axioms

MDPI

Article

γ-Dual Codes over Finite Commutative Chain Rings

Hai Q. Dinh [1], Hiep L. Thi [2,*] and Roengchai Tansuchat [3,*]

[1] Department of Mathematical Sciences, Kent State University, Kent, OH 44240, USA; hdinh@kent.edu
[2] Faculty of Education, Thu Dau Mot University, Binh Duong 55107, Vietnam
[3] Centre of Excellence in Econometrics, Faculty of Economics, Chiang Mai University, Chiang Mai 50200, Thailand
* Correspondence: lthiep@tdmu.edu.vn (H.L.T.); roengchaitan@gmail.com (R.T.)

Abstract: In this article, the notion of γ-dual codes over finite chain rings is introduced as an extension of dual codes over finite chain rings. Various characteristics and properties of γ-dual codes over finite chain rings are explored. We provide both necessary and sufficient conditions for the existence of γ-self-dual codes over finite chain rings. Additionally, we investigate the γ-dual of skew ϕ-α-constacyclic codes over finite chain rings.

Keywords: constacyclic codes; dual codes; chain rings; skew codes

MSC: 94B15; 94B05; 11T71

1. Introduction

Constacyclic codes play a significant role in the field of error-correcting codes as they offer a direct extension of cyclic codes, which have been extensively studied since the late 1950s. Cyclic codes are, in fact, the most thoroughly investigated codes in the realm of error correction.

Many widely recognized codes, such as the BCH, Kerdock, Golay, Reed–Muller, Preparata, Justesen, and binary Hamming codes, either belong to the family of cyclic codes or are derived from them.

Constacyclic codes also find practical applications due to their ability to be efficiently encoded using simple shift registers. Moreover, they possess intricate algebraic structures that enable effective error detection and correction. These characteristics make them highly desirable in engineering applications.

Let F be a finite field of characteristic p, and let α be a nonzero element of F. The α-constacyclic codes of length n over F can be classified as ideals $\langle g(t)\rangle$ in the quotient ring $\frac{F[t]}{\langle t^n-\alpha\rangle}$. Here, $g(t)$ is the unique monic polynomial of minimum degree in the code, which is a divisor of $t^n-\alpha$.

In the context of $\frac{F[t]}{\langle t^n-\alpha\rangle}$, which is a principal ideal ring, every ideal is generated by a unique monic divisor of $t^n-\alpha$. Therefore, the irreducible factorization of $t^n-\alpha$ in $F[t]$ determines all α-constacyclic codes of length n over F.

Traditionally, most of the research focused on the case where the code length n is coprime to the characteristic of the field F. This condition ensures that every root of $t^n-\alpha$ is a simple root in an extension field of F. Consequently, it allows for a description of all such roots and, thus, the α-constacyclic codes through the use of the cyclotomic cosets modulo n (refer to [1,2] for more details).

Cyclic codes with repeated roots, which arise when the code length n is divisible by the field F characteristic p, have been extensively studied by various researchers. The pioneering work on this topic can be traced back to Berman in 1967 [3], followed by contributions from Massey et al. in the 1970s and 1980s [4], Falkner et al. [5], and Roth and Seroussi [6]. However, it was in the 1990s that Castagnoli et al. [7] and van Lint [8]

check for updates

Citation: Dinh, H.Q.; Thi, H.L.; Tansuchat, R. γ-Dual Codes over Finite Commutative Chain Rings. *Axioms* **2024**, *13*, 655. https://doi.org/10.3390/axioms13100655

Academic Editors: José R. Cardoso and Rooney R. A. Coelho

Received: 22 June 2023
Revised: 15 September 2023
Accepted: 22 September 2023
Published: 24 September 2024

Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

investigated repeated-root codes in a more general context. They demonstrated that repeated-root cyclic codes possess a concatenated construction and exhibit poor asymptotic properties. Despite this, there are a few exceptional cases where such codes are optimal, which has motivated researchers to continue exploring this class of codes (for instance, see [9–12]).

In a recent study, Dinh [13] conducted a comprehensive classification and provided detailed structures for all constacyclic codes of length p^s over $\mathcal{R}$. Additionally, in 2012, Dinh [14] extended this analysis to encompass constacyclic codes of length $2p^s$ over $\mathcal{R}$. Furthermore, in 2022, we studied δ-dual codes over finite fields as a generalization of dual codes [15].

In [16], D. Boucher and colleagues introduced a generalization of cyclic codes by utilizing a generator polynomial in noncommutative skew polynomial rings. This novel approach opened up the possibility of exploring a much broader range of codes known as skew cyclic codes. By employing this new class of codes, they were able to conduct systematic searches for codes. A subsequent advancement was made by extending this approach to codes over Galois rings, as described in [17].

In 2010, [18] investigated skew ϕ-α-constacyclic codes over finite chain rings. They were studied in a specific scenario where the codes were generated by monic right divisors of $x^n - \alpha$, where $\phi(\alpha) = \alpha$. If $\phi = id$, then skew ϕ-α-constacyclic codes correspond to α-constacyclic codes over finite chain rings.

Let $\langle \rho \rangle$ be the unique maximal ideal of the finite chain ring R. Then, ρ is a nilpotent ideal of R with a nilpotency index by t. Hence,

$$R = \langle 1 \rangle \supseteq \langle \rho \rangle \supseteq \cdots \supseteq \langle \rho^{t-1} \rangle \supseteq \langle \rho^t \rangle = \langle 0 \rangle.$$

It is well known that $\mathbb{F}_{p^m}[t;\phi]$ is a left and right Euclidean ring. If R is a finite chain ring, $\mathcal{R}[t;\phi]$ is neither left nor right Euclidean. Thus, left and right divisions must be defined. Let $f(t) = \sum_{\iota=0}^{k} a_\iota t^\iota$ and $g(t) = \sum_{\tau=0}^{l} b_\tau x^\tau$, where b_τ is a unit in R and $k \geq l$. From $k \geq l$, $\deg(f(t) - a_k.\phi^{k-l}(b_l^{-1})t^{k-l}g(t)) \leq \deg(f(t))$. By the inductive method, there exist $q(t)$ and $r(t)$ which are skew polynomials satisfying $f(t) = q(t)g(t) + r(t)$ with $\deg(r(t)) < \deg(g(t))$ or $r(t) = 0$. If $r(t) = 0$, then $g(t)|f(t)$. Two polynomials $q(t)$ and $r(t)$ are unique and are called *the right quotient* and *right remainder*, respectively. Note that if $k < l$, then $f(t) = 0.g(t) + f(t)$. This algorithm is called the *Right Division Algorithm* in $R[t;\phi]$. The *Left Division Algorithm* in $R[t;\phi]$ can be defined similarly. A linear code C is said to be skew ϕ-α-constacyclic if C is closed under the ϕ-α-constacyclic shift $v_{\phi,\alpha} : R^n \to R^n$ defined by

$$v_{\phi,\alpha}(a_0, a_1, \ldots, a_{n-1}) = (\phi(\alpha a_{n-1}), \phi(a_0), \ldots, \phi(a_{n-2})).$$

Euclidean and Hermitian inner products are among the most frequently utilized inner products in coding theory. The dual codes corresponding to these products have been extensively explored, and it is well-established that they differ significantly in various contexts, leading to a wide range of applications. For instance, in the construction of quantum codes from classical codes, the well-known CSS construction relies on the Euclidean dual, while the so-called Hermitian construction employs Hermitian dual codes. To further investigate these inner products and derive additional applications, it is beneficial to identify a general inner product that encompasses both the Euclidean and Hermitian inner products as specific instances.

In [19], Fan and Zhang extended the Euclidean and Hermitian inner products to what is known as the Galois inner product, and they examined the Galois self-dual constacyclic codes. The Galois dual of linear codes has been the subject of extensive research ([20–28]). In [19], one such inner product, termed the ℓ-Galois inner product, was introduced. By employing this ℓ-Galois inner product for codes over finite fields, [29] developed new entanglement-assisted quantum error-correcting codes. In this paper, we further generalize the concept to introduce a γ-inner product, which includes the ℓ-Galois, Euclidean, and

Hermitian inner products as specific cases. While this paper primarily focuses on establishing the fundamental properties of the γ-inner product and γ-dual codes, we demonstrate that many properties of ℓ-Galois, Euclidean, and Hermitian dual codes are preserved in γ-dual codes, with necessary modifications. Continuing the research from [20–28], this paper examines the γ-dual codes of codes over finite chain rings and the γ-dual codes of skew ϕ-α-constacyclic codes over finite chain rings.

In this paper, we provide an overview of the preliminary concepts in Section 2. Following that, in Section 3, we give the structure of the γ-dual codes over finite chain rings. We extend our investigation to analyze γ-dual codes of length p^s over the ring $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. The structural properties of these codes are detailed in Theorems 7, 8, 9, and 12. To further expand our study, Section 4 investigates the γ-dual codes of the skew ϕ-α-constacyclic codes over finite chain rings.

2. Preliminaries

Throughout this paper, R is a finite commutative chain ring. The following result is cited in [30] (Proposition 2.1).

Proposition 1 ([30]). *(Proposition 2.1). Let R be a finite commutative chain ring. The following conditions are equivalent:*

(i) $\mathcal{R}$ is a local ring and the maximal ideal M of $\mathcal{R}$ is principal;

(ii) $\mathcal{R}$ is a local principal ideal ring;

(iii) $\mathcal{R}$ is a chain ring.

Let z be a fixed generator of the maximal ideal M of a finite commutative chain ring $\mathcal{R}$. Then, z is nilpotent and we denote its nilpotency index by ω. The ideals of $\mathcal{R}$ form a chain:

$$R = \left\langle z^0 \right\rangle \supsetneq \left\langle z^1 \right\rangle \supsetneq \cdots \supsetneq \left\langle z^{\omega-1} \right\rangle \supsetneq \langle z^{\omega} \rangle = \langle 0 \rangle.$$

Put $\overline{R} = \frac{R}{M}$. By $^- : R[t] \longrightarrow \overline{R}[t]$, we denote the natural ring homomorphism that maps $r \mapsto r + M$ and the variable t to t. The following is a well-known fact about finite commutative chain rings (cf. [31]).

Proposition 2. *Let R be a chain ring and $M = \langle z \rangle$ is a unique maximal ideal, and let ω be the nilpotency of z. Then,*

(a) $|R| = p^k, |\overline{R}| = p^l$, and the characteristics of R and $\overline{R}$ are powers of p, where p is prime, and $k \geq l$ are integers.

(b) For $i = 0, 1, \ldots, \omega$, $|\langle z^i \rangle| = |\overline{R}|^{\omega - i}$. In particular, $|R| = |\overline{R}|^{\omega}$, i.e., $k = l\omega$.

Given n-tuples $v = (v_0, v_1, \ldots, v_{n-1}), d = (d_0, d_1, \ldots, d_{n-1}) \in R^n$, the inner product (or dot product) is defined:

$$v \cdot d = v_0 d_0 + v_1 d_1 + \cdots + v_{n-1} d_{n-1}.$$

If $v \cdot d = 0$, then v and d are orthogonal. For a linear code C over R, its *dual code* $C^{\perp}$ is given by

$$C^{\perp} = \{v \,|v \cdot d = 0, \forall d \in C\}.$$

A code C is called *self-orthogonal* if $C \subseteq C^{\perp}$, and it is called *self-dual* if $C = C^{\perp}$. The following result is well known (cf. [2,30,32]).

Proposition 3. *Let R be a finite chain ring and $|R| = p^{\alpha}$. Then, $|C| = p^k$ and $C^{\perp} = p^{\alpha n - k}$ where $0 \leq k \leq \alpha n$ and $k \in \mathbb{Z}$.*

Given an n-tuple $(d_0, d_1, \ldots, d_{n-1}) \in R^n$, the *cyclic shift* v and *negashift* ν on R^n are defined as usual, i.e.,

$$v(d_0, d_1, \ldots, d_{n-1}) = (d_{n-1}, d_0, d_1, \cdots, d_{n-2}),$$

and

$$\nu(d_0, d_1, \ldots, d_{n-1}) = (-d_{n-1}, d_0, d_1, \cdots, d_{n-2}).$$

A code C is called *cyclic* if $v(C) = C$, and C is called *negacyclic* if $\nu(C) = C$. More generally, if α is a unit of the ring R, then the *α-constacyclic shift* v_α on R^n is the shift

$$v_\alpha(v_0, v_1, \ldots, v_{n-1}) = (\alpha v_{n-1}, v_0, v_1, \cdots, v_{n-2}),$$

and a code C is said to be *α-constacyclic* if $v_\alpha(C) = C$.

The following fact is well known in [2].

Proposition 4. *A linear code C of length n is α-constacyclic over R if and only if C is an ideal of $\frac{R[t]}{\langle t^n - \lambda \rangle}$.*

It is well known from [13] that the dual of an α-constacyclic code is an α^{-1}-constacyclic code.

Proposition 5. *The dual of an α-constacyclic code is an α^{-1}-constacyclic code.*

Analogous to classical constacyclic codes, skew ϕ-α-constacyclic codes are considered as left ideals in $\frac{R[t;\phi]}{\langle t^n - \alpha \rangle}$. We give the conditions of ϕ and α such that $\langle t^n - \alpha \rangle$ is a two-sided ideal.

Theorem 1 ([18]). *(Proposition 2.2). The following statements are equivalent:*

(i) *$t^n - \alpha$ is central in $R[t;\phi]$.*
(ii) *$\langle t^n - \alpha \rangle$ is a two-sided ideal.*
(iii) *n is a multiple of the order of ϕ and α is fixed by ϕ.*

The class $\mathcal{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ is a class of finite chain rings which has been used widely as alphabets in certain constacyclic codes. Some authors ([13,33–37]) studied it. In 1991, Alkhamees [38] completely characterized $\mathrm{Aut}(\mathcal{R})$, where $\mathrm{Aut}(\mathcal{R})$ is the group of the automorphism of R.

Theorem 2. *For $\delta \in \mathrm{Aut}(\mathbb{F}_{p^m})$ and $\xi \in \mathbb{F}_{p^m}^*$, let*

$$\phi_{\delta,\xi} : \mathcal{R} \longrightarrow \mathcal{R}$$

be defined by

$$\phi_{\delta,\xi}(a + bu) = \delta(a) + \xi\delta(b)u.$$

Then, $\mathrm{Aut}(\mathcal{R}) = \{\phi_{\delta,\xi} \mid \delta \in \mathrm{Aut}(\mathbb{F}_{p^m}),\ \xi \in \mathbb{F}_{p^m}^*\}$.

For a skew polynomial $g(t)$ in $\mathcal{R}[t;\phi]$, a left ideal generated by $g(t)$, denoted by $\langle g(t) \rangle$, is in general not a two-sided ideal. However, if $g(t) = t^l h(t) (l \in \mathbb{N}_0)$ satisfying $h(t)$ is central (i.e., commutes with all elements of $\mathcal{R}[t;\phi]$), then $\langle f(t) \rangle$ is a principal two-sided ideal in $\mathcal{R}[t;\phi]$. Thus, the following corollary is a direct consequence.

Corollary 1 ([18]). *(Corollary 2.2). If the skew polynomial $f(t)$ is a monic central and $\deg(f(t)) = n$, then $\frac{\mathcal{R}[t;\phi]}{\langle f(t) \rangle} = \{g(t) \mid \deg(g(t)) < n$, and $g(t)$ is canonical$\}$.*

For ϕ-α-constacyclic codes over finite fields, a code C is a skew ϕ-α-constacyclic code if and only if C is a left ideal $\langle g(t)\rangle \subseteq \frac{\mathbb{F}_{p^m}[t;\phi]}{\langle t^n-\alpha\rangle}$, where $g(t)$ is a right divisor of $t^n-\alpha$.

Theorem 3 ([18]). *(Theorem 2.2). Let n be an integer and $ord(\phi)|n$ and α a unit in R such that $\phi(\alpha)=\alpha$. Then, C is a skew ϕ-α-constacyclic code if and only if $C=\langle g(t)\rangle \subseteq \frac{R[t;\phi]}{\langle t^n-\alpha\rangle}$, where $g(t)$ is a right divisor of $t^n-\alpha$.*

From this theorem, we can find a skew ϕ-α-constacyclic code as a left ideal $\langle g(t)\rangle \subseteq \frac{R[t;\phi]}{\langle t^n-\alpha\rangle}$, where $g(t)$ is a right divisor of $t^n-\lambda$. Nevertheless, it is challenging to enumerate all skew ϕ-α-constacyclic codes because $R[t;\phi]$ does not have unique factorization. As a result, there are significantly more right factors compared to the commutative case, leading to a greater number of skew ϕ-α-constacyclic codes.

Example 1. *Let $\mathcal{R}=\mathbb{F}_3+u\mathbb{F}_3$ be a finite chain ring. We consider the automorphism $\phi_{id,2}$ of $\mathbb{F}_3+u\mathbb{F}_3$, where $\phi_{id,2}(a+ub)=a+2bu$. Then, we have two irreducible factorizations of t^6-1 in $(\mathbb{F}_3+u\mathbb{F}_3)[x;\phi_{id,2}]$:*

$$t^6-1=(t+1)^3(t+2)^3=(t^2+ut+2)^3.$$

Let $f(t)=a_0+a_1t+\cdots+a_rt^r\in R[t]$, where $a_r\neq 0$. Then, the polynomial $f^*(t)=a_r+a_{r-1}t+a_{r-2}t^2+\cdots+a_0t^r$ is called the *reciprocal polynomial* of $f(t)$. It is equivalent that $f^*(t)$ can be expressed by $f^*(t)=t^rf(\frac{1}{t})$. If J is an ideal of R, then $J^*=\{f^*(t)\mid f(t)\in J\}$ is an ideal of R. We define the annihilator of J as

$$\mathcal{A}(J)=\{g(t)\mid f(t)g(t)=0, \text{for any } f(t)\in J\}.$$

The following two results are given in [39].

Lemma 1. *(cf. [40]), (Lemma 2.8).*

(1) If $\deg f\geq \deg g$, then $(f(t)+g(t))^*=f^*(t)+t^{\deg f-\deg g}g^*(t)$.
(2) $(f(t)g(t))^*=f^*(t)g^*(t)$.

Lemma 2. *(cf. [40], (Lemma 2.9)). Let $J=\langle f(t),ug(t)\rangle$ be an ideal of $\mathcal{R}$. Then, $J^*=\{h^*(t)\mid h(t)\in J\}=\langle f^*(t),ug^*(t)\rangle$.*

3. γ-Dual of α-Constacyclic Codes over Finite Chain Rings

Let a,b be two vectors of R^n, i.e., $a=(a_0,a_1,a_2,\cdots,a_{n-1})$ and $b=(b_0,b_1,\cdots,b_{n-1})$. Suppose that γ is an automorphism of R. Then, the γ-inner product in R is defined:

$$\langle a,b\rangle_\gamma=a_0\gamma(b_0)+x_1\gamma(b_1)+\cdots+a_{n-1}\gamma(b_{n-1})=\sum_{i=0}^{n-1}a_i\gamma(b_i).$$

The γ-dual of a code C, denoted by $C^{\perp_\gamma}$, is defined as follows:

$$C^{\perp_\gamma}=\{u|\langle c,u\rangle_\gamma=0,\forall c\in C\}.$$

From the definition of γ-dual, if C is a code of length n over R, then $C^{\perp_\gamma}=\gamma^{-1}(C^\perp)$ and $C^{\perp_\gamma}$ is a linear code. In fact, for any $a\in C^{\perp_\gamma}$, by the definition of $C^{\perp_\gamma}$, we have $\langle c,\gamma(a)\rangle=0$ for all $c\in C$. It implies that $\gamma(a)\in C^\perp$. Therefore, $a\in\gamma^{-1}(C^\perp)$, proving that $C^{\perp_\gamma}\subseteq\gamma^{-1}(C^\perp)$. If $b\in\gamma^{-1}(C^\perp)$, then $\gamma(b)\in C^\perp$. This implies that $\langle c,\gamma(b)\rangle=0$ for all $c\in C$, i.e., $b\in C^{\perp_\gamma}$. Hence, $\gamma^{-1}(C^\perp)\subseteq C^{\perp_\gamma}$. Therefore, $\gamma^{-1}(C^\perp)=C^{\perp_\gamma}$. We have some properties of the γ-dual codes.

Lemma 3. *Let C be a constacyclic code of length n over R. Then,*

(i) $C^{\perp_\gamma}$ *is a linear code of length* n *over* R.
(ii) $C^{\perp_\gamma} = \gamma^{-1}(C^{\perp})$.

Assume that $v \in C^{\perp_\gamma}$ and $d \in C$. Because C is an α-constacyclic code, we have $v_\alpha^{n-1}(d) \in C$. Hence, we have

$$\begin{aligned}
0 &= \langle v_\alpha^{n-1}(d), v\rangle_\gamma \\
&= \langle \alpha(d_1, d_2, \ldots, d_{n-1}, \alpha^{-1}d_0), (v_0, v_1, \ldots, v_{n-1})\rangle_\gamma \\
&= \alpha\left[d_1\gamma(v_0) + \cdots + d_{n-1}\gamma(x_{n-2}) + \alpha^{-1}d_0\gamma(x_{n-1}\right] \\
&= \alpha\left[\alpha^{-1}d_0\gamma(v_{n-1}) + d_1\gamma(v_0) + \cdots + v_{n-1}\gamma(v_{n-1})\right] \\
&= \alpha\left[d_0\gamma(\gamma^{-1}(\alpha^{-1})v_{n-1}) + d_1\gamma(v_0) + \cdots + d_{n-1}\gamma(v_{n-2})\right].
\end{aligned}$$

Thus, we see that $\langle d, v_{\gamma^{-1}(\alpha^{-1})}(v)\rangle_\gamma = 0$ for all $d \in C$. Therefore, $v_{\gamma^{-1}(\lambda^{-1})}(v) \in C^{\perp_\gamma}$. Therefore, the γ-dual code of an α-constacyclic code is a $\gamma^{-1}(\alpha^{-1})$-constacyclic code of length n over R. We summarize our argument in the following proposition.

Proposition 6. *Let* C *be an* α*-constacyclic code of length* n *over* R*. Then, the* γ*-dual code of an* α*-constacyclic code is a* $\gamma^{-1}(\alpha^{-1})$*-constacyclic code of length* n *over* R*. Refs. [13,41] showed some properties of a dual code over* $\mathcal{R}$*.*

Theorem 4. *(cf. [13]). If* $\mu \in \mathbb{F}_{p^m}^*$*, then* $\frac{\mathcal{R}[t]}{\langle t^{p^s}-\mu\rangle}$ *is a local ring (*$\langle u, x-\mu_0\rangle$ *is the maximal ideal), but* $\frac{\mathcal{R}[t]}{\langle t^{p^s}-\mu\rangle}$ *is not a chain ring. The* μ*-constacyclic codes of* p^s *length over* $\mathcal{R}$*, i.e., ideals of the ring* $\frac{\mathcal{R}[x]}{\langle t^{p^s}-\mu\rangle}$*, are*

- *Type* 1: $\langle 0\rangle$, $\langle 1\rangle$.
- *Type* 2: $\langle u(\mu_0 t-1)^\tau\rangle$*, where* $0 \le \tau \le p^s-1$.
- *Type* 3: $\left\langle (\mu_0 t-1)^j + u(\mu_0 t-1)^l h(t)\right\rangle$*, where* $1 \le \tau \le p^s-1$, $0 \le l < \tau$*, and either* $h(t)$ *is* 0 *or* $h(t)$ *is a unit in* $\mathbb{F}_{p^m}[t]$.
- *Type* 4: $\langle (\mu_0 t-1)^\tau + u(\mu_0 t-1)^l\ h(t), u(\mu_0 t-1)^\kappa\rangle$*, with* $h(t)$ *as in Type* 3, $\deg(h) \le \kappa - l - 1$*, and* $\kappa < T$*, where* T *is the smallest integer satisfying* $u(\mu_0 t-1)^T \in \langle (\mu_0 t-1)^\tau + u(\mu_0 t-1)^l h(t)\rangle$*; and* $T = \tau$*, if* $h(t) = 0$*, or otherwise* $T = \min\{\tau, p^s - \tau + l\}$.

Theorem 5 ([41]). *(Theorem 4.8 and Proposition 5.1). Let the* α*-constacyclic code* C *be associated to the ideal* $\langle (\mu_0 t-1)^\iota + u(\mu_0 t-1)^l h(t)\rangle$*, where* $h(t)$ *is* 0 *or* $h(t)$ *is a unit. Then,* $C^\perp$ *associated to the ideal* $\mathcal{A}(C)^*$ *is determined:*

(1) If $h(t)$ *is* 0*, then* $\mathcal{A}(C)^* = \langle (\mu_0 t-1)^{p^s-\iota}\rangle$.

(2) If $h(t)$ *is a unit and* $1 \le \iota \le \frac{p^s+l}{2}$*, then* $\mathcal{A}(C)^* = \langle a(t)\rangle$*, where*

$$a(t) = (\mu_0 t-1)^{p^s-\iota} - u(\mu_0 t-1)^{p^s-2\iota+l}\sum_{\tau=0}^{\iota-l-1} c_\tau(-1)^{\tau+l-\iota}(\mu_0 t-1)^\tau t^{\iota-l-\tau}.$$

(3) If $h(t)$ *is a unit and* $\frac{p^s+l}{2} < \iota \le p^s-1$*, then* $\mathcal{A}(C)^* = \langle b(t), u(\mu_0 t-1)^{p^s-\iota}\rangle$*, where*

$$b(t) = (\mu_0 t-1)^{\iota-l} - u\sum_{\tau=0}^{p^s-\iota-1} c_\tau(-1)^{\tau+l-\iota}(\mu_0 t-1)^\tau t^{\iota-l-\tau}.$$

Theorem 6 ([41]). *(Theorem 4.9). Let the* α*-constacyclic code* C *be associated to the ideal* $\langle (\mu_0 t-1)^\iota + u(\mu_0 t-1)^l h(t), u(\mu_0 t-1)^\omega\rangle$ *(*$h(t)$ *is* 0 *or* $h(t)$ *is a unit). Then, the* $C^\perp$ *associated to the ideal* $\mathcal{A}(C)^*$ *is determined:*

(1) If $h(t) = 0$, *then* $\mathcal{A}(C)^* = \left\langle (\mu_0 t - 1)^{p^s - \omega}, u(\mu_0 t - 1)^{p^s - \iota} \right\rangle$.

(2) If $h(t)$ *is a unit, then* $\mathcal{A}(C)^* = \left\langle c(t), u(\mu_0 t - 1)^{p^s - \iota} \right\rangle$, *where*

$$c(t) = (\mu_0 t - 1)^{p^s - \omega} - u(\mu_0 t - 1)^{p^s - \iota - \omega + l} \sum_{\tau=0}^{\omega - l - 1} c_\tau (-1)^{\tau + l - \iota} (\mu_0 t - 1)^\tau t^{\iota - l - \tau}.$$

In [42], the authors have successfully constructed some quantum maximum-distance-separable codes from Types 2, 3, and 4 constacyclic codes of length p^s over finite fields. Therefore, the ability to compute the structure of γ-dual codes in a more general context holds the promise of helping us discover further quantum maximum-distance-separable codes with improved parameters over $\mathcal{R}$. In order to construct quantum codes from γ-dual codes of length p^s over $\mathcal{R}$, we need to determine the γ-dual codes of μ-constacyclic codes of length p^s over $\mathcal{R}$, where $\mu \in \mathbb{F}_{p^m}^*$. From Theorem 2, we can see that $\mu^{-1} \in \mathrm{Aut}(\mathcal{R})$. Then, there exist $a \in \mathrm{Aut}(\mathbb{F}_{p^m})$ and $b \in \mathbb{F}_{p^m}$ satisfying $\mu^{-1} = \phi_{a^{-1},b}$. By [41] (Theorem 4.7), $C^\perp = \langle ((\mu_0)t - 1)^{p^s - \tau}, u \rangle$. Hence, $C^{\perp_\gamma} = \langle (a^{-1}(\mu_0^{-1})t - 1)^{p^s - \tau}, u \rangle$. Hence, $C^{\perp_\gamma}$ of the Type 2 μ-constacyclic codes is completely determined.

Theorem 7. *Let* $C = \langle u(\mu_0 t - 1)^\tau \rangle$ *be a* μ*-constacyclic code of length* p^s *over* $\mathcal{R}$, *and then* $C^{\perp_\gamma} = \langle (a^{-1}(\mu_0^{-1})t - 1)^{p^s - \tau}, u \rangle$.

Theorem 8. *Let the constacyclic code* C *be associated to the ideal* $\langle (\mu_0 t - 1)^\iota + u(\mu_0 t - 1)^l h(t) \rangle$, *where* $h(t)$ *is* 0 *or* $h(t) = \sum_\tau c_\tau (\mu_0 t - 1)^\tau$ *is a unit, where* $h_\tau \in \mathbb{F}_{p^m}$ *and* $h_0 \neq 0$. *Then,* $C^{\perp_\gamma}$ *can be determined:*

(i) If $h(t)$ *is* 0, *then* $C^{\perp_\gamma} = \langle (a^{-1}(\mu_0^{-1})t - 1)^{p^s - \iota} \rangle$.

(ii) If $h(t)$ *is a unit and* $1 \leq \iota \leq \frac{p^s + l}{2}$, *then* $C^{\perp_\gamma} = \langle m_\gamma(t) \rangle$, *where*

$$m_\gamma(t) = (a^{-1}(\mu_0^{-1})t - 1)^{p^s - \iota} - u(a^{-1}(\mu_0^{-1})t - 1)^{p^s - 2\iota + l} \sum_{\tau=0}^{\iota - l - 1} b a^{-1}(h_\tau)(-1)^{\tau + l - \iota} (a^{-1}(\mu_0^{-1})t - 1)^\tau t^{\iota - l - \tau}.$$

(iii) If $h(t)$ *is a unit and* $\frac{p^s + l}{2} < \iota \leq p^s - 1$, *then* $C^{\perp_\gamma} = \langle \kappa_\gamma(t), u(a^{-1}(\mu_0^{-1})t - 1)^{p^s - \iota} \rangle$, *where*

$$\kappa_\gamma(t) = (a^{-1}(\mu_0^{-1})t - 1)^{\iota - l} - u \sum_{\tau=0}^{p^s - \iota - 1} b a^{-1}(h_\tau)(-1)^{\tau + l - \iota} (a^{-1}(\mu_0^{-1})t - 1)^\tau t^{\iota - l - \tau}.$$

Proof. If $h(t) = 0$, by Theorem 5, then $\mathcal{A}(C)^* = \langle (\mu_0 t - 1)^{p^s - \iota} \rangle$. Hence, $C^{\perp_\gamma} = \langle (\gamma^{-1}(\mu_0^{-1})$ $t - 1)^{p^s - \iota} \rangle$, proving (i). We will prove (ii). Assume that $h(t)$ is a unit and $1 \leq \iota \leq \frac{p^s + l}{2}$. In Theorem 5, we have $\mathcal{A}(C)^* = \langle m(t) \rangle$, where

$$m(t) = (\mu_0 t - 1)^{p^s - \iota} - u(\mu_0 t - 1)^{p^s - 2\iota + l} \sum_{\tau=0}^{\iota - l - 1} c_\tau (-1)^{\tau + l - \iota} (\mu_0 t - 1)^\tau t^{\iota - l - \tau}.$$

From Lemma 3, we see that $C^{\perp_\gamma} = \gamma^{-1}(C^\perp)$. This implies that $C^{\perp_\gamma} = \langle m_\gamma(t) \rangle$, where

$$m_\gamma(tx) = (a^{-1}(\mu_0^{-1})t - 1)^{p^s - \iota} - u(a^{-1}(\mu_0^{-1})t - 1)^{p^s - 2\iota + l} \sum_{\tau=0}^{\iota - l - 1} b a^{-1}(c_\tau)(-1)^{\tau + l - \iota} (a^{-1}(\mu_0^{-1})t - 1)^\tau t^{\iota - l - \tau}.$$

Assume that $h(t)$ is a unit and $\frac{p^s+l}{2} < \iota \leq p^s - 1$. From Theorem 5, $\mathcal{A}(C)^* = \langle b(t), u(t-1)^{p^s-\iota} \rangle$, where

$$\kappa(t) = (\mu_0 t - 1)^{\iota - l} - u \sum_{\tau=0}^{p^s-\iota-1} ba^{-1}(c_\tau)(-1)^{\tau+l-\iota}(\mu_0 t - 1)^\tau t^{\iota-l-\tau}.$$

This shows that $C^{\perp_\gamma} = \langle \kappa_\gamma(t), u(a(\mu_0^{-1})t - 1)^{p^s-\iota} \rangle$, where

$$\kappa_\gamma(t) = (a^{-1}(\mu_0^{-1})t - 1)^{\iota-l} - u \sum_{\tau=0}^{p^s-\tau-1} ba^{-1}(c_\tau)(-1)^{\tau+l-\iota}(a^{-1}(\mu_0^{-1})t - 1)^\tau t^{\iota-l-\tau},$$

showing (iii), as required. □

As mentioned in the paragraph before Theorem 7, in [42], the authors successfully constructed some quantum maximum-distance-separable codes from the Type 4 constacyclic codes of length p^s over finite fields. We are aware that constructing optimal quantum codes from Type 4 constacyclic codes over finite fields requires us to determine the structure of the dual codes of Type 4 constacyclic codes. Therefore, in continuation of that research direction, we aim to build quantum codes from Type 4 constacyclic codes over $\mathcal{R}$ by means of γ-dual codes. Thus, the theorem below enables us to ascertain the structure of γ-dual codes, which is the key to constructing quantum codes over $\mathcal{R}$ by γ-dual codes. Next, we determine the γ-dual code $C^{\perp_\gamma}$ of the Type 4 μ-constacyclic codes, where C is associated to the ideal $\langle (\mu_0 t - 1)^\tau + u(\mu_0 t - 1)^l\ h(t), u(\mu_0 t - 1)^\kappa \rangle$, with $h(t)$ as in Type 3, $\deg(h) \leq \kappa - l - 1$, and $\kappa < T$, where T is the smallest integer satisfying $u(\mu_0 t - 1)^T \in \left\langle (\mu_0 t - 1)^\tau + u(\mu_0 t - 1)^l h(t) \right\rangle$; and $T = \tau$, if $h(t) = 0$, or otherwise $T = \min\{\tau, p^s - \tau + l\}$.

Theorem 9. *Let the constacyclic code C be associated to the ideal $\langle (\mu_0 t - 1)^\tau + u(\mu_0 t - 1)^l\ h(t), u(\mu_0 t - 1)^\kappa \rangle$, where $h(t)$ is 0 or $h(t)$ is a unit. Then, $C^{\perp_\gamma}$ associated to the ideal $\mathcal{A}(C)^{\perp_\gamma}$ is determined:*

(1) If $h(t) = 0$, then $\mathcal{A}(C)^{\perp_\gamma} = \left\langle (a^{-1}(\mu_0^{-1})t - 1)^{p^s-\omega}, u(a^{-1}(\mu_0^{-1})t - 1)^{p^s-\iota} \right\rangle$.

(2) If $h(t)$ is a unit, then $\mathcal{A}(C)^{\perp_\gamma} = \left\langle c_\gamma(t), u(a^{-1}(\mu_0^{-1})t - 1)^{p^s-\iota} \right\rangle$, where

$$c_\gamma(t) = (a^{-1}(\mu_0^{-1})t - 1)^{p^s-\omega} - u(a^{-1}(\mu_0^{-1})t - 1)^{p^s-\iota-\omega+l} \sum_{\tau=0}^{\omega-l-1} ba^{-1}(c_\tau)(-1)^{\tau+l-\iota}(a(\mu_0^{-1})t - 1)^\tau t^{\iota-l-\tau}.$$

Proof. If $h(t) = 0$, then $C = \langle (\mu_0 t - 1)^\iota, u(\mu_0 t - 1)^\omega \rangle$. By Theorem 6,

$$\mathcal{A}(C)^* = \left\langle (\mu_0 t - 1)^{p^s-\omega}, u(\mu_0 t - 1)^{p^s-\iota} \right\rangle.$$

By using Lemma 3, $\mathcal{A}(C)^{\perp_\gamma} = \left\langle (a^{-1}(\mu_0^{-1})t - 1)^{p^s-\omega}, u(a^{-1}(\mu_0^{-1})t - 1)^{p^s-\iota} \right\rangle$, proving (1). For the case that $h(t)$ is a unit, by Theorem 6 again, $\mathcal{A}(C)^* = \left\langle \ell^*(t), u(\mu_0 t - 1)^{p^s-\iota} \right\rangle$, where

$$\ell^*(t) = (-1)^{p^s-\omega}(\mu_0 t - 1)^{p^s-\omega} - u(-1)^{p^s-\iota-\omega+l}(\mu_0 t - 1)^{p^s-\iota-\omega+l} \sum_{\tau=0}^{\omega-l-1} c_\tau(-1)^\tau(\mu_0 t - 1)^\tau t^{\iota-l-\tau}.$$

Using Lemma 3, we have $\mathcal{A}(C)^{\perp_\gamma} = \left\langle c_\gamma(t), u(a^{-1}(\mu_0^{-1})t - 1)^{p^s-\iota} \right\rangle$, where

$$c_\gamma(t) = (a^{-1}(\mu_0^{-1})t - 1)^{p^s-\omega} - u(a^{-1}(\mu_0^{-1})t - 1)^{p^s-\iota-\omega+l} \sum_{\tau=0}^{\omega-l-1} ba^{-1}(c_\tau)(-1)^{\tau+l-\iota}(a(\mu_0^{-1})t - 1)^\tau t^{\iota-l-\tau}.$$

□

We consider the case $\alpha = \nu + u\eta$ for nonzero elements ν, η of $\mathbb{F}_{p^m}$. In [13], $(\nu + u\eta)$-constacyclic codes of length p^s over $\mathcal{R}$ are ideals of the ring

$$\mathcal{R}_{\nu,\eta} = \frac{\mathcal{R}[t]}{\langle x^{p^s} - (\eta + u\gamma)\rangle}.$$

Because $\nu \in \mathbb{F}_{p^m}$, $\nu^{p^m} = \nu$, and so $\nu^{p^{tm}} = \nu$, for any positive integer t. By the Division Algorithm, there exist nonnegative integers ν_q, ν_r satisfying $s = \nu_q m + \nu_r$, and $0 \leq \nu_r \leq m-1$. Let $\nu_0 = \nu^{p^{(\nu_q+1)m-s}} = \nu^{p^{m-\nu_r}}$. Then, $\nu_0^{p^s} = \nu^{p^{(\nu_q+1)m}} = \nu$. [13] proved that $t - \nu_0$ is nilpotent.

Lemma 4. *(cf. [13] (Lemma 4.1)). In $\mathcal{R}_{\nu,\eta}$, $\langle (t-\nu_0)^{p^s}\rangle = \langle u\rangle$. In particular, $t - \nu_0$ is nilpotent with a nilpotency index $2p^s$.*

Theorem 10. *(cf. [13] (Theorem 4.2)). The ring $\mathcal{R}_{\nu,\eta}$ is a chain ring with a maximal ideal $\langle t - \nu_0\rangle$, whose ideals are*

$$\mathcal{R}_{\nu,\eta} = \langle 1\rangle \supseteq \langle x - \eta_0\rangle \supseteq \cdots \supseteq \langle (x-\eta_0)^{2p^s-1}\rangle \supseteq \langle (x-\eta_0)^{2p^s}\rangle = \langle 0\rangle.$$

In other words, $(\nu + u\eta)$-constacyclic codes of length p^s over $\mathcal{R}$ are precisely the ideals $\langle (t-\nu_0)^i\rangle \subseteq \mathcal{R}_{\nu,\eta}$, where $0 \leq \iota \leq 2p^s$. The $(\nu + u\eta)$-constacyclic code $\langle (t-\nu_0)^i\rangle$ has $p^{m(2p^s-\iota)}$ codewords.

Similar to $\mathcal{R}_{\nu,\eta}$, we can prove the following result.

Lemma 5. *The ring $\mathcal{R}_{\eta^{-1},-\eta^{-2}\nu}$ is a chain ring with a maximal ideal $\langle x - \eta_0^{-1}\rangle$, whose ideals are*

$$\mathcal{R}_{\eta^{-1},-\eta^{-2}\nu} = \langle 1\rangle \supseteq \langle t - \eta_0^{-1}\rangle \supseteq \cdots \supseteq \langle (t-\eta_0^{-1})^{2p^s-1}\rangle \supseteq \langle (t-\eta_0^{-1})^{2p^s}\rangle = \langle 0\rangle.$$

In other words, $(\eta^{-1} - u\eta^{-2}\nu)$-constacyclic codes of length p^s over $\mathcal{R}$ are precisely the ideals $\langle (t-\eta_0^{-1})^\iota\rangle \subseteq \mathcal{R}_{\eta^{-1},-\eta^{-2}\gamma}$, $0 \leq \iota \leq 2p^s$. Moreover, each $(\eta^{-1} - u\eta^{-2}\nu)$-constacyclic code $\langle (t-\eta_0^{-1})^\iota\rangle \subseteq \mathcal{R}_{\eta^{-1},-\eta^{-2}\nu}$ contains $p^{m(2p^s-\iota)}$ codewords.

We recall the dual codes of any $(\nu + u\eta)$-constacyclic codes cited in [13].

Theorem 11. *(cf. [13] (Theorem 4.3)). Let C be a $(\nu + u\eta)$-constacyclic code of length p^s over $\mathcal{R}$. Then, $C = \langle (t-\nu_0)^\iota\rangle \subseteq \mathcal{R}_{\nu,\eta}$, for some $\iota \in \{0, 1, \ldots, 2p^s\}$ and $C^\perp$ is the $(\nu^{-1} - u\nu^{-2}\eta)$-constacyclic code*

$$C^\perp = \left\langle (\nu_0^{-1}t - 1)^{2p^s-\iota}\right\rangle \subseteq \mathcal{R}_{\nu^{-1},-\nu^{-2}\eta}.$$

We already know that a nonzero element α in $\mathcal{R}$ must have two forms: $0 \neq \mu \in \mathbb{F}_{p^m}$ and $(\nu + u\eta)$, where $\eta \neq 0$. We have extensively investigated the results regarding the γ-dual codes when $0 \neq \nu \in \mathbb{F}_{p^m}$ in Theorems 7, 8, and 9. The following theorem will allow us to determine the γ-dual codes in the remaining case when α takes the form of $(\nu + u\eta)$, where $\eta \neq 0$.

Theorem 12. *Let C be a $(\nu + u\eta)$-constacyclic code of length p^s over $\mathcal{R}$. Then,*

$$C^{\perp_\gamma} = \left\langle (a^{-1}(\eta_0^{-1})t - 1)^{2p^s-\iota}\right\rangle \subseteq \mathcal{R}_{a^{-1}(\nu_0^{-1}),-\nu\alpha^{-1}(\nu^{-2}\eta)}.$$

Proof. From Theorem 11, we have $C^\perp = \left\langle (\eta_0^{-1}x - 1)^{2p^s-i}\right\rangle$. By applying Lemma 3, we can determine

$$C^{\perp_\gamma} = \left\langle (\alpha^{-1}(\eta_0^{-1})x - 1)^{2p^s-i}\right\rangle \subseteq \mathcal{R}_{\alpha(\nu_0^{-1}),-\nu\alpha(\nu^{-2}\eta)}.$$

□

4. γ-Dual of Skew ϕ-α-Constacyclic Codes over Finite Chain Rings

Given a skew ϕ-α-constacyclic code, we define a γ-dual code of skew ϕ-α-constacyclic codes over finite chain rings as follows.

Definition 1. *Let C be a skew ϕ-α-constacyclic code and γ be an automorphism of R. Then, the γ-inner product is defined by*

$$\langle u, v\rangle_\gamma = \sum_{\iota=0}^{n-1} u_\iota \gamma(v_\iota),$$

for $u = (u_0, u_1, \ldots, u_{n-1})$ and $v = (v_0, v_1, \ldots, v_{n-1})$ in R^n. From the γ-inner product, the definition of the γ-dual code $C^{\perp_\gamma}$ is given by

$$C^{\perp_\gamma} = \{u | \langle c, u\rangle_\gamma = 0, \forall c \in C\}.$$

By Definition 1, we have the following result.

Lemma 6. *Let C be a skew ϕ-α-constacyclic code of length n over R. Then,*

(i) *$C^{\perp_\gamma}$ is a linear code.*

(ii) *$C^{\perp_\gamma} = \gamma^{-1}(C^\perp)$.*

Proof. For any $t_1, t_2 \in R$, $c = (c_0, c_1, \ldots, c_{n-1}) \in C$ and $y = (y_0, y_1, \ldots, y_{n-1}), z = (z_0, z_1, \ldots, z_{n-1}) \in C^{\perp_\gamma}$. We see that

$$\begin{aligned}\langle c, t_1 y + t_2 z\rangle_\gamma &= \sum_{i=0}^{n-1} c_i \delta(t_1 y + t_2 z)\\ &= \gamma(t_1) \sum_{i=0}^{n-1} c_i \delta(y_i) + \gamma(t_2) \sum_{i=0}^{n-1} c_i \delta(z_i)\\ &= \gamma(t_1)\langle c, y\rangle_\gamma + \gamma(t_2)\langle c, z\rangle_\gamma\\ &= 0.\end{aligned}$$

It implies that $t_1 y + t_2 z \in C^{\perp_\gamma}$, proving (i). □

For any $v \in C^{\perp_\gamma}$, i.e., for all $c \in C$, we have $\langle c, \gamma(v)\rangle = 0$. Thus, $\gamma(v) \in C^\perp$. Therefore, $v \in \gamma^{-1}(C^\perp)$, proving that $C^{\perp_\gamma} \subseteq \gamma^{-1}(C^\perp)$. If $y \in \gamma^{-1}(C^\perp)$, then $\gamma(y) \in C^\perp$. Thus, $\langle c, \gamma(y)\rangle = 0$ for all $c \in C$. Hence, $y \in C^{\perp_\gamma}$, i.e., $\gamma^{-1}(C^\perp) \subseteq C^{\perp_\gamma}$. It proves that $\gamma^{-1}(C^\perp) = C^{\perp_\gamma}$, completing the proof of (ii).

The relationship between a skew ϕ-α-constacyclic code of length n over R and its dual is provided in the following result.

Theorem 13. *Let C be a skew ϕ-α-constacyclic code of length n over R, where $ord(\phi)|n$ and $\phi\gamma = \gamma\phi$. Then, the γ-dual code of a skew ϕ-α-constacyclic code is a skew ϕ-$\gamma^{-1}(\alpha^{-1})$-constacyclic code of length n over R. In particular, if $\alpha^2 = 1$ and C is a skew ϕ-α-constacyclic code, then $C^{\perp_\gamma}$ is a skew ϕ-$\gamma^{-1}(\alpha)$-constacyclic code.*

Proof. Because α is fixed by ϕ, we can see that α^{-1} is fixed by ϕ. Let $u = (u_0, u_1, \ldots, u_{n-1}) \in C$ and $v = (v_0, v_1, \ldots, v_{n-1}) \in C^{\perp_\gamma}$. From $u \in C$, we have $\tau_{\phi,\alpha}^{n-1}(u) = \big(\phi^{n-1}(\alpha u_1), \phi^{n-1}(\alpha u_2), \ldots, \phi^{n-1}(\alpha u_{n-1}), \phi^{n-1}(u_0)\big) \in C$. By the definition of $C^{\perp_\gamma}$, we have

$$\begin{aligned} 0 &= \langle \tau_{\phi,\alpha}^{n-1}(u), v\rangle_\gamma \\ &= \lambda\langle \big(\phi^{n-1}(\alpha u_1), \phi^{n-1}(\alpha u_2), \ldots, \phi^{n-1}(\alpha u_{n-1}), \phi^{n-1}(\alpha^{-1} u_0)\big), (v_0, \ldots, v_{n-1})\rangle_\gamma \\ &= \alpha\big(\phi^{n-1}(\alpha^{-1}u_0)\gamma(v_{n-1}) + \sum_{i=1}^{n-1} \phi^{n-1}(u_i)\gamma(v_i)\big). \end{aligned}$$

As n is a multiple of the order of ϕ and α^{-1} is fixed by ϕ, we can see that

$$\begin{aligned} 0 = \phi(0) &= \phi\big[\alpha\big(\phi^{n-1}(\alpha^{-1}u_0)\gamma(v_{n-1}) + \sum_{i=1}^{n-1} \phi^{n-1}(u_i)\gamma(v_{i-1})\big)\big] \\ &= \alpha\big[u_0\phi(\alpha^{-1}\gamma(v_{n-1})) + \sum_{i=1}^{n-1}(u_i)\phi(\gamma(v_{i-1}))\big] \\ &= \alpha\big[u_0\phi(\gamma(\gamma^{-1}(\alpha^{-1})v_{n-1})) + \sum_{i=1}^{n-1} u_i\phi(\gamma(v_{i-1}))\big] \\ &= \alpha\big[u_0\gamma(\phi(\gamma^{-1}(\alpha^{-1})v_{n-1})) + \sum_{i=1}^{n-1} u_i\gamma(\phi(v_{i-1}))\big]. \end{aligned}$$

□

Hence, $\tau_{\phi,\gamma^{-1}(\alpha^{-1})}(v) \in C^{\perp_\gamma}$. This implies that the γ-dual of a skew ϕ-λ-constacyclic code is a skew ϕ-$\gamma^{-1}(\alpha^{-1})$-constacyclic code of length n over R. As an application, if $\alpha^2 = 1$ and C is a skew ϕ-α-constacyclic code, then $C^{\perp_\gamma}$ is a skew ϕ-$\gamma^{-1}(\alpha)$-constacyclic code.

The dual code of a skew ϕ-α-constacyclic code C is determined in the following theorem [18] (Theorem 3.3).

Theorem 14 ([18]). *(Theorem 3.3). We denote that $R[t;\phi]S^{-1}$ is the right localization of $R[t;\phi]$. Assume that $\alpha^2 = 1$. Let $g(t)$ be a right divisor of $t^n - \alpha$ and $h(t) := \frac{t^n-\alpha}{g(t)}$. Let C be the skew ϕ-α-constacyclic code generated by $g(t)$. Then, the following statements hold:*

(i) *The skew polynomial $t^{\deg(h(t))}\varphi(h(t))$ is a right divisor of $t^n - \alpha$.*

(ii) *The dual $C^\perp$ is a skew ϕ-α-constacyclic code generated by $t^{\deg(h(t))}\zeta(h(t))$, where $\zeta : R[t;\phi] \to R[t;\phi]S^{-1}$ is defined by*

$$\zeta(\sum_{\iota=0}^{k} a_\iota t^\iota) = \sum_{\iota=0}^{k} t^{-\iota} a_\iota.$$

We determine the γ-dual code of a skew ϕ-α-constacyclic code as follows.

Theorem 15. *Assume that $\alpha^2 = 1$. Let $g(t)$ be a right divisor of $t^n - \alpha$. Let C be the skew ϕ-α-constacyclic code generated by $g(t)$. Let γ be an automorphism of R such that $\phi\gamma = \gamma\phi$. Then, the γ-dual of the skew ϕ-α-constacyclic code is a skew ϕ-α-constacyclic code generated by $\langle\gamma^{-1}(t^{\deg(h(t))}\zeta(h(t)))\rangle$.*

Proof. Combining Theorem 14 and Lemma 6, we can see that the γ-dual of the skew ϕ-α-constacyclic code is $\gamma^{-1}(C^\perp)$. This shows that $C^{\perp_\gamma} = \langle\gamma^{-1}(t^{\deg(h(t))}\zeta(h(t)))\rangle$. □

Next, we study the γ-dual of skew ϕ-α-constacyclic codes over $\mathcal{R}$. In [18], the left ideals of $\frac{\mathcal{R}[t;\phi]}{\langle tx^n-\alpha\rangle}$ are divided into three types as in the following theorem.

Theorem 16 ([18]). *(Theorem 4.1). Let C be a nonzero left ideal in $\frac{\mathcal{R}[t;\phi]}{\langle t^n-\alpha\rangle}$ and A be the set of all nonzero skew polynomials of minimal degree in C. Then,*

(i) *The left ideals of Type LI-1 are $(\langle 0\rangle, \langle 1\rangle)$ or $C = \langle g(t)\rangle$, where $g(t)$ is the unique skew polynomial.*

(ii) *The left ideals of Type LI-2: $C = \langle g(t)\rangle$, where $g(t) = ug_1(t)$ in A with the leading coefficient u.*

(iii) *The left ideals of Type LI-3 are $C = \langle g(t), f(t)\rangle$, where $g(t) = ug_1(t)$ in A with the leading coefficient u and a unique monic skew polynomial $f(t) = f_0(t) + uf_1(t)$ of minimal degree in C satisfying $\deg(f_1(t)) < \deg(g_1(t))$.*

Next, we provide some properties of the left ideals of each type LI-i (i=1, 2, 3.)

We write $\overleftarrow{f(t)}$ to indicate that $\overleftarrow{f(t)}$ is the skew polynomial such that $f(t)u = u\overleftarrow{f(t)}$.

Lemma 7 ([18]). *(Proposition 4.3). A left ideal of type LI-3 is generated by $\{g(t) = ug_1(t), f(t) = f_0(t) + uf_1(t)\}$, where $f_0(t), f_1(t), g_1(t) \in \mathbb{F}_{p^m}[t;\phi]$ satisfy the following properties:*

(i) *$g_1(t)$, $f_0(t)$ are monic;*

(ii) *$\deg(f_1(t)) < \deg(g_1(t)) < \deg(f_0(t)) < n$;*

(iii) *$g_1(t)$ is a right divisor of $f_0(t)$ in $\mathbb{F}_{p^m}[t;\phi]$;*

(iv) *$f_0(t)$ is a right divisor of $t^n - \bar{\alpha}$ in $\mathbb{F}_{p^m}[t;\phi]$. Moreover, if $\alpha \in \mathbb{F}_{p^m}$, then $g_1(t)$ is a right divisor of $(\overleftarrow{\frac{t^n-\lambda}{f_0(t)}})f_1(t)$ in $\mathbb{F}_{p^m}[t;\phi]$.*

The structure of the dual codes of skew ϕ-cyclic and negacyclic codes over $\mathcal{R}$ is investigated in [18].

Theorem 17 ([18]). *(Theorem 4.2). Let $\alpha \in \{-1, 1\}$. Then, the dual code of a left ideal in $\frac{\mathcal{R}[t;\phi]}{\langle t^n-\alpha\rangle}$ is also a left ideal in $\frac{\mathcal{R}[t;\phi]}{\langle t^n-\alpha\rangle}$, determined as follows:*

$LI-1^{\perp}$. If $C = \langle g_0(x) + ug_1(t)\rangle$, then $C^{\perp} = \langle t^{n-\deg(g_0(t))}\varphi(\frac{t^n-\lambda}{g_0(t)+ug_1(t)})\rangle$.

$LI-2^{\perp}$. If $C = \langle ug_1(t)\rangle$, then $C^{\perp} = \langle u, t^{n-\deg(g_1(t))}\varphi(\frac{t^n-\lambda}{g_1(t)})\rangle$.

$LI-3^{\perp}$. If $C = \langle ug_1(t), f_0(t) + uf_1(t)\rangle$, then there exists $m(t) \in \mathbb{F}_{p^m}[t;\phi]$ such that $m(t)g_1(t) = (\overleftarrow{\frac{t^n-\lambda}{f_0(t)}})f_1(t)$ and

$$C^{\perp} = \langle t^{n-\deg(f_0(t))\varphi(\frac{t^n-\lambda}{f_0(t)}u)}, t^{n-\deg(g_1(t))}\varphi(\frac{t^n-\lambda}{g_1(t)} - um(t))\rangle,$$

where $\zeta : \mathcal{R}[t;\phi] \to \mathcal{R}[t;\phi]S^{-1}$ defined by $\zeta(\sum_{\iota=0}^{k} a_\iota t^\iota) = \sum_{\iota=0}^{k} t^{-\iota}a_\iota$.

Combining Theorem 17 and Lemma 6, the γ-dual code of the skew ϕ-cyclic and negacyclic codes over $\mathcal{R}$ is determined.

Theorem 18. *Let $\alpha \in \{-1, 1\}$. Then, the γ-dual code of a left ideal in $\frac{\mathcal{R}[t;\phi]}{\langle t^n-\alpha\rangle}$ is also a left ideal in $\frac{\mathcal{R}[t;\phi]}{\langle t^n-\alpha\rangle}$, determined as follows:*

$LI-1^{\perp}$. If $C = \langle g_0(t) + ug_1(t)\rangle$, then $C^{\perp_\gamma} = \langle \gamma^{-1}\big(t^{n-\deg(g_0(t)}\zeta(\frac{t^n-\alpha}{g_0(t)+ug_1(t)})\big)\rangle$.

$LI-2^{\perp}$. If $C = \langle ug_1(t)\rangle$, then $C^{\perp_\gamma} = \langle u, \gamma^{-1}(t^{n-\deg(g_1(t))}\zeta(\frac{t^n-\lambda}{g_1(t)}))\rangle$.

$LI-3^{\perp}$. If $C = \langle ug_1(t), f_0(t) + uf_1(t)\rangle$, then there exists $m(t) \in \mathbb{F}_{p^m}[t;\phi]$ such that $m(t)g_1(t) = (\overleftarrow{\frac{t^n-\lambda}{f_0(t)}})f_1(t)$ and

$$C^{\perp_\gamma} = \langle \gamma^{-1}\big(t^{n-\deg(f_0(t))\varphi(\frac{t^n-\lambda}{f_0(t)}u)}, \gamma^{-1}(t^{n-\deg(g_1(t))}\zeta(\frac{t^n-\lambda}{g_1(t)} - um(t))\big)\rangle,$$

where $\varphi : \mathcal{R}[t;\phi] \to \mathcal{R}[t;\phi]S^{-1}$ *defined by* $\zeta(\sum_{\iota=0}^{k} a_\iota x^\iota) = \sum_{\iota=0}^{k} t^{-\iota} a_\iota$.

5. Conclusions

Euclidean and Hermitian inner products lead to different code constructions. Their differences allow researchers to explore diverse code design options, each suited to specific applications and error models. This diversity is critical because different scenarios may require distinct code properties, such as error correction capabilities, encoding and decoding efficiency, and fault tolerance. Seeking a general inner product that encompasses both Euclidean and Hermitian inner products as special cases is essential for simplifying the study of coding theory. Such a generalization provides a unified framework for understanding the relationships between different inner products and allows for the development of more versatile codes. Researchers can apply this generalization to create codes that are adaptable to various communication and computational tasks. While Euclidean and Hermitian inner products have prominent roles in quantum computing, their applications extend to other domains, such as classical information theory, wireless communication, and cryptography. Exploring and understanding the inner products' properties and connections can lead to improvements in these areas as well.

One of the most significant applications of Euclidean and Hermitian inner products is in the field of quantum error correction. Quantum computers are highly sensitive to errors, and the development of error-correcting codes is essential for the reliability of quantum computations. The CSS construction, based on the Euclidean dual, and the Hermitian construction, using Hermitian dual codes, are pivotal in constructing quantum error-correcting codes. These codes are vital for the implementation of quantum computing, which has the potential to revolutionize computation and cryptography. In [19], they introduced an inner product known as the ℓ-Galois inner product. By utilizing this ℓ-Galois inner product for codes defined over finite fields, the authors of [29] were able to derive novel entanglement-assisted quantum error-correcting codes.

In this study, we took this a step further by extending this concept to the γ-inner product, with the ℓ-Galois, Euclidean, and Hermitian inner products as special cases. We study the γ-dual codes of length n over finite chain rings in Section 3. The structure of the γ-dual codes of length p^s over $\mathcal{R}$ is given in Theorems 7, 8, 9, and 12. Section 4 investigates the γ-dual codes of the skew ϕ-constacyclic codes over finite chain rings. We determine the γ-dual of a skew ϕ-α-constacyclic code of length n over finite chain rings in Theorem 13. Theorem 17 gives the γ-dual of the skew ϕ-α-constacyclic codes over $\mathcal{R}$.

For future work, we would like to construct new entanglement-assisted quantum error-correcting codes from γ-dual codes over finite chain rings. It would also be interesting to investigate the quantum error-correcting codes from the γ-dual of skew ϕ-α-constacyclic codes over finite chain rings.

Author Contributions: All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: H.Q. Dinh and R. Tansuchat are grateful to the Centre of Excellence in Econometrics, Faculty of Economics, Chiang Mai University, for partial financial support. This research is partially supported by the Office of Research Administration, Chiang Mai University.

Data Availability Statement: Data are within the article.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. MacWilliams, F.J.; Sloane, N.J.A. *The Theory of Error-Correcting Codes*, 10th ed.; Elsevier: Amsterdam, The Netherlands, 1998.
2. Pless, V.; Huffman, W.C. *Handbook of Coding Theory*; Elsevier: Amsterdam, The Netherlands, 1998.
3. Berman, S.D. Semisimple cyclic and Abelian codes. II. *Kibernetika (Kiev)* **1967**, *3*, 21–30. (In Russian). English translation: *Cybernetics* **1967**, *3*, 17–23.
4. Massey, J.L.; Costello, D.J.; Justesen, J. Polynomial weights and code constructions. *IEEE Trans. Inf. Theory* **1973**, *19*, 101–110.
5. Falkner, G.; Kowol, B.; Heise, W.; Zehendner, E. On the existence of cyclic optimal codes. *Atti Sem. Mat. Fis. Univ. Modena* **1979**, *28*, 326–341.
6. Roth, R.M.; Seroussi, G. On cyclic MDS codes of length q over GF(q). *IEEE Trans. Inform. Theory* **1986**, *32*, 284–285.
7. Castagnoli, G.; Massey, J.L.; Schoeller, P.A.; von Seemann, N. On repeated-root cyclic codes. *IEEE Trans. Inform. Theory* **1991**, *37*, 337–342.
8. van Lint, J.H. Repeated-root cyclic codes. *IEEE Trans. Inform. Theory* **1991**, *37*, 343–345.
9. Abualrub, T.; Ghrayeb, A.; Oehmke, R. A mass formula and rank of $\mathbb{Z}_4$ cyclic codes of length 2^e. *IEEE Trans. Inform. Theory* **2004**, *50*, 3306–3312.
10. Nedeloaia, C.-S. Weight distributions of cyclic self-dual codes. *IEEE Trans. Inform. Theory* **2003**, *49*, 1582–1591.
11. Tang, L.-z.; Soh, C.B.; Gunawan, E. A note on the q-ary image of a q^m-ary repeated-root cyclic code. *IEEE Trans. Inform. Theory* **1997**, *43*, 732–737.
12. Zimmermann, K.-H. On generalizations of repeated-root cyclic codes. *IEEE Trans. Inform. Theory* **1996**, *42*, 641–649.
13. Dinh, H.Q. Constacyclic codes of length p^s over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. *J. Algebra* **2010**, *324*, 940–950.
14. Dinh, H.Q. Repeated-root constacyclic codes of length $2p^s$. *Finite Fields Appl.* **2012**, *18*, 133–143.
15. Dinh, H.Q.; Ha, T.L.; Nguyen, B.T.; Paravee, M. δ-dual codes over finite fields. *Appl. Algebra Eng. Commun. Comput.* **2022**, *125*. https://doi.org/10.1007/s00200-022-00549-4.
16. Boucher, D.; Geiselmann, W.; Ulmer, F. Skew-cyclic codes. *Appl. Algebra Eng. Commun. Comput.* **2007**, *18*, 379–389.
17. Boucher, D.; Ulmer, F. Coding with skew polynomial rings. *J. Symb. Comput.* **2009**, *44*, 1644–1656.
18. Jitman, S.; Ling, S.; Udomkavanich, P. Skew Constacyclic Codes over finite chain rings. *Adv. Math. Commun.* **2012**, *6*, 39–63.
19. Fan, Y.; Zhang, L. Galois self-dual constacyclic codes. *Des. Codes Cryptogr.* **2017**, *84*, 473–492.
20. Cao, M. MDS codes with Galois hulls of arbitrary dimensions and the related entanglement-assisted quantum error correction. *IEEE Trans. Inf. Theory* **2021**, *67*, 7964–7984.
21. Carlet, C.; Mesnager, S.; Tang, C.; Qi, Y. On σ-LCD codes. *IEEE Trans. Inf. Theory* **2019**, *65*, 1694–1704.
22. Ding, Y.; Lu, X. Galois hulls of cyclic codes over finite fields. *IEICE Trans. Commun.* **2020**, *E103-A*, 370–375.
23. Fang, X.; Jin, R.; Luo, J.; Ma, W. New Galois hulls of GRS codes and application to EAQECCs. *Cryptogr. Commun.* **2022**, *14*, 145–159.
24. Liu, X.; Fan, Y.; Liu, H. Galois LCD codes over finite fields. *Finite Fields Appl.* **2018**, *49*, 227–242.
25. Liu, H.; Pan, X. Galois hulls of linear codes over finite fields. *Des. Codes Cryptogr.* **2020**, *88*, 241–255.
26. Mi, J.; Cao, X. Constructing MDS Galois self-dual constacyclic codes over finite fields. *Discrete Math.* **2021**, *344*, 112388.
27. Wu, R.; Shi, M. A note on k-Galois LCD codes over the ring $\mathbb{F}_q + u\mathbb{F}_q$. *Bull. Aust. Math. Soc.* **2021**, *104*, 154–161.
28. Yuan, W. Research of LCD constacyclic codes and LCD MDS codes based on Galois inner product. *J. Systems Sci. Math. Sci.* **2018**, *38*, 1464–1476.
29. Liu, X.; Yu, L.; Hu, P. New entanglement-assisted quantum codes from k-Galois dual codes. *Finite Fields Their Appl.* **2019**, *55*, 21–32.
30. Dinh, H.Q.; López-Permouth, S.R. Cyclic and negacyclic codes over finite chain rings. *IEEE Trans. Inform. Theory* **2004**, *50*, 1728–1744.
31. McDonald, B.R. *Finite Rings with Identity*; Pure and Applied Mathematics, Marcel Dekker: New York, NY, USA, 1974.
32. Huffman, W.C.; Pless, V. *Fundamentals of Error-Correcting Codes*; Cambridge University Press: Cambridge, UK, 2003.
33. Amarra, V.; Nemenzo, R. On $(1-u)$- cyclic codes over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$. *Appl. Math. Lett.* **2008**, *21*, 1129–1133.
34. Bonnecaze, A.; Udaya, P. Cyclic codes and self-dual codes over $\mathbb{F}_2 + u\mathbb{F}_2$. *IEEE Trans. Inform. Theory* **1999**, *45*, 1250–1255.
35. Dinh, H.Q. Constacyclic codes of length 2^s over Galois extension rings of $\mathbb{F}_2 + u\mathbb{F}_2$. *IEEE Trans. Inf. Theory* **2009**, *55*, 1730–1740.
36. Qian, J.F.; Zhang, L.N.; Zhu, S.X. $(1+u)$- cyclic and cyclic codes over the ring $\mathbb{F}_2 + u\mathbb{F}_2$. *Appl. Math. Lett.* **2006**, *19*, 820–823.
37. Bonnecaze, A.; Udaya, P. Decoding of cyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2$. *IEEE Trans. Inf. Theory* **1999**, *45*, 2148–2157.
38. Alkhamees, Y. The determination of the group of automorphisms of a finite chain ring of characteristic p. *Q. J. Math.* **1991**, *42*, 387–391.
39. Dinh, H.Q. Negacyclic codes of length 2^s over Galois rings. *IEEE Trans. Inform. Theory* **2005**, *51*, 4252–4262.
40. Dinh, H.Q.; Wang, L.; Zhu, S. Negacyclic codes of length $2p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. *Finite Fields Appl.* **2015**, *31*, 178–201.

41. Dinh, H.Q.; Fan, Y.; Liu, H.; Liu, X.; Sriboonchitta, S. On self-dual constacyclic codes of length p^s over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. *Discrete Math.* **2017**, *341*, 324–335.
42. Dinh, H.Q.; Nguyen, R. Eldin, Bac.T.; R Tansuchat, MDS repeated-root constacyclic codes of prime power lengths over finite fields and construction of quantum MDS codes. *Int. J. Theor. Physics.* **2020**, *59*, 3043–3078.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.