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From HX-Groups to HX-Polygroups

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Abstract: HX-groups are a natural generalization of groups that are similar in construction to hypergroups. However, they do not have to be considered as hypercompositional structures like hypergroups; instead, they are classical groups. After clarifying this difference between the two algebraic structures, we review the main properties of HX-groups, focusing on the regularity property. An HX-group $\mathcal{G}$ on a group G with the identity e is called regular whenever the identity E of $\mathcal{G}$ contains e. Any regular HX-group may be characterized as a group of cosets, and equivalent conditions for describing this property are established. New properties of HX-groups are discussed and illustrated by examples. These properties are uniformity and essentiality. In the second part of the paper, we introduce a new algebraic structure, that of HX-polygroups on a polygroup. Similarly to HX-groups, we propose some characterizations of HX-polygroups as polygroups of cosets or double cosets. We conclude the paper by proposing several lines of research related to HX-groups.

Keywords: HX-group; hypercompositional structure; coset; double coset; polygroup; regularity; uniformity; essentiality

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1. Introduction

The term *hypergroup* has appeared in the mathematical history in different periods (unfortunately) for defining different generalizations of the classical algebraic concept of groups. This might create some confusion in terminology. The term has been used in a variety of contexts; however, all definitions share some common features, such as the use of a binary operation or a multivalued operation satisfying certain axioms. First, it was introduced in 1934 by the French mathematician F. Marty to define a new algebraic structure that represents the key element of hypercompositional algebra. A hypergroup in the sense of Marty [1] is a non-empty set H endowed with a hyperoperation (this is a multivalued function) $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the collection of all non-empty subsets of H, satisfying two properties: associativity and reproducibility (a formal definition is given in the preliminaries). It is worth mentioning here that the hyperoperation "$\circ$" of the hypergroup H can be extended to a binary operation on $\mathcal{P}^*(H)$ for any two arbitrary non-empty subsets A and B of H as follows: $A \circ B = \bigcup\{a \circ b \mid a \in A, b \in B\}$. In the same period, more exactly between 1937 and 1939, a group of American mathematicians, led by O. Ore, M. Dresher and J.E. Eaton, studied "the grouplike systems with non-unique multiplication" [2], calling them *multigroups* [2–5]. Despite this variation in terminology, these multigroups are in fact Marty's hypergroups and there was no further continuation of these studies.

The name hypergroup has been used also to refer to other different algebraic objects. One belongs to McMullen [6], who defined hypergroups through commutative rings, while the second notion of a hypergroup was introduced by Sunder et al. [7], as distinguished linear bases of a complex unital associative algebra satisfying particular conditions. Hypergroups in harmonic analysis date back to 1973, when C. Dunkl [8] introduced them as locally compact spaces on a convolution structure on their measures. The theories of these hypergroups has been developed nowadays in parallel and separately from that of Marty's hypergroups.

A third moment in the past when the term "hypergroup" was used was around 1985, when three Chinese mathematicians, HongXing Li, QinZhi Duan and PeizHuang Wang, defined an "upgrade structure of a group" [9]. Two years later, Li [10] renamed this algebraic object with the term *HX-group*. Starting from a group $(G, \cdot)$, a non-empty subset $\mathcal{G}$ of $\mathcal{P}^*(G)$ is called an *HX-group* on G if it is a group with respect to the binary operation $\circ : \mathcal{P}^*(G) \times \mathcal{P}^*(G) \longrightarrow \mathcal{P}^*(G)$ defined by $A \circ B = \{a \cdot b \mid a \in A, b \in B\}$. Notice the similarity between this operation and the one extending from a hyperoperation, that, on one hand, led to some confusion, that have already been clarified by Cristea et al. [11]. On the other hand, this similarity was a source of inspiration for P. Corsini, the leader of the Italian school of hypergroup theory and the author of the pioneering book [12] on this topic, who noticed a natural link between HX-groups and algebraic hypercompositional structures. To any HX-group $(\mathcal{G}, \circ)$ with the underlying set G, Corsini [13] associated a hypergroupoid $(G^*, \hat{\circ})$, where $G^* = \cup_{A \in \mathcal{G}} A$ and $x \hat{\circ} y = \cup_{A,B \in \mathcal{G}} \{x \circ y \mid x \in A, y \in B\}$. He called this structure a *Chinese hypergroupoid* and studied it for the group $\mathbb{Z}/n\mathbb{Z}$ [14], finding a condition under which it is a hypergroup. HX-groups with the underlying set being the dihedral group D_n have been investigated by Sonea [15], who has calculated the fuzzy grade [16,17] and the commutativity degree [18] of their associated Chinese hypergroups.

One of the most important types of hypergroups is represented by *polygroups*, introduced by Comer [19] in 1984 in relation to color schemes and relation algebras. He proved that the algebra associated with a color scheme is in fact a polygroup and that the system formed with the double cosets of a group G modulo an arbitrary subgroup H is again a polygroup (see the construction at the end of Section 2). He also presented a method to obtain polygroups from cogroups [20]. Polygroups can be obtained also from groups, as Jafarpour et al. showed in [21]. The same term, polygroup, appeared also in [22], but without a future development. Polygroups are regular, reversible hypergroups with a unique scalar identity. The same structure had already appeared in the literature but with a different name, i.e., *quasicanonical hypergroups* [23,24]. Its commutative version, i.e., the *canonical hypergroup*, dates back to the beginning of 1970s, when Mittas [25] studied it as an independent structure in the framework of valuation theory, and not just as the additive structure of a hyperfield. In fact, this was the way that canonical hypergroups appeared in the first studies of Krasner [26] and have continued to be investigated as the additive structure of the Krasner hyperfields and the hypercompositional structure with the most applications in different areas, e.g., valuation theory [27–29], algebraic geometry [30], number theory, affine algebraic group schemes [31], matroids theory [32], tropical geometry [33], and hypermodules [34]. The state of the art in hyperfield theory was included in an article recently published by Ch. Massouros and G. Massouros [35], with many detailed answers to several fundamental questions emerging in recent decades about Krasner hyperfields. The foundations of hypergroup theory are excellently recalled in the review paper of the previously mentioned two authors, where a lot of examples and constructions of hypergroups are proposed and explained to "highlight the particularity of the hypergroup theory versus the abstract group theory" [36]. The article also contains a well-documented bibliography that can be used to obtain an in-depth insight into hypercompositional algebra. Both manuscripts, refs. [35,36], are a good resource for someone who wants to start to learn about hypergroups and hyperfield theories, since they are open access and contain the fundamental notions and results of these theories, supported by plenty of interesting examples and comments related to their meaning, origins and applications.

The main characteristic of HX-groups is the one of being (under some conditions) groups of cosets, and thus quotient groups, and this property is called regularity. After a brief preliminary section where we fix the terminology and recall the fundamental concepts related to HX-groups and polygroups, Section 3 discusses in depth the equivalent conditions under which an HX-group is regular, presenting also examples of HX-groups not satisfying the regularity property. The study continues then with the investigation of the other two properties of HX-groups, uniformity and essentiality, concluding that any uniform regular HX-group can be written as a group of cosets (see Corollary 2). In addition, any regular HX-group satisfying the essentiality condition is called strong and any strong HX-group $(G, \mathcal{G})$ is a quotient group, if $G^* = \cup\{A \mid A \in \mathcal{G}\}$ is a subgroup of G (see Corollary 3). The second part of the paper, covered in Section 4, is dedicated to the introduction of the concept of *HX-polygroups*, having a polygroup as a support. We characterize the HX-polygroups as polygroups of double cosets in the sense of Dresher–Ore [4] (more details to follow in the next section). The paper ends with some conclusive ideas and three concrete proposals for future work: connections with soft set theory, analysis of the properties of the direct product of two HX-groups, and extension to HX-rings.

2. Preliminaries

In this section, we will briefly review the theory of HX-groups and that of polygroups by recalling the main definitions and theorems and presenting several nontrivial examples. For more details, the readers are referred to the pioneering articles [9,10,37] and the fundamental book [38] on polygroup theory.

2.1. HX-Groups

Since the first results on the theory of HX-groups were not consistent in terms of notation, this problem was solved in [11], and here we will use the same terminology and notation proposed there. From the beginning, we clarify that "$\cdot$" represents a mapping $G \times G \longrightarrow G$ on an arbitrary set G, while "$\circ$" is used to define a mapping $\mathcal{P}^*(G) \times \mathcal{P}^*(G)$ on the set $\mathcal{P}^*(G) = \mathcal{P}(G) \setminus \{\emptyset\}$, representing the set of all non-empty subsets of G. Thus, both mappings are binary operations and the second one should not be confused with a multivalued operation that we will use to define (in the next subsection) a hypergroup or a polygroup.

Definition 1 ([9,11])**.** *Consider an arbitrary group $(G, \cdot)$ and the set $\mathcal{P}^*(G)$ of all non-empty subsets of G on which we define an operation by the law $A \circ B = \{a \cdot b \mid a \in A, b \in B\}$ for any $A, B \in \mathcal{P}^*(G)$. A non-empty subset $\mathcal{G}$ of $\mathcal{P}^*(G)$ is called an HX-group on G if $(\mathcal{G}, \circ)$ is a group, with the neutral element denoted by E.*

We will denote by e the neutral element of the group G and, as mentioned before, by E the neutral element of the the group $\mathcal{G}$. Similarly, the inverse of the element g in the group G is denoted by g^{-1} and A^{-1} stays for the inverse of the element A in the group $\mathcal{G}$, while $A^{\ominus} = \{a^{-1} \mid a \in A\}$ is the inverse set of A. Moreover, we define $G^* = \cup\{A \mid A \in \mathcal{G}\}$ and it is easy to see that E and G^* are both subsemigroups of the group G, as stated in [10].

Let us now illustrate these notions in the following examples.

Example 1 ([9])**.** *Consider the additive finite group*

$$G = \mathbb{Z}_2 \times \mathbb{Z}_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\},$$

with the neutral element $e = (0,0)$. Set $B_0 = \{(0,0), (1,0)\}$; $B_1 = \{(0,0), (0,1), (0,2)\}$; $B_2 = \{(0,0), (0,2)\}$; $B_3 = \{(0,1), (1,1)\}$; $B_4 = \{(0,2), (1,2)\}$; $B_5 = \{(1,0), (1,1), (1,2)\}$. Then, we obtain the following HX-groups:

$B_{1,5}$	B_1	B_5
B_1	B_1	B_5
B_5	B_5	B_1

with the neutral element $E = B_1$, where ${B_5}^{-1} = B_5 = B_5^{\ominus}$, and

$B_{0,3,4}$	B_0	B_3	B_4
B_0	B_0	B_3	B_4
B_3	B_3	B_4	B_0
B_4	B_4	B_0	B_3

Its neutral element is $E = B_0$, while ${B_3}^{-1} = B_4 = B_3^{\ominus}$.

As we can see in the next example, the property $A^{-1} = A^{\ominus}$ does not hold for any subset A of $\mathcal{G}$.

Example 2 ([11]). *Let $(G, \cdot) = (\mathbb{R}^+, \cdot)$ be the multiplicative group of positive reals. For any positive rational number a, take $G_a = [a, \infty)$. Then, $\mathcal{G} = \{G_a \mid a \in \mathbb{Q}^+\}$ is a regular HX-group on G with the neutral element $E = [1, \infty)$. We immediately notice that $E^{-1} = E$, while $E^{\ominus} = (0, 1]$.*

These examples motivate the introduction of the following type of HX-group.

Definition 2 ([37]). *An HX-group $\mathcal{G}$ on G is called uniform if for any subset $A \in \mathcal{G}$ the equality $A^{-1} = A^{\ominus}$ holds.*

The class of uniform HX-groups is characterized as follows.

Theorem 1 ([37]). *An HX-group on G is uniform if and only if its neutral element E is a subgroup of G.*

The regularity property of HX-groups, introduced in [9], was then studied in [10]. We recall that an HX-group $(G, \mathcal{G})$ is called *regular* whenever $e \in E$. Moreover, if E is finite, then $(G, \mathcal{G})$ is a regular HX-group.

One of the main problems in the theory of HX-groups is the construction of the HX-groups as quotient structures. This idea is illustrated in the following result.

Proposition 1 ([10,37]). *Let H be a subgroup of an arbitrary group G and E be an idempotent subset of G, i.e., $E^2 = E$. If for all $a \in H$ there is $aE = Ea$, then $\mathcal{G} = \{aE \mid a \in H\}$ is an HX-group with the neutral element E. Moreover, if E is a normal subgroup of G, then the HX-group $\mathcal{G} = \{aE \mid a \in G\}$ is regular and uniform.*

Indeed, the surjection $f : H \longrightarrow \mathcal{G}$ defined by $f(a) = aE$, for any $a \in H$, is a group homomorphism, i.e., $f(ab) = (ab)E = (aE)(bE) = f(a)f(b)$, meaning that $H/Kerf$ is isomorphic with $\mathcal{G}$. The set $aE = \{a \cdot x \mid x \in E\}$ is called the left quasi-coset of E, while the right quasi-coset is $Ea = \{x \cdot a \mid x \in E\}$.

In the finite case, the following properties are essential when one constructs an HX-group with the support a finite group.

Theorem 2 ([10]). *Let $\mathcal{G}$ be an HX-group with the support G and the neutral element E. Then,*

(i) For any subset A of $\mathcal{G}$, $|A| = |E|$, where $|A|$ means the cardinality of the set A.
(ii) For any two subsets A and B in $\mathcal{G}$ such that $A \cap B \neq \emptyset$, it follows that $|A \cap B| = |E|$.

These properties are better illustrated in the following example of an HX-group with the support the dihedral group D_4, formed with four rotations and four reflections.

Example 3 ([15]). *On the dihedral group $D_4 = \langle \rho, \sigma \mid \rho^4 = \sigma^2 \rangle = \{e, \rho, \rho^2, \rho^3, \sigma, \rho\sigma, \rho^2\sigma, \rho^3\sigma\}$, consider the HX-group $\mathcal{G}_1 = \{\{e, \rho^2\}, \{\rho, \rho^3\}, \{\sigma, \rho^2\sigma\}, \{\rho\sigma, \rho^3\sigma\}\}$ represented by the following Cayley's table*

$\star$	$\{e, \rho^2\}$	$\{\rho, \rho^3\}$	$\{\sigma, \rho^2\sigma\}$	$\{\rho\sigma, \rho^3\sigma\}$
$\{e, \rho^2\}$	$\{e, \rho^2\}$	$\{\rho, \rho^3\}$	$\{\sigma, \rho^2\sigma\}$	$\{\rho\sigma, \rho^3\sigma\}$
$\{\rho, \rho^3\}$	$\{\rho, \rho^3\}$	$\{e, \rho^2\}$	$\{\rho\sigma, \rho^3\sigma\}$	$\{\sigma, \rho^2\sigma\}$
$\{\sigma, \rho^2\sigma\}$	$\{\sigma, \rho^2\sigma\}$	$\{\rho\sigma, \rho^3\sigma\}$	$\{e, \rho^2\}$	$\{\rho, \rho^3\}$
$\{\rho\sigma, \rho^3\sigma\}$	$\{\rho\sigma, \rho^3\sigma\}$	$\{\sigma, \rho^2\sigma\}$	$\{\rho, \rho^3\}$	$\{e, \rho^2\}$

Every element of the HX-group $\mathcal{G}_1$ has cardinality 2 and its neutral element $E = \{e, \rho^2\}$ contains the neutral element e of the support group, so $\mathcal{G}_1$ is a regular and uniform HX-group because $E = E^{-1} = E^{\ominus}$.

In addition, it is worth noting that $\mathcal{G}_1$ is not the only HX-group with the support the dihedral group D_4, but there exist other two HX-groups $\mathcal{G}_2 = \{\{e, \rho, \rho^2, \rho^3\}, \{\sigma, \rho\sigma, \rho^2\sigma, \rho^3\sigma\}\}$ and $\mathcal{G}_3 = \{\{e\}, \{\rho\}, \{\rho^2\}, \{\rho^3\}, \{\sigma\}, \{\rho\sigma\}, \{\rho^2\sigma\}, \{\rho^3\sigma\}\}$, both satisfying the conditions of Theorem 2. The general construction of HX-groups with the support dihedral group D_n is presented in [15].

2.2. Polygroups

In this subsection, we will briefly recall the terminology and notations related to polygroups. New properties have been recently investigated by Sonea [39] and Al Tahan et al. [40] in connection with the commutativity degree of finite polygroups. This property for complete hypergroups has been investigated in [18].

Let H be a non-empty set and $\mathcal{P}^*(H)$ be the set of all non-empty subsets of H. Let "$\circ$" be a *hyperoperation* on H; that is, "$\circ$" is a function from $H \times H$ into $\mathcal{P}^*(H)$. If $(a, b) \in H \times H$, its image under "$\circ$" in $\mathcal{P}^*(H)$ is denoted by $a \circ b$, or briefly by ab if there is no confusion and it is called the hyperproduct of the elements a and b. The hyperoperation is extended to subsets of H in a natural way, that is, for non-empty subsets A, B of H, $A \circ B = \cup\{a \circ b \mid a \in A, b \in B\}$. The notation $a \circ A$ is used for $\{a\} \circ A$ and $A \circ a$ for $A \circ \{a\}$. Generally, the singleton $\{a\}$ is identified with its member a. The structure $(H, \circ)$ is called a *semihypergroup* whenever $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$, i.e., whenever the hyperoperation is associative. A semihypergroup is a *hypergroup* whenever $a \circ H = H \circ a = H$ for all $a \in H$, meaning that the reproduction axiom holds.

Since the polygroups have a similar structure to the one of groups, we will denote the hyperoperation on a polygroup by "$\cdot$". This is a multivalued operation and it will not be confused with the operation defined on a group and denoted in the same way. The hypergroup $(P, \cdot)$ is called a *polygroup* whenever the following conditions hold:

(1) P has a scalar identity e (i.e., $e \cdot x = x \cdot e = x$, for every $x \in P$);
(2) Every element x of P has a unique inverse x^{-1} in P (i.e., $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$);
(3) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$.

It is clear that any group can be viewed as a polygroup and that the following elementary properties hold: $e^{-1} = e$ and $(x^{-1})^{-1} = x$. A non-empty subset K of a polygroup $(P, \cdot)$ is a *subpolygroup* of P whenever $x, y \in K$ implies $x \cdot y \in K$ and $x \in K$ implies $x^{-1} \in K$. A subpolygroup N of a polygroup $(P, \cdot)$ is *normal* whenever $N \cdot x = x \cdot N$ for all $x \in P$. The concept of the normal subhypergroup of a quasicanonical hypergroup, equivalently with that of a normal subpolygroup of a polygroup, was defined for the first time by Massouros in [24]. A very detailed and well-explained analysis of the relevant mathematical background that led to its definition is presented. Just a few years later, Corsini introduced the concept of an invariant or normal part of a hypergroup and as a consequence also that of a normal subhypergroup. In both studies, the above mentioned definition is given. However, a different definition appears in [38] (Definition 3.3.2), saying that a subhypergroup N of a polygroup P is normal if $x^{-1} \cdot N \cdot x \subseteq N$, but without implying the equality $x \cdot N = N \cdot x$, for any x in P. Indeed, let N be a subhypergroup of a polygroup P satisfying the property

$x^{-1} \cdot N \cdot x \subseteq N$. Since $x^{-1} \cdot N \cdot x$ is a subset of N, it follows that there exists an element y in N such that $y \notin x^{-1} \cdot N \cdot x$. Suppose now by absurd that $x \cdot N = N \cdot x$. Then, there exists an element z in N such that $x \cdot y \cap z \cdot x \neq \emptyset$. Thereby, $y \in x^{-1} \cdot N \cdot x$, which is a contradiction. Thus, in order to not create confusion, we propose using the original definition given by Massouros, and the one proposed by Davvaz should be used to define a new type of subhypergroup/subpolygroup.

Given an arbitrary group G and an arbitrary subgroup (not necessarily normal) H of G, the system $G /\!\!/ H$ formed with all double cosets of G modulo H is the motivating example of a polygroup (in fact it is a chromatic polygroup, being isomorphic with the algebra of a color scheme, as proven by Comer [41] in his first studies on polygroups), already appearing in a paper by Dresher and Ore [4]. Indeed, $G /\!\!/ H = \{HgH; g \in G\}$ equipped with the hyperoperation $(Hg_1H) \cdot (Hg_2H) = \{Hg_1hg_2H \mid h \in H\}$ is a polygroup with the scalar identity $H = HeH$ (where e is the neutral element of the group G), where the inverse of the element HgH is $(HgH)^{-1} = Hg^{-1}H$. This construction remembers the one of a quotient group.

3. Regularity, Uniformity and Essentiality Properties in HX-Groups

In this section, we propose some equivalent conditions in that an HX-group can be represented as a group of cosets. Also, we illustrate some examples of HX-groups that are not groups of cosets.

We start this section with a general characterization of uniform HX-groups. For a non-empty subset H of a group G, by $H \leqslant G$, we mean that H is a subgroup of G.

Theorem 3. *For an HX-group $\mathcal{G}$ on G with the neutral element E, the following assertions are equivalent:*

(i) *There exists $X \in \mathcal{G}$ such that $X \subseteq E^{\ominus}$.*
(ii) $E^{\ominus} \in \mathcal{G}$.
(iii) $E^{\ominus} \leqslant G$.
(iv) $E \leqslant G$.
(v) *For all $A \in \mathcal{G}$, $A^{\ominus} = A^{-1}$ holds, meaning that $\mathcal{G}$ is a uniform HX-group.*
(vi) *$G^* \leqslant G$ and for all $A, B \in \mathcal{G}$ such that $A \subseteq B$, it follows that $A = B$.*
(vii) *For all $A \in \mathcal{G}$, $A^{\ominus} \in \mathcal{G}$ holds.*

Proof. $(i) \to (ii)$ Since $X \neq \emptyset$, there exists $x_0 \in X$ such that $x_0 \in E^{\ominus}$. Therefore, $x_0^{-1} \in E$ and hence $e = x_0x_0^{-1} \in x_0E \subseteq XE = X$. This implies that $E = Ee \subseteq EX = X$. Thus, $E \subseteq E^{\ominus}$ and therefore $E^{\ominus} \subseteq (E^{\ominus})^{\ominus} = E$, meaning that $E = E^{\ominus}$, and so $E^{\ominus} \in \mathcal{G}$.

$(ii) \to (iii)$ Let $z_1, z_2 \in G$ such that $z_1, z_2 \in E^{\ominus}$. This means that $z_1^{-1}, z_2^{-1} \in E$ and then $(z_1z_2)^{-1} = z_2^{-1}z_1^{-1} \in E^2 = E$, implying that $z_1z_2 \in E^{\ominus}$. Thereby, $E^{\ominus}$ is closed under the group operation. Since $E \in \mathcal{G}$, it follows that $E \neq \emptyset$ and so there exists $a \in E$. Thus, $e = aa^{-1} \in EE^{\ominus} = E^{\ominus}$, because $E^{\ominus} \in \mathcal{G}$ and E is the neutral element of the group $\mathcal{G}$. Finally, for any $z \in G$ such that $z \in E^{\ominus}$, it follows that $z^{-1} \in E$ and then $z^{-1}E^{\ominus} \subseteq EE^{\ominus} = E^{\ominus}$. We can conclude that $z^{-1} = z^{-1}e \in z^{-1}E^{\ominus} \subseteq E^{\ominus}$. Thus, clearly $E^{\ominus} \leqslant G$.

$(iii) \to (iv)$ This is a clear implication that follows immediately from group properties.

$(iv) \leftrightarrow (v)$ This is the assertion in Theorem 1.

$(iv) \to (vi)$ Since $E \leqslant G$, it follows that $G^* \leqslant G$, according to [10,11]. Let us prove now the second part of the assertion, supposing that $A, B \in \mathcal{G}$, such that $A \subseteq B$. Then, $B^{-1}A \subseteq B^{-1}B = E$ and therefore $B^{-1}A(A^{-1}) \subseteq EA^{-1} = A^{-1}$. Using the equivalence between (iv) and (v), we can rewrite the last equality as $B^{\ominus} \subseteq A^{\ominus}$, implying that $B = (B^{\ominus})^{\ominus} \subseteq (A^{\ominus})^{\ominus} = A$, concluding that $A = B$.

$(vi) \to (iv)$ Since E is closed under the group operation, because $E^2 = E$, it is enough to prove that any inverse of an element in E is again in E. Take an arbitrary x in E. Since $x \in E \in G^*$ and $G^* \leqslant G$, it follows that $x^{-1} \in G^*$, meaning that there exists $A \in \mathcal{G}$ such that $x^{-1} \in A$. It follows that $x^{-1}A^{-1} \subseteq AA^{-1} = E$, implying that $A^{-1} \subseteq xE \subseteq EE = E$

and then, by the hypothesis, we have $A^{-1} = E$, equivalently with $E = A$. Since $x^{-1} \in A$, it follows that $x^{-1} \in E$ and therefore $E \leqslant G$.

$(v) \to (vii)$ This is an obvious implication.

$(vii) \to (ii)$ By the hypothesis, we know there exists $A \in \mathcal{G}$ such that $A^{\ominus} \in \mathcal{G}$. Thus, we have $A^{\ominus}(A^{\ominus})^{-1} = E$. In addition, since $A = AE$ and therefore $A^{\ominus} = E^{\ominus}A^{\ominus}$, we can write $A^{\ominus}(A^{\ominus})^{-1} = E^{\ominus}[A^{\ominus}(A^{\ominus})^{-1}]$, meaning that $E = E^{\ominus}E$. Thereby, $E^{\ominus} = (E^{\ominus}E)^{\ominus} = E^{\ominus}E$, leading to the equality $E = E^{\ominus}$. Clearly, now we have $E^{\ominus} \in \mathcal{G}$. $\square$

Proposition 2. *Let G be a torsion group (i.e., every element of G has a finite order) and $(G, \mathcal{G})$ be an HX-group. Then, $(G, \mathcal{G})$ is regular.*

Proof. Since $EE = E$, we conclude that E is closed under the product of G. In addition, G is a torsion group; thus, each element has a finite order and hence $e \in E$. Thus, $(G, \mathcal{G})$ is a regular HX-group. $\square$

Corollary 1. *Any HX-group constructed on a finite group is regular.*

The following result characterizes the regular HX-groups as groups of cosets.

Proposition 3. *Let $(G, \mathcal{G})$ be an HX-group. Then, the following assertions are equivalent:*

(i) $(G, \mathcal{G})$ *is regular.*

(ii) *For any $A \in \mathcal{G}$, there exists $a \in A$ such that $A = aE = Ea$.*

Proof. $(i) \to (ii)$ Let A be an arbitrary element in $\mathcal{G}$. Then, $E = AA^{-1} = A^{-1}A$, meaning that, for $e \in E$, there exist $a \in A$ and $b \in A^{-1}$ such that $e = ab$. It follows that $bA \subseteq A^{-1}A = E$ and therefore $abA \subseteq aE$. Since we can write $A = eA = abA \subseteq aE$, it implies that $A \subseteq aE \subseteq AE = A$, concluding that $A = aE$. Similarly, one proves that $A = Ea$.

$(ii) \to (i)$ Since $E \in \mathcal{G}$, there exists $a \in E$ such that $E = aE$. Thus, for $a \in E$, there exists $x \in E$ such that $a = ax$, meaning that $x = e$. Therefore, $e \in E$, so $(G, \mathcal{G})$ is a regular HX-group. $\square$

Corollary 2. *Any uniform regular HX-group can be written as a group of cosets.*

Proof. Let $(G, \mathcal{G})$ be a uniform regular HX-group. For an arbitrary $a \in G^* = \cup\{A \mid A \in \mathcal{G}\}$, there exists $A \in \mathcal{G}$ such that $a \in A$. By Theorem 3(v), we know that $A^{\ominus} = A^{-1}$; thus, we can write:

$$aEa^{-1} \subseteq AEA^{\ominus} = AEA^{-1} = AA^{-1} = E,$$

meaning that E is a normal subgroup of G^*. Since $(G, \mathcal{G})$ is regular, by Proposition 3, for each $A \in \mathcal{G}$, there exists $a \in A$ such that $A = aE$. Thus, $\mathcal{G} = \frac{G^*}{E} = \{aE \mid a \in G^*\}$. $\square$

In the following, we introduce and discuss the property of minimality/maximality of the neutral element E of an HX-group.

Definition 3. *In an arbitrary HX-group $(G, \mathcal{G})$, we say that the neutral element E is $\subseteq$-minimal (or $\subseteq$-maximal) in $\mathcal{G}$ whenever E is a minimal (or maximal) element of $\mathcal{G}$ with respect to the inclusion.*

Example 4 ([10]). *Let $(G, +) = (\mathbb{R}, +)$ be the additive group of all real numbers and H be the set of all integers and take $E = \{r \in \mathbb{R} \mid r > 0\} = (0, +\infty)$. Then, $\mathcal{G} = \{n + E \mid n \in H\}$ is an HX-group on G with the neutral element E. We note that the elements of $\mathcal{G}$ may form a countable chain $\ldots \supset (-2, +\infty) \supset (-1, +\infty) \supset E \supset (1, +\infty) \supset (2, +\infty) \supset \ldots$, meaning that E is not either $\subseteq$-minimal or $\subseteq$-maximal in $\mathcal{G}$.*

Example 5. *If the HX-group $\mathcal{G}$ with the neutral element E can be written as a subset of cosets, it means it can be written as a partition, and thus $A \cap E = \emptyset$ for any proper subset A of $\mathcal{G}$, $A \neq E$. In other words, E is a $\subseteq$-minimal and $\subseteq$-maximal element of $\mathcal{G}$.*

Theorem 4. *Let $(G, \mathcal{G})$ be an HX-group. The following conditions are equivalent:*

(i) E is $\subseteq$-minimal in $\mathcal{G}$.
(ii) E is $\subseteq$-maximal in $\mathcal{G}$.
(iii) For any $A, B \in \mathcal{G}$ such that $A \cap B \in \mathcal{G}$, $A = B$ holds.
(iv) For any $A, B \in \mathcal{G}$ such that $A \cup B \in \mathcal{G}$, $A = B$ holds.

Proof. $(i) \to (iii)$ Let $A, B \in \mathcal{G}$ such that $A \cap B \in \mathcal{G}$. For simplicity, denote $C = A \cap B$. Then, $CA^{-1} \subseteq E$ and $CB^{-1} \subseteq E$, implying that $CA^{-1} = CB^{-1} = E$, since E is $\subseteq$-minimal in $\mathcal{G}$. Therefore, $C = CE = CA^{-1}A = EA = A$ and similarly $C = B$, leading to the equality $A = B$.

$(iii) \to (i)$ Let $A \in \mathcal{G}$ such that $A \subseteq E$. Thus, $A \cap E = A \in \mathcal{G}$, implying that $A = E$, and so E is $\subseteq$-minimal in $\mathcal{G}$.

$(ii) \to (iii)$ Let $A, B \in \mathcal{G}$ such that $A \cap B \in \mathcal{G}$. Taking $C = A \cap B$, we may write $E = CC^{-1} \subseteq AC^{-1}$ and $E = CC^{-1} \subseteq BC^{-1}$. It follows that $E = AC^{-1} = BC^{-1}$ and hence $C = EC = AC^{-1}C = AE = A$, and similarly, we obtain $C = B$, implying that $A = B$.

$(iii) \to (ii)$ Let $A \in \mathcal{G}$ such that $E \subseteq A$. Thus, $A \cap E = E \in \mathcal{G}$, implying that $A = E$, so E is $\subseteq$-maximal in $\mathcal{G}$.

$(ii) \to (iv)$ Let $A, B \in \mathcal{G}$ such that $A \cup B \in \mathcal{G}$. Denoting $D = A \cup B$, we obtain that $E \subseteq A^{-1}D$ and $E \subseteq B^{-1}D$, leading to the equalities $A^{-1}D = B^{-1}D = E$, because E is $\subseteq$-maximal in $\mathcal{G}$. It follows immediately that $D = ED = AA^{-1}D = AE = A$. Similarly, one obtains $D = B$ and thus $A = B$.

$(iv) \to (ii)$ Let $A \in \mathcal{G}$ such that $E \subseteq A$. Then, $A \cup E = A \in \mathcal{G}$ implies that $A = E$ and therefore E is $\subseteq$-maximal in $\mathcal{G}$. $\square$

Theorem 5. *Let $(G, \mathcal{G})$ be an HX-group with $E \cap E^{\ominus} \neq \emptyset$. Then, E is $\subseteq$ -minimal in $\mathcal{G}$ if and only if for any $A \in \mathcal{G}$ such that $A \cap E^{\ominus} \neq \emptyset$, it follows that $A = E$.*

Proof. Let E be $\subseteq$-minimal in $\mathcal{G}$ and $A \in \mathcal{G}$ such that $A \cap E^{\ominus} \neq \emptyset$. There exists $a \in A \cap E^{\ominus}$, and for $x \in E$ it follows that $xa \in EA = A$, which implies that $x \in Aa^{-1}$, with $a^{-1} \in E$. Thus, $x \in AE = A$. Therefore, $E \subseteq A$ and hence $A^{-1}E \subseteq A^{-1}A$ = E. Thus, $A^{-1} \subseteq E$, which, by the minimality of E, leads to $A^{-1} = E$. This implies that $E = AA^{-1} = AE = A$.

Conversely, let $A \in \mathcal{G}$ such that $A \subseteq E$. Thus, $E = A^{-1}A \subseteq A^{-1}E = A^{-1}$. Since $E \cap E^{\ominus} \neq \emptyset$, it follows that $A^{-1} \cap E^{\ominus} \neq \emptyset$. By the hypothesis, this implies that $A^{-1} = E$, because $A^{-1} \in \mathcal{G}$. Therefore, $E = A$ and hence E is $\subseteq$-minimal in $\mathcal{G}$. $\square$

Remark 1. *If $(G, \mathcal{G})$ is a regular HX-group, the conditions of Theorem 5 are clearly satisfied.*

Proposition 4. *Let $(G, \mathcal{G})$ be an HX-group such that E is $\subseteq$-maximal in $\mathcal{G}$. If $G^* \leqslant G$, then $E \leqslant G$ too, and therefore $\mathcal{G}$ is a uniform HX-group.*

Proof. Since $e \in G^*$, there exists $A \in \mathcal{G}$ such that $e \in A$. Then, $E = eE \subseteq AE = A$ and by the maximality of E, it follows that $E = A$ and therefore $e \in E$, so $(G, \mathcal{G})$ is a regular HX-group. Applying Proposition 3, we know that for any $X \in \mathcal{G}$ there exists $x_0 \in X$ such that $X = x_0E$.

Let $A, B \in \mathcal{G}$ such that $A \subseteq B$. Accordingly with the above mentioned property, there exist $a \in A$ and $b \in B$ such that $A = aE$ and $B = bE$. Thus, $aE \subseteq bE$ and therefore $E \subseteq a^{-1}bE$. Based on the maximality of E, we have $E = a^{-1}bE$, equivalently with $aE = bE$. Applying Theorem 3(vi), we conclude that $E \leqslant G$. $\square$

Example 6. *(i)* *Let $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and G be the multiplicative group $(\mathbb{R}^*, \cdot)$. Take $\mathcal{G} = \{(-\infty, -1), (1, +\infty)\}$. The following assertions hold:*
(i-1) $(G, \mathcal{G})$ is an HX-group with the neutral element $E = (1, +\infty)$.
(i-2) E is $\subseteq$-maximal in $\mathcal{G}$.
(i-3) $1 \notin E$ (and thus $\mathcal{G}$ is not regular) and for all $a \in G^*$, $aE \notin \mathcal{G}$.

(ii) *Let $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$ and G be the multiplicative group $(\mathbb{Q}^*, \cdot)$. For each $n \in \mathbb{Z}$, take $A_n = 2^n\mathbb{N}$ and let $\mathcal{G} = \{A_n \mid n \in \mathbb{Z}\}$. Then, clearly we have:*
(ii-1) $(G, \mathcal{G})$ is an HX-group with the neutral element $E = \mathbb{N}$.
(ii-2) $E \nleq G$.
(ii-3) $1 \in E$ (and thus $\mathcal{G}$ is regular) and E is not $\subseteq$-maximal in $\mathcal{G}$.
(ii-4) Each element of $\mathcal{G}$ is a coset of E.

The following auxiliary result suggests the introduction of a new concept, i.e., that of *essentiality*.

Lemma 1. *Let $(G, \mathcal{G})$ be a regular HX-group. For each $X \in \mathcal{G}$ with the property that $X \cap E \neq \emptyset$, $XE^{\ominus} = EE^{\ominus}$ holds.*

Proof. For an arbitrary element $X \in \mathcal{G}$ such that $X \cap E \neq \emptyset$, there exists $x \in X \cap E$ and since $e = xx^{-1}$ with $x^{-1} \in E^{\ominus}$, we have $e \in XE^{\ominus}$. Thus, $X^{-1} \subseteq X^{-1}XE^{\ominus}$, equivalently with $X^{-1} \subseteq EE^{\ominus}$. Then, $XX^{-1} \subseteq (XE)E^{\ominus} = XE^{\ominus}$, leading to $E \subseteq XE^{\ominus}$. Then, on one hand, it follows that $EE^{\ominus} \subseteq XE^{\ominus}E^{\ominus} = X(EE)^{\ominus} = XE^{\ominus}$.

On the other hand, since $(G, \mathcal{G})$ is regular, by Proposition 3, there exists $x_0 \in X$ such that $X = x_0E = Ex_0$. Since $e \in EE^{\ominus} \subseteq XE^{\ominus}$, it follows that $e \in EX^{\ominus}$ and thus $X \subseteq XEX^{\ominus} = XX^{\ominus}$. Then, $X^{-1}X \subseteq X^{-1}XX^{\ominus}$, meaning that $E \subseteq EX^{\ominus}$, with $X = x_0E$. So, $E \subseteq EE^{\ominus}x_0^{-1}$, implying that $Ex_0 \subseteq EE^{\ominus}$, i.e., $X \subseteq EE^{\ominus}$. From here, we immediately obtain the other inclusion of the requested equality, i.e., $XE^{\ominus} \subseteq EE^{\ominus}E^{\ominus} = E(EE)^{\ominus} = EE^{\ominus}$, concluding the proof. $\square$

Definition 4. *Let $(G, \mathcal{G})$ be an HX-group.*

(i) *We say that $E^{\ominus}$ is essential in $EE^{\ominus}$, and denote this by $E^{\ominus} \lhd_{\mathcal{G}} EE^{\ominus}$, whenever for an arbitrary $A \in \mathcal{G}$ such that $A \subseteq EE^{\ominus}$, it follows that $A \cap E^{\ominus} \neq \emptyset$.*
(ii) *The HX-group $(G, \mathcal{G})$ is called strong whenever it is regular and $E^{\ominus} \lhd_{\mathcal{G}} EE^{\ominus}$.*

Example 7 ([11] Example 3.10)**.** *Consider the set $G = \{(a_1, a_2, a_3) \mid a_1 \in \mathbb{R} \setminus \{0\}, a_2, a_3 \in \mathbb{Z}\}$ with the operation*

$$(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = (a_1 \cdot b_1, a_2 + b_2, a_3 + b_3),$$

where "+" and "·" are the usual addition and multiplication of real numbers. Clearly $(G, \cdot)$ is a group with the neutral element $e = (1, 0, 0)$. For each $n \in \mathbb{Z}$, take $A_n = \{(1, m, n) \mid m \in \mathbb{Z}\}$ and define on $\mathcal{G} = \{A_n \mid n \in \mathbb{Z}\}$ the operation $A_{n_1} \circ A_{n_2} = A_{n_1+n_2}$, for all $n_1, n_2 \in \mathbb{Z}$. Thus, $(G, \mathcal{G})$ is an HX-group with the neutral element $E = A_0 \ni e$ and hence $E = E^{-1} = E^{\ominus}$. This implies that $EE^{\ominus} = E$, so $(G, \mathcal{G})$ is a strong HX-group.

Example 8 ([11] Example 3.11)**.** *Let G be the group $(\mathbb{C} \setminus \{0\}, \cdot)$. For each $r \in \mathbb{R} \setminus \{0\}$, consider the sets*

$$z_r = \{re^{i\varphi} | \varphi = \frac{n}{2^k}\pi, k, n \in \mathbb{N}\}.$$

Taking $\mathcal{G} = \{z_r \mid r \in \mathbb{R} \setminus \{0\}\}$, for each $r, s \in \mathbb{R} \setminus \{0\}$, define $z_r \circ z_s = z_{rs}$. Thus, $(G, \mathcal{G})$ is an HX-group with the neutral element $E = z_1$, where $z_1^{\ominus} = \{e^{-i\varphi} \mid \varphi = \frac{n}{2^k}\pi, k, n \in \mathbb{N}\}$. Thus, for each $z_r \in \mathcal{G}$ such that $z_r \subseteq EE^{\ominus}$, it follows that $|r| = 1$. Since z_1 and z_{-1} have non-empty intersections with $E^{\ominus}$, we conclude that $(G, \mathcal{G})$ is a strong HX-group.

Theorem 6. *Let $(G, \mathcal{G})$ be a regular HX-group. The following assertions are equivalent:*

(i) *For any $A \in \mathcal{G}$ such that $A \cap E \neq \emptyset$, it follows that $A = E$.*
(ii) $E^{\ominus} \lhd_{\mathcal{G}} EE^{\ominus}$.

Proof. $(i) \to (ii)$ Let $A \in \mathcal{G}$ such that $A \subseteq EE^{\ominus}$. Suppose by the absurd that $A \cap E^{\ominus} = \emptyset$. This means that $A \cap E = \emptyset$, because otherwise, by the hypothesis, we would get $A = E$, with $E \cap E^{\ominus} \ni e$, since $\mathcal{G}$ is regular. Having $A \subseteq EE^{\ominus}$ and $A \neq \emptyset$, there exists $a \in A$ such that $a \in EE^{\ominus}$. This implies that $a = e_1 e_2^{-1}$ for some $e_1, e_2 \in E$. Thus, $e_1 = ae_2$ and hence $e_1 \in AE = A$. Therefore, $e_1 \in A \cap E$, which is a contradiction.

$(ii) \to (i)$ Let $A \in \mathcal{G}$ such that $A \cap E \neq \emptyset$. By Lemma 1, we have $AE^{\ominus} = EE^{\ominus}$. On one hand, for $e \in E^{\ominus}$, it follows that $A = Ae \subseteq AE^{\ominus} = EE^{\ominus}$, implying that $A \cap E^{\ominus} \neq \emptyset$. Thus, there exists $e_1 \in E$ such that $e_1^{-1} \in A$. Thereby, $e \in Ae_1 \subseteq AE = A$, which leads to $E = Ee \subseteq EA = A$.

On the other hand, using the hypothesis, there exists $e_2 \in A \cap E$ and thus $A^{-1}e_2 \subseteq A^{-1}A = E$. This implies that $A^{-1} \subseteq Ee_2^{-1} \subseteq EE^{\ominus}$, meaning that $A^{-1} \cap E^{\ominus} \neq \emptyset$. Similarly, as stated above, $E \subseteq A^{-1}$. Therefore, $A = AE \subseteq AA^{-1} = E$, concluding that $A = E$. □

We conclude this section with a sufficient condition under which a strong HX-group $\mathcal{G}$ can be written as a group of cosets modulo the neutral element E of $\mathcal{G}$.

Corollary 3. *Let $(G, \mathcal{G})$ be a strong HX-group.*

(i) *For any $A \in \mathcal{G}$ such that $A \cap E^{\ominus} \neq \emptyset$, it follows that $A = E$.*
(ii) *If $G^* \leqslant G$, then $\mathcal{G} = \frac{G^*}{E}$.*

Proof. (i) Since $A \cap E^{\ominus} \neq \emptyset$, it follows that $A^{\ominus} \cap E \neq \emptyset$. Thus, there exists $z \in A^{\ominus} \cap E$. Having $e = z^{-1}z$, we may conclude that $e \in AE = A$ and therefore $e \in A \cap E$. According to Theorem 6, we immediately get $A = E$.

(ii) Using Corollary 2, it is enough to show that $E \leqslant G$. Let $a \in E$ be an arbitrary element, so $a \in G^*$ and hence $a^{-1} \in G^*$. Thus, there exists $A \in \mathcal{G}$ such that $a^{-1} \in A$. This implies that $e \in aA \subseteq EA = A$ and hence $a = ae \in EA = A$. Therefore, $a \in E \cap A$. Now, by Theorem 6, we get $A = E$. Thus, $a^{-1} \in E$ and the proof is complete. □

4. HX-Polygroups

The aim of this section is to extend the notion of HX-groups to the class of polygroups. We then obtain conditions that characterize an HX-polygroup as a polygroup of cosets or as a double coset polygroup.

Let $(P, \cdot)$ be a polygroup and $\mathcal{H}$ br a non-empty subset of $\mathcal{P}^*(P)$. We say that $\mathcal{H}$ is an *HX-subset* of P if for all $(A, B) \in \mathcal{H}^2$, there exists $C \in \mathcal{H}$ such that $C \cap A \cdot B \neq \emptyset$.

Definition 5. *Let $(P, \cdot)$ be a polygroup and $\mathcal{H}$ be an HX-subset of P. We say that $\mathcal{H}$ is an HX-polygroup on P if $(\mathcal{H}, \odot)$ is a polygroup with the hyperoperation defined as follows: for any $(A, B) \in \mathcal{H}^2$,*

$$A \odot B = \{C \in \mathcal{H} \mid C \cap A \cdot B \neq \emptyset\}.$$

We denote the polygroup $(\mathcal{H}, \odot)$ by $(P, \mathcal{H})$.

Example 9. *Let P be a polygroup with the scalar identity e and N be a normal subpolygroup of P. Then, $\mathcal{H} = \{Nx \mid x \in P\} = \dfrac{P}{N}$ is the quotient polygroup and clearly $(\mathcal{H}, P)$ is an HX-polygroup with the hyperoperation $Nx_1 \odot Nx_2 = \{Ny \mid Ny \cap Nx_1x_2 \neq \emptyset\}$.*

Proposition 5. *Let $(G, \cdot)$ be a torsion group and $\mathcal{G}$ be an HX-group on G. Then, the hyperoperation in Definition 5 reads $A \odot B = \{A \cdot B\}$, for all $(A, B) \in \mathcal{G}^2$.*

Proof. According to Proposition 2, every HX-group derived from a torsion group is regular, so it is a group of cosets. Thus, there exists a unique $C \in \mathcal{G}$ such that $C \cap A \cdot B \neq \emptyset$ for all $(A, B) \in \mathcal{G}^2$, and hence $A \odot B = \{A \cdot B\}$ for all $(A, B) \in \mathcal{G}^2$. □

Example 10. *Let H be a subgroup of an arbitrary group $(G,\cdot)$. The system of all double cosets $G /\!\!/ H = \{HxH \mid x \in G\}$ is a polygroup. Since*

$$HxH \odot HyH = \{HzH \mid z \in G, HzH \cap HxH \cdot HyH \neq \emptyset\} = \{HxhyH \mid h \in H\},$$

we conclude that $\mathcal{G} = \{HxH \mid x \in G\}$ is an HX-polygroup on G that we will call the double coset HX-polygroup.

According to Corollary 1, every HX-group on a finite group is regular, so it can be seen as a group of cosets. A natural question arises: *does this property also apply for HX-polygroups?* The following example answers this question negatively, so there exist finite polygroups on which we may construct an HX-polygroup which is not a polygroup of cosets.

Example 11. *On the set $P = \{e,a,b,c\}$, define the hyperoperation $\cdot$ as in the Cayley's table below:*

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	b	c
b	b	b	c	e,a
c	c	c	e,a	b

It is easy to check that $(P,\cdot)$ is a polygroup with the scalar identity e. Considering the set $H = \{\{e,a\},\{b,c\}\}$, we obtain that $\mathcal{H} = (H,\odot)$ is the HX-polygroup defined by the following Cayley's table:

$\odot$	$\{e,a\}$	$\{b,c\}$
$\{e,a\}$	$\{e,a\}$	$\{b,c\}$
$\{b,c\}$	$\{b,c\}$	$\{e,a\},\{b,c\}$

Moreover, since $\{e,a\} \cdot x \neq \{b,c\}$, for all $x \in P$, we conclude that $\mathcal{H}$ is not a polygroup of cosets. In addition, it is a double coset HX-polygroup. Indeed, $H = HeH = HaH = \{e,a\}$ and $HbH = HcH = \{b,c\}$.

Example 12. *Let $G = \{0,1,2\}$ be a cyclic group of order 3. Endow the set $P = \{\{0\},\{1,2\}\}$ with the hyperoperation $\odot$ defined in the next Cayley's table:*

$\odot$	$\{0\}$	$\{1,2\}$
$\{0\}$	$\{0\}$	$\{1,2\}$
$\{1,2\}$	$\{1,2\}$	P

Then, $(P,\odot)$ is an HX-polygroup on G, which is not a polygroup of cosets nor a double coset HX-polygroup.

Proposition 6. *Let $(P,\cdot)$ be a polygroup and $\mathcal{H}$ be an HX-polygroup on P which satisfies the following conditions:*

(i) The identity E of $\mathcal{H}$ is a subpolygroup of P.
(ii) $x \cdot i(A) \cap E \neq \emptyset$ for all $A \in \mathcal{H}$ and $x \in A$.
(iii) $A \cdot E \in \mathcal{H}$ for all $A \in \mathcal{H}$.

Then, $i(A) = \{x^{-1} \mid x \in A\}$ for all $A \in \mathcal{H}$.

Proof. According to Condition (ii), for an arbitrary $A \in \mathcal{H}$ and $x \in A$, there exists $y \in i(A)$ such that $x \cdot y \cap E \neq \emptyset$. Now, let $u \in x \cdot y \cap E$. Because E is a subpolygroup of P, there exists $u^{-1} \in E$ such that $e \in u \cdot u^{-1} \subseteq x \cdot (y \cdot u^{-1})$. Hence, $x^{-1} \in y \cdot u^{-1} \subseteq A^{\ominus} \cdot E$. According to Condition (iii), we have $i(A) \cdot E \in \mathcal{H}$, implying that $i(A) \cdot E \subseteq i(A) \odot E = E$ (since E is the identity of $\mathcal{H}$) and thus $x^{-1} \in i(A)$. Consequently, $\{x^{-1} \mid x \in A\} \subseteq i(A)$ for all $A \in \mathcal{H}$. Now, let an arbitrary $a \in i(A)$. Since $i(A) \in \mathcal{H}$ and $i(i(A)) = A$, applying Condition (ii), we have $a \cdot A \cap E \neq \emptyset$. Thus, there exist $b \in E$ and $x \in A$ such that $b \in a \cdot x$ and therefore $a^{-1} \in x \cdot b^{-1}$. Since E is a polygroup of P, it follows that $b^{-1} \in E$ and hence $a^{-1} \in x \cdot E \subseteq A \cdot E$, where $A \cdot E \in \mathcal{H}$, and therefore $A \cdot E \subseteq A \odot A = A$. Therefore, $a^{-1} \in A$ and thereby $a = (a^{-1})^{-1} \in \{x^{-1} \mid x \in A\}$. Thus, the proof is complete. $\square$

Theorem 7. *Let $(P, \cdot)$ be a polygroup and $\mathcal{H}$ be an HX-polygroup on P with the identity E which satisfies the following conditions:*

(i) $i(A) = \{a^{-1} \mid a \in A\}$ *for all* $A \in \mathcal{H}$.
(ii) $A \cdot i(A) = E$ *for all* $A \in \mathcal{H}$.
(iii) $K = \bigcup\limits_{A \in \mathcal{H}} A$ *is a subpolygroup of* P.
(iv) $E \cdot A \in \mathcal{H}$ *for all* $A \in \mathcal{H}$.

Then, $(\mathcal{H}, \odot)$ is a polygroup of cosets.

Proof. First note that, for each $L \in \mathcal{H}$, Condition (iv) implies $E \cdot L = L$, because $\{L\} = E \odot L = \{C \in \mathcal{H} \mid C \cap E \cdot L \neq \emptyset\} \supseteq E \cdot L$. In particular, it holds that $E \cdot E = E = i(E)$ and thus E is a subpolygroup of P.

Now, suppose that $A \in \mathcal{H}$ and $x \in A$. We will prove that $E \cdot x = A = x \cdot E$. Let $y \in A$. Since $K \cdot x = K$, there exists $u \in K$ such that $y \in u \cdot x$. Therefore, $u \in y \cdot x^{-1} \subseteq A \cdot i(A) = E$ and hence $y \in E \cdot x$. Thus, $A \subseteq E \cdot x$. Since $E \cdot A = A$, it follows that $E \cdot x \subseteq A$. Consequently, $E \cdot x = A$. Now, we prove that $x \cdot E = A$. Since $i(A) \in \mathcal{H}$, by Condition (iii), we have $i(A) \cdot i(i(A)) = E$ and hence $i(A) \cdot A = E$. Because $E \cdot i(A) = i(A)$, we may write $A \cdot E \cdot i(A) = A \cdot i(A) = E$. This implies $A \cdot E \cdot i(A) \cdot A = E \cdot A = A$ and hence $A \cdot E \cdot E = A$. Thus, $A \cdot E = A$ and since $x \in A$, it follows that $x \cdot E \subseteq A$. Similarly, one can prove that $A \subseteq x \cdot E$. Now, we show that E is normal in K. For this reason, let $x \in K$, and so there exists $A \in \mathcal{H}$ such that $x \in A$. According to what we have proved above, $E \cdot x = x \cdot E = A$. Thus, E is normal in K and $\mathcal{H} = \frac{K}{E}$, which shows that $\mathcal{H}$ is a polygroup of cosets. $\square$

Theorem 8. *Let $(G, \cdot)$ be a group and $\mathcal{H}$ be an HX-polygroup on G. If $K = \bigcup_{A \in \mathcal{H}} A$ is a subgroup of G, then we have*

(i) $e \in E$, *where e is the identity element of G and E is the scalar identity of $\mathcal{H}$.*
(ii) *If $X \cap E \neq \emptyset$, then $X = E$ for all $X \in \mathcal{H}$.*
(iii) *If $X \cap Y \neq \emptyset$, then $X = Y$ for all $X, Y \in \mathcal{H}$.*

Proof. (i) Since K is a subgroup of G, there exists $A \in \mathcal{H}$ such that $e \in A$. Thus, $E = e \cdot E \subseteq A \cdot E$ and so $A \cdot E \cap E = E \neq \emptyset$. Therefore, $E \in A \odot E = \{A\}$ and hence $E = A \ni e$.
(ii) If $x \in X \cap E$, then there exists Y in $\mathcal{H}$ such that $x^{-1} \in Y$. Therefore, $e = x^{-1} \cdot x \in Y \cdot E$ and hence $Y \cdot E \cap E \neq \emptyset$. Thus, $E \in Y \odot E = \{Y\}$, so $E = Y$. Since $x \in E$, $x = x \cdot e \in X \cdot E$ and hence $X \cdot E \cap E \neq \emptyset$. Thus, $X = E$.
(iii) Now, suppose that $a \in X \cap Y$. Since K is a subgroup of G, there exists $Z \in \mathcal{H}$ such that $a^{-1} \in Z$. Thus, $e = a^{-1} \cdot a \in Z \cdot X$ and hence, based on Item (i), we conclude that $Z \cdot X \cap E \neq \emptyset$. This implies that $E \in Z \odot X$, and so $Z = i(X)$. Similarly, we have $Z = i(Y)$ and therefore $X = Y$. $\square$

The following theorem states a necessary and sufficient condition such that an HX-polygroup constructed on a group G is a double coset polygroup.

Theorem 9. *Let $(G, \cdot)$ be a group and $\mathcal{H}$ be an HX-polygroup on G with the scalar identity E. If $K = \bigcup_{A \in \mathcal{H}} A$ is a subgroup of G, then $\mathcal{H}$ is a double coset HX-polygroup if and only if $ExE \in \mathcal{H}$ for every $x \in K$.*

Proof. First suppose that $\mathcal{H}$ is a double coset HX-polygroup. This means that $\mathcal{H} = \{HxH \mid x \in G\}$ for some subgroup H of G. Since E is the scalar identity of $\mathcal{H}$, it follows that $e \in H \cap E$ and according to Theorem 8, we have $E = H$. Therefore, $ExE \in \mathcal{H}$ for every $x \in K$.

Conversely, let $X \in \mathcal{H}$ with $x \in X$. By Theorem 8, we have $ExE \in \mathcal{H}$ and $ExE \cap X \neq \emptyset$. Thereby, $X = ExE$. $\square$

5. Conclusions and Some Open Problems

Polygroups are a type of hypergroup with a very similar behavior to groups that can also be generated from groups or again from other polygroups. They were introduced by Comer, motivated by the example of double cosets of a group modulo a subgroup, without knowing the initial works that appeared in Europe, in particular in Italy and Greece, conducted by P. Bonansinga, C. Massouros and S. Ioulidis. Thus, it is natural to extend the notion of HX-groups on a group to that of HX-polygroups on a polygroup and investigate the regularity property and find conditions under which they can be characterized as a double coset polygroup. In this paper, we have managed to do this for HX-polygroups with groups as their support. In the future, we will continue our investigation for HX-polygroups on polygroups that are not groups.

In our opinion, this work may open several new lines of research and here we will suggest three of them.

1. Since polygroups have also recently been studied in relation to soft sets [42], it is worth developing the theory of soft HX-groups [43] and its extension to soft HX-polygroups.
2. Another studied problem related to HX-groups is the one involving a direct product. In particular, it will be interesting to know the relationship between the direct product of the HX-groups $(\mathcal{G}_1, G_1)$ and $(\mathcal{G}_2, G_2)$ and the HX-group $(\mathcal{G}, G_1 \times G_2)$ formed on the direct product group $G_1 \times G_2$. This aspect was very well studied in [44]. The same problem can be extended and investigated for HX-polygroups, considering the direct product $P_1 \times P_2$ of two polygroups $(P_1, \cdot_1)$ and $(P_2, \cdot_2)$ equipped with the hyperoperation defined in [38] by

$$(a_1, b_1) \circ (a_2, b_2) = \{(x, y) \in P_1 \times P_2 \mid x \in a_1 \cdot_1 a_2, y \in b_1 \cdot_2 b_2\}.$$

   Then, one can study the relationship existing between the direct product of the HX-polygroups $(\mathcal{H}_1, P_1)$ and $(\mathcal{H}_2, P_2)$ and the HX-polygroup $(\mathcal{H}, P_1 \times P_2)$ created on the direct product polygroup $P_1 \times P_2$.
3. The final problem that we would like to propose for future work is related to HX-rings. They were introduced in 1988 by Hong Xing Li, the same author who defined the HX-groups. Considering a ring $(R, +, \cdot)$, two binary operations can be defined on $\mathcal{P}^*(R)$:

$$\begin{aligned} A \oplus B &= \{a + b \mid a \in A, b \in B\} \text{ and} \\ A \odot B &= \{a \cdot b \mid a \in A, b \in B\} \end{aligned}$$

   endowing $\mathcal{P}^*(R)$ with a semigroup structure with respect to each of these two operations. Note that these operations are defined in the same way as the operation on an HX-group. Moreover, the above-defined operations do not satisfy the distributivity law, i.e., $(A \oplus B) \odot C = (A \odot C) \oplus (B \odot C)$ and $A \odot (B \oplus C) = (A \odot B) \oplus (A \odot C)$, but just the inclusive distributive laws, i.e., $(A \oplus B) \odot C \subset (A \odot C) \oplus (B \odot C)$ and $A \odot (B \oplus C) \subset (A \odot B) \oplus (A \odot C)$. Therefore, H. X. Li defined a non-empty subset $\mathcal{F}$ of $\mathcal{P}^*(R)$ as a distributive class on R if, for any $A, B, C \in \mathcal{F}$, the distributivity property of the operations "$\oplus$" and "$\odot$" holds. In addition, he noticed that a distributivity class may not be closed with respect to operations and defined the concept of HX-rings as follows.

   Definition 6 ([45]). *If a distributive class on a ring R with the operations "$\oplus$" and "$\odot$" forms a ring, then it is called an HX-ring. Denote by Q its zero element, i.e., the neutral element with respect to the addition "$\oplus$". An HX-ring is called regular if $0 \in Q$, where 0 is the zero element of the ring R.*

The main result in [45] states that any regular HX-ring $\mathcal{R}$ on a ring R can be written as a set of cosets $\mathcal{R} = \{a + I \mid a \in H\}$ with the zero element $Q = I$, where H is a subring of R and I is semi-ideal with respect to H, meaning that I is a subsemigroup of $(R, +)$ and $IH \cup HI \subset I$. We strongly believe that this topic deserves further and deeper investigation, first by providing significant examples of HX-rings (that unfortunately are completely missing from the original paper) and then by finding new properties similar to the uniformity or essentiality defined for HX-groups.

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