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Bivariate Length-Biased Exponential Distribution under Progressive Type-II Censoring: Incorporating Random Removal and Applications to Industrial and Computer Science Data

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Abstract: In this paper, we address the analysis of bivariate lifetime data from a length-biased exponential distribution observed under Type II progressive censoring with random removals, where the number of units removed at each failure time follows a binomial distribution. We derive the likelihood function for the progressive Type II censoring scheme with random removals and apply it to the bivariate length-biased exponential distribution. The parameters of the proposed model are estimated using both likelihood and Bayesian methods for point and interval estimators, including asymptotic confidence intervals and bootstrap confidence intervals. We also employ different loss functions to construct Bayesian estimators. Additionally, a simulation study is conducted to compare the performance of censoring schemes. The effectiveness of the proposed methodology is demonstrated through the analysis of two real datasets from the industrial and computer science domains, providing valuable insights for illustrative purposes.

Keywords: bivariate length-biased exponential distribution; farlie-gumbel-morgenstern copula; maximum likelihood estimation; bayesian estimation; progressive type-II censoring; loss function; computer science analysis; bootstrap confidence interval; asymptotic confidence interval

MSC: 62F15; 62F40; 62H05; 62N02; 62P30

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1. Introduction

In numerous life test studies, accurately recording the lifetimes of test units can be challenging. An experimenter may choose to end the life test prematurely to save time or reduce costs. Consequently, the test is deemed censored, meaning the collected data includes precise failure times for the failed units and running times for the non-failed units. For example, accidental breakage of some experimental units or subjects dropping out for personal reasons during clinical trials can occur midway through the trial. In certain life-testing studies involving expensive units, it may be advantageous to remove some units early from the test, allowing their use in other tests. Additionally, in some clinical trials, the survival duration after treatment may span several years, prompting the experimenter to conclude the study before observing the survival durations for all individuals. Censored data emerges in these scenarios when the experimenter does not acquire complete information for all units or individuals under study.

Various types of censoring arise depending on the data collection methodology in a life-testing experiment. Let's examine a life-testing experiment involving n items under examination. Assume that the experiment must be concluded at a predetermined time,

denoted as T. In this scenario, one can only acquire failure times that are less than or equal to T, resulting in what is known as Type-I censored data. Alternatively, instead of predetermining the total time of the experiment, the experimenter may decide to conclude the experiment after observing the first r failures. In such cases, the data are referred to as Type-II censored. The two most common censoring schemes, Type-I and Type-II, have a limitation in that systems can only be removed from the test at the final termination time. However, in many practical applications, systems need to be removed at different points during the test, which is known as progressive censoring. Progressive censoring allows flexibility in the removal of systems throughout the testing process, beyond just the final termination time. For further details, refer to Balakrishnan and Aggarwala [1], Balakrishnan [2], Balakrishnan and Cramer [3], and El-Sherpieny et al. [4].

The copula function has garnered considerable attention as a contemporary area of study in the realm of statistics. Its significance lies in its ability to serve as a powerful and efficient tool for modeling bivariate and multivariate distributions. This versatility finds extensive practical applications across diverse disciplines, including economics, finance, hydrology, computer science, sports, psychology, social sciences, climatology, meteorology, environmental studies, and medical data analysis. A copula is a type of multivariate distribution function in which the individual margins are uniformly distributed on the interval [0, 1]. In this context, we focus solely on the bivariate copula distribution. To explore copulas and their properties further, comprehensive information can be found in the works of Nelsen [5]. Recent contributions to bivariate models incorporating copula functions and their applications can be found in Flores [6], Suzuki et al. [7], Verril et al. [8], Peres et al. [9], Peres et al. [10], Samanthi et al. [11], Elgawad et al. [12], Barakat et al. [13], Qura et al. [14], Fayomi et al. [15,16], Chesneau et al. [17], and Almetwally et al. [18], among others.

In the domain of bivariate censoring, Balakrishnan and Kim [19] laid the foundation by introducing the likelihood function for a bivariate distribution, specifically focusing on Type-II censoring. Subsequently, Balakrishnan and Kim [20,21] extended their work by presenting the likelihood function based on a Progressive Type-II censoring scheme for a bivariate distribution. Kim et al. [22] examined the use of a Type-II censoring scheme in a bivariate context for the estimation of unknown parameters in the bivariate generalized exponential distribution. Bai et al. [23] developed Type-I progressive interval censoring schemes for the Marshall-Olkin bivariate Weibull (MOBW) distribution, focusing on how percentile bootstrap methods impact the construction of confidence intervals for unknown parameters. Aly et al. [24] derived the likelihood function for progressive Type-I censoring and applied it to the Marshall-Olkin bivariate Kumaraswamy distribution. Muhammed [25] derived the likelihood function for a bivariate Weibull distribution, which is part of the bivariate hazard power parameter family of distributions, based on complete, Type-I, and Type-II censored samples.

In the context of progressive Type-II censored samples, EL-Morshedy et al. [26] have explored both maximum likelihood and Bayesian methods for estimating parameters of the bivariate Burr X-G family with various distributions. Their discussion is specifically centered around Type-II censored data.El-Sherpieny et al. [27] acquired Bayesian and non-Bayesian estimations for the parameters of the bivariate generalized Rayleigh distribution, utilizing the Clayton copula function within the framework of progressive Type-II censoring schemes. El-Sherpieny et al. [4] investigated the point and interval estimation of unknown parameters in the Farlie-Gumbel-Morgenstern (FGM) bivariate Weibull distribution based on progressive Type-II censored samples. Additionally, the study introduced two bootstrap confidence intervals. Muhamed [28] developed a generalized likelihood function for the Marshal-Olkin bivariate class of distributions under progressive Type II censoring, applied to the bivariate Dagum distribution, and focused on maximum likelihood estimation for the unknown parameters.

In this paper, we extend the work of Arun et al. [29] by considering the estimation of the bivariate length-biased exponential distribution using the Farlie-Gumbel-Morgenstern

copula (FGM-BLBE) parameters under a progressive Type-II censoring scheme with random removal, based on the maximum likelihood and Bayesian estimation methods. Furthermore, the interval estimation of the FGM-BLBE parameters is obtained using asymptotic and bootstrap confidence intervals under the progressive Type-II censoring scheme. The performance of the proposed estimators is evaluated through extensive simulations. The optimal censoring scheme is proposed based on three distinct optimality criteria: mean squared error (MSE), bias, and length of the confidence interval (L.CI). Ultimately, this effort aims to establish a comprehensive guideline for identifying the most effective censoring scheme, providing valuable insights for statisticians. Applications of two real datasets are introduced to confirm the validity of the model.

The rest of this paper is organized as follows. The description and formulation of the model are provided in Section 2. The maximum likelihood estimation (MLE), along with approximate confidence intervals, is presented in Section 3. Section 4 covers Bayesian estimates and the corresponding highest posterior density (HPD) credible intervals. In Section 5, the construction of confidence intervals for unknown parameters uses both asymptotic and bootstrap methods. In Section 6, the Monte Carlo simulation results and data analysis are presented. Section 7 introduces and analyzes two real datasets to evaluate the model. Finally, concluding remarks are given in Section 8.

2. Description and Formulation of the Model

Consider random variables Y_1 and Y_2 with their joint cumulative distribution function (CDF) denoted as $F_{Y_1,Y_2}(y_1, y_2)$ and their joint probability density function (PDF) as $f_{Y_1,Y_2}(y_1, y_2)$. Additionally, let $F_{Y_1}(y_1)$ and $F_{Y_2}(y_2)$ represent the marginal CDFs of Y_1 and Y_2, respectively, and their marginal PDFs denoted as $f_{Y_1}(y_1)$ and $f_{Y_2}(y_2)$, respectively. Sklar [30] introduced the concept of a copula as a linking function that connects the joint distribution function to its marginal distribution functions using the relationship $F_{Y_1,Y_2}(y_1, y_2) = P(Y_1 \leq y_1, Y_2 \leq y_2) = C(F_{Y_1}(y_1), F_{Y_2}(y_2))$, and the associated joint density is $f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2) c(F_{Y_1}(y_1), F_{Y_2}(y_2))$, where c is the copula density.

In the literature, various types of copulas have been extensively discussed, each possessing distinct properties and characteristics. One prominent example is the Farlie-Gumbel-Morgenstern copula, which belongs to a widely recognized parametric copula family introduced by Gumbel [31]. The joint CDF and PDF of the Farlie-Gumbel-Morgenstern (FGM) copula are provided as follows:

$$C(u, v) = u\, v\, (1 + \theta(1 - u)(1 - v)), \quad -1 \leq \theta \leq 1, \tag{1}$$

and

$$c(u, v) = 1 + \theta(1 - 2u)(1 - 2v), \quad -1 \leq \theta \leq 1, \tag{2}$$

where $u = F_{Y_1}(y_1, \Psi_1)$, $v = F_{Y_2}(y_2, \Psi_2)$, and Ψ_1 and Ψ_2 are vectors of the parameters for the variables Y_1 and Y_2, respectively. The parameter θ serves as a dependence parameter that relies on the underlying random variables, and when set to $\theta = 0$, it represents the independent case, signifying complete absence of dependence between the variables. The FGM copula is thus simple, flexible, and possesses tail dependence, allowing for parametric control over the level of dependence. Furthermore, its analytical properties are advantageous, and its historical significance as one of the earliest introduced copula families contributes to the development and understanding of copula theory.

Dara and Ahmad [32] introduced the length-biased exponential (LBE) model, also referred to as the moment exponential (ME) model, by assigning weights to the exponential (E) model. Their research demonstrated that the LBE distribution offers greater adaptability compared to the E model. The CDF, probability density function (PDF) and hazard rate function (HRF) of the LBE model with scale parameter λ are as follows:

$$F_Y(y; \lambda) = 1 - \left(1 + \frac{y}{\lambda}\right) e^{-\frac{y}{\lambda}}, \quad y > 0, \quad \lambda > 0, \tag{3}$$

$$f_Y(y;\lambda) = \frac{y}{\lambda^2} e^{-\frac{y}{\lambda}}, \quad y > 0, \quad \lambda > 0, \tag{4}$$

and

$$h_Y(y;\lambda) = \frac{y}{\lambda^2\left(1+\frac{y}{\lambda}\right)}. \tag{5}$$

Arun et al. [29] introduced the bivariate length-biased exponential distribution using the Farlie-Gumbel-Morgenstern approach. This distribution, denoted as the FGM-BLBE distribution, serves as a bivariate counterpart to the length-biased exponential distribution.

The CDF, PDF, and HRF of an FGM-BLBE distribution are obtained as follows:

$$\begin{aligned} F_{Y_1,Y_2}(y_1,y_2) = &\left[1-\left(1+\frac{y_1}{\lambda_1}\right)e^{-\frac{y_1}{\lambda_1}}\right]\left[1-\left(1+\frac{y_2}{\lambda_2}\right)e^{-\frac{y_2}{\lambda_2}}\right] \\ &\left\{1+\theta\left(1+\frac{y_1}{\lambda_1}\right)e^{-\frac{y_1}{\lambda_1}}\left(1+\frac{y_2}{\lambda_2}\right)e^{-\frac{y_2}{\lambda_2}}\right\}, \quad y_1,y_2>0, \lambda_1,\lambda_2>0, -1\le\theta\le 1, \end{aligned} \tag{6}$$

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) = &\frac{y_1}{\lambda_1^2}e^{-\frac{y_1}{\lambda_1}}\frac{y_2}{\lambda_2^2}e^{-\frac{y_2}{\lambda_2}} \\ &\left\{1+\theta\left[2\left(1+\frac{y_1}{\lambda_1}\right)e^{-\frac{y_1}{\lambda_1}}-1\right]\left[2\left(1+\frac{y_2}{\lambda_2}\right)e^{-\frac{y_2}{\lambda_2}}-1\right]\right\}, \end{aligned} \tag{7}$$

and

$$h_{Y_1,Y_2}(y_1,y_2) = \frac{f_{Y_1,Y_2}(y_1,y_2)}{S_{Y_1,Y_2}(y_1,y_2)}, \tag{8}$$

where

$$\begin{aligned} S_{Y_1,Y_2}(y_1,y_2) = &\left(1+\frac{y_1}{\lambda_1}\right)e^{-\frac{y_1}{\lambda_1}}\left(1+\frac{y_2}{\lambda_2}\right)e^{-\frac{y_2}{\lambda_2}} \\ &\left\{1+\theta\left[1-\left(1+\frac{y_1}{\lambda_1}\right)e^{-\frac{y_1}{\lambda_1}}\right]\left[1-\left(1+\frac{y_2}{\lambda_2}\right)e^{-\frac{y_2}{\lambda_2}}\right]\right\}, \end{aligned} \tag{9}$$

S is the survival function (SF), which is expressed as $S_{Y_1,Y_2}(y_1,y_2) = C\left(S_{Y_1}(y_1), S_{Y_2}(y_2)\right)$.

The hazard rate function quantifies the probability of an event (such as failure) occurring per unit time, given that the system has survived up until that moment. By analyzing variations in the hazard rate over time or across different values of x and y, we can evaluate the dynamic behavior and risk profile of the FGM-BLBE distribution, as illustrated in Figures 1 and 2.

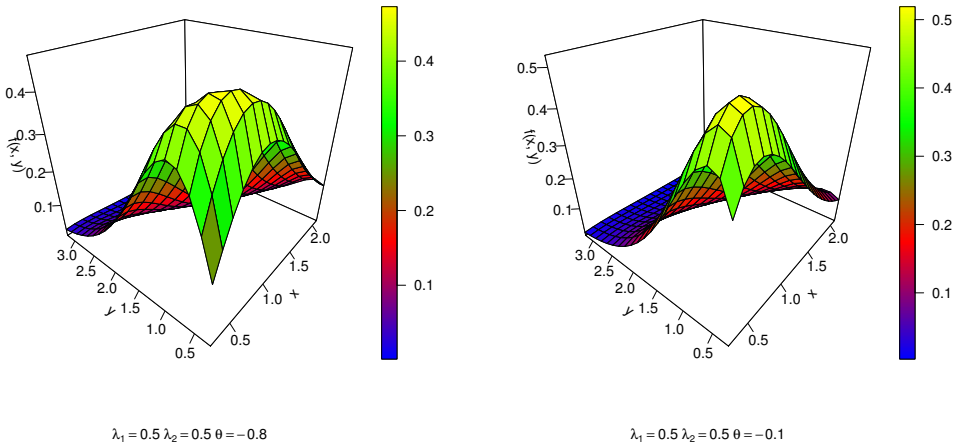


Figure 1. *Cont.*

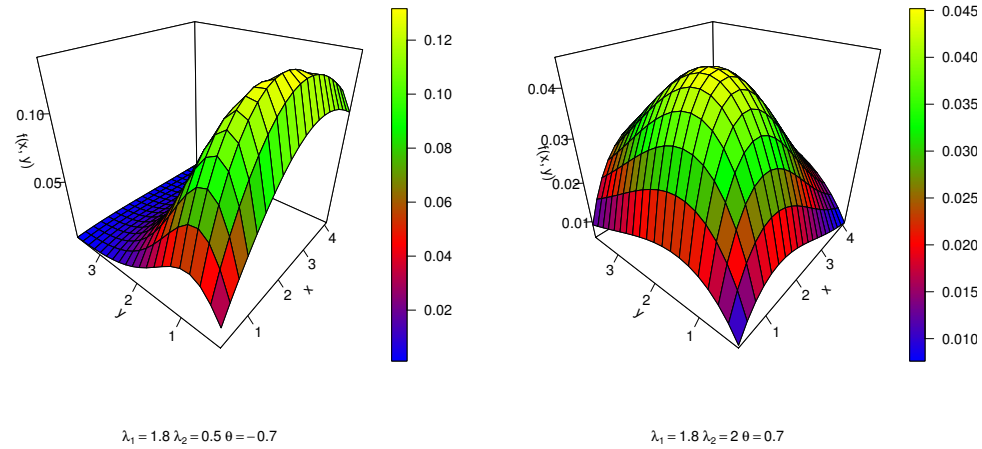


Figure 1. Plots of joint density of FGM-BLBE distribution for some parameter values.

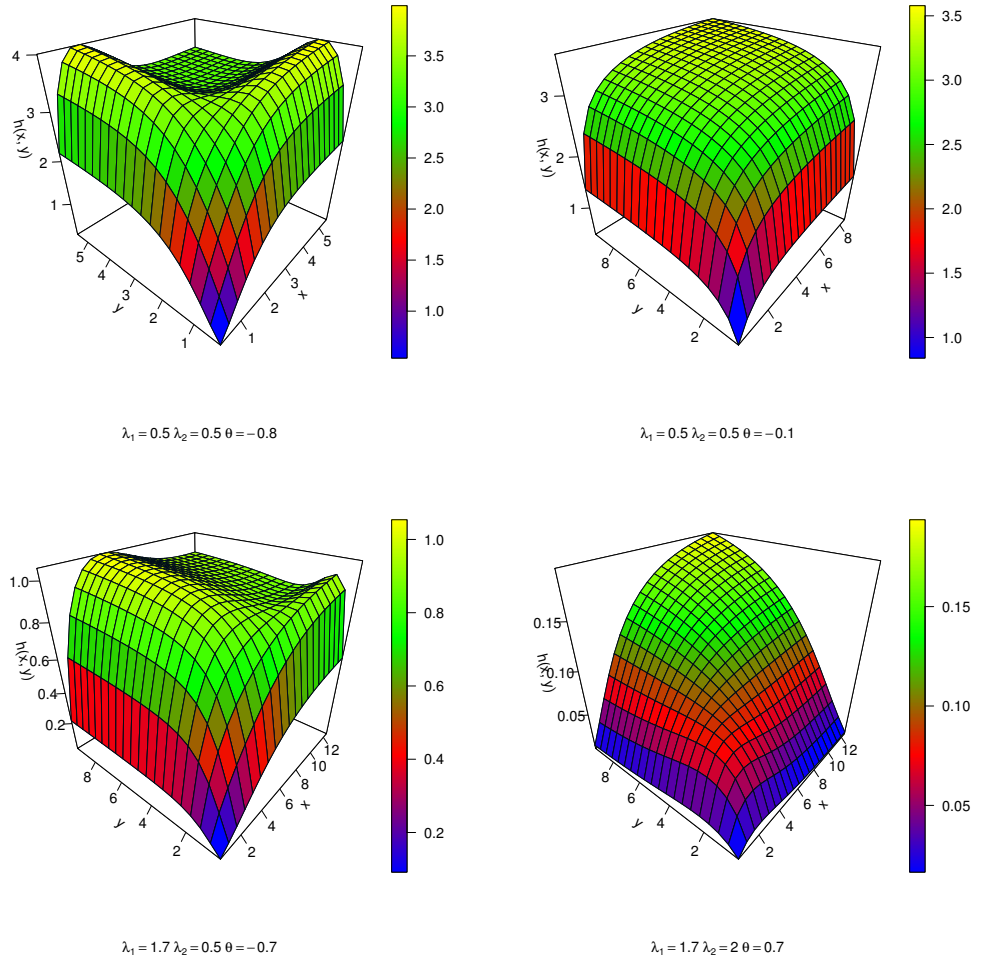


Figure 2. Plots of joint hazard rate of FGM-BLBE distribution for some parameter values.

Consider a random sample $(y_{1_{1:n}}, y_{2_{1:n}}), (y_{1_{2:n}}, y_{2_{2:n}}), \ldots, (y_{1_{n:n}}, y_{2_{n:n}})$ drawn from a bivariate distribution with cumulative distribution function $F_{Y_1,Y_2}(y_1, y_2)$ and probability density function $f_{Y_1,Y_2}(y_1, y_2)$. When arranging the values y_{1i}, where $i = 1, 2, \ldots, n$, in ascending order, we obtain $y_{1_{1:n}} < y_{1_{2:n}} < \ldots < y_{1_{n:n}}$. Here, the term $y_{2_{[i:n]}}$ represents the concomitant of the i^{th} order statistic. The consideration of these concomitants becomes crucial when choosing bivariate and prediction techniques that rely on the rankings of Y_1.

To delve deeper into the topic of concomitants of order statistics, you can refer to the work of David and Nagaraja [33] and Ke Wang [34].

Under the progressive type-II censoring scheme, the following assumptions will be employed and can be described as follows:

1. Suppose there are n independent units undergoing life testing, and these units have the FGM-BLBE distribution.
2. Generate progressive samples as $(y_{1_{1:m:n}}, y_{2_{[1:m:n]}}) < (y_{1_{2:m:n}}, y_{2_{[2:m:n]}}) < \cdots < (y_{1_{m:m:n}}, y_{2_{[m:m:n]}})$, subject to a progressive censoring scheme denoted as R_i, where $i = 1, 2, \ldots, m$ (which is less than or equal to n) and m is predetermined as the number of complete failures to be observed.
3. Identify the m completely observed failure times as $Y_{1_{i:m:n}}^{(R_i)}$, where $i = 1, \ldots, m$. At the time of the first failure, $(y_{1_{1:m:n}}, y_{2_{[1:m:n]}})$, R_1 follows a binomial$(n - m, p)$ distribution, representing the number of units randomly removed from the remaining $(n - 1)$ surviving items. At the time of the second failure, $(y_{1_{2:m:n}}, y_{2_{[2:m:n]}})$, R_2 follows a binomial$(n - m - R_1, p)$ distribution, indicating the number of units randomly removed from the remaining $(n - 2 - R_1)$ units, and so on. The process continues until the m-th failure occurs at any time, at which point all the remaining $(n - m - R_1 - R_2 - \ldots - R_{m-1})$ units are removed.
4. Assume that removing an individual unit from Y_1 during the test is independent of the others, and all units have the same removal probability, denoted as p. In such a scenario, the number of units removed at each failure time follows a binomial distribution. Then, the distribution of units removed is denoted as $R_i \sim$ binomial$(n - m - \sum_{j=1}^{m-1} R_j, p)$, and $R_m = n - m - \sum_{j=1}^{m-1} R_j$ for $j = 2, 3, \ldots, m - 1$.

The full likelihood function, represented as $L(y_{1_{i:m:n}}, y_{2_{[i:m:n]}}, \nabla)$, based on a progressive type-II censored sample with binomial removals, is defined for any vector of parameters ∇, as:

$$L(y_{1_{i:m:n}}, y_{2_{[i:m:n]}}, \nabla) = L_1(y_{1_{i:m:n}}, y_{2_{[i:m:n]}}, \nabla) P(R = r). \tag{10}$$

Balakrishnan and Kim [20] define the likelihood function $L_1(y_{1i:m:n}, y_{2[i:m:n]})$ for the case of progressive type II censored samples with R removal as follows:

$$L_1(y_{1_{i:m:n}}, y_{2_{[i:m:n]}}, \nabla) = C \prod_{i=1}^{m} f_{Y_1,Y_2}(y_{1_{i:m:n}}, y_{2_{[i:m:n]}}) \left[1 - F_{Y_1}(y_{1_{i:m:n}})\right]^{R_i}, \tag{11}$$

where C is a constant which does not depend on vector parameter ∇ and is determined by $C = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \ldots (n - R_1 - R_2 - \cdots - R_{m-1} - m + 1)$. The likelihood function of the binomial distribution is given by:

$$\begin{aligned} P(R = r) =& P(R_1 = r_1, R_2 = r_2, \ldots, R_{m-1} = r_{m-1}), \\ =& \frac{(n - m)!}{\left(n - m - \sum_{j=1}^{m-1} r_j\right)! \prod_{j=1}^{m-1} r_j} p^{\sum_{j=1}^{m-1} r_j} (1 - p)^{(m-1)(n-m) - \sum_{j=1}^{m-1} (m-j) r_j}, \end{aligned} \tag{12}$$

where p denotes a fixed constant probability, it is determined by the experimenter's level of experience regarding the number of removals.

We observe that, as $L_1(y_{1_{i:m:n}}, y_{2_{[i:m:n]}}, \nabla)$ does not rely on the binomial parameter p, the Maximum Likelihood Estimator (MLE) of p can be derived directly by maximizing Equation (12). Hence, the MLE of p follows as

$$\hat{p} = \frac{\sum_{i=1}^{m-1} r_i}{(m - 1)(n - m) - \sum_{i=1}^{m-1} (m - i - 1) r_i}. \tag{13}$$

Similarly, $P(R = r)$ does not depend on the parameter vector ∇, then the MLE of ∇ can be readily obtained by maximizing $L_1(y_{1_{i:m:n}}, y_{2_{[i:m:n]}}, \nabla)$.

3. MLE of the FGM-BLBE Distribution under Progressive Type-II Censoring

In this section, we focus on the estimation of FGM-BLBE parameters in the presence of progressive Type-II censored samples, employing the Maximum Likelihood Estimators (MLEs).

Let $(Y_{1_{1:m:n}}, Y_{2_{1:m:n}}), (Y_{1_{2:m:n}}, Y_{2_{2:m:n}}), \ldots, (Y_{1_{m:m:n}}, Y_{2_{m:m:n}})$ represent a progressively Type-II censored sample with random removal from the FGM-BLBE distribution, characterized by the PDF in Equation (7), and defined by the parameter vector $\nabla = (\lambda_1, \lambda_2, \theta)$. The log-likelihood function is structured as follows:

$$\begin{aligned}\ell(\nabla) = &-2m\Big[\ln(\lambda_1) + \ln(\lambda_2)\Big] + \sum_{i=1}^{m} \ln(y_{1_{i:m:n}}) + \sum_{i=1}^{m} \ln(y_{2_{[i:m:n]}}) - \frac{1}{\lambda_1}\sum_{i=1}^{m} y_{1_{i:m:n}} \\ &- \frac{1}{\lambda_2}\sum_{i=1}^{m} y_{2_{[i:m:n]}} + \sum_{i=1}^{m} \ln\Big\{1 + \theta\Big[2\phi(y_{1_{i:m:n}}, \lambda_1) - 1\Big]\Big[2\eta(y_{2_{[i:m:n]}}, \lambda_2) - 1\Big]\Big\} \\ &+ \sum_{i=1}^{m} R_i \ln\Big[\phi(y_{1_{i:m:n}}, \lambda_1)\Big],\end{aligned} \tag{14}$$

where

$$\phi(y_{1_{i:m:n}}, \lambda_1) = \left(1 + \frac{y_{1_{i:m:n}}}{\lambda_1}\right) e^{-\frac{y_{1_{i:m:n}}}{\lambda_1}},$$

and

$$\eta(y_{2_{[i:m:n]}}, \lambda_2) = \left(1 + \frac{y_{2_{[i:m:n]}}}{\lambda_2}\right) e^{-\frac{y_{2_{[i:m:n]}}}{\lambda_2}}.$$

The partial derivatives of Equation (14) with respect to the unknown parameters can be computed as follows:

$$\frac{\partial \ell(\nabla)}{\partial \lambda_1} = -\frac{2m}{\lambda_1} + \frac{1}{\lambda_1^2}\sum_{i=1}^{m} y_{1_{i:m:n}} + \sum_{i=1}^{m} \frac{2\theta\Big(2\eta(y_{2_{[i:m:n]}}, \lambda_2) - 1\Big) B_{1_{i:m:n}}(y_{1_{i:m:n}}, \lambda_1)}{1 + \theta\Big(2\phi(y_{1_{i:m:n}}, \lambda_1) - 1\Big)\Big(2\eta(y_{2_{[i:m:n]}}, \lambda_2) - 1\Big)} + \sum_{i=1}^{m} R_i \frac{B_{1_{i:m:n}}(y_{1_{i:m:n}}, \lambda_1)}{\Big[\phi(y_{1_{i:m:n}}, \lambda_1)\Big]}, \tag{15}$$

$$\frac{\partial \ell(\nabla)}{\partial \lambda_2} = -\frac{2m}{\lambda_2} + \frac{1}{\lambda_2^2}\sum_{i=1}^{m} y_{2_{[i:m:n]}} + \sum_{i=1}^{m} \frac{2\theta\Big(2\phi(y_{1_{i:m:n}}, \lambda_1) - 1\Big) B_{2_{[i:m:n]}}(y_{2_{[i:m:n]}}, \lambda_2)}{1 + \theta\Big(2\phi(y_{1_{i:m:n}}, \lambda_1) - 1\Big)\Big(2\eta(y_{2_{[i:m:n]}}, \lambda_2) - 1\Big)}, \tag{16}$$

and

$$\frac{\partial \ell(\nabla)}{\partial \theta} = \sum_{i=1}^{m} \frac{\Big(2\phi(y_{1_{i:m:n}}, \lambda_1) - 1\Big)\Big(2\eta(y_{2_{[i:m:n]}}, \lambda_2) - 1\Big)}{1 + \theta\Big(2\phi(y_{1_{i:m:n}}, \lambda_1) - 1\Big)\Big(2\eta(y_{2_{[i:m:n]}}, \lambda_2) - 1\Big)}, \tag{17}$$

where

$$B_{1_{i:m:n}}(y_{1_{i:m:n}}, \lambda_1) = \frac{\partial}{\partial \lambda_1}\phi(y_{1_{i:m:n}}, \lambda_1) = \frac{y_{1_{i:m:n}}^2}{\lambda_1^3} e^{-\frac{y_{1_{i:m:n}}}{\lambda_1}},$$

and

$$B_{2_{[i:m:n]}}(y_{2_{[i:m:n]}}, \lambda_2) = \frac{\partial}{\partial \lambda_2}\eta(y_{2_{[i:m:n]}}, \lambda_2) = \frac{y_{2_{[i:m:n]}}^2}{\lambda_2^3} e^{-\frac{y_{2_{[i:m:n]}}}{\lambda_2}}.$$

The MLE of the parameters in the FGM-BLBE model, denoted by $\hat{\nabla}$, under the progressive Type-II censoring scheme is obtained by solving a set of nonlinear equations derived from the likelihood function. Due to the complexity of these equations, finding an analytical solution is often infeasible. As a result, numerical techniques, such as the Newton-Raphson method, are employed to approximate the optimal values of the parameters.

4. Bayesian Estimation

In this segment, we explore the Bayesian inference of undisclosed parameters in the context of latent failure times following the FGM-BLBE distribution, utilizing a progressive

Type-II censoring scheme. Bayesian estimates (BEs) and Highest Posterior Density (HPD) credible intervals were computed for the undisclosed parameters.

4.1. Prior Distributions

In this subsection, we made the assumption that the prior distributions of λ_1, and λ_2, are independent and are characterized by Gamma distributions with parameters (a_1, b_1), and (a_2, b_2), respectively. The copula parameter θ is assumed to follow a uniform distribution over the interval $(-1, 1)$. Consequently, the prior distributions for λ_1, λ_2, and θ take the following forms:

$$\pi(\lambda_1) = \frac{b_1^{a_1}}{\Gamma a_1}\lambda_1^{a_1-1}e^{-b_1\lambda_1} \quad \lambda_1 > 0 \quad and \quad a_1, b_1 > 0,$$

$$\pi(\lambda_2) = \frac{b_2^{a_2}}{\Gamma a_2}\lambda_2^{a_2-1}e^{-b_2\lambda_2} \quad \lambda_2 > 0 \quad and \quad a_2, b_2 > 0,$$

and

$$\pi(\theta) = \frac{1}{2} \quad -1 < \theta < 1,$$

then, the joint prior is given by

$$\begin{aligned}\pi(\lambda_1, \lambda_2, \theta) &= \pi(\lambda_1)\pi(\lambda_2)\pi(\theta) \\ &= \frac{1}{2}\frac{b_1^{a_1}}{\Gamma a_1}\lambda_1^{a_1-1}e^{-b_1\lambda_1}\frac{b_2^{a_2}}{\Gamma a_2}\lambda_2^{a_2-1}e^{-b_2\lambda_2}.\end{aligned} \tag{18}$$

4.2. Posterior Distribution

Using the combined equations of prior and likelihood, the posterior density function for the parameters λ_1, λ_2, and θ in the context of the FGM-BLBE model, employing a progressive Type-II censoring scheme, can be expressed in the following manner:

$$\begin{aligned}\pi(\lambda_1, \lambda_2, \theta|x) &= \frac{L(y_1, y_2|\lambda_1, \lambda_2, \theta)\pi(\lambda_1, \lambda_2, \theta)}{\int_0^\infty\int_0^\infty\int_0^\infty L(y_1, y_2|\lambda_1, \lambda_2, \theta)\pi(\lambda_1, \lambda_2, \theta)d\lambda_1 d\lambda_2 d\theta} \\ &\propto \lambda_1^{-2m+a_1}\lambda_2^{-2m+a_2}e^{-\lambda_1\left(\sum_{i=1}^m \frac{y_{1_{i:m:n}}}{\lambda_1^2}+b_1\right)}e^{-\lambda_2\left(\sum_{i=1}^m \frac{y_{2_{i:m:n}}}{\lambda_2^2}+b_2\right)}\prod_{i=1}^m\left[\phi(y_{1_{i:m:n}}, \lambda_1)\right]^{R_i} \\ &\quad \prod_{i=1}^m\left\{1+\theta\left[2\left(1+\frac{y_{1_{i:m:n}}}{\lambda_1}\right)e^{-\frac{y_{1_{i:m:n}}}{\lambda_1}}-1\right]\left[2\left(1+\frac{y_{2_{i:m:n}}}{\lambda_2}\right)e^{-\frac{y_{2_{i:m:n}}}{\lambda_2}}-1\right]\right\}.\end{aligned} \tag{19}$$

The Bayesian estimator for any function involving variables λ_1, λ_2, and θ, denoted as $\zeta(\lambda_1, \lambda_2, \theta)$ within the context of the square error loss (SEL) function, is represented by the posterior mean. This is expressed as $\tilde{\zeta}(\lambda_1, \lambda_2, \theta)$ and can be derived in the following manner:

$$\tilde{\zeta}(\lambda_1, \lambda_2, \theta) = \frac{\int_0^\infty\int_0^\infty\int_0^\infty \zeta(\lambda_1, \lambda_2, \theta)L(y_1, y_2|\lambda_1, \lambda_2, \theta)\pi(\lambda_1, \lambda_2, \theta)d\lambda_1 d\lambda_2 d\theta}{\int_0^\infty\int_0^\infty\int_0^\infty L(y_1, y_2|\lambda_1, \lambda_2, \theta)\pi(\lambda_1, \lambda_2, \theta)d\lambda_1 d\lambda_2 d\theta}. \tag{20}$$

The Bayesian estimator for $\zeta(\lambda_1, \lambda_2, \theta)$ lacks a closed-form expression, necessitating the utilization of an approximation method to compute the estimate presented in Equation (20). We suggest employing the Markov Chain Monte Carlo (MCMC) method to obtain Bayesian estimators (BEs) and Highest Posterior Density (HPD) credible intervals for the unknown parameters. This method proves particularly advantageous in Bayesian inference due to its focus on posterior distributions that often pose challenges for mathematical analysis. The Metropolis-Hastings (MH) algorithm is employed, initiating with the generation of a candidate sample ζ^* from the proposal distribution $q(.)$. It's crucial to note that samples

from the proposal distribution aren't automatically accepted as posterior samples; rather, acceptance is determined probabilistically based on the acceptance probability.

The MH sampling algorithm, introduced by Metropolis et al. [35] and further developed by Hastings (1970), can be elucidated in the following manner:

Step 1: Commence by initializing any initial estimate ∇, denoted as $\zeta^{(0)} = (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\theta})$.
Step 2: Set the iteration index as $t = 1$.
Step 3: Generate a candidate point ζ^* using a normal proposal distribution $q(\zeta) = N(\hat{\zeta}, \mathrm{var}(\hat{\zeta}))$.
Step 4: For the given candidate point ζ^*, compute the acceptance probability.

$$A(\zeta^{(t-1)}, \zeta^*) = \min\left[1, \frac{\pi(\zeta^*|\underline{x})q(\zeta^{(t-1)}|\zeta^*)}{\pi(\zeta^{(t-1)}|\underline{x})q(\zeta^*|\zeta^{(t-1)})}\right].$$

Step 5: Generate a sample from a uniform distribution, i.e., $u \sim U(0,1)$.

$$\text{If}\begin{cases} u \le A(\zeta^{(t-1)}, \zeta^*) & Accept \quad \zeta^* = \zeta^{(t)}, \\ u \ge A(\zeta^{(t-1)}, \zeta^*) & Accept \quad \zeta^{(t)} = \zeta^{(t-1)}. \end{cases}$$

Step 6: Increment the iteration index: $t = t + 1$, and repeat steps 2–5 M times until obtaining M samples, resulting in $(\lambda_1^{(t)}, \lambda_2^{(t)}, \theta^{(t)})$ for $t = 1, 2, \ldots M$.

From the random samples of size M drawn from the posterior density, a subset of initial samples may be discarded (burn-in), and the remaining samples can be utilized to calculate Bayesian estimators (BEs). The BEs for ζ concerning the SEL function are given by:

$$\tilde{\zeta} = \frac{\sum_{t=\Upsilon+1}^{M} \zeta^{(t)}}{(M - \Upsilon)}. \tag{21}$$

Here, Υ represents the number of burn-in samples, and the summation is performed over $t = \Upsilon + 1, \ldots, M$.

Step 7: Under the LINEX loss function, the BEs of ζ are obtained as:

$$\zeta = \frac{1}{c} \ln\left[\frac{\sum_{t=\Upsilon+1}^{M} e^{-c\zeta^{(t)}}}{(M - \Upsilon)}\right]. \tag{22}$$

5. Confidence Intervals

In this section, two different approaches for constructing confidence intervals (CIs) for the unknown parameters of the FGM-BLBE model under progressive Type-II censoring are introduced. These techniques encompass the asymptotic confidence interval (ACI) and the bootstrap confidence interval, which is subdivided into percentile bootstrap and bootstrap-t methods for λ_j, where $j = 1, 2$, and θ.

5.1. Asymptotic Confidence Intervals

One of the most widely used approaches for establishing confidence intervals for parameters involves leveraging the asymptotic normality of the maximum likelihood estimator (MLE). Specifically, this approach relies on the asymptotic variance-covariance matrix of the MLE of the parameters, denoted as the Fisher information matrix $I(\nabla)$. This matrix is computed from the negative second derivatives of the natural logarithm

of the likelihood function, evaluated at the estimated parameter values $\hat{\nabla} = (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\theta})$. The asymptotic variance–covariance matrix of the parameter vector ∇ can be represented as

$$I(\hat{\nabla}) = -\begin{bmatrix} I_{\hat{\lambda}_1\hat{\lambda}_1} & & \\ I_{\hat{\lambda}_2\hat{\lambda}_1} & I_{\hat{\lambda}_2\hat{\lambda}_2} & \\ I_{\hat{\theta}\hat{\lambda}_1} & I_{\hat{\theta}\hat{\lambda}_2} & I_{\hat{\theta}\hat{\theta}} \end{bmatrix}, \tag{23}$$

where $V(\hat{\nabla}) = I^{-1}(\hat{\nabla})$. Confidence intervals for parameter ∇ can be constructed based on the asymptotic normality of the MLE. Specifically, a $100(1-\alpha)\%$ confidence interval for parameter ∇ can be calculated as $\hat{\lambda}_l \pm Z_{0.025}\sqrt{I_{\hat{\lambda}_l\hat{\lambda}_l}}$ and $\hat{\theta} \pm Z_{0.025}\sqrt{I_{\hat{\theta}\hat{\theta}}}$, where $l = 1, 2$ and $Z_{0.025}$ represents the percentile of the standard normal distribution with a right tail probability of $\frac{\alpha}{2}$.

5.2. Highest Posterior Density

To obtain the Highest Posterior Density (HPD) credible intervals for ζ, arrange the samples $\zeta^t = (\theta^t, \lambda_1^t, \lambda_2^t)$ for $t = 1, 2, \ldots M$ in ascending order, as $\zeta^{[1]}, \zeta^{[2]}, \ldots \zeta^{[M]}$ after burn-in, specifically as $\zeta^{[\Upsilon+1]}, \zeta^{[\Upsilon+2]}, \ldots \zeta^{[M]}$. For any arbitrary $0 < \alpha < 1$, the $100(1-\alpha)\%$ two-sided HPD credible interval of ζ can be computed as:

$$\left(\zeta^{[(\alpha/2)(M-\Upsilon)]}, \zeta^{[(1-(\alpha/2))(M-\Upsilon)]}\right).$$

Subsequently, the HPD credible interval of ζ^t is determined by selecting the interval with the smallest width.

6. Simulation

The simulation section in statistical analysis is a powerful tool for estimating the parameters of a distribution. Simulations involve generating artificial data based on the FGM-BLBE distribution, allowing researchers to explore the behavior of these models and estimate the associated parameters. This method is particularly useful when analytical solutions are challenging or unavailable. The primary goals of the simulation section are to assess the performance of statistical methods (MLE and Bayesian with different loss functions), evaluate the robustness of parameter estimation techniques, and gain insights into the potential variability of results under different scenarios. By mimicking the underlying data generation process, simulations provide a controlled environment for testing the statistical properties of estimation procedures. The objectives of this section are as follows:

- Assess the accuracy and precision of parameter estimates of FGM-BLBE distribution based on progressive Type-II Censoring.
- Compare the performance of various estimation methods and conclude the best estimation method for parameter of FGM-BLBE distribution based on progressive Type-II Censoring.
- Investigate the impact of sample size, size of censored sample and distributional assumptions on estimation results.

Simulation Design

The simulation design of this study assesses the effectiveness of theoretical outcomes, encompassing both point and interval estimators, through different estimation methods such as maximum likelihood and Bayesian methodologies based on symmetric and asymmetric loss functions. A Monte Carlo simulation study is conducted, employing initial parameter values of the FGM-BLBE distribution, specifically $(\lambda_1, \lambda_2, \theta)$ = (2.4, 2.8, 0.4), (0.6, 2.8, 0.4), (0.6, 1.4, 0.4), and $(1.5, 2, 0.8)$. By varying combinations of n (sample sizes), m (effective sample size), and p (parameter of the binomial removal pattern), a comprehensive set of 5000 progressively Type-II censored samples is generated, incorporating binomial removal from the FGM-BLBE distribution.

Moreover, varying values of (n, m, p) are considered. Within each configuration, the MLEs, Bayesian estimators (BEs), and their associated ACIs or HPD intervals are examined and assessed. Additionally, various combinations of (n, m) such as (50, 35), (50, 45), (100, 75), and (100, 90) are considered, with fixed values of $p = 0.3, 0.5$, and 0.8. For each configuration, the MLEs, Bayesian estimates, and their associated asymptotic confidence intervals (ACIs) or HPD intervals are assessed with a confidence level of 95%.

In the simulation study, it's important to highlight that the effectiveness of the suggested point estimates is evaluated by assessing biases, and mean squared errors (MSEs). As for the proposed interval estimates, their performance is evaluated by considering average interval lengths (AILs) (LACI and LCCI, for MLE and Bayesian, respectively) and coverage probability (CP).

In the simulation study, Bayesian estimators (BEs) are computed using the SELF and LINEX functions, specifically for c values of 0.5 and 2. It is essential to note that the hyperparameter values are determined through an elicitation method, where the choice of the hyperparameter method relies on prior knowledge about the unknown parameters. These informative priors are derived from the MLEs for (λ_1, λ_2), obtained by equating the mean and variance of $(\hat{\lambda}_1, \hat{\lambda}_2)$ to the mean and variance of the considered priors (Gamma priors). Here, t ranges from 1 to M, where M is the number of samples available from the FGM-BLBE distribution. By equating the mean and variance of $(\lambda_1^t, \lambda_2^t)$ with those of the Gamma priors, the hyperparameters can be determined, as described by Dey et al. [36].

Based on the generated data, Maximum Likelihood Estimates (MLEs) and their associated 95% ACIs or HPD are computed. It's noteworthy that, when obtaining MLEs, the initial guess values are assumed to be identical to the true parameter values. Subsequently, the hyperparameter values are derived. These hyperparameter values are then incorporated to calculate the desired estimates. Finally, utilizing the MH algorithm for BEs, 2000 burn-in samples are excluded from the total 12,000 MCMC samples generated from the posterior density. Tables 1–4 provide insights into these observations as follows:

- BEs for parameters are calculated using two distinct loss functions, namely, the SEL and LINEX functions. Estimates associated with the LINEX function are obtained for c values of 0.5 and 2.
- All average estimates and their corresponding MSEs for both methods are presented.
- In the case of fixed censoring schemes (with identical parameters for binomial removal), as the effective size increases (i.e., n or m, or their combinations), the Bias, MSEs, and AILs of both MLEs and BEs for parameters of FGM-BLBE decrease.
- For fxed value of n, when effective sample size m increases, the simulated MSEs decreases for most cases.
- Based on Bias and MSEs, the Bayes estimates under SEL and LINEX provide better results than other estimates for MLEs.
- As the sample size (n) and effective sample size (m) increase, the AILs for intervals related to the parameters of FGM-BLBE decrease.
- The HPD credible intervals outperform the confidence intervals of the MLEs based on AIL and Coverage Probability (CP).
- In terms of CP, the confidence intervals of the MLEs exhibit an average coverage probability of 95The performance of both classical and Bayesian estimates is deemed quite satisfactory.
- Notably, the performance of BEs is better under the LINEX loss function compared to the SEL function.

Table 1. MLE and Bayesian: $\lambda_1 = 2.4, \lambda_2 = 2.8, \theta = 0.4$.

$\lambda_1 = 2.4, \lambda_2 = 2.8, \theta = 0.4$				MLE				SELF			LINEX 1 (c = 0.5)			LINEX 2 (c = 2)		
n	**p**	**m**		**Bias**	**MSE**	**LACI**	**CP**	**Bias**	**MSE**	**LCCI**	**Bias**	**MSE**	**LCCI**	**Bias**	**MSE**	**LCCI**
50	0.3	35	λ_1	−0.0328	0.0859	1.1424	95.00%	0.0299	0.0230	0.5356	−0.0273	0.0199	0.4970	0.0928	0.0349	0.5771
			λ_2	0.0124	0.1292	1.4090	96.10%	0.0653	0.0393	0.6593	−0.0261	0.0296	0.6033	0.1566	0.0676	0.7453
			θ	0.0849	0.4679	2.6619	95.40%	0.0631	0.1578	1.5195	−0.0765	0.1716	1.5299	0.0776	0.1586	1.4898
		45	λ_1	−0.0061	0.0623	0.9785	95.50%	0.0280	0.0193	0.4932	−0.0086	0.0160	0.4624	0.0883	0.0283	0.5255
			λ_2	−0.0098	0.1065	1.2793	97.70%	0.0407	0.0304	0.5656	−0.0221	0.0262	0.5319	0.1092	0.0445	0.6137
			θ	0.0197	0.2538	1.9743	95.60%	−0.0279	0.1384	1.4122	−0.0700	0.1552	1.4293	0.0432	0.1342	1.3645
	0.5	35	λ_1	−0.0099	0.0939	1.2015	95.20%	0.0444	0.0257	0.5542	−0.0138	0.0208	0.5239	0.1085	0.0398	0.5948
			λ_2	0.0179	0.1289	1.4062	95.80%	0.0689	0.0398	0.7032	−0.0193	0.0297	0.6377	0.1609	0.0692	0.7777
			θ	0.0650	1.2308	4.3435	95.10%	−0.0366	0.1660	1.5987	−0.1181	0.1864	1.6172	0.0442	0.1615	1.5679
		45	λ_1	−0.0081	0.0567	0.9319	95.70%	0.0330	0.0169	0.4774	−0.0129	0.0142	0.4506	0.0827	0.0248	0.5081
			λ_2	−0.0054	0.0830	1.1295	95.90%	0.0443	0.0259	0.5930	−0.0184	0.0213	0.5612	0.1128	0.0407	0.6421
			θ	0.0086	0.2612	2.0043	95.20%	−0.0348	0.1441	1.4432	−0.1091	0.1603	1.4888	0.0383	0.1406	1.4236
	0.8	35	λ_1	−0.0192	0.1010	1.2444	95.20%	0.0396	0.0224	0.5497	−0.0188	0.0182	0.5120	0.1039	0.0359	0.5991
			λ_2	−0.0052	0.1598	1.5679	95.20%	0.0559	0.0366	0.6698	−0.0239	0.0290	0.6135	0.1449	0.0618	0.7362
			θ	0.1475	2.1214	3.6423	96.30%	−0.0373	0.1654	1.5142	−0.1207	0.1856	1.5342	0.0444	0.1618	1.4888
		45	λ_1	−0.0122	0.0666	1.0111	97.30%	0.0353	0.0188	0.5002	−0.0108	0.0158	0.4725	0.0852	0.0274	0.5342
			λ_2	−0.0015	0.0933	1.1983	97.70%	0.0467	0.0268	0.6009	−0.0161	0.0219	0.5627	0.1152	0.0418	0.6486
			θ	0.0317	0.2982	2.1382	99.90%	−0.0198	0.1373	1.4085	−0.0928	0.1524	1.4512	0.0425	0.1352	1.3700
100	0.3	75	λ_1	−0.0148	0.0381	0.7629	95.20%	0.0184	0.0106	0.3849	−0.0094	0.0096	0.3733	0.0476	0.0133	0.3999
			λ_2	−0.0033	0.0553	0.9218	94.60%	0.0277	0.0158	0.4686	−0.0147	0.0140	0.4552	0.0680	0.0211	0.4926
			θ	−0.0051	0.1240	1.3809	95.30%	−0.0222	0.0964	1.1995	−0.0753	0.1056	1.2145	0.0301	0.0936	1.1848
		90	λ_1	−0.0096	0.0301	0.6790	95.90%	0.0169	0.0081	0.3380	−0.0063	0.0074	0.3268	0.0410	0.0100	0.3473
			λ_2	−0.0027	0.0411	0.7945	95.60%	0.0207	0.0112	0.4087	−0.0111	0.0102	0.3962	0.0541	0.0146	0.4212
			θ	−0.0045	0.0999	1.2391	96.10%	−0.0203	0.0820	1.1113	−0.0736	0.0906	1.1468	0.0260	0.0796	1.0764
	0.5	75	λ_1	−0.0175	0.0379	0.7609	94.80%	0.0162	0.0102	0.3824	−0.0114	0.0094	0.3709	0.0451	0.0128	0.3982
			λ_2	−0.0050	0.0497	0.8742	94.50%	0.0258	0.0136	0.4494	−0.0121	0.0121	0.4313	0.0659	0.0185	0.4711
			θ	0.0121	0.1616	1.5759	96.00%	−0.0179	0.1041	1.2251	−0.0732	0.1137	1.2536	0.0371	0.1016	1.2036
		90	λ_1	0.0017	0.0331	0.7131	95.40%	0.0152	0.0097	0.3500	−0.0010	0.0088	0.3447	0.0417	0.0120	0.3572
			λ_2	−0.0003	0.0438	0.8208	95.80%	0.0250	0.0127	0.4004	−0.0069	0.0113	0.3879	0.0583	0.0164	0.4152
			θ	0.0017	0.1049	1.2700	96.60%	−0.0153	0.0826	1.0660	−0.0707	0.0914	1.0988	0.0193	0.0795	1.0361
	0.8	75	λ_1	−0.0115	0.0400	0.7836	95.80%	0.0211	0.0110	0.3793	−0.0067	0.0099	0.3694	0.0502	0.0139	0.3950
			λ_2	−0.0065	0.0493	0.8706	94.60%	0.0248	0.0141	0.4378	−0.0132	0.0126	0.4186	0.0648	0.0190	0.4604
			θ	0.0168	0.1316	1.4213	94.60%	−0.0269	0.1036	1.2091	−0.0812	0.1136	1.2111	0.0270	0.1001	1.2003
		90	λ_1	−0.0093	0.0314	0.6936	96.70%	0.0174	0.0088	0.3454	−0.0060	0.0080	0.3379	0.0418	0.0108	0.3530
			λ_2	−0.0030	0.0404	0.7884	94.90%	0.0224	0.0115	0.3998	−0.0093	0.0104	0.3856	0.0554	0.0149	0.4137
			θ	0.0164	0.1066	1.2791	94.70%	−0.0065	0.0869	1.1233	−0.0555	0.0937	1.1434	0.0194	0.0857	1.1097

Table 2. MLE and Bayesian: $\lambda_1 = 0.6, \lambda_2 = 2.8, \theta = 0.4$.

$\lambda_1 = 0.6, \lambda_2 = 2.8, \theta = 0.4$				MLE				SELF			LINEX 1 (c = 0.5)			LINEX 2 (c = 2)		
n	**p**	**m**		**Bias**	**MSE**	**LACI**	**CP**	**Bias**	**MSE**	**LCCI**	**Bias**	**MSE**	**LCCI**	**Bias**	**MSE**	**LCCI**
50	0.3	35	λ_1	−0.0047	0.0054	0.2870	95.00%	0.0114	0.0017	0.1461	0.0077	0.0016	0.1429	0.0153	0.0019	0.1495
			λ_2	0.0277	0.1033	1.2557	95.00%	0.0624	0.0366	0.6941	−0.0196	0.0281	0.6428	0.1549	0.0642	0.7735
			θ	0.0297	0.3360	2.2704	94.20%	−0.0415	0.1594	1.5305	−0.1210	0.1812	1.5669	0.0379	0.1528	1.4824
		45	λ_1	0.0034	0.0040	0.2487	95.40%	0.0104	0.0013	0.1258	0.0074	0.0012	0.1242	0.0134	0.0014	0.1275
			λ_2	0.0241	0.0912	1.1806	95.20%	0.0507	0.0296	0.6108	−0.0131	0.0240	0.5708	0.1205	0.0458	0.6623
			θ	0.0274	0.2796	2.0531	95.70%	−0.0086	0.1372	1.4309	−0.0795	0.1486	1.4811	0.0261	0.1381	1.4265
	0.5	35	λ_1	0.0010	0.0047	0.2689	95.10%	0.0120	0.0015	0.1346	0.0082	0.0014	0.1326	0.0159	0.0017	0.1364
			λ_2	0.0254	0.1134	1.3169	94.20%	0.0563	0.0373	0.6965	−0.0251	0.0300	0.6514	0.1478	0.0630	0.7502
			θ	0.0730	0.4035	2.4747	94.40%	−0.0334	0.1610	1.5234	−0.1114	0.1765	1.5283	0.0439	0.1600	1.4947
		45	λ_1	−0.0007	0.0040	0.2472	95.40%	0.0096	0.0012	0.1262	0.0066	0.0011	0.1245	0.0126	0.0013	0.1278
			λ_2	0.0077	0.0917	1.1872	94.40%	0.0449	0.0270	0.5973	−0.0180	0.0223	0.5626	0.1137	0.0418	0.6484
			θ	0.0647	0.2887	2.0919	95.10%	−0.0159	0.1390	1.4569	−0.0857	0.1526	1.4726	0.0415	0.1370	1.4280
	0.8	35	λ_1	0.0021	0.0062	0.3089	94.70%	0.0130	0.0016	0.1390	0.0092	0.0015	0.1377	0.0169	0.0018	0.1412
			λ_2	0.0440	0.1558	1.5383	94.40%	0.0684	0.0404	0.7062	−0.0138	0.0306	0.6567	0.1612	0.0702	0.7839
			θ	0.1084	0.9564	3.8119	94.70%	−0.0072	0.1535	1.5017	−0.0851	0.1680	1.5414	0.0703	0.1543	1.4590
		45	λ_1	−0.0013	0.0038	0.2432	94.80%	0.0091	0.0011	0.1220	0.0061	0.0011	0.1208	0.0121	0.0012	0.1236
			λ_2	0.0216	0.0900	1.1733	94.70%	0.0525	0.0294	0.6045	−0.0114	0.0235	0.5767	0.1227	0.0462	0.6564
			θ	0.0608	0.2444	1.9243	95.50%	−0.0071	0.1334	1.3996	−0.0771	0.1458	1.4296	0.0607	0.1336	1.3723
100	0.3	75	λ_1	−0.0014	0.0024	0.1936	95.10%	0.0054	0.0007	0.0986	0.0037	0.0007	0.0979	0.0072	0.0007	0.0990
			λ_2	0.0189	0.0494	0.8712	94.00%	0.0273	0.0148	0.4558	−0.0111	0.0131	0.4388	0.0679	0.0199	0.4729
			θ	0.0104	0.1328	1.4288	95.30%	−0.0252	0.1019	1.2475	−0.0805	0.1120	1.2396	0.0292	0.0987	1.2122
		90	λ_1	−0.0005	0.0020	0.1765	95.70%	0.0045	0.0006	0.0895	0.0030	0.0005	0.0887	0.0060	0.0006	0.0901
			λ_2	0.0137	0.0462	0.8413	95.20%	0.0248	0.0135	0.4145	−0.0072	0.0122	0.4039	0.0582	0.0172	0.4269
			θ	0.0104	0.1020	1.2441	95.70%	−0.0150	0.0856	1.1287	−0.0642	0.0935	1.1731	0.0283	0.0833	1.0873
	0.5	75	λ_1	0.0038	0.0026	0.1985	95.80%	0.0057	0.0007	0.1006	0.0039	0.0007	0.0992	0.0075	0.0008	0.1014
			λ_2	0.0197	0.0517	0.8882	95.30%	0.0320	0.0153	0.4358	−0.0064	0.0132	0.4220	0.0727	0.0209	0.4541
			θ	0.0148	0.1216	1.3665	94.90%	−0.0231	0.1041	1.2150	−0.0777	0.1153	1.2561	0.0311	0.0991	1.1716
		90	λ_1	−0.0029	0.0020	0.1739	96.80%	0.0039	0.0005	0.0863	0.0024	0.0005	0.0856	0.0054	0.0006	0.0867
			λ_2	0.0191	0.0403	0.7833	95.50%	0.0286	0.0118	0.4000	−0.0035	0.0104	0.3908	0.0621	0.0157	0.4125
			θ	0.0110	0.0985	1.2298	95.10%	−0.0225	0.0825	1.0821	−0.0746	0.0911	1.1093	0.0240	0.0795	1.0663
	0.8	75	λ_1	0.0019	0.0023	0.1880	94.10%	0.0070	0.0006	0.0936	0.0053	0.0006	0.0930	0.0089	0.0007	0.0945
			λ_2	0.0203	0.0451	0.8295	95.00%	0.0329	0.0146	0.4401	−0.0053	0.0125	0.4232	0.0733	0.0201	0.4602
			θ	0.0391	0.1234	1.3689	95.10%	−0.0212	0.0942	1.1688	−0.0655	0.1031	1.1980	0.0419	0.0924	1.1256
		90	λ_1	−0.0012	0.0019	0.1690	96.10%	0.0049	0.0005	0.0835	0.0034	0.0005	0.0825	0.0064	0.0005	0.0842
			λ_2	0.0181	0.0402	0.7835	96.50%	0.0238	0.0117	0.4021	−0.0048	0.0105	0.3873	0.0574	0.0153	0.4194
			θ	0.0253	0.1052	1.2682	95.70%	−0.0201	0.0841	1.0994	−0.0610	0.0913	1.1156	0.0266	0.0822	1.0762

Table 3. MLE and Bayesian: $\lambda_1 = 0.6, \lambda_2 = 1.4, \theta = 0.4$.

$\lambda_1 = 0.6, \lambda_2 = 1.4, \theta = 0.4$				MLE				SELF			LINEX 1 (c = 0.5)			LINEX 2 (c = 2)		
n	**p**	**m**		**Bias**	**MSE**	**LACI**	**CP**	**Bias**	MSE	**LCCI**	**Bias**	**MSE**	**LCCI**	**Bias**	**MSE**	**LCCI**
50	0.3	35	λ_1	−0.0132	0.0051	0.2796	94.80%	0.0117	0.0016	0.1415	0.0080	0.0014	0.1390	0.0156	0.0017	0.1436
			λ_2	0.0073	0.0278	0.6533	95.40%	0.0299	0.0088	0.3372	0.0090	0.0073	0.3242	0.0522	0.0115	0.3513
			θ	0.0947	0.3987	2.4484	94.90%	−0.0308	0.1763	1.5871	−0.1097	0.1909	1.5719	0.0575	0.1759	1.5793
		45	λ_1	−0.0082	0.0039	0.2432	94.90%	0.0090	0.0012	0.1290	0.0060	0.0011	0.1276	3.0000	0.0013	0.1312
			λ_2	0.0067	0.0215	0.5718	95.70%	0.0275	0.0072	0.2959	0.0081	0.0061	0.2858	0.0447	0.0089	0.3084
			θ	0.0558	0.2393	1.9062	95.30%	−0.0166	0.1351	1.3935	−0.0847	0.1498	1.4207	0.0514	0.1307	1.3536
	0.5	35	λ_1	0.0045	0.0060	0.3045	94.50%	0.0125	0.0015	0.1403	0.0087	0.0014	0.1380	0.0164	0.0017	0.1430
			λ_2	0.0112	0.0308	0.6872	94.30%	0.0301	0.0090	0.3430	0.0093	0.0075	0.3261	0.0523	0.0117	0.3608
			θ	0.0847	1.0215	3.9499	94.20%	−0.0264	0.1598	1.5356	−0.1044	0.1781	1.6191	0.0519	0.1554	1.4894
		45	λ_1	−0.0011	0.0041	0.2526	94.80%	0.0105	0.0013	0.1256	0.0075	0.0012	0.1238	0.0136	0.0014	0.1276
			λ_2	0.0027	0.0229	0.5933	95.20%	0.0208	0.0068	0.2958	0.0048	0.0060	0.2882	0.0375	0.0082	0.3069
			θ	0.0527	0.2348	1.8892	94.80%	−0.0260	0.1297	1.4052	−0.0996	0.1446	1.4422	0.0397	0.1256	1.3729
	0.8	35	λ_1	−0.0034	0.0055	0.2904	94.50%	0.0118	0.0017	0.1460	0.0080	0.0016	0.1439	0.0157	0.0019	0.1487
			λ_2	−0.0013	0.0297	0.6754	95.60%	0.0276	0.0095	0.3473	0.0070	0.0080	0.3302	0.0495	0.0121	0.3645
			θ	0.1188	0.3647	2.3222	94.60%	−0.0095	0.1613	1.5605	−0.0893	0.1733	1.5790	0.0695	0.1647	1.5288
		45	λ_1	−0.0028	0.0037	0.2376	95.20%	0.0084	0.0011	0.1240	0.0054	0.0011	0.1225	0.0115	0.0012	0.1259
			λ_2	0.0010	0.0219	0.5790	95.83%	0.0261	0.0071	0.3006	0.0061	0.0060	0.2878	0.0432	0.0088	0.3129
			θ	0.0730	0.2703	2.0189	95.50%	−0.0081	0.1367	1.4128	−0.0828	0.1485	1.4442	0.0577	0.1368	1.4020
100	0.3	75	λ_1	−0.0007	0.0021	0.1808	94.70%	0.0055	0.0006	0.0953	0.0037	0.0006	0.0943	0.0073	0.0007	0.0963
			λ_2	0.0111	0.0128	0.4413	94.60%	0.0170	0.0038	0.2220	0.0072	0.0035	0.2184	0.0271	0.0045	0.2295
			θ	0.0314	0.1218	1.3629	95.00%	−0.0209	0.0990	1.2019	−0.0759	0.1085	1.2219	0.0333	0.0962	1.1645
		90	λ_1	0.0006	0.0020	0.1735	95.40%	0.0054	0.0006	0.0869	0.0030	0.0005	0.0862	0.0071	0.0006	0.0874
			λ_2	0.0010	0.0102	0.3969	95.30%	0.0101	0.0030	0.2049	0.0021	0.0028	0.2010	0.0183	0.0033	0.2077
			θ	0.0305	0.1068	1.2664	95.20%	−0.0178	0.0885	1.1292	−0.0659	0.0956	1.1423	0.0299	0.0862	1.1189
	0.5	75	λ_1	−0.0018	0.0023	0.1865	95.10%	0.0055	0.0006	0.0934	0.0037	0.0006	0.0929	0.0073	0.0007	0.0939
			λ_2	0.0073	0.0131	0.4481	95.20%	0.0150	0.0038	0.2275	0.0052	0.0035	0.2216	0.0251	0.0044	0.2321
			θ	0.0370	0.1244	1.3758	93.70%	−0.0269	0.0968	1.1932	−0.0797	0.1074	1.2178	0.0257	0.0928	1.1561
		90	λ_1	0.0013	0.0021	0.1788	96.50%	0.0051	0.0006	0.0894	0.0034	0.0006	0.0888	0.0071	0.0006	0.0903
			λ_2	0.0012	0.0107	0.4024	95.80%	0.0146	0.0033	0.2145	0.0047	0.0030	0.2118	0.0229	0.0037	0.2182
			θ	0.0244	0.1077	1.2837	94.50%	−0.0231	0.0857	1.1155	−0.0748	0.0966	1.1466	0.0179	0.0808	1.0828
	0.8	75	λ_1	−0.0014	0.0024	0.1913	95.90%	0.0058	0.0007	0.0947	0.0040	0.0006	0.0937	0.0076	0.0007	0.0958
			λ_2	0.0034	0.0115	0.4210	95.70%	0.0131	0.0034	0.2171	0.0034	0.0032	0.2144	0.0230	0.0039	0.2224
			θ	0.0099	0.1223	1.3711	95.40%	−0.0400	0.0976	1.1740	−0.0984	0.1121	1.2206	0.0176	0.0916	1.1566
		90	λ_1	−0.0012	0.0020	0.1775	96.70%	0.0046	0.0006	0.0881	0.0031	0.0005	0.0872	0.0061	0.0006	0.0886
			λ_2	0.0019	0.0105	0.4014	95.90%	0.0114	0.0030	0.2068	0.0030	0.0028	0.2029	0.0197	0.0033	0.2108
			θ	0.0061	0.1011	1.2470	96.30%	−0.0351	0.0844	1.1097	−0.0970	0.0959	1.1398	−0.0024	0.0788	1.0932

Table 4. MLE and Bayesian: $\lambda_1 = 1.5, \lambda_2 = 2, \theta = 0.8$.

				MLE				SELF			LINEX 1 (c = 0.5)			LINEX 2 (c = 2)		
n	**p**	**m**		**Bias**	**MSE**	**LACI**	**CP**	**Bias**	**MSE**	**LCCI**	**Bias**	**MSE**	**LCCI**	**Bias**	**MSE**	**LCCI**
50	0.3	35	λ_1	−0.0070	0.0343	0.7260	94.20%	0.0312	0.0093	0.3403	0.0085	0.0076	0.3263	0.0552	0.0124	0.3597
			λ_2	0.0064	0.0722	1.0535	94.70%	0.0391	0.0176	0.4679	−0.0015	0.0143	0.4377	0.0830	0.0253	0.5004
			θ	0.2461	1.1824	4.1540	94.10%	−0.0742	0.2469	1.9682	−0.2396	0.3021	2.0128	0.0818	0.2529	1.9908
		45	λ_1	−0.0059	0.0276	0.6514	95.70%	0.0230	0.0078	0.3178	0.0052	0.0067	0.3104	0.0417	0.0095	0.3326
			λ_2	−0.0010	0.0472	0.8517	95.40%	0.0333	0.0136	0.4088	0.0013	0.0113	0.3912	0.0669	0.0184	0.4356
			θ	0.1490	0.4781	2.6481	95.50%	−0.0708	0.2022	1.7733	−0.2310	0.2546	1.7841	0.0521	0.1997	1.7729
	0.5	35	λ_1	−0.0039	0.0330	0.7123	94.40%	0.0292	0.0096	0.3521	0.0065	0.0079	0.3327	0.0532	0.0126	0.3718
			λ_2	0.0092	0.0661	1.0076	94.60%	0.0487	0.0198	0.4806	0.0075	0.0154	0.4495	0.0935	0.0290	0.5141
			θ	0.1821	0.8007	3.4361	94.80%	−0.0944	0.2629	2.0018	−0.2660	0.3235	1.9956	0.0660	0.2730	1.9872
		45	λ_1	−0.0035	0.0256	0.6267	95.00%	0.0247	0.0077	0.3167	0.0061	0.0066	0.3063	0.0435	0.0096	0.3271
			λ_2	−0.0029	0.0478	0.8572	95.10%	0.0338	0.0139	0.4287	0.0022	0.0116	0.4070	0.0674	0.0188	0.4513
			θ	0.1231	0.4419	2.5621	95.00%	−0.0881	0.2149	1.7729	−0.2358	0.2646	1.8031	0.0486	0.2146	1.7874
	0.8	35	λ_1	0.0062	0.0416	0.7999	94.60%	0.0317	0.0104	0.3627	0.0089	0.0086	0.3461	0.0558	0.0135	0.3863
			λ_2	0.0173	0.0730	1.0576	94.50%	0.0506	0.0199	0.4789	0.0095	0.0154	0.4520	0.0950	0.0289	0.5167
			θ	0.2036	1.8335	5.2502	94.80%	−0.0930	0.2693	1.9472	−0.2637	0.3301	1.9601	0.0664	0.2705	1.9056
		45	λ_1	−0.0051	0.0258	0.6291	94.70%	0.0233	0.0071	0.2963	0.0056	0.0062	0.2890	0.0418	0.0089	0.3123
			λ_2	0.0107	0.0505	0.8800	95.60%	0.0408	0.0146	0.4183	0.0088	0.0118	0.3969	0.0749	0.0201	0.4439
			θ	0.1335	0.4812	2.6697	96.50%	−0.0888	0.2104	1.7752	−0.2319	0.2525	1.7374	0.0436	0.2133	1.7664
100	0.3	75	λ_1	−0.0044	0.0154	0.4861	95.50%	0.0137	0.0043	0.2412	0.0030	0.0040	0.2357	0.0246	0.0049	0.2473
			λ_2	−0.0041	0.0291	0.6689	95.70%	0.0188	0.0079	0.3236	−0.0008	0.0072	0.3140	0.0385	0.0095	0.3345
			θ	0.0690	0.1587	1.5388	95.00%	−0.0646	0.1221	1.3426	−0.1612	0.1511	1.3890	0.0235	0.1138	1.3015
		90	λ_1	0.0035	0.0117	0.4245	96.20%	0.0125	0.0034	0.2116	0.0026	0.0031	0.2085	0.0244	0.0039	0.2154
			λ_2	−0.0008	0.0212	0.5707	96.10%	0.0152	0.0058	0.2821	−0.0007	0.0053	0.2769	0.0315	0.0068	0.2913
			θ	0.0421	0.1103	1.2922	96.40%	−0.0572	0.0915	1.1658	−0.1539	0.1178	1.2033	0.0032	0.0811	1.1233
	0.5	75	λ_1	−0.0037	0.0150	0.4810	94.80%	0.0147	0.0043	0.2391	0.0040	0.0039	0.2336	0.0257	0.0049	0.2447
			λ_2	0.0034	0.0262	0.6345	95.30%	0.0191	0.0072	0.3163	0.0030	0.0065	0.3090	0.0387	0.0088	0.3248
			θ	0.0701	0.1601	1.5450	95.10%	−0.0723	0.1109	1.1736	−0.1670	0.1410	1.2443	0.0142	0.1013	1.1556
		90	λ_1	−0.0025	0.0119	0.4280	95.40%	0.0123	0.0033	0.2112	0.0033	0.0030	0.2077	0.0214	0.0037	0.2152
			λ_2	0.0031	0.0210	0.5677	95.70%	0.0187	0.0059	0.2850	0.0028	0.0053	0.2791	0.0351	0.0070	0.2924
			θ	0.0490	0.1192	1.3405	95.60%	−0.0697	0.0959	1.1814	−0.1506	0.1213	1.2189	0.0049	0.0861	1.1466
	0.8	75	λ_1	0.0018	0.0142	0.4671	94.70%	0.0166	0.0043	0.2501	0.0057	0.0039	0.2437	0.0279	0.0050	0.2547
			λ_2	−0.0079	0.0491	0.8683	95.10%	0.0314	0.1665	0.3152	−0.0053	0.0067	0.3057	0.0786	0.0261	0.3280
			θ	0.0335	0.1402	1.4628	95.00%	−0.0968	0.1217	1.3807	−0.1922	0.1557	1.4195	−0.0085	0.1087	1.3358
		90	λ_1	−0.0007	0.0112	0.4147	95.20%	0.0122	0.0032	0.2046	0.0032	0.0029	0.2011	0.0214	0.0036	0.2089
			λ_2	0.0061	0.0236	0.6018	95.70%	0.0217	0.0068	0.3032	0.0051	0.0061	0.2959	0.0382	0.0081	0.3118
			θ	0.0304	0.1095	1.2874	96.00%	−0.0855	0.0924	1.1046	−0.1685	0.1213	1.1597	−0.0082	0.0798	1.0759

7. Applications

In this section, we discussed two bivariate real data sets. The FGM-BLBE distribution has been used based on progressive Type-II censoring with random removal.

7.1. Iron Material Jobs Data

The Iron material jobs data set has been obtained from Dasgupta [37]. In the context of tasks involving iron sheets, we employ a perforation procedure that involves drilling a total of four holes, two on each arm, in an L-shaped rectangular sheet measuring 100 mm by 150 mm. This efficient process is conducted on a 100-ton press operating at a speed of 250 strokes per hour, known as piercing. Simultaneously creating two holes at a time, the perforated L-shaped iron sheet is intended for use in mini- or light-duty truck chassis.

Following the piercing, a burr is formed in a circular shape around the hole on the opposite side of the metal sheet. The intense pressure applied during the piercing causes an irregular shift in the contact surface, leading to uneven and aggressive elevation of metal granules along the rim of the circular hole. The size of the burr is influenced by the characteristics of the metal grains and the piercing load. Skrotzki et al. [38] noted that factors such as composition, melting point, cooling pace, thermal and constitutional under-cooling, and convection all impact the grain structure and texture of metals.

Subsequently, the burr is removed through chamfering using a drill. The dial gauge, employed to measure burr in the provided data sets, has a minimum count of 20 microns (m), equivalent to 0.02 mm. For the first set of 50 observations on burr, the hole diameter and sheet thickness are 12 mm and 3.15 mm, respectively. In the second data set of 50 observations, the hole diameter and sheet thickness are 9 mm and 2 mm, respectively.

During the actual works, one hole is selected and positioned based on a predetermined orientation, and readings of the hole diameter are taken in relation to that specific hole. It is important to note that the two data sets are associated with the comparison of two computers. For additional details regarding data on iron material jobs, refer to Dasgupta [37]. Now, we used this data.

The MLE estimates and different measures for each bivariate models for iron material jobs data and MLE and Bayesian estimation under progressive type II with random removal for Iron material jobs data are presented in Tables 5 and 6, respectively, depicting various instances of the progressive type II scheme. Specifically, when m = 35 and p = 0.3, and 0.8, the corresponding value of R is shown as follows:

Data 1: m = 35, p = 0.3:
x: 0.02, 0.02, 0.04, 0.04, 0.06, 0.08, 0.08, 0.08, 0.12, 0.12, 0.12, 0.14, 0.14, 0.14, 0.14, 0.14, 0.14, 0.16, 0.16, 0.16, 0.16, 0.16, 0.18, 0.18, 0.18, 0.22, 0.22, 0.24, 0.24, 0.24, 0.24, 0.26, 0.26, 0.26, 0.32
y: 0.12, 0.16, 0.06, 0.04, 0.24, 0.16, 0.32, 0.18, 0.26, 0.06, 0.04, 0.14, 0.24, 0.22, 0.02, 0.24, 0.14, 0.06, 0.16, 0.22, 0.06, 0.16, 0.02, 0.14, 0.32, 0.24, 0.18, 0.16, 0.18, 0.14, 0.16, 0.22, 0.18, 0.26, 0.04
R: 4 3 2 1 1 1 0 3 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
m = 35, p = 0.8:
x: 0.02, 0.02, 0.04, 0.06, 0.08, 0.08, 0.08, 0.08, 0.12, 0.12, 0.12, 0.14, 0.14, 0.14, 0.14, 0.14, 0.16, 0.16, 0.16, 0.16, 0.16, 0.16, 0.18, 0.22, 0.22, 0.22, 0.24, 0.24, 0.24, 0.24, 0.24, 0.26, 0.26, 0.32, 0.32
y: 0.12, 0.16, 0.06, 0.24, 0.16, 0.32, 0.02, 0.18, 0.26, 0.06, 0.04, 0.14, 0.24, 0.02, 0.24, 0.14, 0.06, 0.16, 0.14, 0.06, 0.06, 0.16, 0.32, 0.08, 0.24, 0.14, 0.16, 0.12, 0.18, 0.14, 0.16, 0.22, 0.18, 0.22, 0.04
R: 3 10 2 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

Given the absence of information on the unknown parameters, we employ non-informative priors with a1 = b1 = a2 = b2 = 0 in this illustrative example. Additionally, we presume informative gamma priors for the parameters of FGM-BLBE. MLEs and BEs, along with their standard errors, are computed for both case 1 and case 2, and the results are presented in Table 6. This table reveals the point estimates for the unknown parameters λ_1, λ_2, and θ obtained through maximum likelihood and Bayesian methods.

The findings in Table 6 highlight that the HPD intervals are marginally shorter than the other confidence intervals.

The FGM-BLBE distribution is compared to the following bivariate models in Table 5 as FGM Kavya-Manoharan Pareto (FGMKMP) (Fayomi et al. [16]), Frank bivariate X Lindely (FBXL), Gumbel bivariate X Lindely (GBXL), Clayton bivariate X Lindely (CBXL), Clayton Kavya-Manoharan exponential (CKME) (Fayomi et al. [16]), FGM Fréchet (FGMF) (Almetwally and Muhammed [39]), AMH Fréchet (AMHF) (Almetwally and Muhammed [39]), and Bivariate FGM Generalized Exponential (FGMGE) (Abd Elaal and Jarwan [40]). Table 5 describes the MLEs of the distribution's parameters and presents the goodness-of-fit metrics: Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), Cramér-von Mises (CVM), and Anderson-Darling (AD). According to Table 5 findings, the FGM-BLBE distribution performs better than the FGMKMP, FBXL, CBXL, GBXL, FGMF, AMHF, and UEPFGMGE distributions for the iron material jobs data set.

Figure 3 presents histograms for each variable of the iron material jobs data, which are displayed in the diagonal elements, where the X-set, ranging from 0.02 to 0.32, exhibits variability with mean around 0.16. Distribution skewness is apparent, showcasing a diverse spread of values with some concentration around the median. While for Y-set, spanning values from 0.02 to 0.32, displays variability with a mean around 0.15. It exhibits some concentration around the median, indicating a skewed distribution with instances of recurring values, notably around 0.14 and 0.22. In the upper triangle of the matrix, numerical values represent the correlation coefficients between pairs of variables of iron material jobs data as 0.16. While in the lower triangle of the matrix, pairs panels function is particularly useful for exploring relationships between multiple variables simultaneously, providing a visual summary of the iron material jobs data to identify patterns, trends, and potential correlations.

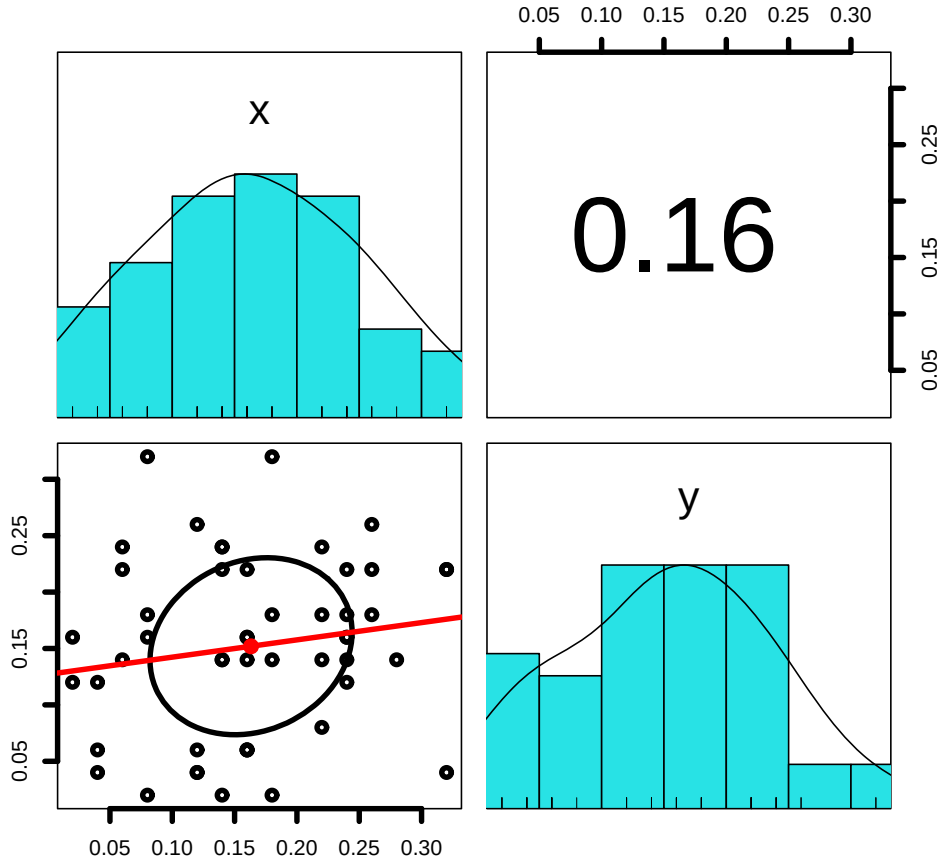


Figure 3. Matrix for iron material jobs data: Histograms, Correlation, and pairs panels.

Figure 4 presents scatter, contour, and density plots for the FGM (Farlie-Gumbel-Morgenstern) distribution applied to iron material jobs data under different censoring schemes. The scatter plots provide a visual representation of the relationship between variables under different censoring scenarios, showing how the data points distribute across the given parameters. The density plots offer insights into the probability density function of the data, highlighting areas with higher concentrations of probability mass, while the contour plots delineate the levels of the density function, helping to visualize the gradient and spread of the data across the parameter space.

The scatter plot in R visually depicts the relationship between variables X and Y of iron material jobs data in a three-dimensional space with different binomial removal as ($p = 0$, $p = 0.3$, and $p = 0.8$). The red dots represent data points, revealing patterns and trends in the distribution. This 3D scatter plot offers an insightful view of the dataset's structure and inter-variable interactions of iron material jobs data. The density plot of FGM copula, showcasing the FGM copula for variables X and Y, offers a captivating three-dimensional representation. This visualization vividly communicates the dependence structure, allowing for a nuanced understanding of the joint behavior of X and Y within the FGM copula framework with different copula parameter under different binomial removal as ($p = 0$, $p = 0.3$, and $p = 0.8$). The contour plot effectively visualizes the density of the FGM copula for variables X and Y. Contour lines delineate regions of similar density, revealing intricate patterns in the joint distribution. This plot provides valuable insights into the dependence structure between X and Y within the FGM copula framework for iron material jobs data.

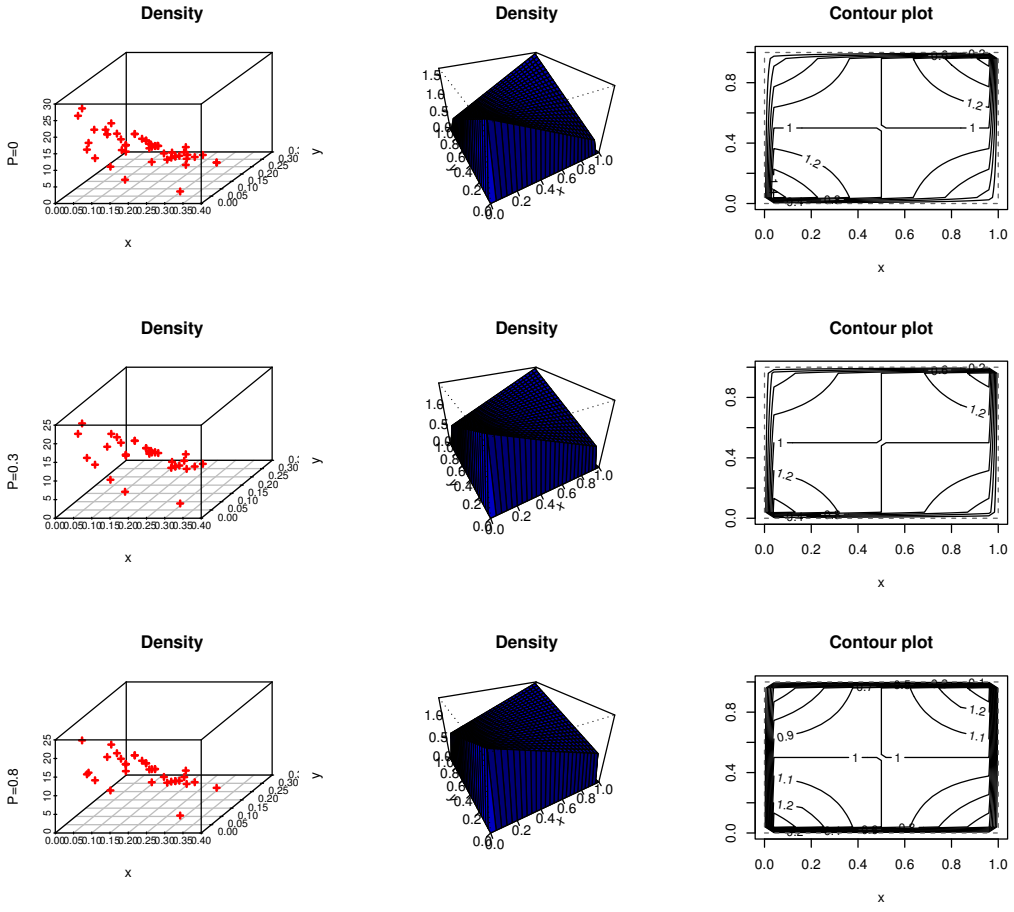


Figure 4. Scatter, contour, and density plots For FGM with iron material jobs data under different censoring schemes.

Table 5. MLE estimates and different measures for each bivariate models: iron material jobs data.

		λ_1	β_1	λ_2	β_2	θ	AIC	CAIC	BIC	HQIC	CVM	AD
FGM-BLBE	Estimtes St.Er	0.07959 0.00780	-	0.07398 0.00728	-	0.79705 0.46327	**−205.98594**	**−205.46420**	**−200.24987**	**−203.80161**	**7.30001**	**42.32165**
FGMKMP	Estimtes St.Er	6.01178 0.84166	-	6.59510 0.90998	-	2.34381 0.48686	−147.11384	−146.59210	−141.37777	−144.92951	7.30934	42.69029
FBXL	Estimtes St.Er	6.66253 0.91581	-	7.22629 0.99686	-	3.52154 1.16191	−170.55050	−170.02876	−164.81443	−168.36617	7.34487	42.41380
CBXL	Estimtes St.Er	6.82193 0.88092	-	7.43917 0.96401	-	1.00952 0.37817	−171.76838	−171.24664	−166.03231	−169.58405	7.34104	42.45320
GBXL	Estimtes St.Er	9.00178 0.97309	-	11.74399 1.01774	-	1.55245 0.16223	−84.14827	−83.62654	−78.41221	−81.96395	7.31618	42.37532
CKME	Estimtes St.Er	4.79516 0.73600	-	5.28292 0.81263	-	1.17668 0.41577	−158.81737	−158.29563	−153.08130	−156.63304	7.35802	42.49229
FGMF	Estimtes St.Er	0.09275 0.01131	1.24845 0.11868	0.08217 0.01049	1.19709 0.11698	0.89482 0.62744	−144.68363	−143.31999	−135.12351	−141.04308	7.32562	42.49152
AMHF	Estimtes St.Er	0.09391 0.01139	1.24202 0.11837	0.08291 0.01050	1.19088 0.11656	0.44762 0.28883	−144.12488	−142.76125	−134.56477	−140.48434	7.31517	42.40317
FGMGE	Estimtes St.Er	3.14750 0.70756	11.46258 1.56305	2.72129 0.59101	11.50583 1.61405	0.64172 0.42020	−203.87218	−202.50854	−194.31207	−200.23163	7.32889	42.31873

Table 6. MLE and Bayesian estimation under progressive type II with random removal: iron material jobs data.

		MLE			**Bayesian Gamma Prior**				**Bayesian Non-Informative Prior**			
p		**Estimtes**	**St.Er**	**LACI**	**Estimtes**	**St.Er**	**LCCI**	**CP**	**Estimtes**	**St.Er**	**LCCI**	**CP**
0.3	λ_1	0.0798	0.0094	0.0367	0.0817	0.0093	0.0357	0.9497	0.0853	0.0146	0.0559	0.9500
	λ_2	0.0779	0.0093	0.0363	0.0795	0.0091	0.0357	0.9495	0.0845	0.0155	0.0590	0.9498
	θ	0.5898	0.5932	2.3255	0.1853	0.4779	1.6533	0.9496	0.3857	0.6206	2.3137	0.9497
0.8	λ_1	0.0812	0.0097	0.0380	0.0827	0.0093	0.0368	0.9498	0.0863	0.0153	0.0577	0.9499
	λ_2	0.0736	0.0089	0.0348	0.0748	0.0087	0.0314	0.9498	0.0806	0.0155	0.0588	0.9498
	θ	0.4429	0.6454	2.5301	0.2317	0.4834	1.6612	0.9493	0.1951	0.7245	2.6355	0.9499

7.2. Computers Times

The data was obtained from Oliveira et al. [41]. The dataset comprises 50 simulated primitive computer series systems, each equipped with a processor and memory. The operational state of the computer relies on the proper functioning of both components. Assuming that the system undergoes a latent deterioration process, the deterioration advances rapidly over a short time frame, rendering the system more vulnerable to shocks. This heightened susceptibility allows a lethal shock to randomly damage either the first, second, or both components. Due to the possibility of a fatal shock affecting both components simultaneously, the reliability of the independence premise is questionable. Consequently, we employed the FGM copula to investigate this matter.

The FGM-BLBE distribution is compared to the following bivariate models in Table 7 as FGM Kavya-Manoharan Pareto (FGMKMP) (Fayomi et al. [16]), FGM Fréchet (FGMF) (Almetwally and Muhammed [39]), and AMH Fréchet (AMHF) (Almetwally and Muhammed [39]). Table 7 described the MLEs of the distribution's parameters and showed the goodness of fit metrics, AIC, CAIC, BIC, HQIC, CVM, and AD. According to Table 7 findings, the FGM-BLBE distribution performs better than the FGMKMP, FGMP, FGMF, and AMHF distributions for the computer times data set.

To identify outliers in the processor and memory data, we generated Figure 5 and employed matrix histogram, scatter plot with a boxplot to distinguish between different groups. Conversely, Figure 5 delved into the dependent plot of processor and memory data, highlighting dependencies and the density distribution of numerical data. The examination of Figure 5 reveals that the data exhibits right-skewed shapes, asymmetry, and a weak negative correlation.

The MLE estimates and various metrics for individual bivariate models in computer times data, as well as the MLE and Bayesian estimates under the progressive Type II with random removal, are displayed in Tables 7 and 8. These tables represent different scenarios of the progressive Type II scheme, particularly when m = 40 and p = 0.3, and 0.8. The associated value of R is provided for each case.

Data set 2: Case 1: m = 40 p = 0.3:
x: 0.1058, 0.1058, 0.1115, 0.1955, 0.3309, 0.4614, 0.5520, 0.6270, 0.7947, 0.8584, 0.9096, 0.9867, 0.9867, 0.9938, 1.0051, 1.0833, 1.0833, 1.0986, 1.1739, 1.1917, 1.3288, 1.3482, 1.5254, 1.6465, 1.7494, 1.9292, 2.1000, 2.1396, 2.3364, 2.3757, 2.5913, 3.3887, 3.5202, 3.6621, 3.6621, 3.6621, 4.3435, 5.0533, 5.7561, 5.7561.
y: 0.1058, 0.1058, 0.1115, 0.1955, 0.3309, 0.8584, 0.5520, 1.7289, 0.7947, 1.9556, 0.6214, 0.9867, 0.9867, 1.7689, 1.0051, 3.3059, 3.3059, 1.0986, 3.3857, 0.0801, 0.9689, 1.9705, 4.4088, 2.0197, 2.3643, 3.9291, 2.0513, 2.1548, 0.1624, 2.7953, 2.5913, 1.9796, 1.4095, 0.0026, 0.0026, 0.0026, 1.0001, 2.3238, 0.3212, 0.3212.
R: 3 2 1 1 0 1 0 2 0 0 0

peaks around 3.3059 and 4.0043. Mean-centered at 1.3983, the distribution suggests diverse patterns. In the upper triangle of the matrix, numerical values represent the correlation coefficients between pairs of variables of computers times data as −0.03 (negative week correlation). While in the lower triangle of the matrix, pairs panels function is particularly useful for exploring relationships between multiple variables simultaneously, providing a visual summary of the computers times data to identify patterns, trends, and potential correlations. Figure 6 discussed Scatter, contour, and density plots For FGM copula for computers times data under different censoring schemes.

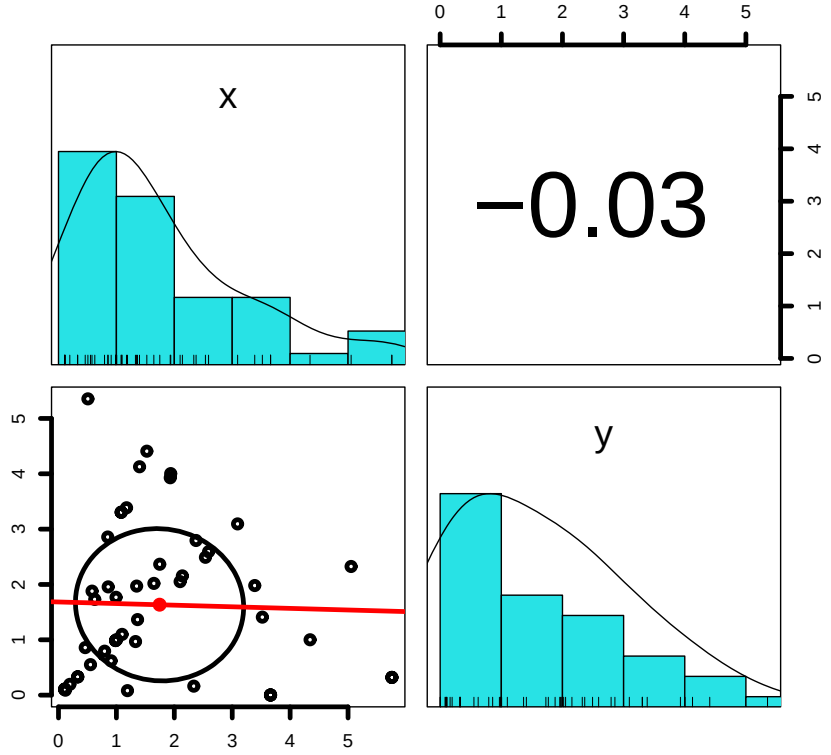


Figure 5. Correlation matrix for computers times data II

Table 7. MLE estimates and different measures for each bivariate models: Data II.

	FGM-BLBE		**FGMKMP**		**FGMP**		**FGMF**		**AMHF**	
	Estimtes	**St.Er**	**Estimtes**	**St.Er**	**Estimtes**	**St.Er**	**Estimtes**	**St.Er**	**Estimtes**	**St.Er**
λ_1	0.85614	0.08562	0.84594	0.12999	1.19089	0.16104	0.63528	0.11099	0.62308	0.10579
β_1	-		-		-		0.88858	0.08887	0.88787	0.08895
λ_2	0.81578	0.08100	0.97325	0.15444	1.31475	0.17978	0.24555	0.08925	0.28082	0.10113
β_2	-		-		-		0.42012	0.03889	0.41604	0.03737
θ	0.35529	0.26528	0.94248	0.34217	0.85556	0.32445	0.72727	0.45871	0.67755	0.18285
AIC	350.69989		354.05397		352.84962		386.88554		385.38092	
CAIC	351.22163		354.57571		363.37136		388.24918		386.74456	
BIC	356.43596		359.79004		358.58569		396.44566		394.94104	
HQIC	352.88422		356.23830		355.03395		390.52609		389.02147	
CVM	7.67624		7.71419		7.72638		7.91517		7.81652	
AD	42.49362		42.73824		42.56708		43.05157		42.91217	

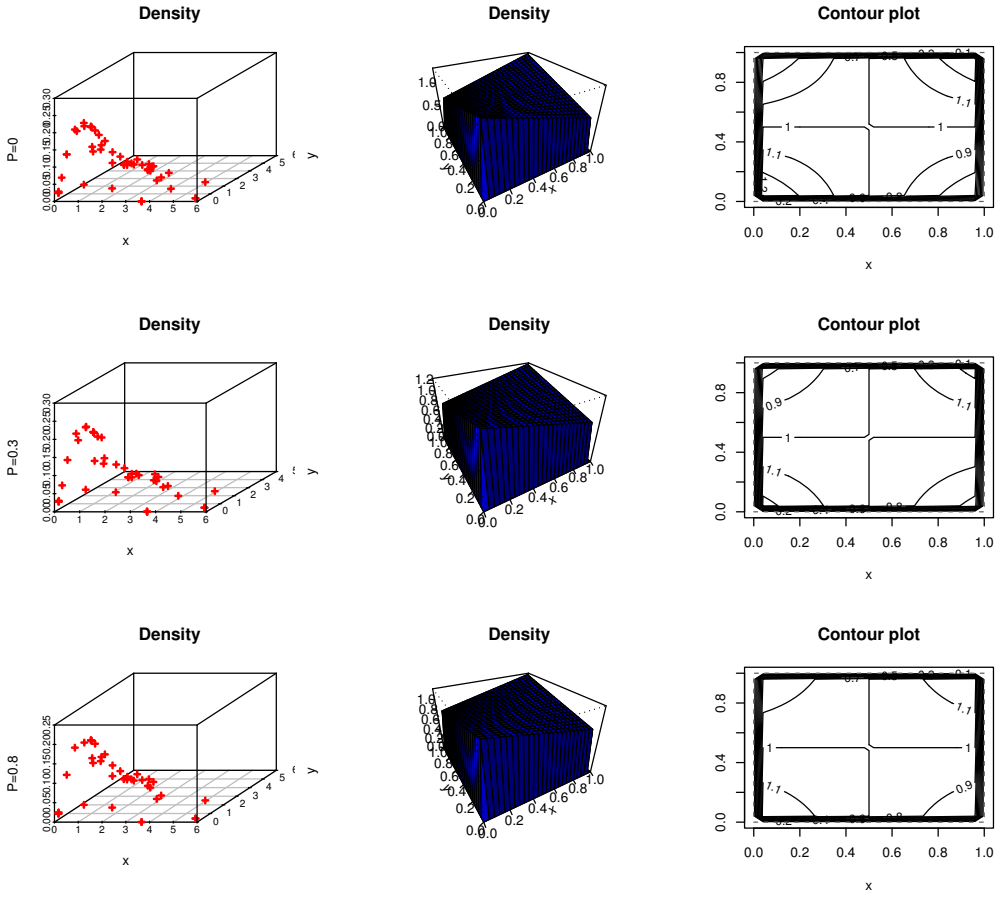


Figure 6. Scatter, countour, and density plots For FGM with computer times data II under different censoring schemes.

Table 8. MLE and Bayesian estimation under progressive type II with random removal: Data 2.

		MLE			**Bayesian Gamma Prior**				**Bayesian Non-informative Prior**			
p		**Estimtes**	**St.Er**	**LACI**	**Estimtes**	**St.Er**	**LCCI**	**CP**	**Estimtes**	**St.Er**	**LCCI**	**CP**
0.3	λ_1	0.9280	0.1039	0.4074	0.9413	0.1053	0.4092	0.9498	0.9899	0.1639	0.6295	0.9500
	λ_2	0.7037	0.0784	0.3075	0.7122	0.0772	0.3004	0.9499	0.7403	0.1224	0.4713	0.9499
	θ	0.2755	0.3158	1.2378	0.3349	0.3382	1.2785	0.9496	0.1189	0.3314	1.2519	0.9499
0.8	λ_1	0.8469	0.0955	0.3743	0.8618	0.0955	0.3754	0.9499	0.9042	0.1529	0.5727	0.9499
	λ_2	0.8499	0.0948	0.3717	0.8616	0.0953	0.3767	0.9499	0.8886	0.1429	0.5405	0.9500
	θ	0.2316	0.3180	1.2464	0.2318	0.0201	1.4030	0.9499	0.2316	0.0203	1.2366	0.9499

8. Conclusions

In this paper, we explored both Bayesian and non-Bayesian methods for estimating the parameters of the FGM-BLBE distribution when data is subject to progressive Type-II censoring with random removal. The binomial removal method was employed to eliminate failure times and generate censored samples. Our findings are relevant to various real-world applications. Beyond traditional confidence intervals, we examined bootstrap confidence intervals using both the percentile and bootstrap-t methods. Our simulations revealed that Bayesian estimation using MCMC—particularly under the LINEX loss function—yielded accurate parameter estimates for the FGM-BLBE distribution. This conclusion is supported by low bias, MSE, and narrow confidence intervals. Importantly, increasing the number of censored samples (m) further improved the accuracy of all estimators. Notably, Bayesian estimation outperformed non-Bayesian methods, and Bayes credible intervals proved

more reliable than asymptotic confidence intervals. To illustrate practical applications, we presented two real-world examples from industry and computer science, demonstrating successful parameter estimation within the FGM-BLBE distribution.

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