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Some New Algebraic Method Developments in the Characterization of Matrix Equalities

Yongge Tian

College of Business and Economics, Shanghai Business School, Shanghai 201400, China; tyg@sbs.edu.cn

Abstract: Algebraic expressions and equalities can be constructed arbitrarily in a given algebraic framework according to the operational rules provided, and thus it is a prominent and necessary task in mathematics and applications to construct, classify, and characterize various simple general algebraic expressions and equalities. As an update to this prominent topic in matrix algebra, this article reviews and improves upon the well-known block matrix methodology and matrix rank methodology in the construction and characterization of matrix equalities. We present a collection of fundamental and useful formulas for calculating the ranks of a wide range of block matrices and then derive from these rank formulas various valuable consequences. In particular, we present several groups of equivalent conditions in the characterizations of the Hermitian matrix, the skew-Hermitian matrix, the normal matrix, etc.

Keywords: block matrix; Hermitian matrix; generalized inverse; matrix equality; normal matrix; range; rank

MSC: 15A09; 15A24

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1. Introduction

Throughout this article, the symbol $\mathbb{C}^{m\times n}$ denotes the collection of all $m \times n$ matrices over the field of complex numbers, A^* denotes the conjugate transpose of $A \in \mathbb{C}^{m\times n}$, $r(A)$ denotes the rank of $A \in \mathbb{C}^{m\times n}$, i.e., the maximum order of the invertible submatrix of A, and $\mathscr{R}(A) = \{Ax \,|\, x \in \mathbb{C}^n\}$ denotes the range (column space) of a matrix $A \in \mathbb{C}^{m\times n}$. A matrix $A \in \mathbb{C}^{m\times m}$ is said to be Hermitian if and only if $A = A^*$; to be skew-Hermitian if and only if $A = -A^*$; to be normal if and only if $AA^* = A^*A$; to be EP if and only if $\mathscr{R}(A) = \mathscr{R}(A^*)$. The Moore–Penrose generalized inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the four Penrose equations:

$$AXA = A, \;\; XAX = X, \;\; (AX)^* = AX, \;\; (XA)^* = XA. \tag{1}$$

A square matrix $A \in \mathbb{C}^{m\times m}$ is said to be group invertible if and only if there exists an $X \in \mathbb{C}^{m\times m}$ satisfying the following three matrix equations:

$$AXA = A, \;\; XAX = X, \;\; AX = XA. \tag{2}$$

The matrix X satisfying the three equations, called the group inverse of A, is unique and is denoted by $X = A^\#$. As we know, the theory of generalized inverses of matrices, as ubiquitous tool to deal with singular matrices, has been widely utilized to approach various complicated theoretical and computational problems in mathematics and applications. The most important fact is that we can use generalized inverses of matrices in the case when ordinary inverses of matrices do not exist in order to deal with general matrix equations. As shown above, generalized inverses can be defined to be common solutions to one or more algebraic equations as certain extensions of the ordinary inverses of matrices. In comparison, the Moore–Penrose inverse and the group inverse of a matrix are two highly

recognized generalized inverses, which are known to have many remarkable algebraic and computational properties, and thus they have been widely studied in mathematics and other applications. For more detailed information regarding generalized inverses of matrices, we refer the reader to [1–3].

Let us start with the introduction of the problem considered in this article. Given two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same size, the matrix equality $A = B$ is directly defined by the fact that $a_{ij} = b_{ij}$ holds for all elements in A and B. In matrix theory and applications, we may encounter various problems with solving a simple or general matrix equation $f(A, B) = g(A, B)$, where $f(\cdot)$ and $g(\cdot)$ are certain algebraic operations of two given matrices A and B of appropriate size. One of the specified cases related to this type of problem can be formulated as:

$$f(A,\, B) = 0 \Leftrightarrow A = B; \tag{3}$$

namely, $A = B$ is the unique solution to the matrix equation on the left-hand side. Due to the arbitrariness of the matrix expression $f(A,\, B)$, there does not exist a general and significant method to construct matrix equations such as (3) or to find their solutions, except in some special cases (cf. [4,5]).

Increasingly complex research on algebraic expressions and equalities has created a need for sophisticated methods that go beyond conventional operations of elements in a given algebraic framework. The aim of this article is to review some well-known matrix rank equalities in the theory of generalized inverses and to establish a diversity of new expansion formulas for calculating ranks of many specified block matrices. We also provide some novel and insightful methods for constructing and characterizing matrix equalities and use them in the study of problems such as (3).

The article is organized as follows. In Section 2, we introduce some established useful formulas, results, and facts regarding ranks, ranges, and generalized inverses of matrices in linear algebra and matrix theory. In Sections 3 and 4, we give a series of well-known or novel concrete block matrices, calculate their ranks, and derive various consequences and applications from the rank formulas, including the characterization of the Hermitian/skew-Hermitian matrix, normal matrix, etc. Conclusions are given in Section 5.

2. Preliminary Results

In this section, we recall some basic or well-known simple general formulas, results, and facts related to ranks, ranges, and generalized inverses of matrices in linear algebra and matrix theory that we need to use in the remainder of the paper. We first describe the simple but intrinsic idea of studying matrix equalities that have been frequently used by algebraists in the past several decades. Recall that the rank of a matrix, or the dimension of the subspaces generated by the row or column of a matrix, is one of the most fundamental concepts in linear algebra. It is straightforward to know from the definition of the rank of matrix that $A = 0$ if and only if $r(A) = 0$. As a consequence, we see that the matrix equality $A = B$ holds if and only if $r(A - B) = 0$ holds for two given matrices of the same size. In order to use this proposed assertion in the characterization of the matrix equalities in this paper, we need to use the following well-known results on matrix rank equalities and their consequences.

Lemma 1 ([6]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, and $D \in \mathbb{C}^{l\times k}$. Then, the following rank equalities hold*:

$$r[\, A,\, B\,] = r(A) + r(B - AA^{\dagger}B) = r(B) + r(A - BB^{\dagger}A), \tag{4}$$

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r\begin{bmatrix} A & B - AA^{\dagger}B \\ C - CA^{\dagger}A & D - CA^{\dagger}B \end{bmatrix} = r(A) + r\begin{bmatrix} 0 & B - AA^{\dagger}B \\ C - CA^{\dagger}A & D - CA^{\dagger}B \end{bmatrix}, \tag{5}$$

and the following facts hold:

$$r[A,\ B] = r(A) = r(B) \Leftrightarrow \mathscr{R}(A) = \mathscr{R}(B) \Leftrightarrow AA^{\dagger} = BB^{\dagger}, \tag{6}$$

$$r[A,\ B] = r(A) + r(B) \Leftrightarrow \mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}, \tag{7}$$

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) \Leftrightarrow AA^{\dagger}B = B,\ \ CA^{\dagger}A = C \text{ and } CA^{\dagger}B = D. \tag{8}$$

In particular, if $\mathscr{R}(A) \supseteq \mathscr{R}(B)$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(C^*)$, *then*

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} - r(A) = r(D - CA^{\dagger}B), \tag{9}$$

and the following fact holds:

$$CA^{\dagger}B = D \Leftrightarrow r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A). \tag{10}$$

Lemma 2 ([7]). *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, *and* $D \in \mathbb{C}^{l\times k}$. *Then, the following rank equality holds*:

$$r(D - CA^{\dagger}B) = r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} - r(A), \tag{11}$$

and following fact holds:

$$CA^{\dagger}B = D \Leftrightarrow r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} = r(A).$$

The facts and results in the following three lemmas are obvious or well-known in matrix theory (cf. [1,8]).

Lemma 3. *Let* $A_1 \in \mathbb{C}^{m\times n_1}$, $B_1 \in \mathbb{C}^{m\times p_1}$, $A_2 \in \mathbb{C}^{m\times n_2}$, *and* $B_2 \in \mathbb{C}^{m\times p_2}$. *If* $\mathscr{R}(A_1) = \mathscr{R}(A_2)$ *and* $\mathscr{R}(B_1) = \mathscr{R}(B_2)$, *then* $r[A_1,\ B_1] = r[A_2,\ B_2]$ *holds.*

Lemma 4. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{m\times p}$. *Then, the following three statements are equivalent*:

(a) *The matrix equation* $AX = B$ *is solvable for* X.
(b) $\mathscr{R}(A) \supseteq \mathscr{R}(B)$.
(c) $r[A,\ B] = r(A)$.

Lemma 5. *The following results hold*:

(I) *Let* $A, B \in \mathbb{C}^{m\times m}$ *be two normal matrices of the same size. Then, A and B can simultaneously be unitarily diagonalizable if and only if* $AB = BA$.
(II) $\mathscr{R}(A) = \mathscr{R}(A^*) \Leftrightarrow AA^{\dagger} = A^{\dagger}A$.
(III) $r(A^2) = r(A) \Leftrightarrow A^k(A^k)^{\dagger} = AA^{\dagger} \Leftrightarrow (A^k)^{\dagger}A^k = A^{\dagger}A$ *for* $k \geq 2$.

3. How to Establish Equalities for Ranks of Block Matrices

As described at the beginning of Section 2, the usefulness of matrix rank formulas in the characterization of matrix equalities has been noted by some earlier mathematicians. In particular, in an earlier paper [6], a wide variety of fundamental matrix rank equalities were systematically established, and various remarkable applications of the matrix rank methodology were proposed and established in the characterizations of matrix equalities composed of generalized inverses. In the past several decades, the original work developed in the seminal paper [6] has been viewed as one of the essential contributions to the current development of matrix theory and applications, and indeed, some important and valuable studies have been done in this domain. As a fruitful investigation in this respect, we intend to establish in this section a wide range of simple and useful formulas for calculating the

ranks of certain specified partitioned matrices and present their essential consequences in characterizing a series of basic matrix equalities.

Theorem 1. *Let A, $B \in \mathbb{C}^{m\times n}$. Then, the following rank equalities hold*:

$$\begin{aligned} r(A - AB^{\dagger}A) &= r\begin{bmatrix} B^*BB^* & B^*A \\ AB^* & A \end{bmatrix} - r(B) \\ &= r(B^*AB^* - B^*BB^*) + r(A) - r(B). \end{aligned} \tag{12}$$

If $\mathscr{R}(A) \supseteq \mathscr{R}(B)$ and $\mathscr{R}(A^*) \supseteq \mathscr{R}(B^*)$, then

$$r(A - AB^{\dagger}A) = r\begin{bmatrix} B & A \\ A & A \end{bmatrix} - r(B) = r(A - B) + r(A) - r(B). \tag{13}$$

If $\mathscr{R}(A) = \mathscr{R}(B)$ and $\mathscr{R}(A^*) = \mathscr{R}(B^*)$, then

$$r(A - AB^{\dagger}A) = r(B - BA^{\dagger}B) = r(A - B), \tag{14}$$

$$r(A^{\dagger} - A^{\dagger}BA^{\dagger}) = r(B^{\dagger} - B^{\dagger}AB^{\dagger}) = r(A^{\dagger} - B^{\dagger}). \tag{15}$$

In particular, for $A \in \mathbb{C}^{m\times m}$, the following rank equality holds:

$$\begin{aligned} r\begin{bmatrix} A & A(A^{\dagger})^* \\ (A^{\dagger})^*A & (A^{\dagger})^* \end{bmatrix} &= r(A) + r((A^{\dagger})^* - (A^{\dagger})^*A(A^{\dagger})^*) \\ &= r(A) + r(A - A(A^{\dagger})^*A), \end{aligned} \tag{16}$$

$$r(A - A(A^{\dagger})^*A) = r(A^3 - AA^*A). \tag{17}$$

Hence, $A(A^{\dagger})^*A = A \Leftrightarrow (A^{\dagger})^*A(A^{\dagger})^* = (A^{\dagger})^* \Leftrightarrow A^3 = AA^*A$.

Proof. The first equality in (12) directly follows from (11). Consequently, simplifying it leads to the second equality in (12). Equation (13) directly follows from (9). Note that $\mathscr{R}(A) = \mathscr{R}(B)$ and $\mathscr{R}(A^*) = \mathscr{R}(B^*)$ are equivalent to $AA^{\dagger} = BB^{\dagger}$, $A^{\dagger}A = B^{\dagger}B$, and $r(A) = r(B)$ by (6). Thus, (14) and (15) follow from (13). Equation (16) follows from (9). Equation (17) follows from (12) and from the simple fact $r((A^{\dagger})^*) = r(A^{\dagger}) = (A)$. □

Theorem 2. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, and $D \in \mathbb{C}^{l\times k}$. Then, the following facts hold*:

(a) $\mathscr{R}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \mathscr{R}\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} = \mathscr{R}\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \Leftrightarrow \mathscr{R}\begin{bmatrix} A^* \\ B^* \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} C^* \\ D^* \end{bmatrix} = \{0\}$, $\mathscr{R}(A) = \mathscr{R}(B)$, *and* $\mathscr{R}(C) = \mathscr{R}(D)$.

(b) $r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix} = r\begin{bmatrix} B \\ D \end{bmatrix} = r[A,\ B] = r[C,\ D] \Leftrightarrow r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) = r(B) = r(C) = r(D)$.

(c) *If* $r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) = r(B) = r(C)$, *then* $r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(D)$.

Proof. It is easily seen that the first range equalities in Result (a) imply:

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r(D) = r(B) + r(C), \tag{18}$$

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r\begin{bmatrix} A & 0 & 0 & B \\ 0 & D & C & 0 \end{bmatrix} = r[A,\ B] + r[C,\ D], \tag{19}$$

where (19) implies $\mathscr{R}\begin{bmatrix}A^*\\B^*\end{bmatrix} \cap \mathscr{R}\begin{bmatrix}C^*\\D^*\end{bmatrix} = \{0\}$ by (7). Combining (18) and (19) leads to:

$$r[A, B] + r[C, D] = r(A) + r(D) = r(B) + r(C),$$

which further implies that:

$$r[A, B] = r(A) = r(B), \;\; r[C, D] = r(C) = r(D),$$

by noticing that $r[A, B] \geq r(A)$, $r[A, B] \geq r(B)$, $r[C, D] \geq r(C)$, and $r[C, D] \geq r(D)$. Therefore, the last two range equalities in Result (a) hold. Conversely, by (7), the three range equalities on the right-hand side of Result (a) imply that:

$$r\begin{bmatrix}A & B\\C & D\end{bmatrix} = r[A, B] + r[C, D] = r(A) + r(D) = r(B) + r(C).$$

Correspondingly,

$$r\begin{bmatrix}A & B & A & 0\\C & D & 0 & D\end{bmatrix} = r\begin{bmatrix}A & 0\\0 & D\end{bmatrix} = r(A) + r(D) = r\begin{bmatrix}A & B\\C & D\end{bmatrix},$$

$$r\begin{bmatrix}A & B & 0 & B\\C & D & C & 0\end{bmatrix} = r\begin{bmatrix}0 & B\\C & 0\end{bmatrix} = r(B) + r(C) = r\begin{bmatrix}A & B\\C & D\end{bmatrix}.$$

These rank equalities imply that the first two range equalities in Result (a) hold by (6).

The first four rank equalities in Result (b) imply:

$$\mathscr{R}\begin{bmatrix}A\\C\end{bmatrix} = \mathscr{R}\begin{bmatrix}B\\D\end{bmatrix}, \;\; \mathscr{R}\begin{bmatrix}A^*\\B^*\end{bmatrix} = \mathscr{R}\begin{bmatrix}C^*\\D^*\end{bmatrix}$$

by (6), and further imply $\mathscr{R}(A) = \mathscr{R}(B)$, $\mathscr{R}(C) = \mathscr{R}(D)$, $\mathscr{R}(A^*) = \mathscr{R}(C^*)$, and $\mathscr{R}(B^*) = \mathscr{R}(D^*)$. In this case, the first four rank equalities are reduced to the rank equalities on the right-hand side of Result (b). Conversely, combining the following simple rank inequalities:

$$r\begin{bmatrix}A & B\\C & D\end{bmatrix} \geq r[A, B] \geq r(A), \;\; r\begin{bmatrix}A & B\\C & D\end{bmatrix} \geq r[A, B] \geq r(B),$$
$$r\begin{bmatrix}A & B\\C & D\end{bmatrix} \geq r[C, D] \geq r(C), \;\; r\begin{bmatrix}A & B\\C & D\end{bmatrix} \geq r[C, D] \geq r(D),$$
$$r\begin{bmatrix}A & B\\C & D\end{bmatrix} \geq r\begin{bmatrix}A\\C\end{bmatrix} \geq r(A), \quad r\begin{bmatrix}A & B\\C & D\end{bmatrix} \geq r\begin{bmatrix}A\\C\end{bmatrix} \geq r(C),$$
$$r\begin{bmatrix}A & B\\C & D\end{bmatrix} \geq r\begin{bmatrix}B\\D\end{bmatrix} \geq r(B), \quad r\begin{bmatrix}A & B\\C & D\end{bmatrix} \geq r\begin{bmatrix}B\\D\end{bmatrix} \geq r(D)$$

with the rank equalities on the right-hand side of Result (b) yields the first four rank equalities on the left-hand side of Result (b). Result (c) is a direct consequence of Result (b). □

As we know, constructive methodology in the field of mathematics is an effective strategy when solving certain specified mathematical proof problems. This method is particularly suitable for approaching the given topics that are difficult to deal with by means of conventional analysis methods. The methodology usually requires observing, analyzing, and understanding research objects from new perspectives and viewpoints based on the characteristics and properties of the problem conditions and conclusions. By firmly grasping the intrinsic connection between the conditions and conclusions that reflect the given problem, utilizing the characteristics of the problem and known mathematical relationships and theories as tools, a mathematical object that satisfies the conditions or conclusions is then constructed properly. In this way, the implicit relationships and proper-

ties in the original problem are clearly presented in the newly constructed mathematical object, making it convenient and efficient to solve mathematical problems.

In what follows, we construct a selection of block matrices, calculate their ranks, and derive various consequences from the rank formulas.

Theorem 3. *Let A, $B \in \mathbb{C}^{m\times n}$. Then, the following rank equalities hold*:

$$r\begin{bmatrix} A & B \\ B & A \end{bmatrix} = r(A+B) + r(A-B), \tag{20}$$

$$r\begin{bmatrix} A & -B \\ B & A \end{bmatrix} = r(A+iB) + r(A-iB), \tag{21}$$

$$r\begin{bmatrix} A+B & A-B \\ A-B & A+B \end{bmatrix} = r(A) + r(B), \tag{22}$$

$$r\begin{bmatrix} A+B & B-A \\ A-B & A+B \end{bmatrix} = r((1+i)A+(1-i)B) + r((1-i)A+(1+i)B), \tag{23}$$

where $i = \sqrt{-1}$.

Proof. Equation (20) follows from the following block matrix decomposition equality:

$$\begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix}\begin{bmatrix} A & B \\ B & A \end{bmatrix}\begin{bmatrix} I_n & I_n \\ I_n & -I_n \end{bmatrix} = 2\begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix} \tag{24}$$

and the fact that the block matrices $\begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix}$ and $\begin{bmatrix} I_n & I_n \\ I_n & -I_n \end{bmatrix}$ are obviously nonsingular. Replacing B in (20) with iB and noting that

$$r\begin{bmatrix} A & iB \\ iB & A \end{bmatrix} = r\begin{bmatrix} A & i^2B \\ B & A \end{bmatrix} = r\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

leads to (21). Replacing A and B in (20) and (21) with $A+B$ and $A-B$ and simplifying leads to (22) and (23), respectively. □

Observe that the construction of matrix equality in (24) is simple and explicit so that the reader can properly understand the occurrence of the two basic rank expansion formulas in (20)–(23) and their consequences. Obviously, (20)–(23) reveal certain essential links between the two specified block matrices and the operations of their sub-matrices so that we are able to derive from them a series of possible new matrix rank equalities by choosing A and B as certain specified forms. As some illustrative examples, we give below diverse expansion formulas for calculating the ranks of several concrete block matrices.

Corollary 1. *Let A, $B \in \mathbb{C}^{m\times m}$ and let integer $k \geq 2$. Then, the following rank equalities hold*:

$$r\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}^k\right) = r((A+B)^k) + r((A-B)^k), \tag{25}$$

$$r\left(\begin{bmatrix} A & -B \\ B & A \end{bmatrix}^k\right) = r((A+iB)^k) + r((A-iB)^k), \tag{26}$$

$$r\left(\begin{bmatrix} A+B & A-B \\ A-B & A+B \end{bmatrix}^k\right) = r(A^k) + r(B^k), \tag{27}$$

$$r\left(\begin{bmatrix} A+B & B-A \\ A-B & A+B \end{bmatrix}^k\right) = r(((1+i)A+(1-i)B)^k) + r(((1-i)A+(1+i)B)^k). \tag{28}$$

Proof. Follows from expanding the left-hand sides of the matrices in (25)–(28) and applying (20)–(23). □

Theorem 4. *Let $A, B \in \mathbb{C}^{m\times n}$, and $C, D \in \mathbb{C}^{n\times p}$. Then, the following rank equalities hold*:

$$r\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\begin{bmatrix} C & D \\ D & C \end{bmatrix}\right) = r((A+B)(C+D)) + r((A-B)(C-D)), \tag{29}$$

$$r\left(\begin{bmatrix} A & -B \\ B & A \end{bmatrix}\begin{bmatrix} C & -D \\ D & C \end{bmatrix}\right) = r((A+iB)(C+iD)) + r((A-iB)(C-iD)). \tag{30}$$

Proof. Follows from expanding the left-hand sides of the matrices in (29) and (30) and applying (20) and (21). □

Theorem 5. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, and $D \in \mathbb{C}^{l\times k}$. Then, the following rank equalities hold*:

$$r\begin{bmatrix} A & B & A & 0 \\ C & D & 0 & 0 \\ A & 0 & A & B \\ 0 & 0 & C & D \end{bmatrix} = r\begin{bmatrix} 2A & B \\ C & D \end{bmatrix} + r\begin{bmatrix} 0 & B \\ C & D \end{bmatrix}, \tag{31}$$

$$r\begin{bmatrix} A & B & 0 & 0 \\ C & D & 0 & D \\ A & 0 & A & B \\ 0 & D & C & D \end{bmatrix} = r\begin{bmatrix} A & B \\ C & 2D \end{bmatrix} + r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \tag{32}$$

$$r\begin{bmatrix} A & B & A & 0 \\ C & D & 0 & D \\ A & 0 & A & B \\ 0 & D & C & D \end{bmatrix} = r\begin{bmatrix} 2A & B \\ C & 2D \end{bmatrix} + r(B) + r(C), \tag{33}$$

$$r\begin{bmatrix} A & B & -A & 0 \\ C & D & 0 & D \\ -A & 0 & A & B \\ 0 & D & C & D \end{bmatrix} = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + r\begin{bmatrix} 0 & B \\ C & D \end{bmatrix}, \tag{34}$$

$$r\begin{bmatrix} A & B & 0 & B \\ C & D & C & 0 \\ 0 & B & A & B \\ C & 0 & C & D \end{bmatrix} = r\begin{bmatrix} A & 2B \\ 2C & D \end{bmatrix} + r(A) + r(D), \tag{35}$$

$$r\begin{bmatrix} A & B & 0 & -B \\ C & D & C & 0 \\ 0 & -B & A & B \\ C & 0 & C & D \end{bmatrix} = r\begin{bmatrix} A & 0 \\ C & D \end{bmatrix} + r\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}, \tag{36}$$

$$r\begin{bmatrix} A & B & 0 & B \\ C & D & C & D \\ 0 & B & A & B \\ C & D & C & D \end{bmatrix} = r\begin{bmatrix} A & 2B \\ 2C & 2D \end{bmatrix} + r(A), \tag{37}$$

$$r\begin{bmatrix} A & B & A & B \\ C & D & C & 0 \\ A & B & A & B \\ C & 0 & C & D \end{bmatrix} = r\begin{bmatrix} 2A & 2B \\ 2C & D \end{bmatrix} + r(D), \tag{38}$$

$$r\begin{bmatrix} 2A & B & A & B \\ C & 2D & C & D \\ A & B & 2A & B \\ C & D & C & 2D \end{bmatrix} = r\begin{bmatrix} 3A & 2B \\ 2C & 3D \end{bmatrix} + r(A) + r(D), \tag{39}$$

$$r\begin{bmatrix} 2A & B & A & 2B \\ C & 2D & 2C & D \\ A & 2B & 2A & B \\ 2C & D & C & 2D \end{bmatrix} = 2r\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{40}$$

$$r\begin{bmatrix} A & B & -A & B \\ C & D & C & D \\ -A & B & A & B \\ C & D & C & D \end{bmatrix} = r\begin{bmatrix} 0 & B \\ C & D \end{bmatrix} + r(A), \tag{41}$$

$$r\begin{bmatrix} A & B & -A & B \\ C & D & C & -D \\ -A & B & A & B \\ C & -D & C & D \end{bmatrix} = r(A) + r(B) + r(C) + r(D), \tag{42}$$

$$r\begin{bmatrix} 3A & B & -A & B \\ C & 3D & C & -D \\ -A & B & 3A & B \\ C & -D & C & 3D \end{bmatrix} = r\begin{bmatrix} A & B \\ C & D \end{bmatrix} + r(A) + r(D), \tag{43}$$

$$r\begin{bmatrix} A & B & -A & 0 \\ C & D & 0 & 0 \\ A & 0 & A & B \\ 0 & 0 & C & D \end{bmatrix} = r\begin{bmatrix} (1+i)A & B \\ C & D \end{bmatrix} + r\begin{bmatrix} (1-i)A & B \\ C & D \end{bmatrix}, \tag{44}$$

$$r\begin{bmatrix} A & B & 0 & 0 \\ C & D & 0 & -D \\ 0 & 0 & A & B \\ 0 & D & C & D \end{bmatrix} = r\begin{bmatrix} A & B \\ C & (1+i)D \end{bmatrix} + r\begin{bmatrix} A & B \\ C & (1-i)D \end{bmatrix}, \tag{45}$$

$$r\begin{bmatrix} A & B & -A & 0 \\ C & D & 0 & -D \\ A & 0 & A & B \\ 0 & D & C & D \end{bmatrix} = r\begin{bmatrix} (1+i)A & B \\ C & (1+i)D \end{bmatrix} + r\begin{bmatrix} (1-i)A & B \\ C & (1-i)D \end{bmatrix}, \tag{46}$$

$$r\begin{bmatrix} A & B & 0 & -B \\ C & D & -C & 0 \\ 0 & B & A & B \\ C & 0 & C & D \end{bmatrix} = r\begin{bmatrix} A & (1+i)B \\ (1+i)C & D \end{bmatrix} + r\begin{bmatrix} A & (1-i)B \\ (1-i)C & D \end{bmatrix}, \tag{47}$$

$$r\begin{bmatrix} A & B & 0 & -B \\ C & D & -C & -D \\ 0 & B & A & B \\ C & D & C & D \end{bmatrix} = r\begin{bmatrix} A & (1+i)B \\ C & D \end{bmatrix} + r\begin{bmatrix} A & (1-i)B \\ C & D \end{bmatrix}, \tag{48}$$

$$r\begin{bmatrix} A & B & -A & -B \\ C & D & -C & 0 \\ A & B & A & B \\ C & 0 & C & D \end{bmatrix} = r\begin{bmatrix} A & B \\ (1+i)C & D \end{bmatrix} + r\begin{bmatrix} A & B \\ (1-i)C & D \end{bmatrix}, \tag{49}$$

$$r\begin{bmatrix} A & B & -A & B \\ C & D & C & -D \\ A & -B & A & B \\ -C & D & C & D \end{bmatrix} = r\begin{bmatrix} (1+i)A & (1-i)B \\ (1-i)C & (1+i)D \end{bmatrix} + r\begin{bmatrix} (1-i)A & (1+i)B \\ (1+i)C & (1-i)D \end{bmatrix}, \tag{50}$$

$$r\begin{bmatrix} A & B & -A & -B \\ C & D & -C & D \\ A & B & A & B \\ C & -D & C & D \end{bmatrix} = r\begin{bmatrix} (1+i)A & (1+i)B \\ (1+i)C & (1-i)D \end{bmatrix} + r\begin{bmatrix} (1-i)A & (1-i)B \\ (1-i)C & (1+i)D \end{bmatrix}. \quad (51)$$

Proof. Follows from replacing the two matrices in (20) and (21) with the given 2×2 sub-block matrices on the left-hand sides of (37)–(51). □

Theorem 6. *Let $A, B, C, D \in \mathbb{C}^{m\times n}$. Then, the following rank equalities hold*:

$$r\begin{bmatrix} A & B & C & D \\ B & A & D & C \\ C & D & A & B \\ D & C & B & A \end{bmatrix} = r(A+B+C+D) + r(A+B-C-D) + r(A-B+C-D) + r(A-B-C+D), \quad (52)$$

$$r\begin{bmatrix} A & B & -C & -D \\ B & A & -D & -C \\ C & D & A & B \\ D & C & B & A \end{bmatrix} = r(A+B+iC+iD) + r(A+B-iC-iD) + r(A-B+iC-iD) + r(A-B-iC+iD), \quad (53)$$

$$r\begin{bmatrix} A & B & C & D \\ D & A & B & C \\ C & D & A & B \\ B & C & D & A \end{bmatrix} = r(A+B+C+D) + r(A-B+C-D) + r(A+iB-C-iD) + r(A-iB-C+iD). \quad (54)$$

For $A, B, C, D \in \mathbb{C}^{m\times m}$, and $k \geq 2$, the following rank equalities hold:

$$r\left(\begin{bmatrix} A & B & C & D \\ B & A & D & C \\ C & D & A & B \\ D & C & B & A \end{bmatrix}^k\right) = r((A+B+C+D)^k) + r((A+B-C-D)^k) + r((A-B+C-D)^k) + r((A-B-C+D)^k), \quad (55)$$

$$r\left(\begin{bmatrix} A & B & -C & -D \\ B & A & -D & -C \\ C & D & A & B \\ D & C & B & A \end{bmatrix}^k\right) = r((A+B+iC+iD)^k) + r((A+B-iC-iD)^k) + r((A-B+iC-iD)^k) + r((A-B-iC+iD)^k), \quad (56)$$

$$r\left(\begin{bmatrix} A & B & C & D \\ D & A & B & C \\ C & D & A & B \\ B & C & D & A \end{bmatrix}^k\right) = r((A+B+C+D)^k) + r((A-B+C-D)^k) + r((A+iB-C-iD)^k) + r((A-iB-C+iD)^k). \quad (57)$$

Proof. Follows from replacing the two matrices in (20) and (21) with the given 2×2 sub-block matrices on the left-hand sides of (55)–(57). □

Theorem 7. *Let A, $B \in \mathbb{C}^{m\times m}$. Then, the following rank equalities hold*:

$$r\begin{bmatrix} A^2 & AB & B^2 & BA \\ BA & B^2 & AB & A^2 \\ B^2 & BA & A^2 & AB \\ AB & A^2 & BA & B^2 \end{bmatrix} = r((A+B)^2) + r((A-B)^2) \tag{58}$$

$$+ r((A+B)(A-B)) + r((A-B)(A+B)), \tag{59}$$

$$r\begin{bmatrix} -A^2 & AB & B^2 & BA \\ BA & -B^2 & AB & A^2 \\ B^2 & BA & -A^2 & AB \\ AB & A^2 & BA & -B^2 \end{bmatrix} = r((A+iB)^2) + r((A-iB)^2) \tag{60}$$

$$+ r((A+iB)(A-iB)) + r((A-iB)(A+iB)). \tag{61}$$

Proof. Follows from replacing the two matrices in (20) and (21) with the given 2×2 sub-block matrices on the left-hand sides of (59) and (61). □

The rank expansion formulas in the following lemma are well-known in elementary linear algebra, which can be viewed as entry-level examples for illustrating the constructive matrix methodology.

Lemma 6. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times m}$, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be scalars. Then, the following rank equalities hold*:

$$\begin{aligned} &r\begin{bmatrix} I_m & A \\ B & I_n \end{bmatrix} = m + r(I_n - BA) = n + r(I_m - AB), \\ &r\begin{bmatrix} I_m & \lambda_1 A + \lambda_2 ABA + \cdots + \lambda_k (AB)^{k-1}A \\ B & I_n \end{bmatrix} \\ &= n + r(I_m + \lambda_1 AB + \lambda_2 (AB)^2 + \cdots + \lambda_k (AB)^k) \\ &= m + r(I_n + \lambda_1 BA + \lambda_2 (BA)^2 + \cdots + \lambda_k (BA)^k). \end{aligned}$$

If $m = n$, then the following rank equalities hold:

$$\begin{aligned} r(I_m - AB) &= r(I_m - BA), \\ r(I_m + \lambda_1 AB + \lambda_2 (AB)^2 + \cdots + \lambda_k (AB)^k) &= r(I_m + \lambda_1 BA + \lambda_2 (BA)^2 + \cdots + \lambda_k (BA)^k). \end{aligned}$$

Theorem 8. *Let $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times p}$, $Y \in \mathbb{C}^{q\times m}$, and $B \in \mathbb{C}^{p\times q}$. Then, the following rank equalities hold*:

$$r\begin{bmatrix} A & AXB \\ BYA & B \end{bmatrix} = r(A) + r(B - BYAXB) = r(B) + r(A - AXBYA). \tag{62}$$

In particular, if $r(A) = r(B)$, then $r(B - BYAXB) = r(A - AXBYA)$ holds. Hence, the two equalities $AXBYA = A$ and $BYAXB = B$ are equivalent under $r(A) = r(B)$.

Proof. Applying (9) to the block matrix $\begin{bmatrix} A & AXB \\ BYA & B \end{bmatrix}$ and simplifying, we first obtain

$$r\begin{bmatrix} A & AXB \\ BYA & B \end{bmatrix} = r\begin{bmatrix} A & 0 \\ 0 & B - BYAXB \end{bmatrix} = r(A) + r(B - BYAXB),$$

$$r\begin{bmatrix} A & AXB \\ BYA & B \end{bmatrix} = r\begin{bmatrix} A - AXBYA & 0 \\ 0 & B \end{bmatrix} = r(B) + r(A - AXBYA).$$

Combining the two rank equalities yields (62). □

As immediate applications of (62), we are able to obtain the following concrete matrix rank formulas and their consequences.

Theorem 9. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$. Then, the following rank equalities hold*:

$$\begin{aligned}
r\begin{bmatrix} A & AB \\ BA & B \end{bmatrix} &= r(A) + r(B - BAB) \\
&= r(B) + r(A - ABA), \\
r\begin{bmatrix} A & (AB)^2 \\ BA & B \end{bmatrix} &= r(A) + r(B - B(AB)^2) \\
&= r(B) + r(A - (AB)^2A), \\
r\begin{bmatrix} A & (AB)^2 \\ (BA)^2 & B \end{bmatrix} &= r(A) + r(B - B(AB)^3) \\
&= r(B) + r(A - (AB)^3A), \\
r\begin{bmatrix} A & AB + (AB)^2 \\ BA & B \end{bmatrix} &= r(A) + r(B - BAB - B(AB)^2) \\
&= r(B) + r(A - ABA - (AB)^2A), \\
r\begin{bmatrix} A & AB + (AB)^2 \\ (BA)^2 & B \end{bmatrix} &= r(A) + r(B - B(AB)^2 - B(AB)^3) \\
&= r(B) + r(A - (AB)^2A - (AB)^3A).
\end{aligned}$$

In consequence, the following facts hold:

$$\begin{aligned}
ABA = A &\Leftrightarrow r(B - BAB) = r(B) - r(A), \\
BAB = B &\Leftrightarrow r(A - ABA) = r(A) - r(B), \\
r(A) = r(B) \text{ and } ABA = A &\Leftrightarrow r(A) = r(B) \text{ and } BAB = B, \\
(AB)^2A = A &\Leftrightarrow r(B - B(AB)^2) = r(B) - r(A), \\
B(AB)^2 = B &\Leftrightarrow r(A - (AB)^2A) = r(A) - r(B), \\
r(A) = r(B) \text{ and } (AB)^2A = A &\Leftrightarrow r(A) = r(B) \text{ and } B(AB)^2 = B, \\
(AB)^2A + ABA = A &\Leftrightarrow r(B - BAB - B(AB)^2) = r(B) - r(A), \\
B(AB)^2 + BAB = B &\Leftrightarrow r(A - ABA - (AB)^2A) = r(A) - r(B), \\
r(A) = r(B) \text{ and } (AB)^2A + ABA = A &\Leftrightarrow r(A) = r(B) \text{ and } B(AB)^2 + BAB = B, \\
(AB)^3A + (AB)^2A = A &\Leftrightarrow r(B - B(AB)^2 - B(AB)^3) = r(B) - r(A), \\
B(AB)^3 + B(AB)^2 = B &\Leftrightarrow r(A - (AB)^2A - (AB)^3A) = r(A) - r(B), \\
r(A) = r(B) \text{ and } (AB)^3A + (AB)^2A = A &\Leftrightarrow r(A) = r(B) \text{ and } B(AB)^3 + B(AB)^2 = B.
\end{aligned}$$

All the rank expansion formulas in the following theorems can directly be established by (9) and elementary block matrix operations and, therefore, their derivations are omitted.

Theorem 10. *Let A, $B \in \mathbb{C}^{m\times m}$. Then, the following rank equalities hold*:

$$\begin{aligned}
r\begin{bmatrix} AB & AB^2 \\ BA^2 & BA \end{bmatrix} &= r\begin{bmatrix} AB & 0 \\ BA^2 & BA - BA^2B \end{bmatrix} = r\begin{bmatrix} AB - AB^2A & AB^2 \\ 0 & BA \end{bmatrix}, \\
r\begin{bmatrix} AB & B^2A \\ A^2B & BA \end{bmatrix} &= r\begin{bmatrix} AB & B^2A \\ 0 & BA - AB^2A \end{bmatrix} = r\begin{bmatrix} AB - BA^2B & B^2A \\ 0 & BA \end{bmatrix},
\end{aligned}$$

and the following facts hold (cf. [9]):

$$r(AB) = r(BA) \text{ and } AB^2A = AB \Leftrightarrow r(AB) = r(BA) \text{ and } BA^2B = BA,$$
$$r(AB) = r(BA) \text{ and } AB^2A = BA \Leftrightarrow r(AB) = r(BA) \text{ and } BA^2B = AB.$$

Theorem 11. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times n}$, and let λ be a nonzero scalar. Then, the following rank equalities hold*:

$$\begin{aligned} r\begin{bmatrix} AB & ABA \\ BAB & \lambda BA \end{bmatrix} &= r(AB) + r(\lambda BA - (BA)^2) \\ &= r(BA) + r(\lambda AB - (AB)^2), \\ r\begin{bmatrix} AB & (AB)^2A \\ BAB & \lambda BA \end{bmatrix} &= r(AB) + r(\lambda BA - (BA)^3) \\ &= r(BA) + r(\lambda AB - (AB)^3), \\ r\begin{bmatrix} AB & (AB)^2A \\ (BA)^2B & \lambda BA \end{bmatrix} &= r(AB) + r(\lambda BA - (BA)^4) \\ &= r(BA) + r(\lambda AB - (AB)^4), \\ r\begin{bmatrix} AB & ABA + (AB)^2A \\ BAB & \lambda BA \end{bmatrix} &= r(AB) + r(\lambda BA - (BA)^2 - (BA)^3) \\ &= r(BA) + r(\lambda AB - (AB)^2 - (AB)^3). \end{aligned}$$

In consequence, the following facts hold:

$$\begin{aligned} (AB)^2 = \lambda AB &\Leftrightarrow r(\lambda BA - (BA)^2) = r(BA) - r(AB), \\ (BA)^2 = \lambda BA &\Leftrightarrow r(\lambda AB - (AB)^2) = r(AB) - r(BA), \\ r(AB) = r(BA) \text{ and } (AB)^2 = \lambda AB &\Leftrightarrow r(AB) = r(BA) \text{ and } (BA)^2 = \lambda BA. \\ (AB)^3 = \lambda AB &\Leftrightarrow r(\lambda BA - (BA)^3) = r(BA) - r(AB), \\ (BA)^3 = \lambda BA &\Leftrightarrow r(\lambda AB - (AB)^3) = r(AB) - r(BA), \\ r(AB) = r(BA) \text{ and } (AB)^3 = \lambda AB &\Leftrightarrow r(AB) = r(BA) \text{ and } (BA)^3 = \lambda BA. \\ (AB)^4 = \lambda AB &\Leftrightarrow r(\lambda BA - (BA)^4) = r(BA) - r(AB), \\ (BA)^4 = \lambda BA &\Leftrightarrow r(\lambda AB - (AB)^4) = r(AB) - r(BA), \\ r(AB) = r(BA) \text{ and } (AB)^4 = \lambda AB &\Leftrightarrow r(AB) = r(BA) \text{ and } (BA)^4 = \lambda BA \end{aligned}$$

and

$$\begin{aligned} &r(AB) = r(BA) \text{ and } (AB)^3 + (AB)^2 = \lambda AB \\ &\Leftrightarrow r(AB) = r(BA) \text{ and } (BA)^3 + (BA)^2 = \lambda AB. \end{aligned}$$

Theorem 12. *Let $A \in \mathbb{C}^{m\times n}$ $B \in \mathbb{C}^{m\times n}$, and let λ be a nonzero scalar. Then, the following rank equalities hold*:

$$\begin{aligned} r\begin{bmatrix} AB & (AB)^2A \\ (BA)^2B & \lambda(BA)^2 \end{bmatrix} &= r(AB) + r(\lambda(BA)^2 - (BA)^4) \\ &= r((BA)^2) + r(\lambda AB - (AB)^3), \\ r\begin{bmatrix} (AB)^2 & (AB)^2A \\ (BA)^2B & \lambda(BA)^2 \end{bmatrix} &= r((AB)^2) + r(\lambda(BA)^2 - (BA)^3) \\ &= r((BA)^2) + r(\lambda(AB)^2 - (AB)^3), \end{aligned}$$

and the following facts hold:

$$\begin{aligned}
(AB)^3 = \lambda AB &\Leftrightarrow r(\lambda(BA)^2 - (BA)^4) = r((BA)^2) - r(AB),\\
(BA)^4 = \lambda(BA)^2 &\Leftrightarrow r(\lambda AB - (AB)^3) = r(AB) - r((BA)^2),\\
(AB)^3 = \lambda(AB)^2 &\Leftrightarrow r(\lambda(BA)^2 - (BA)^3) = r((BA)^2) - r((AB)^2),\\
(BA)^3 = \lambda(BA)^2 &\Leftrightarrow r(\lambda(AB)^2 - (AB)^3) = r((AB)^2)) - r((BA)^2)
\end{aligned}$$

and

$$\begin{aligned}
&r((BA)^2) = r(AB) \text{ and } (AB)^3 = \lambda AB\\
&\Leftrightarrow r((BA)^2) = r(AB) \text{ and } (BA)^4 = \lambda(BA)^2,\\
&r((BA)^2) = r((BA)^2) \text{ and } (AB)^3 = \lambda(AB)^2\\
&\Leftrightarrow r((BA)^2) = r((BA)^2) \text{ and } (BA)^3 = \lambda(BA)^2.
\end{aligned}$$

Theorem 13. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times n}$, and let λ be a nonzero scalar. Then, the following rank equalities hold*:

$$\begin{aligned}
r\begin{bmatrix} ABA & (AB)^2 \\ (BA)^2 & \lambda BAB \end{bmatrix} &= r(ABA) + r(\lambda BAB - (BA)^2B)\\
&= r(BAB) + r(\lambda ABA - (AB)^2A),\\
r\begin{bmatrix} ABA & (AB)^3 \\ (BA)^2 & \lambda BAB \end{bmatrix} &= r(ABA) + r(\lambda BAB - (BA)^3B)\\
&= r(BAB) + r(\lambda ABA - (AB)^3A),\\
r\begin{bmatrix} ABA & (AB)^2 + (AB)^3 \\ (BA)^2 & \lambda BAB \end{bmatrix} &= r(ABA) + r(\lambda BAB - B(AB)^2 - B(AB)^3)\\
&= r(BAB) + r(\lambda ABA - (AB)^2A - (AB)^3A),\\
r\begin{bmatrix} ABA & (AB)^2 + (AB)^3 \\ (BA)^3 & \lambda BAB \end{bmatrix} &= r(ABA) + r(\lambda BAB - B(AB)^3 - B(AB)^4)\\
&= r(BAB) + r(\lambda ABA - (AB)^3A - (AB)^4A),
\end{aligned}$$

and the following facts hold:

$$\begin{aligned}
(AB)^2A = \lambda ABA &\Leftrightarrow r(\lambda BAB - (BA)^2B) = r(BAB) - r(ABA),\\
(BA)^2B = \lambda BAB &\Leftrightarrow r(\lambda ABA - (AB)^2A) = r(ABA) - r(BAB)
\end{aligned}$$

and

$$\begin{aligned}
&r(ABA) = r(BAB) \text{ and } (AB)^2A = \lambda ABA\\
&\Leftrightarrow r(ABA) = r(BAB) \text{ and } (BA)^2B = \lambda BAB,\\
&r(ABA) = r(BAB) \text{ and } (AB)^3A = \lambda ABA\\
&\Leftrightarrow r(ABA) = r(BAB) \text{ and } (BA)^3B = \lambda BAB,\\
&r(ABA) = r(BAB) \text{ and } (AB)^3A + (AB)^2A = \lambda ABA\\
&\Leftrightarrow r(ABA) = r(BAB) \text{ and } B(AB)^3 + B(AB)^2 = \lambda BAB,\\
&r(ABA) = r(BAB) \text{ and } (AB)^4A + (AB)^3A = \lambda ABA\\
&\Leftrightarrow r(ABA) = r(BAB) \text{ and } B(AB)^4 + B(AB)^3 = \lambda BAB.
\end{aligned}$$

Moreover, we have the following expansion formulas for calculating the ranks of block matrices composed of multiple products of matrices.

Theorem 14. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, $C \in \mathbb{C}^{p\times m}$, and let λ be a nonzero scalar. Then, the following rank equalities hold*:

$$\begin{aligned}
r\begin{bmatrix} AB & ABC \\ BCAB & \lambda BC \end{bmatrix} &= r(AB) + r(\lambda BC - BCABC) \\
&= r(BC) + r(\lambda AB - ABCAB), \\
r\begin{bmatrix} AB & (ABC)^2 \\ BCAB & \lambda BC \end{bmatrix} &= r(AB) + r(\lambda BC - BC(ABC)^2) \\
&= r(BC) + r(\lambda AB - (ABC)^2AB), \\
r\begin{bmatrix} AB & ABC \\ (BCA)^2B & \lambda BC \end{bmatrix} &= r(AB) + r(\lambda BC - BC(ABC)^2) \\
&= r(BC) + r(\lambda AB - (ABC)^2AB), \\
r\begin{bmatrix} AB & (ABC)^2 \\ (BCA)^2B & \lambda BC \end{bmatrix} &= r(AB) + r(\lambda BC - BC(ABC)^3) \\
&= r(BC) + r(\lambda AB - (ABC)^3AB), \\
r\begin{bmatrix} AB & ABC + (ABC)^2 \\ BCAB & \lambda BC \end{bmatrix} &= r(AB) + r(\lambda BC - BCABC - BC(ABC)^2) \\
&= r(BC) + r(\lambda AB - ABCAB - (ABC)^2AB), \\
r\begin{bmatrix} AB & ABC + (ABC)^2 \\ (BCA)^2B & \lambda BC \end{bmatrix} &= r(AB) + r(\lambda BC - BC(ABC)^2 - BC(ABC)^3) \\
&= r(BC) + r(\lambda AB - (ABC)^2AB - (ABC)^3AB).
\end{aligned}$$

Remark 1. *The versatile and engaging findings in the above theorems actually reveal the remarkable fact that we are able to establish certain nontrivial links between different algebraic matrix equalities composed of general matrices through certain constructions of block matrices and the calculations of the ranks of the block matrices. Undoubtedly, numerous expressions of block matrices and the corresponding rank expansion formulas can be reasonably established for two or more given matrices and their algebraic operations, and consequently, many kinds of essential equivalent facts concerning matrix equalities can be obtained from the rank formulas as more and deeper approaches are devoted to this research topic.*

4. Characterizations of Matrix Equalities by the Matrix Rank Methodology

In this section, we continue to construct some block matrices, calculate their ranks, and use them to characterize various matrix equalities related to the Hermitian matrix, the skew-Hermitian matrix, the normal matrix, etc.

Theorem 15. *Let $A \in \mathbb{C}^{n\times n}$ and $B, C \in \mathbb{C}^{m\times n}$. Then, the following five conditions are equivalent*:

(a) $r\begin{bmatrix} A & A^* \\ BA^*A & CAA^* \end{bmatrix} = r(A)$.

(b) $r\begin{bmatrix} AA^\dagger & A^\dagger A \\ BA^* & CA \end{bmatrix} = r(A)$.

(c) $\mathscr{R}\begin{bmatrix} A \\ BA^*A \end{bmatrix} = \mathscr{R}\begin{bmatrix} A^* \\ CAA^* \end{bmatrix}$.

(d) $\mathscr{R}\begin{bmatrix} AA^\dagger \\ BA^* \end{bmatrix} = \mathscr{R}\begin{bmatrix} A^\dagger A \\ CA \end{bmatrix}$.

(e) $\mathscr{R}(A) = \mathscr{R}(A^*)$ *and* $BA^* = CA$.

Proof. Applying (5) to the block matrix $\begin{bmatrix} A & A^* \\ BA^*A & CAA^* \end{bmatrix}$, we first obtain

$$r\begin{bmatrix} A & A^* \\ BA^*A & CAA^* \end{bmatrix} = r(A) + r\begin{bmatrix} 0 & A^* - AA^\dagger A^* \\ 0 & CAA^* - BA^*A^* \end{bmatrix},$$
$$r\begin{bmatrix} A & A^* \\ BA^*A & CAA^* \end{bmatrix} = r(A^*) + r\begin{bmatrix} A - A^\dagger A^2 & 0 \\ BA^*A - CA^2 & 0 \end{bmatrix}.$$

In this case, the matrix rank equality in Condition (a) is equivalent to the following facts:

$$A^2A^\dagger = A^\dagger A^2 = A, \ \ BA^*A = CA^2, \ \ B(A^*)^2 = CAA^* \tag{63}$$

by (8). Obviously, the first two matrix equalities in (63) are equivalent to $\mathscr{R}(A) = \mathscr{R}(A^*)$ by (6). Next, post-multiplying the second equality with $A^\dagger$ and simplifying by the first pair of equalities leads to $BA^* = CA$, thus establishing the equivalence of Conditions (a) and (e).

Also, by (4), we are able to obtain the following two rank equalities

$$r\begin{bmatrix} AA^\dagger & A^\dagger A \\ BA^* & CA \end{bmatrix} = r(A) + r\begin{bmatrix} 0 & A^\dagger A - AA^\dagger A^\dagger A \\ 0 & CA - BA^* \end{bmatrix},$$
$$r\begin{bmatrix} AA^\dagger & A^\dagger A \\ BA^* & CA \end{bmatrix} = r(A) + r\begin{bmatrix} AA^\dagger - AA^\dagger A^\dagger A & 0 \\ BA^* - CA & 0 \end{bmatrix}.$$

Consequently, Condition (b) is equivalent to $AA^\dagger A^\dagger A = AA^\dagger = A^\dagger A$ and $BA^* = CA$ by (8), where $AA^\dagger = A^\dagger A$ is well-known to be equivalent to $\mathscr{R}(A) = \mathscr{R}(A^*)$ (cf. [1]). Thus, Conditions (b) and (e) are equivalent.

Note that the two rank equalities in Conditions (a) and (b) can also be written as

$$r\begin{bmatrix} A & A^* \\ BA^*A & CAA^* \end{bmatrix} = r\begin{bmatrix} A \\ BA^*A \end{bmatrix} = r\begin{bmatrix} A^* \\ CAA^* \end{bmatrix}, \ \ r\begin{bmatrix} AA^\dagger & A^\dagger A \\ BA^* & CA \end{bmatrix} = r\begin{bmatrix} AA^\dagger \\ BA^* \end{bmatrix} = r\begin{bmatrix} A^\dagger A \\ CA \end{bmatrix}.$$

In such cases, applying (6) to the two pairs of equalities leads to the equivalences of Conditions (a)–(d). □

For different choices of B and C in Theorem 15, we are able to derive various concrete equivalent facts concerning matrix equalities that involve a matrix and its conjugate transpose. We next present five corollaries that display how rank and range equalities of block matrices are involved in the characterizations of some well-known matrix equalities composed of matrix products of a matrix and its conjugate transpose (cf. [2,10–17]).

As we know, Hermitian matrices possess many elegant and pleasing formulas and facts, have many significant applications in the research areas of both theoretical and applied mathematics, and have already been recognized as one of the basic conceptual objects and building materials in matrix theory and its applications.

Corollary 2. *Let $A \in \mathbb{C}^{m\times m}$. Then, the following five conditions are equivalent*:

(a) $r\begin{bmatrix} A & A^* \\ A^*A & AA^* \end{bmatrix} = r(A).$

(b) $r\begin{bmatrix} AA^\dagger & A^\dagger A \\ A^* & A \end{bmatrix} = r(A).$

(c) $\mathscr{R}\begin{bmatrix} A \\ A^*A \end{bmatrix} = \mathscr{R}\begin{bmatrix} A^* \\ AA^* \end{bmatrix}.$

(d) $\mathscr{R}\begin{bmatrix} AA^\dagger \\ A^* \end{bmatrix} = \mathscr{R}\begin{bmatrix} A^\dagger A \\ A \end{bmatrix}.$

(e) $A = A^*.$

Proof. Replacing $B = C = I_m$ in Theorem 15 leads to the equivalences of Conditions (a)–(e), where $A = A^*$ obviously implies $\mathscr{R}(A) = \mathscr{R}(A^*)$ and thus is removed from Condition (e) of Theorem 15. □

Given the above preparations, we are able to derive our main results as follows.

Theorem 16. *Let $A, B \in \mathbb{C}^{m\times n}$. Then, the following five conditions are equivalent*:

(a) $A = B$.
(b) $\mathscr{R}(A) = \mathscr{R}(B), \mathscr{R}(A^*) = \mathscr{R}(B^*)$ *and* $AB^\dagger A = A$.
(c) $\mathscr{R}(A) = \mathscr{R}(B), \mathscr{R}(A^*) = \mathscr{R}(B^*)$ *and* $BA^\dagger B = B$.
(d) $\mathscr{R}(A) = \mathscr{R}(B), \mathscr{R}(A^*) = \mathscr{R}(B^*)$ *and* $A^\dagger BA^\dagger = A^\dagger$.
(e) $\mathscr{R}(A) = \mathscr{R}(B), \mathscr{R}(A^*) = \mathscr{R}(B^*)$ *and* $B^\dagger AB^\dagger = B^\dagger$.

Proof. The equivalences of Conditions (a)–(e) follow from (14) and (15). □

Theorem 17. *Let $A, B \in \mathbb{C}^{m\times m}$ be two Hermitian matrices. Then, the following three conditions are equivalent*:

(a) $A = B$.
(b) $r(AB) = r(A) = r(B)$, $AB = BA$, *and* $ABA = BAB$.
(c) $r(AB) = r(A) = r(B)$, $ABA = BAB$, *and* $(AB)^2 = (BA)^2$.

Specifically, for two invertible Hermitian matrices A and B of the same size, the following three conditions are equivalent:

(d) $A = B$.
(e) $AB = BA$ *and* $ABA = BAB$.
(f) $ABA = BAB$ *and* $(AB)^2 = (BA)^2$.

Proof. Condition (a) obviously implies Conditions (b) and (c) under the Hermitian matrix assumption. Under Condition (b), we apply the well-known Frobenius rank inequality $r(PNQ) \geq r(PN) + r(NQ) - r(N)$ for the product of any three matrices of appropriate sizes to the two products ABA and BAB to obtain:

$$r(ABA) \geq r(AB) + r(BA) - r(B) = r(B) = r(A),$$

$$r(BAB) \geq r(BA) + r(AB) - r(A) = r(A) = r(B).$$

Combining the facts with the two regular rank inequalities $r(ABA) \leq r(A)$ and $r(BAB) \leq r(B)$ leads to:

$$r(ABA) = r(BAB) = r(A) = r(B),$$

which further implies that the following range equalities

$$\mathscr{R}(ABA) = \mathscr{R}(A) \text{ and } \mathscr{R}(BAB) = \mathscr{R}(B)$$

hold as well, since $\mathscr{R}(ABA) \subseteq \mathscr{R}(A)$ and $\mathscr{R}(BAB) \subseteq \mathscr{R}(B)$ always hold. In this case, we obtain, by $AB = BA$, Lemma 3, and elementary block matrix operations, that

$$\begin{aligned} r\begin{bmatrix} ABA & BAB \\ A & B \end{bmatrix} &= r\begin{bmatrix} ABA - ABA & BAB - ABB \\ A & B \end{bmatrix} \\ &= r\begin{bmatrix} 0 & 0 \\ A & B \end{bmatrix} = r[A,\ B] = r[ABA,\ BAB], \end{aligned}$$

which implies that

$$X[ABA,\ BAB] = [XABA,\ XBAB] = [A,\ B]$$

holds for matrix X by Lemma 4. Finally, pre-multiplying $ABA = BAB$ by X and simplifying yields $A = B$, thus establishing Condition (a).

Under Condition (c), we apply the Frobenius rank inequality to the two products $(AB)^2$ and $(BA)^2$ to obtain

$$r((AB)^2) \geq r(ABA) + r(BAB) - r(BA) = r(A) = r(B),$$

$$r((BA)^2) \geq r(BAB) + r(ABA) - r(AB) = r(A) = r(B),$$

which further imply

$$r((AB)^2) = r((BA)^2) = r(A) = r(B),$$

and thus,

$$\mathscr{R}((AB)^2) = \mathscr{R}(A) \text{ and } \mathscr{R}((BA)^2) = \mathscr{R}(B)$$

hold. In this case, we obtain, by $ABA = BAB$, Lemma 3, and elementary block matrix operations, that

$$\begin{aligned} r\begin{bmatrix} (AB)^2 & (BA)^2 \\ B & A \end{bmatrix} &= r\begin{bmatrix} (AB)^2 - ABAB & (BA)^2 - ABAA \\ B & A \end{bmatrix} \\ &= r\begin{bmatrix} 0 & 0 \\ B & A \end{bmatrix} = r[A,\ B] = r[(AB)^2,\ (BA)^2], \end{aligned}$$

which implies that the following equalities

$$X[(AB)^2,\ (BA)^2] = [X(AB)^2,\ X(BA)^2] = [B,\ A]$$

hold for matrix X by Lemma 4. Finally, pre-multiplying $(AB)^2 = (BA)^2$ with X and simplifying yields $B = A$, thus establishing (a).

If A and B are invertible, then $r(AB) = r(A) = r(B)$ hold naturally. Thus, Conditions (a), (b), and (c) are reduced to Conditions (d), (e), and (f), respectively. □

As direct consequences, we replace A with $\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$ and B with $\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$ in Theorem 17, where A and B are any two $m \times n$ matrices, to obtain the following facts.

Corollary 3. *Let $A,\ B \in \mathbb{C}^{m\times n}$. Then, the following three statements are equivalent*:

(a) $A = B$.
(b) $r(AB^*) = r(A^*B) = r(A) = r(B)$, $AB^* = BA^*$, $A^*B = B^*A$, *and* $AB^*A = BA^*B$.
(c) $r(AB^*) = r(A^*B) = r(A) = r(B)$, $AB^*A = BA^*B$, $(AB^*)^2 = (BA^*)^2$, *and* $(A^*B)^2 = (B^*A)^2$.

Similar to the preceding results and facts, the equivalent facts in Theorems 16 and 17 and their derivations also reveal some essential links between the matrix equality $A = B$ and some other weaker conditions composed of different matrix rank equalities and algebraic matrix equalities for A and B, where $ABA = BAB$ is known as the Yang–Baxter matrix equation in the literature (cf. [18–21]).

Theorem 18. *Let $A \in \mathbb{C}^{m\times m}$. Then, the following 12 conditions are equivalent*:

(a) $r\begin{bmatrix} A & A^* \\ (A^*A)^2 & (AA^*)^2 \end{bmatrix} = r(A)$.
(b) $r\begin{bmatrix} AA^\dagger & A^\dagger A \\ A^*AA^* & AA^*A \end{bmatrix} = r(A)$.
(c) $\mathscr{R}\begin{bmatrix} A \\ (A^*A)^2 \end{bmatrix} = \mathscr{R}\begin{bmatrix} A^* \\ (AA^*)^2 \end{bmatrix}$.
(d) $\mathscr{R}\begin{bmatrix} AA^\dagger \\ A^*AA^* \end{bmatrix} = \mathscr{R}\begin{bmatrix} A^\dagger A \\ AA^*A \end{bmatrix}$.
(e) $A^3 = A^*AA^*$ *and* $A^5 = (A^*A)^2A^*$.

(f) A *is Hermitian*.
(g) AA^*A *is Hermitian*.
(h) $(AA^*)^2A$ *is Hermitian*.
(i) $r(A^2) = r(A)$, A^2 *and* A^3 *are Hermitian*.
(j) $r(A^2) = r(A)$, A^2 *and* A^5 *are Hermitian*.
(k) $r(A^2) = r(A)$, A^3 *and* A^4 *are Hermitian*.
(l) $r(A^2) = r(A)$, A^3 *and* A^5 *are Hermitian*.

Proof. Replacing $B = A^*A$ and $C = AA^*$ in Theorem 15 leads to the equivalences of Conditions (a)–(d) and (g), where $A^*AA^* = AA^*A$ implies $\mathscr{R}(A) = \mathscr{R}(A^*)$ and thus is removed in Condition (g).

The equivalences of Conditions (f), (g) and (h) follow from [4], Theorem 7.

The first equality in (e) implies $\mathscr{R}(A) \supseteq \mathscr{R}(A^3) = \mathscr{R}(A^*AA^*) = \mathscr{R}(A^*)$. So that $\mathscr{R}(A) = \mathscr{R}(A^*)$ holds by noting $r(A) = r(A^*)$. Hence, $AA^\dagger = A^\dagger A$ and $(A^4)^\dagger A^4 = A^\dagger A$ hold by Lemma 5 (II) and (III). Now substituting the first equality into the second in (e) leads to $A^5 = A^4A^*$. Pre-multiplying the equality with $(A^4)^\dagger$ and simplifying with the above two equalities, we obtain $A = (A^\dagger A)A = (A^4)^\dagger A^5 = (A^4)^\dagger A^4A^* = A^\dagger AA^* = A^*$. Thus, Condition (e) implies Condition (f). Conversely, Condition (f) implies Condition (e) as well.

From $r(A^2) = r(A)$ and $A^2 = (A^2)^*$ in Condition (i), we first obtain $\mathscr{R}(A) = \mathscr{R}(A^2) = \mathscr{R}((A^2)^*) = \mathscr{R}(A^*)$. Hence, $AA^\dagger = A^\dagger A$ and $(A^2)^\dagger A^2 = A^\dagger A$ hold by Lemma 5 (II) and (III). In this case, pre-multiplying $A^3 = (A^2)A^*$ with $(A^2)^\dagger$ and simplifying yields $A = (A^2)^\dagger A^3 = (A^2)^\dagger A^2A^* = A^\dagger AA^* = A^*$, thus Condition (i) implies Condition (f). The equivalences of Conditions (f), (j), (k), and (l) can be established by a similar approach. □

Theorem 19. *Let* $A \in \mathbb{C}^{m\times m}$. *Then, the following 12 conditions are equivalent*:

(a) $r\begin{bmatrix} A & -A^* \\ (A^*A)^2 & (AA^*)^2 \end{bmatrix} = r(A)$.
(b) $r\begin{bmatrix} AA^\dagger & A^\dagger A \\ A^*AA^* & -AA^*A \end{bmatrix} = r(A)$.
(c) $\mathscr{R}\begin{bmatrix} A \\ (A^*A)^2 \end{bmatrix} = \mathscr{R}\begin{bmatrix} -A^* \\ (AA^*)^2 \end{bmatrix}$.
(d) $\mathscr{R}\begin{bmatrix} AA^\dagger \\ A^*AA^* \end{bmatrix} = \mathscr{R}\begin{bmatrix} A^\dagger A \\ -AA^*A \end{bmatrix}$.
(e) $A^3 = A^*AA^*$ *and* $A^5 = -(A^*A)^2A^*$.
(f) A *is skew-Hermitian*.
(g) AA^*A *is skew-Hermitian*.
(h) $(AA^*)^2A$ *is skew-Hermitian*.
(i) $r(A^2) = r(A)$, A^2 *is Hermitian, and* A^3 *is skew-Hermitian*.
(j) $r(A^2) = r(A)$, A^2 *is Hermitian, and* A^5 *is skew-Hermitian*.
(k) $r(A^2) = r(A)$, A^3 *is skew-Hermitian, and* A^4 *is Hermitian*.
(l) $r(A^2) = r(A)$, *and* A^3 *and* A^5 *are skew-Hermitian*.

Recently, the present author showed in [22] the following equivalent facts:

$$A^3 = \pm AA^*A \Leftrightarrow A = \pm A^*.$$

Combining this result with (17) and Theorems 1 and 16, we are able to obtain the following consequences.

Theorem 20. *Let* $A \in \mathbb{C}^{m\times m}$. *Then, the following 21 conditions are equivalent*:

(a) $A = A^*$.
(b) $A^3 = AA^*A$.

(c) $A^\dagger = (A^\dagger)^*$.
(d) $A^*A^\dagger A^* = A^*$.
(e) $A^\dagger A^* A^\dagger = A^\dagger$.
(f) $r(A^2) = r(A)$ *and* $A^\dagger = (A^\#)^*$.
(g) $r(A^2) = r(A)$ *and* $A^\# = (A^\#)^*$.
(h) $r(A^2) = r(A)$ *and* $A^*A^\#A^* = A^*$.
(i) $r(A^2) = r(A)$ *and* $A^\dagger A^* A^\dagger = A^\#$.
(j) $r(A^2) = r(A)$ *and* $A^\#A^*A^\# = A^\dagger$.
(k) $r(A^2) = r(A)$ *and* $A^\#A^*A^\# = A^\#$.
(l) $r(A^2) = r(A)$ *and* $A^*A^\dagger A^* A^\dagger A^* = A^*A^\dagger A^*$.
(m) $r(A^2) = r(A)$ *and* $A^*A^\#A^*A^\#A^* = A^*A^\#A^*$.
(n) $r(A^2) = r(A)$ *and* $A^\dagger A^* A^\dagger A^* A^\dagger = A^\dagger A^* A^\dagger$.
(o) $r(A^2) = r(A)$ *and* $A^\dagger A^* A^\dagger A^* A^\dagger = A^\# A^* A^\#$.
(p) $r(A^2) = r(A)$ *and* $A^\dagger A^* A^\# A^* A^\dagger = A^\dagger A^* A^\dagger$.
(q) $r(A^2) = r(A)$ *and* $A^\dagger A^* A^\# A^* A^\dagger = A^\# A^* A^\#$.
(r) $r(A^2) = r(A)$ *and* $A^\# A^* A^\dagger A^* A^\# = A^\dagger A^* A^\dagger$.
(s) $r(A^2) = r(A)$ *and* $A^\# A^* A^\dagger A^* A^\# = A^\# A^* A^\#$.
(t) $r(A^2) = r(A)$ *and* $A^\# A^* A^\# A^* A^\# = A^\dagger A^* A^\dagger$.
(u) $r(A^2) = r(A)$ *and* $A^\# A^* A^\# A^* A^\# = A^\# A^* A^\#$.

Theorem 21. *Let* $A \in \mathbb{C}^{m\times m}$. *Then, the following 21 conditions are equivalent*:
(a) $A = -A^*$.
(b) $A^3 = -AA^*A$.
(c) $A^\dagger = -(A^\dagger)^*$.
(d) $A^*A^\dagger A^* = -A^*$.
(e) $A^\dagger A^* A^\dagger = -A^\dagger$.
(f) $r(A^2) = r(A)$ *and* $A^\dagger = -(A^\#)^*$.
(g) $r(A^2) = r(A)$ *and* $A^\# = -(A^\#)^*$.
(h) $r(A^2) = r(A)$ *and* $A^*A^\#A^* = -A^*$.
(i) $r(A^2) = r(A)$ *and* $A^\dagger A^* A^\dagger = -A^\#$.
(j) $r(A^2) = r(A)$ *and* $A^\#A^*A^\# = -A^\dagger$.
(k) $r(A^2) = r(A)$ *and* $A^\#A^*A^\# = -A^\#$.
(l) $r(A^2) = r(A)$ *and* $A^*A^\dagger A^* A^\dagger A^* = -A^*A^\dagger A^*$.
(m) $r(A^2) = r(A)$ *and* $A^*A^\#A^*A^\#A^* = -A^*A^\#A^*$.
(n) $r(A^2) = r(A)$ *and* $A^\dagger A^* A^\dagger A^* A^\dagger = -A^\dagger A^* A^\dagger$.
(o) $r(A^2) = r(A)$ *and* $A^\dagger A^* A^\dagger A^* A^\dagger = -A^\# A^* A^\#$.
(p) $r(A^2) = r(A)$ *and* $A^\dagger A^* A^\# A^* A^\dagger = -A^\dagger A^* A^\dagger$.
(q) $r(A^2) = r(A)$ *and* $A^\dagger A^* A^\# A^* A^\dagger = -A^\# A^* A^\#$.
(r) $r(A^2) = r(A)$ *and* $A^\# A^* A^\dagger A^* A^\# = -A^\dagger A^* A^\dagger$.
(s) $r(A^2) = r(A)$ *and* $A^\# A^* A^\dagger A^* A^\# = -A^\# A^* A^\#$.
(t) $r(A^2) = r(A)$ *and* $A^\# A^* A^\# A^* A^\# = -A^\dagger A^* A^\dagger$.
(u) $r(A^2) = r(A)$ *and* $A^\# A^* A^\# A^* A^\# = -A^\# A^* A^\#$.

As applications of the above results, we present below a group of characterizations of normal matrix.

Theorem 22. *Let* $A \in \mathbb{C}^{m\times m}$. *Then, the following 13 conditions are equivalent*:
(a) $r\begin{bmatrix} A & A^* \\ AA^*A & A^*AA^* \end{bmatrix} = r(A)$.
(b) $r\begin{bmatrix} AA^\dagger & A^\dagger A \\ AA^* & A^*A \end{bmatrix} = r(A)$.

(c) $\mathscr{R}\begin{bmatrix} A \\ AA^*A \end{bmatrix} = \mathscr{R}\begin{bmatrix} A^* \\ A^*AA^* \end{bmatrix}$.
(d) $\mathscr{R}\begin{bmatrix} AA^\dagger \\ AA^* \end{bmatrix} = \mathscr{R}\begin{bmatrix} A^\dagger A \\ A^*A \end{bmatrix}$.
(e) $r(A^2) = r(A)$, $A(A^*)^2A = A^*A^2A^*$, *and* $A(A^*)^2A^2A^* = A^*A^2(A^*)^2A$.
(f) $r(A^2) = r(A)$, $A(A^*)^2A^2A^* = A^*A^2(A^*)^2A$, *and* $(A(A^*)^2A)^2 = (A^*A^2A^*)^2$.
(g) $r(A^2) = r(A)$, $A^2A^* = A^*A^2$, *and* $A^3A^* = A^*A^3$.
(h) $AA^* = A^*A$, *namely*, *A is normal*.
(i) $r(A^2) = r(A)$, *and* A^2 *and* A^3 *are normal*.
(j) $r(A^2) = r(A)$, *and* A^2 *and* A^5 *are normal*.
(k) $r(A^2) = r(A)$, *and* A^3 *and* A^4 *are normal*.
(l) $r(A^2) = r(A)$, *and* A^3 *and* A^5 *are normal*.
(m) $r(A^2) = r(A)$, *and* A^4 *and* A^5 *are normal*.

Proof. Replacing $B = A$ and $C = A^*$ in Theorem 15 leads to the equivalences of Conditions (a)–(d) and (h), where $AA^* = A^*A$ also implies $\mathscr{R}(A) = \mathscr{R}(A^*)$ and thus is removed from Condition (h). The equivalences of Conditions (e), (f), and (h) follow from replacing A and B with AA^* and A^*A in Theorem 17 (a), (b), and (c).

Note that $r(A^2) = r(A)$ is obviously equivalent to the fact that $A = A^2X = YA^2$ hold for certain matrices X and Y. In this case, substituting the second given equality in Condition (g) into both sides of the third given equality in Condition (g), respectively, yields $A^3A^* = A^2A^*A$ and $A^*A^3 = AA^*A^2$. Consequently, pre-and post-multiplying the two equalities with the above two matrices Y and X respectively yields $A^2A^* = AA^*A$ and $A^*A^2 = AA^*A$. Next pre-and post-multiplying these two equalities with A^* respectively yields $A^*A^2A^* = (AA^*)^2 = (A^*A)^2 = A(A^*)^2A$. In this case, $(AA^* - A^*A)(AA^* - A^*A)^* = (AA^*)^2 + (A^*A)^2 - A(A^*)^2A - A^*A^2A^* = 0$, which further implies $AA^* - A^*A = 0$, i.e., (h) holds. Conversely, pre-multiplying $AA^* = A^*A$ with A and A^2 and applying the given condition yield $A^2A^* = AA^*A = A^*A^2$ and $A^3A^* = A^2A^*A = A^*A^3$, as required for the second and third equalities on the right-hand side. Pre-and post-multiplying $AA^* = A^*A$ with A and A^* yields $A^2(A^2)^* = (AA^*)^2$, which implies that $r(A^2) = r(A^2(A^2)^*) = r((AA^*)^2) = r(AA^*) = r(A)$, since AA^* and $A^2(A^2)^*$ are positive semi-definite. Thus, Condition (h) implies Condition (g).

Note from Condition (i) that the two normal matrices A^2 and A^3 commute. Then, we first obtain from Lemma 5 (I) that there exists an unitary matrix P such that $A^2 = PBP^*$ and $A^3 = PCP^*$ hold, where B and C are two diagonal matrices. In this case, $(A^2)^\dagger = PB^\dagger P^*$ by the definition of the Moore–Penrose inverse, where $B^\dagger$ is diagonal as well. Further, we obtain from $r(A^2) = r(A)$ and $A^2(A^2)^* = (A^2)^*A^2$ that $\mathscr{R}(A) = \mathscr{R}(A^2) = \mathscr{R}(A^2(A^2)^*) = \mathscr{R}((A^2)^*A^2) = \mathscr{R}((A^2)^*) = \mathscr{R}(A^*)$. Hence, $AA^\dagger = A^\dagger A$ and $(A^2)^\dagger A^2 = A^\dagger A$ hold by Lemma 5 (II) and (III). Consequently, we obtain from the above facts the following result:

$$A = AA^\dagger A = A^\dagger A^2 = (A^2)^\dagger A^3 = PB^\dagger P^*PCP^* = PB^\dagger CP^*,$$

where the product $B^\dagger C$ of the two diagonal matrices is also diagonal. This fact means that $AA^* = P(B^\dagger C)^*BCP^* = PBC(B^\dagger C)^*P^* = A^*A$ holds. Thus, Condition (i) implies Condition (h). The equivalences of Conditions (h), (j), (k), (l), and (m) can be established by a similar approach. □

Corollary 4. *Let* $A \in \mathbb{C}^{m\times m}$ *and denote* $B = A(A^2)^*A$. *Then*, $r(B^2) = r(B) = r(A^2)$ *holds, and the following 36 conditions are equivalent*:

(a) $(AA^*)(A^*A) = (A^*A)(AA^*)$, *namely*, $A(A^2)^*A$ *is Hermitian*.
(b) BB^*B *is Hermitian*.
(c) B^2 *and* B^3 *are Hermitian*.
(d) B^2 *and* B^5 *are Hermitian*.
(e) B^3 *and* B^4 *are Hermitian*.

(f) B^3 *and* B^5 *are Hermitian.*
(g) $(BB^*)^3 = (B^*B)^3$ *and* $(BB^*)^4B = (B^*B)^4B^*$.
(h) $(BB^*)^4B = (B^*B)^4B^*$ *and* $(BB^*)^6 = (B^*B)^6$.
(i) $B^3 = BB^*B$.
(j) $B^\dagger = (B^\dagger)^*$.
(k) $B^*B^\dagger B^* = B^*$.
(l) $B^\dagger B^* B^\dagger = B^\dagger$.
(m) $B^\dagger = (B^\#)^*$.
(n) $B^\# = (B^\#)^*$.
(o) $B^*B^\#B^* = B^*$.
(p) $B^\dagger B^* B^\dagger = B^\#$.
(q) $B^\# B^* B^\# = B^\dagger$.
(r) $B^\# B^* B^\# = B^\#$.
(s) $B^*B^\dagger B^*B^\dagger B^* = B^*B^\dagger B^*$.
(t) $B^*B^\#B^*B^\#B^* = B^*B^\#B^*$.
(u) $B^\dagger B^* B^\dagger B^* B^\dagger = B^\dagger B^* B^\dagger$.
(v) $B^\dagger B^* B^\dagger B^* B^\dagger = B^\# B^* B^\#$.
(w) $B^\dagger B^* B^\# B^* B^\dagger = B^\dagger B^* B^\dagger$.
(x) $B^\dagger B^* B^\# B^* B^\dagger = B^\# B^* B^\#$.
(y) $B^\# B^* B^\dagger B^* B^\# = B^\dagger B^* B^\dagger$.
(z) $B^\# B^* B^\dagger B^* B^\# = B^\# B^* B^\#$.
(a1) $B^\# B^* B^\# B^* B^\# = B^\dagger B^* B^\dagger$.
(b1) $B^\# B^* B^\# B^* B^\# = B^\# B^* B^\#$.
(c1) $r\begin{bmatrix} B & B^* \\ B^*B & BB^* \end{bmatrix} = r(B)$.
(d1) $r\begin{bmatrix} BB^\dagger & B^\dagger B \\ B^* & B \end{bmatrix} = r(B)$.
(e1) $\mathscr{R}\begin{bmatrix} B \\ B^*B \end{bmatrix} = \mathscr{R}\begin{bmatrix} B^* \\ BB^* \end{bmatrix}$.
(f1) $\mathscr{R}\begin{bmatrix} BB^\dagger \\ B^* \end{bmatrix} = \mathscr{R}\begin{bmatrix} B^\dagger B \\ B \end{bmatrix}$.
(g1) $r\begin{bmatrix} B & B^* \\ (B^*B)^2 & (BB^*)^2 \end{bmatrix} = r(B)$.
(h1) $r\begin{bmatrix} BB^\dagger & B^\dagger B \\ B^*BB^* & BB^*B \end{bmatrix} = r(B)$.
(i1) $\mathscr{R}\begin{bmatrix} B \\ (B^*B)^2 \end{bmatrix} = \mathscr{R}\begin{bmatrix} B^* \\ (BB^*)^2 \end{bmatrix}$.
(j1) $\mathscr{R}\begin{bmatrix} BB^\dagger \\ B^*BB^* \end{bmatrix} = \mathscr{R}\begin{bmatrix} B^\dagger B \\ BB^*B \end{bmatrix}$.

Proof. Replacing A with $A(A^*)^2A$ in Theorems 2, 18, and 20 leads to the equivalences of Conditions (a)–(j1). □

Corollary 5. *Let* $A \in \mathbb{C}^{m\times m}$. *Then, the following five conditions are equivalent*:

(a) $r\begin{bmatrix} A & A^* \\ A(A^*)^2A^2 & A^*A^2(A^*)^2 \end{bmatrix} = r(A)$.
(b) $r\begin{bmatrix} AA^\dagger & A^\dagger A \\ A(A^*)^2A & A^*A^2A^* \end{bmatrix} = r(A)$.
(c) $\mathscr{R}\begin{bmatrix} A \\ A(A^*)^2A^2 \end{bmatrix} = \mathscr{R}\begin{bmatrix} A^* \\ A^*A^2(A^*)^2 \end{bmatrix}$.

(d) $\mathscr{R}\begin{bmatrix} AA^\dagger \\ A(A^*)^2A \end{bmatrix} = \mathscr{R}\begin{bmatrix} A^\dagger A \\ A^*A^2A^* \end{bmatrix}$.
(e) $\mathscr{R}(A) = \mathscr{R}(A^*)$ *and* $(AA^*)(A^*A) = (A^*A)(AA^*)$.

Proof. Replacing $B = A(A^*)^2$ and $C = A^*A^2$ in Theorem 15 leads to the equivalences of Conditions (a)–(e). □

Corollary 6. *Let* $A \in \mathbb{C}^{m\times m}$. *Then, the following five conditions are equivalent*:

(a) $r\begin{bmatrix} A & A^* \\ A^kA^*A & A^*A^kA^* \end{bmatrix} = r(A)$ *for any integer* $k \geq 1$.
(b) $r\begin{bmatrix} AA^\dagger & A^\dagger A \\ A^kA^* & A^*A^k \end{bmatrix} = r(A)$ *for any integer* $k \geq 1$.
(c) $\mathscr{R}\begin{bmatrix} A \\ A^kA^*A \end{bmatrix} = \mathscr{R}\begin{bmatrix} A^* \\ A^*A^kA^* \end{bmatrix}$ *for any integer* $k \geq 1$.
(d) $\mathscr{R}\begin{bmatrix} AA^\dagger \\ A^kA^* \end{bmatrix} = \mathscr{R}\begin{bmatrix} A^\dagger A \\ A^*A^k \end{bmatrix}$ *for any integer* $k \geq 1$.
(e) $\mathscr{R}(A) = \mathscr{R}(A^*)$ *and* $A^kA^* = A^*A^k$ *for any integer* $k \geq 1$.

Proof. Replacing $B = A^k$ and $C = A^*A^{k-1}$ in Theorem 15 leads to the equivalences of Conditions (a)–(e). □

Finally, we present a group of identifying conditions for the commutativity of two Hermitian matrices and leave their proofs for the reader.

Corollary 7. *Let* A, $B \in \mathbb{C}^{m\times m}$ *be two Hermitian matrices, and denote* $C = AB$. *Then,* $r(C^2) = r(C)$ *holds, and the following 41 statements are equivalent*:

(a) $AB = BA$, *namely*, AB *is Hermitian*.
(b) $A^\dagger B = BA^\dagger$.
(c) $AB^\dagger = B^\dagger A$.
(d) $A^\dagger B^\dagger = B^\dagger A^\dagger$.
(e) $A^2BA = ABA^2$ *and* $\mathscr{R}(BA) \subseteq \mathscr{R}(A)$.
(f) $B^2AB = BAB^2$ *and* $\mathscr{R}(AB) \subseteq \mathscr{R}(B)$.
(g) $(AB)^2$ *and* $(AB)^3$ *are Hermitian*.
(h) $(AB)^2$ *and* $(AB)^5$ *are Hermitian*.
(i) $(AB)^3$ *and* $(AB)^4$ *are Hermitian*.
(j) $(AB)^3$ *and* $(AB)^5$ *are Hermitian*.
(k) $CC^*C = C^*CC^*$.
(l) $(CC^*)^3 = (C^*C)^3$ *and* $(CC^*)^4C = (C^*C)^4C^*$.
(m) $(CC^*)^4C = (C^*C)^4C^*$ *and* $(CC^*)^6 = (C^*C)^6$.
(n) $C^3 = CC^*C$.
(o) $C^\dagger = (C^\dagger)^*$.
(p) $C^*C^\dagger C^* = C^*$.
(q) $C^\dagger C^*C^\dagger = C^\dagger$.
(r) $C^\dagger = (C^\#)^*$.
(s) $C^\# = (C^\#)^*$.
(t) $C^*C^\#C^* = C^*$.
(u) $C^\dagger C^*C^\dagger = C^\#$.
(v) $C^\#C^*C^\# = C^\dagger$.
(w) $C^\#C^*C^\# = C^\#$.
(x) $C^*C^\dagger C^*C^\dagger C^* = C^*C^\dagger C^*$.
(y) $C^*C^\#C^*C^\#C^* = C^*C^\#C^*$.
(z) $C^\dagger C^*C^\dagger C^*C^\dagger = C^\dagger C^*C^\dagger$.

(a1) $C^{\dagger}C^{*}C^{\dagger}C^{*}C^{\dagger} = C^{\#}C^{*}C^{\#}$.
(b1) $C^{\dagger}C^{*}C^{\#}C^{*}C^{\dagger} = C^{\dagger}C^{*}C^{\dagger}$.
(c1) $C^{\dagger}C^{*}C^{\#}C^{*}C^{\dagger} = C^{\#}C^{*}C^{\#}$.
(d1) $C^{\#}C^{*}C^{\dagger}C^{*}C^{\#} = C^{\dagger}C^{*}C^{\dagger}$.
(e1) $C^{\#}C^{*}C^{\dagger}C^{*}C^{\#} = C^{\#}C^{*}C^{\#}$.
(f1) $C^{\#}C^{*}C^{\#}C^{*}C^{\#} = C^{\dagger}C^{*}C^{\dagger}$.
(g1) $C^{\#}C^{*}C^{\#}C^{*}C^{\#} = C^{\#}C^{*}C^{\#}$.
(h1) $r\begin{bmatrix} C & C^{*} \\ C^{*}C & CC^{*} \end{bmatrix} = r(C)$.
(i1) $r\begin{bmatrix} CC^{\dagger} & C^{\dagger}C \\ C^{*} & C \end{bmatrix} = r(C)$.
(j1) $\mathscr{R}\begin{bmatrix} C \\ C^{*}C \end{bmatrix} = \mathscr{R}\begin{bmatrix} C^{*} \\ CC^{*} \end{bmatrix}$.
(k1) $\mathscr{R}\begin{bmatrix} CC^{\dagger} \\ C^{*} \end{bmatrix} = \mathscr{R}\begin{bmatrix} C^{\dagger}C \\ C \end{bmatrix}$.
(l1) $r\begin{bmatrix} C & C^{*} \\ (C^{*}C)^{2} & (CC^{*})^{2} \end{bmatrix} = r(C)$.
(m1) $r\begin{bmatrix} CC^{\dagger} & C^{\dagger}C \\ C^{*}CC^{*} & CC^{*}C \end{bmatrix} = r(C)$.
(n1) $\mathscr{R}\begin{bmatrix} C \\ (C^{*}C)^{2} \end{bmatrix} = \mathscr{R}\begin{bmatrix} C^{*} \\ (CC^{*})^{2} \end{bmatrix}$.
(o1) $\mathscr{R}\begin{bmatrix} CC^{\dagger} \\ C^{*}CC^{*} \end{bmatrix} = \mathscr{R}\begin{bmatrix} C^{\dagger}C \\ CC^{*}C \end{bmatrix}$.

5. Conclusions

As one of the fundamental problems in mathematics, algebraists love to construct, classify, and characterize various possible algebraic equalities according to the pre-assumed definitions and operation rules in a given algebraic framework. However, increasingly complicated problems related to these problems in matrix equalities have created the essential need for sophisticated methods that go beyond standard contents in matrix theory. As a new attempt, we gave an innovative approach to this kind of problem by establishing a variety of explicit expansion formulas for calculating the ranks of block matrices and displaying their fruitful consequences and applications in the complex matrix theory. As displayed in the preceding sections, establishing non-trivial exact matrix rank equalities properly requires careful designs and experiences. We believe that the algebraic methodologies and the fruitful results and facts developed in this paper can bring some profound mathematical insights to the performance and properties of matrix expressions and matrix equalities and hope that they can be employed in the algebraic analysis of various matrix expression and equality problems that have occurred in mathematics and other applications. In addition, we expect to get more mileage out of the ideas and findings by searching for other situations in which the work can be used or extended. As one such reasonable consideration, it is not difficult to symbolically extend the main contributions in this paper to some general algebraic settings, such as rings and operator algebras, in which generalized inverses of elements can be defined accordingly.

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