathcal{M}$.*

From this it is also easy to obtain, for any smooth manifold $\mathcal{M}$, a smooth tensor density field $A \in \mathcal{E}^r_s(\alpha)$, by taking any smooth r,s-tensor field T, and considering $A = T \otimes \varepsilon$.

Now that we know that every manifold has at least one positive-valued smooth α-density field, it is worth noting the following property:

Proposition 5. *Let $\varepsilon \in \mathcal{E}(\alpha)$ be such that $\varepsilon(p)$ is positive-valued for any $p \in \mathcal{M}$. Let $\delta : \mathcal{M} \to |\Lambda|^\alpha\mathcal{M}$ be such that $\delta(p) \in |\Lambda|^\alpha(T_p\mathcal{M})\ \forall p \in \mathcal{M}$. Then $\delta \in \mathcal{E}(\alpha)$ if and only if there exists $f \in \mathscr{C}^\infty(\mathcal{M})$ such that $\delta = f \cdot \varepsilon$.*

Proof. For each vector space E, $|\Lambda|^\alpha(E)$ is always a one-dimensional vector space. This means that, since $\varepsilon(p)$ is never identically 0, it generates the whole $|\Lambda|^\alpha(T_p\mathcal{M})$ at any point. Then, given any other smooth α-density field $\delta \in \mathcal{E}(\alpha)$, there exists a function $f : \mathcal{M} \to \mathbb{R}$ such that $f(p)\cdot\varepsilon(p) = \delta(p)$.

In any coordinate chart $(\mathcal{U}, (x^i))$, we will have $f(p)\varepsilon_0(p)\cdot|Dx_p|^\alpha = \delta_0(p)\cdot|Dx_p|^\alpha$, which means that for any $p \in \mathcal{U}$, $f(p) = \delta_0(p)/\varepsilon_0(p)$. Because δ is a smooth density by hypothesis, and $\varepsilon_0(p)$ is never 0 for any chart, it follows that f is a smooth function.

Conversely, given a smooth function $f \in \mathscr{C}^\infty(\mathcal{M})$, we can define $\delta : \mathcal{M} \to |\Lambda|^\alpha\mathcal{M}$ as $\delta(p) := f(p)\cdot\varepsilon(p)$. Then, for any coordinates we have $\tilde{\delta}_0(x^1,\ldots,x^n) = (f\circ\varphi^{-1})\tilde{\varepsilon}_0(x^1,\ldots,x^n)$, which is $\mathscr{C}^\infty$. By Lemma 2, δ is smooth. □

From now on, we assume that $(\mathcal{M}, g)$ is a pseudo-Riemannian manifold. In that case, we can define a map:

$$\begin{aligned} g^{\frac{\alpha}{2}} : \mathcal{M} &\longrightarrow |\Lambda|^\alpha\mathcal{M}\\ p &\mapsto g^{\frac{\alpha}{2}}(p) := |\det(g_{ij}(p))|^{\frac{\alpha}{2}}\cdot|Dx_p|^\alpha\end{aligned}$$

where $g_{ij}(p)$ is the matrix representation of g at p, for some chart such that $p \in \mathcal{U}$; and $|Dx_p|^\alpha$ is the α-density induced by the same chart.

To see that this map is well-defined, we should verify that it does not depend on the chosen coordinate chart. For some fixed $p \in \mathcal{M}$, consider two coordinate charts $(\mathcal{U}, \varphi)$ and $(\mathcal{V}, \psi)$ such that $p \in \mathcal{U}\cap\mathcal{V}$. We will write $|Dx_p|^\alpha$, $|Dy_p|^\alpha$ for their respective induced densities; and g_{ab}, $\bar{g}_{ab}$ for the metric matrix at p in each coordinate system. If we define the matrix $J_p := \left(\frac{\partial y^i}{\partial x^j}\big|_{\varphi(p)}\right)$ (i for the rows, j for the columns), their interrelation is

$(g_{ij}) = J_p \cdot (\bar{g}_{ij}) \cdot J_p$, from which $\det(g_{ij}) = \det(J_p)^2 \det(\bar{g}_{ij})$. If we write $\det(J_p)$ as $\Delta(p)$, as before, we have by expression (4) that:

$$|\det(g_{ij})|^{\frac{\alpha}{2}} \cdot |Dx_p|^\alpha = |\Delta(p)^2 \det(\bar{g}_{ij})|^{\frac{\alpha}{2}} \cdot |Dx_p|^\alpha = |\det(\bar{g}_{ij})|^{\frac{\alpha}{2}} \cdot |Dy_p|^\alpha.$$

We can therefore conclude that $g^{\frac{\alpha}{2}}$ is well-defined.

By definition, we have that the coefficient of $g^{\frac{\alpha}{2}}$ in any coordinate chart $(\mathcal{U}, \varphi)$ will be exactly $|\det(g_{ij})|^{\frac{\alpha}{2}}$. When composed with φ^{-1}, this is a $\mathscr{C}^\infty$ function from $\varphi(\mathcal{U}) \subset \mathbb{R}^n$ to $\mathbb{R}$, as the coefficients of a tensor are smooth, the determinant of a matrix is a $\mathscr{C}^\infty$ function over its coefficients, and $\det(g_{ij}) \neq 0$ at every point, by the non-degeneracy of pseudo-Riemannian metrics. It is also trivial to see that $g_p^{\frac{\alpha}{2}} \in |\Lambda|^\alpha(T_p\mathcal{M})$. By Lemma 2, $g^{\frac{\alpha}{2}}$ is smooth, and thus $g^{\frac{\alpha}{2}} \in \mathcal{E}(\alpha)$.

Finally, it is easy to see that $g_p^{\frac{\alpha}{2}}$ is a positive-valued density for every $p \in \mathcal{M}$, since it can be written as $|\det(g_{ij}(p))|^{\frac{\alpha}{2}} |Dx_p|^\alpha$, and since $|Dx_p|^\alpha$ is positive-valued for every $p \in \mathcal{U}$.

We conclude:

Proposition 6. *For any pseudo-Riemannian manifold $(\mathcal{M}, g)$, and any $\alpha \in \mathbb{R}$, there exists a "Riemannian" density $g^{\frac{\alpha}{2}} \in \mathcal{E}(\alpha)$ such that for any coordinate chart $(\mathcal{U}, \varphi)$, $g^{\frac{\alpha}{2}}(p) = |\det(g_{ij}(p))|^{\frac{\alpha}{2}} |Dx_p|^\alpha$. Moreover, $g^{\frac{\alpha}{2}}(p)$ is positive-valued for every $p \in \mathcal{M}$.*

The above Riemannian density appears quite often in the literature, be it in the case where $\alpha = 1$. In the literature on general relativity it appears with a negative signature (i.e., $\det(g) < 0$): $(-\det(g))^{\frac{1}{2}}$ (see [7]). It can also be found in the context of integration of differential forms. For example, in Proposition 7.2.10 of [13] and Proposition 16.45 of [1], the Riemannian density is presented as the density that has value 1 for any orthonormal basis of $T_p\mathcal{M}$ (that is, any basis $\{e_1, \ldots, e_n\} \subset T_p\mathcal{M}$ such that $g_p(e_i, e_j) = \delta_i^j$). For $\alpha = 1$, this is a very useful object, as it permits us to integrate functions over manifolds that are not necessarily oriented (see, e.g., Theorem 16.48 in [1]).

5. Covariant Derivatives for Densities and Tensor Densities

In this section, we will make use of the concept of a covariant derivative on a smooth vector bundle (see, for example, paragraph 6.1.1 in [15]). If the vector bundle is the tangent bundle of $\mathcal{M}$, these are also called affine connections on $\mathcal{M}$, and we will specifically denote a covariant derivative by $\nabla^{\mathcal{M}}$. In the particular case of the vector bundle of tensor densities, we may define them in the following way:

Definition 4. *A covariant derivative operator on the bundle of (r, s)-tensor densities of weight α is a mapping:*

$$\begin{aligned} \nabla : \mathfrak{X}(\mathcal{M}) \times \mathcal{E}_s^r(\alpha) &\longrightarrow \mathcal{E}_s^r(\alpha) \\ (X, T) &\mapsto \nabla(X, T) = \nabla_X T \end{aligned}$$

that satisfies:

- *∇ is an $\mathbb{R}$-bilinear mapping,*
- *∇ is tensorial in its first variable: $\nabla_{fX} T = f \cdot \nabla_X T$,*
- *∇ is a derivation in its second variable: $\nabla_X(f \cdot T) = f \cdot \nabla_X T + X(f) \cdot T$.*

For any vector bundle, covariant derivatives are of local character (see, e.g., Lemma's 4.1 and 4.2 of [16]). This means that the value of $\nabla_X T$ at some $p \in \mathcal{M}$ only depends on the value of $X(p)$, and on the values of T over an arbitrarily small neighborhood of p. As a consequence, a covariant derivative operator can be evaluated over smooth local sections of $\mathcal{E}_s^r(\alpha)$. For such a section T over an open set $\mathcal{U}$, and X a smooth vector field over $\mathcal{U}$, $\nabla_X T$ is a well-defined smooth local section of $\mathcal{E}_s^r(\alpha)$ over $\mathcal{U}$. This local dependence allows us to give a local description of any covariant derivative ∇.

In the first part of this section, we only consider the case of α-density fields and covariant derivatives on $\mathcal{E}(\alpha)$.

If $\mathcal{U}$ is a coordinate chart, with coordinate basis of $T\mathcal{U}$ given by $\{\partial/\partial x^i\}_{i=1}^n$, we know that it induces a local frame $|Dx|^\alpha$ for α-densities. If we set $\nabla_{\partial/\partial x^a}|Dx|^\alpha = \Gamma_a|Dx|^\alpha$, the smooth functions $\Gamma_a : \mathcal{U} \to \mathbb{R}$ are the analogues of the Christoffel symbols, but in the case of α-densities. They completely characterize ∇ in $\mathcal{U}$, and we will call them *density symbols*. For any locally smooth α-density δ:

$$\nabla_X \delta = \nabla_{X^i\partial/\partial x^i}(\delta_0 \cdot |Dx|^\alpha) = X^i\left(\frac{\partial \delta_0}{\partial x^i} + \delta_0\Gamma_i\right)\cdot |Dx|^\alpha. \tag{5}$$

We now compute how these density symbols change for different coordinate charts. Given two coordinate charts $\left(\mathcal{U}, (x^i)\right)$ and $\left(\mathcal{V}, (y^i)\right)$, we have two local frames of $\mathcal{E}(\alpha)$ given by $|Dx|^\alpha$ and $|Dy|^\alpha$. Expression (4) shows that the transition matrix from $|Dx|^\alpha$ to $|Dy|^\alpha$ is just $|\det(\partial y^i/\partial x^j)|^{-\alpha} = |\Delta|^{-\alpha}$. In, e.g., [17,18], we may find the transformation rule for linear connections on a general vector bundle. When applied to our current vector bundle of densities, the relation between the density symbols $\tilde{\Gamma}_a$ and Γ_b of the charts $\left(\mathcal{U}, (x^i)\right)$ and $\left(\mathcal{V}, (y^i)\right)$, respectively, becomes

$$\tilde{\Gamma}_a = |\Delta|^\alpha \frac{\partial(|\Delta|^{-\alpha})}{\partial x^a} + \frac{\partial y^k}{\partial x^a}\Gamma_k, \quad a = 1, \ldots, n. \tag{6}$$

We will use the following result:

Lemma 3. *Let $A(t)$ be a matrix function with coefficients $A^i_j(t) : I \subset \mathbb{R} \to \mathbb{R}$ (i rows, j columns), smooth on some t_0. Then, if $\det\left(A(t_0)\right) \neq 0$, we note by $\hat{A}(t_0)$ the inverse matrix of $A(t_0)$, and $\hat{A}^i_j(t_0)$ for its coefficients.*

Then, $\left|\det(A(t))\right|^\alpha$ is differentiable at t_0 for any $\alpha \in \mathbb{R}$, and:

$$\frac{d}{dt}\left(\left|\det(A(t))\right|^\alpha\right)\Big|_{t_0} = \alpha\left|\det(A(t_0))\right|^\alpha \hat{A}^i_j(t_0)\frac{dA^j_i}{dt}\Big|_{t_0}$$

The proof of this property is immediate and it essentially relies on Jacobi's formula for the determinant:

$$\frac{d}{dt}\det(A(t)) = \det(A(t))\mathrm{tr}\left(A^{-1}(t)\cdot\frac{dA}{dt}\right).$$

By this Lemma, we have that:

$$\begin{aligned}\frac{\partial|\Delta|^{-\alpha}}{\partial y^a} &= \frac{\partial}{\partial y^a}\left(\left|\det\left(\frac{\partial x^i}{\partial y^j}\right)\right|^{-\alpha}\right) = -\alpha\left|\det\left(\frac{\partial x^i}{\partial y^j}\right)\right|^{-\alpha}\frac{\partial y^k}{\partial x^l}\frac{\partial}{\partial y^a}\left(\frac{\partial x^l}{\partial y^k}\right)\\ &= -\alpha|\Delta|^{-\alpha}\frac{\partial y^k}{\partial x^l}\frac{\partial^2 x^l}{\partial y^a \partial y^k}.\end{aligned}$$

After plugging this in expression (6), we finally obtain:

$$\tilde{\Gamma}_a = -\alpha\frac{\partial y^k}{\partial x^l}\frac{\partial^2 x^l}{\partial y^a \partial y^k} + \frac{\partial y^k}{\partial x^a}\Gamma_k, \qquad a = 1, \ldots, n. \tag{7}$$

It is well-known that an affine connection $\nabla^{\mathcal{M}}$ on $\mathcal{M}$ (or, equivalently, a covariant derivative of vector fields) induces a covariant derivative operator on tensor fields (also denoted here by $\nabla^{\mathcal{M}}$). We could therefore ask whether this remains the case for densities and tensor densities. In Proposition 9 below, we will show that to answer this question, it is, in fact, enough to find a coherent extension of the affine connection $\nabla^{\mathcal{M}}$ to a covariant derivative ∇ for α-densities $\mathcal{E}(\alpha)$. The reason is that, locally, any tensor density can be

written as the product of a tensor and an α-density. We will show at the end of the paper that we can define an extension ∇ for tensor densities by imposing that it should satisfy the rule $\nabla(T \otimes \delta) = \nabla^{\mathcal{M}}(T) \otimes \delta + T \otimes \nabla(\delta)$.

First, we consider, again, only covariant derivatives for densities. This is not a very common topic in the literature, but some constructions of an extension can be found, especially in older sources ([3,5]). There, the extension is deduced from a comparison of the law of transformation for the Christoffel symbols with the one for densities (3), and it is described in coordinates. Given an affine connection $\nabla^{\mathcal{M}}$ with Christoffel symbols Γ^a_{bc}, the induced covariant derivative ∇ for densities is defined so that, locally, (in the formalism that we have presented here so far):

$$\nabla_{\partial/\partial x^a} |Dx|^\alpha = -\alpha \Gamma^i_{ai} \cdot |Dx|^\alpha, \tag{8}$$

or: $\Gamma_a = -\alpha \Gamma^i_{ai}$ in the notations above. This expression may not seem to be intuitive, at first. However, it is possible to give an interpretation in light of the relation between α-densities and n-forms.

We consider the locally smooth n-form $\omega := (dx^1 \wedge \cdots \wedge dx^n)$. With the usual extension of $\nabla^{\mathcal{M}}$ to tensor fields, we have: $\nabla^{\mathcal{M}}_{\partial/\partial x^a} \omega = -\Gamma^i_{ai} \omega$. With these, we could rewrite expression (8)—be it very informally—as:

$$\nabla_{\partial/\partial x^a} |\omega|^\alpha = -\alpha \frac{\nabla_{\partial/\partial x^a} \omega}{\omega} \cdot |\omega|^\alpha = \pm \alpha \cdot |\omega|^{\alpha - 1} \cdot |\nabla_{\partial/\partial x^a} \omega|.$$

This expression is, of course, not well-defined in any way, but it suggests that the definition in expression (8) boils down to a derivative that, more or less, follows the usual rules for the differentiation of a power. In what follows, we will use this intuition to define a natural extension of $\nabla^{\mathcal{M}}$ to α-densities in a formal way, and show later that it locally yields Equation (8).

However, if we want to extend this to general n-forms, a difficulty may arise. We recall here Lemma 1, which tells us that alternating $(n, 0)$-tensors can be used to construct densities. Similarly, if we are given a locally smooth differential n-form ω over some open set $\mathcal{U} \subset \mathcal{M}$, we can consider $|\omega|^\alpha : \mathcal{U} \to |\Lambda|^\alpha \mathcal{M}$ defined as $|\omega|^\alpha(p) = |\omega_p|^\alpha$. If we consider a coordinate chart $(\mathcal{U}, (x^i))$, where we have that $\omega = \omega_0 dx^1 \wedge \cdots \wedge dx^n$, it is obvious that $|\omega|^\alpha = |\omega_0|^\alpha |Dx|^\alpha$. But $|\omega_0|^\alpha$ will not be smooth unless $\omega_0 \neq 0$! If we exclude that case, the coefficients of $|\omega|^\alpha$ will always be smooth, and by Lemma 2, $|\omega|^\alpha$ will be a smooth section of the bundle of α-densities.

In view of these considerations, we pose the following proposition and definition:

Proposition 7. *Let $\nabla^{\mathcal{M}}$ be an affine connection on a smooth manifold $\mathcal{M}$, with its usual extension to tensor fields. For any $\alpha \in \mathbb{R}$, there exists a unique covariant derivative ∇ for $\mathcal{E}(\alpha)$, which is such that for any $p \in \mathcal{M}$, $X \in \mathfrak{X}(\mathcal{M})$ and non-zero smooth n-form ω over a neighborhood $\mathcal{V}$ of p, we have*

$$(\nabla_X |\omega|^\alpha)_p = \alpha \lambda \cdot |\omega_p|^\alpha,$$

where $\lambda \in \mathbb{R}$ is such that $(\nabla^{\mathcal{M}}_X \omega)_p = \lambda \omega_p$.

Definition 5. *The natural extension ∇ of $\nabla^{\mathcal{M}}$ to $|\Lambda|^\alpha \mathcal{M}$ is the covariant derivative that is guaranteed by Proposition 7.*

Proof. In order to prove Proposition 7, we will show first that the property completely fixes the density symbols of ∇ to be $\Gamma_a = -\alpha \Gamma^i_{ai}$, with Γ^a_{bc} the Christoffel symbols of $\nabla^{\mathcal{M}}$. Then, we will check that these functions form indeed a well-defined covariant derivative.

Let us suppose first that such a ∇ exists. Let $p \in \mathcal{M}$, and a coordinate chart $(\mathcal{U}, (x^i))$ be such that $p \in \mathcal{U}$. Then, $\omega := dx^1 \wedge \cdots \wedge dx^n$ is a non-zero smooth n-form over $\mathcal{U}$, with $(\nabla^{\mathcal{M}}_X \omega)_p = \lambda \omega_p$ for $\lambda = -X^i(p) \Gamma^i_{ji}(p)$.

Now, let us consider $|\omega|^\alpha = |Dx|^\alpha$, which is a smooth local section of the bundle of α-densities. Given the property in the statement of the proposition, we obtain:

$$(\nabla_X |Dx|^\alpha)_p = \alpha\lambda \cdot |Dx_p|^\alpha = -\alpha X^i(p)\Gamma^i_{ji}(p) \cdot |Dx_p|^\alpha.$$

From this, it follows that the density symbols Γ_a of ∇ are necessarily $\Gamma_a = -\alpha\Gamma^i_{ai}$. In conclusion, if ∇ exists, it is uniquely determined.

Next, we verify that $\Gamma_a = -\alpha\Gamma^i_{ji}(p)$ yields a valid covariant derivative for $\mathcal{E}(\alpha)$. To show this, we need to check first whether it satisfies relation (7) under coordinate changes. This is the case, because the standard transformation rule for Christoffel symbols is

$$\tilde{\Gamma}^a_{bc} = \frac{\partial y^a}{\partial x^k}\frac{\partial^2 x^k}{\partial y^b \partial y^c} + \frac{\partial y^a}{\partial x^k}\frac{\partial x^i}{\partial y^b}\frac{\partial x^j}{\partial y^c}\Gamma^k_{ij}.$$

When we take $a = b = l$ and sum, we obtain

$$\tilde{\Gamma}^l_{bl} = \frac{\partial y^l}{\partial x^k}\frac{\partial^2 x^k}{\partial y^b \partial y^l} + \frac{\partial x^i}{\partial y^b}\Gamma^l_{il},$$

which becomes (7) after multiplying both sides with $-\alpha$.

Let now $X \in \mathfrak{X}(\mathcal{M})$, $\delta \in \mathcal{E}(\alpha)$ and $p \in \mathcal{M}$. We choose a coordinate chart $(\mathcal{U}, (x^i))$ such that $p \in \mathcal{U}$, and thus we have $X = X^i \frac{\partial}{\partial x^i}$ and $\delta = \delta_0 |Dx|^\alpha$ in $\mathcal{U}$. Then, we can define $\left(\hat{\nabla}_X \delta\right)(p)$ in the following way:

$$(\nabla_X \delta)(p) = X^i(p)\left(\left.\frac{\partial \delta_0}{\partial x^i}\right|_p - \alpha\delta_0(p)\Gamma^k_{ik}(p)\right) \cdot |Dx_p|^\alpha.$$

It is easy to check that this operator satisfies the required properties of a covariant derivative in Definition 4. Remark also that $\nabla_X \delta$ is a smooth section by Lemma 2 since its coefficient is smooth when δ and X are. □

We continue this section with yet another property of the extension ∇, which argues in favor of its naturality. The property is valid in the case of a pseudo-Riemannian manifold, which comes equipped with the Levi-Civita connection $\nabla^{\mathcal{M}}$ having $\nabla^{\mathcal{M}}_X g = 0$ for all $X \in \mathfrak{X}(\mathcal{M})$. We have seen in Proposition 6 that any metric g defines, for any $\alpha \in \mathbb{R}$, a positive α-density field, $g^{\alpha/2}$. It is reasonable to expect that the "natural" extension of ∇ also gives $\nabla_X g^{\alpha/2} = 0$. This turns out to be, indeed, true:

Proposition 8. *Given a pseudo-Riemannian manifold $\mathcal{M}$ with Levi-Civita connection $\nabla^{\mathcal{M}}$, the natural extension of $\nabla^{\mathcal{M}}$ to $\mathcal{E}(\alpha)$ is the only covariant derivative for $\mathcal{E}(\alpha)$ such that $\nabla_X g^{\frac{\alpha}{2}} = 0$ for all $X \in \mathfrak{X}(\mathcal{M})$.*

Proof. We only need to show that the property $\nabla_X g^{\frac{\alpha}{2}} = 0$ fixes the density symbols to be exactly $-\alpha\Gamma^i_{ai}$, where Γ^a_{bc} are the Christoffel symbols of the Levi-Civita connection $\nabla^{\mathcal{M}}$.

For any coordinate chart $(\mathcal{U}, \varphi)$, we have $g^{\frac{\alpha}{2}} = |\det(g_{ij})|^{\frac{\alpha}{2}} \cdot |Dx|^\alpha$. We know that, for any $p \in \mathcal{U}$, there exist vector fields $\partial x_a \in \mathfrak{X}(\mathcal{M})$ such that $\partial x_a = \frac{\partial}{\partial x^a}$ in a neighborhood around p. Then, we have that, in this neighborhood of p:

$$\nabla_{\partial x_a} g^{\frac{\alpha}{2}} = \nabla_{\partial x_a}\left(|\det(g_{ij})|^{\frac{\alpha}{2}} \cdot |Dx|^\alpha\right) = \left(\frac{\partial}{\partial x^a}|\det(g_{ij})|^{\frac{\alpha}{2}} + \Gamma_a |\det(g_{ij})|^{\frac{\alpha}{2}}\right) \cdot |Dx|^\alpha.$$

Since g is a pseudo-Riemannian metric, $\det(g_{ij})$ is a non-zero function. Therefore, we can apply Lemma 3:

$$\frac{\partial}{\partial x^a}|\det(g_{ij})|^{\frac{\alpha}{2}} = \frac{\alpha}{2}|\det(g_{ij})|^{\frac{\alpha}{2}} \cdot g^{ij}\frac{\partial g_{ji}}{\partial x^a},$$

which, in turn, gives:

$$\begin{aligned}\nabla_{\partial x_a}(\det(g))^{\frac{\alpha}{2}} &= \left(\frac{\alpha}{2}|\det(g_{ij})|^{\frac{\alpha}{2}}\cdot g^{ij}\frac{\partial g_{ji}}{\partial x^a}+\Gamma_a|\det(g_{ij})|^{\frac{\alpha}{2}}\right)\cdot|Dx|^\alpha\\ &= |\det(g_{ij})|^{\frac{\alpha}{2}}\left(\alpha\frac{1}{2}g^{ij}\frac{\partial g_{ji}}{\partial x^a}+\Gamma_a\right)\cdot|Dx|^\alpha.\end{aligned}$$

After imposing that this is exactly 0, and using $\det(g_{ij})\neq 0$, we obtain:

$$-\alpha\frac{1}{2}g^{ij}\frac{\partial g_{ji}}{\partial x^a}=\Gamma_a.$$

But, using the formula for the Christoffel symbols of the Levi-Civita connection, the symmetry of g, and some changes of indices:

$$\Gamma^i_{ai}=\frac{1}{2}g^{ik}\left(\frac{\partial g_{ak}}{\partial x^i}+\frac{\partial g_{ki}}{\partial x^a}-\frac{\partial g_{ai}}{\partial x^k}\right)=\frac{1}{2}\left(g^{ik}\frac{\partial g_{ak}}{\partial x^i}+g^{ik}\frac{\partial g_{ki}}{\partial x^a}-g^{ki}\frac{\partial g_{ak}}{\partial x^i}\right)=\frac{1}{2}g^{ik}\frac{\partial g_{ki}}{\partial x^a}.$$

In conclusion, we have $\Gamma_a(p)=-\alpha\Gamma^i_{ai}(p)$, as required. □

We end the paper with the extension of an affine connection $\nabla^{\mathcal{M}}$ to tensor densities in general, as we had announced in the beginning of the section.

Proposition 9. *Let $\nabla^{\mathcal{M}}$ be an affine connection on a smooth manifold $\mathcal{M}$. Then, for any $\alpha\in\mathbb{R}$, there exists a unique covariant derivative ∇ for $\mathcal{E}^r_s(\alpha)$ such that, for any $T\in\mathcal{E}^r_s(\alpha)$ that can be decomposed as $T=t\otimes\delta$ (for some $t\in\mathfrak{T}^r_s(\mathcal{M})$ and $\delta\in\mathcal{E}(\alpha)$), we have that:*

$$\nabla_X T=(\nabla^{\mathcal{M}}_X t)\otimes\delta+t\otimes(\nabla_X\delta),\qquad\forall\, X\in\mathfrak{X}(\mathcal{M}),$$

where $\nabla^{\mathcal{M}}$ in the first term represents the natural extension of $\nabla^{\mathcal{M}}$ to tensor fields and ∇ in the second term is the natural extension to densities of Proposition 7.

Proof. The key of this proof is the fact that, although it is not true that any tensor density can be written globally as a product of a tensor and a density, locally it is possible to do so.

Let $T\in\mathcal{E}^r_s(\alpha)$, $p\in\mathcal{M}$ and $X\in\mathfrak{X}(\mathcal{M})$. Then, for some coordinate chart $(\mathcal{U},(x^i))$ such that $p\in\mathcal{U}$, we can write $T=T^{j_1\dots j_s}_{i_1\dots i_r}|D^{i_1\dots i_r}_{j_1\dots j_s}x|^\alpha$, with smooth coefficient functions $T^{b_1\dots b_s}_{a_1\dots a_r}:\mathcal{U}\to\mathbb{R}$.

Let $A\subset\mathcal{U}$ be a closed neighborhood of p. Then, by Lemma 10.12 in [1], there exist smooth global extensions $t\in\mathfrak{T}^r_s(\mathcal{M})$, $\delta\in\mathcal{E}(\alpha)$ such that:

$$t|_A=T^{j_1\dots j_s}_{i_1\dots i_r}dx^{i_1}\otimes\cdots\otimes dx^{i_r}\otimes\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_s}}\qquad\text{and}\qquad\delta|_A=|Dx|^\alpha.$$

It is clear from this definition that $T|_A=(t\otimes\delta)|_A$. Then, if ∇ exists for $\mathcal{E}^r_s(\alpha)$ (with the properties of the Proposition), since $(\nabla_X T)(p)$ only depends on the behavior of T in a neighborhood of p, we must have that:

$$(\nabla_X T)(p)=(\nabla_X(t\otimes\delta))(p)=(\nabla^{\mathcal{M}}_X t)(p)\otimes\delta(p)+t(p)\otimes(\nabla_X\delta)(p).$$

This proves the uniqueness of ∇, in case it exists.

We can now use the above equation to define $(\nabla_X T)(p)$. Then, $\nabla_X T$ does not depend on the particular extensions t and δ, since they are all the same in a neighborhood A of p. To show that $(\nabla_X T)(p)$ is well-defined, we only need to prove that the construction does not depend on the chosen system of coordinates.

If $(\mathcal{V}, (y^i))$ is another coordinate chart such that $p \in \mathcal{V}$, we define the corresponding $\tilde{t}$ and $\tilde{\delta}$ for this chart. Then, after comparing the relation between $\tilde{T}^{j_1\dots j_s}_{i_1\dots i_r}$ in (3) to the coordinate change for tensor fields, we can easily see that:

$$\begin{aligned} t &= T^{j_1\dots j_s}_{i_1\dots i_r} dx^{i_1} \otimes \cdots \otimes dx^{i_r} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}} \\ &= \tilde{T}^{l_1\dots l_s}_{k_1\dots k_r} \frac{\partial y^{k_1}}{\partial x^{i_1}} \cdot \ldots \cdot \frac{\partial y^{k_r}}{\partial x^{i_r}} \cdot \frac{\partial x^{j_1}}{\partial y^{l_1}} \cdot \ldots \cdot \frac{\partial x^{j_s}}{\partial y^{l_s}} \cdot |\Delta(p)|^\alpha dx^{i_1} \otimes \cdots \otimes dx^{i_r} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}} \\ &= \tilde{T}^{l_1\dots l_s}_{k_1\dots k_r} |\Delta(p)|^\alpha dy^{k_1} \otimes \cdots \otimes dy^{k_r} \otimes \frac{\partial}{\partial y^{l_1}} \otimes \cdots \otimes \frac{\partial}{\partial y^{l_s}} \\ &= |\Delta(p)|^\alpha \tilde{t}. \end{aligned}$$

Recall that $|\Delta(p)|^\alpha := \left|\det\left(\frac{\partial y^i}{\partial x^j}\big|_{\varphi(p)}\right)\right|^\alpha$ and that $\left|\det\left(\frac{\partial x^i}{\partial y^j}\big|_{\varphi(p)}\right)\right|^\alpha = |\Delta(p)|^{-\alpha}$. Using this, and the relation between $|Dx|^\alpha$ and $|Dy|^\alpha$, we obtain (when hiding the p dependence for the sake of readability):

$$\begin{aligned} \nabla_X T &= \left(\nabla^{\mathcal{M}}_X t\right) \otimes |Dx|^\alpha + t \otimes (\nabla_X |Dx|^\alpha) \\ &= \left(\nabla^{\mathcal{M}}_X (|\Delta|^\alpha \tilde{t})\right) \otimes |Dx|^\alpha + |\Delta|^\alpha \tilde{t} \otimes \left(\nabla_X \left(|\Delta|^{-\alpha} |Dy|^\alpha\right)\right). \end{aligned}$$

By the properties of the covariant derivative, we obtain:

$$\begin{aligned} \nabla_X T &= \left(\nabla^{\mathcal{M}}_X \tilde{t}\right) |\Delta|^\alpha \otimes |Dx|^\alpha + \tilde{t} \otimes (\nabla_X |Dy|^\alpha) + |\Delta|^\alpha X\left(|\Delta|^{-\alpha}\right) \tilde{t} \otimes |Dy|^\alpha \\ &\qquad + X(|\Delta|^\alpha) \tilde{t} \otimes |Dx|^\alpha \\ &= \left(\nabla^{\mathcal{M}}_X \tilde{t}\right) \otimes |Dy|^\alpha + \tilde{t} \otimes (\nabla_X |Dy|^\alpha) + \left(|\Delta|^\alpha X\left(|\Delta|^{-\alpha}\right) + |\Delta|^{-\alpha} X(|\Delta|^\alpha)\right) \tilde{t} \otimes |Dy|^\alpha \\ &= \left(\nabla^{\mathcal{M}}_X \tilde{t}\right)(p) \otimes |Dy_p|^\alpha + \tilde{t}(p) \otimes (\nabla_X |Dy|^\alpha)(p), \end{aligned}$$

which is the same as the definition of $(\nabla_X T)(p)$, when written in the new coordinate chart.

To finalize the proof, we must verify that $\nabla_X T$ (as we have defined) is a smooth section of $\mathcal{E}^r_s(\alpha)$, and that ∇ fulfills the conditions of Definition 4.

On a neighborhood A of p, $\nabla_X T = (\nabla^{\mathcal{M}}_X t) \otimes \delta + t \otimes (\nabla_X \delta)$. Both $(\nabla^{\mathcal{M}}_X t)$ and $(\nabla_X \delta)$ are smooth, and so are their tensorial products with δ and t. This means that $\nabla_X T$ is necessarily smooth. It is clear that $(\nabla_X T)(p) \in T^r_s |\Lambda|^\alpha(T_p \mathcal{M})$, so that $\nabla_X T$ is indeed a globally smooth section of $T^r_s |\Lambda|^\alpha \mathcal{M}$.

As for the algebraic properties in Definition 4, we can see that they are easily inherited from the natural extension. The third condition, for example, is:

$$\begin{aligned} (\nabla_X fT)(p) &= \left(\nabla^{\mathcal{M}}_X ft\right)(p) \otimes |Dx_p|^\alpha + ft(p) \otimes (\nabla_X |Dx|^\alpha)(p) \\ &= X(f)(p) t(p) \otimes |Dx_p|^\alpha + f(p) \left(\nabla^{\mathcal{M}}_X t\right)(p) \otimes |Dx_p|^\alpha \\ &\qquad + ft(p) \otimes (\nabla_X |Dx|^\alpha)(p) \\ &= X(f)(p) T(p) + f(p)(\nabla_X T)(p) \end{aligned}$$

The other conditions follow analogously, and the proof is complete. □

6. Conclusions

In this paper, we have defined tensor densities of arbitrary weight on a manifold, in a global way. We have shown that they correspond to the objects which appear in the literature under the same name, but are defined by means of coordinate transformation rules. We have constructed the smooth vector bundle of tensor densities and we have

shown that a globally smooth tensor density field exists, as well a globally smooth metric density in case the manifold is endowed with a pseudo-Riemannian metric.

We have defined a extension of the action of an affine connection, from tensor fields to densities, and we have explained in what sense it is a natural extension. We have ended the paper with an equivalent characterization, for the case of a pseudo-Riemannian manifold.

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References

1. Lee, J.M. *Introduction to Smooth Manifolds*, 2nd ed.; Springer: Berlin/Heidelberg, Germany, 2013.
2. Einstein, A. *Zur Allgemeinen Relativitätstehorie*; Verlag der Königlich-Preussischen Akademie der Wissenschaften: Berlin, Germany, 1923; pp. 32–34.
3. Takeno, H. On relative tensors. *J. Sci. Hiroshima Univ. Ser. A* **1934**, *5*, 31–41. [CrossRef]
4. Veblen, O.; Thomas, T.Y. Extensions of relative tensors. *Trans. Am. Math. Soc.* **1924**, *26*, 373–377. [CrossRef]
5. Lovelock, D.; Rund, H. *Tensors, Differential Forms, and Variational Principles*; Dover Publications Inc.: Garden City, NY, USA, 1989.
6. Spivak, M. *A Comprehensive Introduction to Differential Geometry*; Publish or Perish, Inc.: Lombard, IL, USA, 1999; Volume 1.
7. d'Inverno, R. *Introducing Einstein's Relativity*; Oxford University Press: Oxford, UK, 1993.
8. Eastwood, M. Notes on projective differential geometry. In *Symmetries and Overdetermined Systems of Partial Differential Equations. The IMA Volumes in Mathematics and Its Applications*; Springer: New York, NY, USA, 2008; Volume 144, pp. 41–60.
9. Eastwood, M.; Matveev, V. Metric connections in projective differential geometry. In *Symmetries and Overdetermined Systems of Partial Differential Equations. The IMA Volumes in Mathematics and Its Applications*; Springer: New York, NY, USA, 2008; Voume 144, pp. 339–351.
10. George, J. Projective connections and the algebra of densities. *AIP Conf. Proc.* **2008**, *1079*, 142–148.
11. Khudaverdian, H.M. Kaluza-klein theory revisited: Projective structures and differential operators on algebra of densities. *J. Physics Conf. Ser.* **2014**, *496*, 012032. [CrossRef]
12. Matveev, V.S. Geodesically equivalent metrics in general relativity. *J. Geom. Phys.* **2012**, *62*, 675–691. [CrossRef]
13. Abraham, R.; Marsden, J.E.; Ratiu, T. *Manifolds, Tensor Analysis and Applications*, 2nd ed.; Springer: Berlin/Heidelberg, Germany, 2003.
14. Brickell, F.; Clark, R.S. *Differentiable Manifolds*; Van Nostrand: New York, NY, USA, 1970.
15. Szilasi, J.; Lovas, R.L.; Kertész, D.C. *Connections, Sprays and Finsler Structures*; World Scientific Publishing: Hackensack, NJ, USA, 2014.
16. Lee, J.M. *Riemannian Manifolds: An Introduction to Curvature*; Springer: Berlin/Heidelberg, Germany, 1997.
17. Darling, R.W.R. *Differential Forms and Connections*; Cambridge University Press: Cambridge , UK, 1994.
18. Miron, R; Anastasiei, M. *The Geometry of Lagrange Spaces: Theory and Applications*; Springer: Berlin/Heidelberg, Germany, 1994.

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