

Article

Results of Third-Order Strong Differential Subordinations

Madan Mohan Soren ¹, Abbas Kareem Wanas ²  and Luminița-Ioana Cotîrlă ^{3,*} 

¹ Department of Mathematics, Berhampur University (Bhanja Bihar), Brahmapur 760007, India; mms.math@buodisha.edu.in

² Department of Mathematics, College of Science, University of Al-Qadisiyah, Al Diwaniyah 58001, Iraq; abbas.kareem.w@qu.edu.iq

³ Department of Mathematics, Technical University of Cluj-Napoca, 400114 Cluj-Napoca, Romania

* Correspondence: luminita.cotirla@math.utcluj.ro

Abstract: In this paper, we present and investigate the notion of third-order strong differential subordinations, unveiling several intriguing properties within the context of specific classes of admissible functions. Furthermore, we extend certain definitions, presenting novel and fascinating results. We also derive several interesting properties of the results of third-order strong differential subordinations for analytic functions associated with the Srivastava–Attiya operator.

Keywords: admissible function; analytic function; strong differential subordination; dominants; multivalent function

MSC: 30C45



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1. Introduction and Definitions

Differential subordination is a fundamental technique in geometric function theory of complex analysis used by many authors in investigations to obtain interesting new results. The notion of strong differential subordination was first used by Antonino and Romaguera [1] (see [2]) to study Briot–Bouquet’s strong differential subordination. They introduced this concept as an extension of the classical notion of differential subordination, due to Miller and Mocanu [3] (see [4]). The concept was beautifully developed for the theory of strong differential subordination in 2009 [5], where the authors extended the concepts familiar to the established theory of differential subordination [4]. There have been many interesting and fruitful usages of a wide variety of first-order and second-order strong differential subordinations for analytic functions. Recently, many researchers have worked in this direction and proved several significant results that can be seen in [6–8]. Various strong differential subordinations were established by linking different types of operators to the study. The Sălăgean differential operator was employed for introducing a new class of analytic functions in [9], and the Ruscheweyh differential operator in [10] for defining a new class of univalent functions and for studying strong differential subordinations. The Sălăgean and Ruscheweyh operators were used together in the study presented in [11], and a multiplier transformation provided new strong differential subordinations in [12–14]. The Komatu integral operator was applied for obtaining new strong differential subordinations results [15,16], and other differential operators proved effective for studying strong differential subordinations [17]. The fractional derivative operator was used in [18], and the fractional integral of the extended Dziok–Srivastava operator was used in [19]. Multivalent meromorphic functions and the Liu–Srivastava operator were involved in obtaining strong differential subordinations in [20]. The topic remains of interest at present, as proven by recently published works (see, for details, [21–23]). Thus, in this current paper, we introduced and investigated the concept of third-order strong differential subordinations, unveiling several intriguing properties within the context of specific classes of admissible functions.

Let \mathbb{N} denote the set of positive integers. Suppose $\mathcal{H} = \mathcal{H}(\mathcal{U})$ denotes the class of analytic functions in the open unit disc

$$\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\},$$

where \mathbb{C} is the set of complex numbers. For $n \in \mathbb{N}$, $b \in \mathbb{C}$, define the class of functions

$$\mathcal{H}[b, n] := \left\{ f : f \in \mathcal{H}; f(z) = b + b_n z^n + b_{n+1} z^{n+1} + \dots \right\}.$$

Given $f, F \in \mathcal{H}$. The function f is *subordinate* to F , denoted by $f(z) \prec F(z)$, if there exists an analytic function ω in \mathcal{U} satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$ so that $f(z) = F(\omega(z))$ ($z \in \mathcal{U}$). Further, if the function F is univalent in \mathcal{U} , then (see [3,4]) $f \prec F \iff f(0) = F(0)$ and $f(\mathcal{U}) \subset F(\mathcal{U})$. Suppose that $\mathcal{F}(z, \zeta)$ is analytic in $\mathcal{U} \times \overline{\mathcal{U}}$ and $f(z)$ is analytic and univalent in \mathcal{U} . We say that $\mathcal{F}(z, \zeta)$ is strongly subordinate to $f(z)$. Simply write

$$\mathcal{F}(z, \zeta) \prec\prec f(z),$$

if $\mathcal{F}(z, \zeta)$ ($\zeta \in \overline{\mathcal{U}}$) as a function of z is subordinate to $f(z)$. Here, also observe that (cf. [2,5,24])

$$\mathcal{F}(z, \zeta) \prec\prec f(z) \iff \mathcal{F}(0, \zeta) = f(0) \text{ and } \mathcal{F}(\mathcal{U} \times \overline{\mathcal{U}}) \subset f(\mathcal{U}).$$

For $p \in \mathbb{N}$, we denote $\mathcal{A}(p)$ as the class of analytic functions defined by

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}. \quad (1)$$

Mishra and Gochhayat [25] introduced and studied the *fractional differintegral operator*. For $f \in \mathcal{A}(p)$, the transform

$$\mathcal{I}_{p,\delta}^{\lambda} : \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$$

is expressed by

$$\begin{aligned} \mathcal{I}_{p,\delta}^{\lambda} f(z) &:= z^p + \sum_{k=1}^{\infty} \left(\frac{p+\delta}{p+k+\delta} \right)^{\lambda} a_{p+k} z^{p+k} \\ (p+\delta) &\in \mathbb{C} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- := \{0, -1, -2, \dots\}; \lambda \in \mathbb{C}. \end{aligned} \quad (2)$$

The operator $\mathcal{I}_{p,\delta}^{\lambda}$ can be seen as a generalization of the Srivastava–Attiya operator [26] (see [27–29]); it is also popularly known as the Srivastava–Attiya operator for multivalent functions (see, for example, [30–32]). Furthermore, $\mathcal{I}_{p,\delta}^{\lambda}$ generalizes several previously studied familiar differential operators as well as integral operators by Bernardi [33], Cho and Kim [34], Jung et al. [35], Libera [36], Sălăgean [37] and Uralegaddi and Somanatha [38]. For a detailed discussion [25], also see [39–41].

They [25] derived from (2) the relation

$$z \left(\mathcal{I}_{p,\delta}^{\lambda} f(z) \right)' = (p+\delta) \mathcal{I}_{p,\delta}^{\lambda-1} f(z) - \delta \mathcal{I}_{p,\delta}^{\lambda} f(z). \quad (3)$$

In terms of the third order, there have been only three articles [1,42–44] for the corresponding third-order implication connected to a special case. Let Π and Δ be sets in \mathbb{C} . Suppose p is an analytic function in \mathcal{U} and

$$\Xi(r_1, s_1, t_1, u_1; z, \zeta) : \mathbb{C}^4 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}.$$

We have determined properties of the function p that imply the following inequality holds:

$$\left\{ \Xi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z, \zeta) \right\} \subset \Pi \implies p(\mathcal{U}) \subset \Delta. \quad (4)$$

A natural question arises as to what conditions on Ξ , Π and Δ are needed so that the implication (4) holds.

In this present article, we consider conditions on Π , Δ and Ξ so that the inequality (4) holds. We see that there are three different cases to consider in analyzing this inequality's truth:

Problem 1. Given Π and Δ , we find Ξ so that (4) holds, and Ξ is an admissible function.

Problem 2. Given Ξ and Π , we find the 'smallest' Δ that satisfies (4).

Problem 3. Given Ξ and Δ , we find the Π that satisfies (4). Furthermore, we find the 'largest' such Π .

The relation (4) can be rephrased in strong subordination terms, when either Π or Δ is a simply connected domain. If Δ is a simply connected domain with $\Delta \neq \mathbb{C}$, and $p(z)$ is analytic in \mathcal{U} , then a conformal mapping $q(z)$ of \mathcal{U} onto Δ can be performed so that $q(0) = p(0)$. In such case, (4) can be written as follows:

$$\left\{ \Xi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z, \zeta) \right\} \subset \Pi \implies p \prec q. \quad (5)$$

Similarly, if Π is a simply connected domain, then there is a conformal mapping h of \mathcal{U} onto Π so that $h(0) = \Xi(p(0), 0, 0, 0; 0, 0)$. If

$$\Xi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z, \zeta)$$

is analytic in \mathcal{U} , then (5) can be reduced to

$$\left\{ \Xi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z, \zeta) \right\} \prec h(z) \implies p \prec q. \quad (6)$$

There are three key ingredients in a differential implication of the form of (5): the Ξ , the set Π and the dominating function q . If two of these entities were given, one would hope to find conditions on the third so that (6) would be satisfied. In this present article, we start with a given set Π and a given q , and determine a set of admissible operators Ξ so that inequality (4) holds. This leads to some of the definitions that will be used in our main results.

Definition 1. Suppose $\Xi : \mathbb{C}^4 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$ and h is univalent in \mathcal{U} . If $p \in \mathcal{H}$ and satisfies the third-order strong differential subordination

$$\Xi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z, \zeta) \prec h(z), \quad (7)$$

then p is said to be a solution of the strong differential subordination. Moreover, if $p \prec q$ for all p satisfying (7), then the univalent function q is a dominant of the solutions for the strong differential subordination. A dominant \tilde{q} that satisfies $\tilde{q} \prec q$ for all dominants q of (7) is the best dominant of (7).

For $\Pi \subset \mathbb{C}$, with Ξ and p given in Definition 1, relation (7) can be written as follows:

$$\left\{ \Xi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z, \zeta) \right\} \subset \Pi. \quad (8)$$

Condition (8) will also be referred to as strong differential subordination, and can be further extended to the definitions of the solution, dominant and best dominant.

Definition 2 ([1]). Let \mathcal{Q} denote the collection of all injective and analytic functions q on $\overline{\mathcal{U}} \setminus E(q)$, where

$$E(q) = \left\{ \zeta : \zeta \in \partial\mathcal{U} \text{ and } \lim_{z \rightarrow \zeta} q(z) = \infty \right\},$$

and $\min |q'(\zeta)| = \rho > 0$ ($\zeta \in \partial\mathcal{U} \setminus E(q)$). Also, $\mathcal{Q}(b)$ is the class of functions q with $q(0) = b$.

We will use the following lemmas from the third-order differential subordinations to find dominants of strong differential subordinations.

Lemma 1 ([1]). Let $\mathcal{U}_{r_0} = \{z : |z| < r_0\}$, with $0 < r_0 < 1$. Let $p(z) = b + b_n z^n + b_{n+1} z^{n+1} + \dots$ be analytic in \mathcal{U} with $n \geq 2$ and $p(z) \neq b$, and let $q \in \mathcal{Q}(b)$. If there exist points $z_0 = r_0 e^{i\theta_0} \in \mathcal{U}$ and $\xi_0 \in \partial\mathcal{U} \setminus E(q)$ such that $p(z_0) = q(\xi_0)$, $p(\overline{\mathcal{U}}_{r_0}) \subset q(\mathcal{U})$,

$$\Re \frac{\xi_0 q''(\xi_0)}{q'(\xi_0)} \geq 0, \text{ and} \quad (9)$$

$$\left| \frac{z p'(z)}{q'(\xi)} \right| \leq n \quad (10)$$

where $z \in \overline{\mathcal{U}}_{r_0}$ and $\xi \in \partial\mathcal{U} \setminus E(q)$, then there exists a real constant $k \geq n \geq 2$ such that

$$z_0 p'(z_0) = n \xi_0 q'(\xi_0), \quad (11)$$

$$\Re \left(\frac{z_0 p''(z_0)}{p'(z_0)} + 1 \right) \geq n \left[\Re \frac{\xi_0 q''(\xi_0)}{q'(\xi_0)} + 1 \right], \quad (12)$$

$$\Re \left(\frac{z_0^2 p'''(z_0)}{p'(z_0)} + 1 \right) \geq n^2 \left[\Re \frac{\xi_0^2 q'''(\xi_0)}{q'(\xi_0)} + 1 \right],$$

or

$$\Re \left(\frac{z_0^2 p'''(z_0)}{p'(z_0)} \right) \geq n^2 \left[\Re \frac{\xi_0^2 q'''(\xi_0)}{q'(\xi_0)} \right]. \quad (13)$$

Consider a special case when q is univalent in Lemma 1. If

$$q(w) = M \frac{Mw + b}{M + \bar{b}w}, \quad (14)$$

with $|b| < M$, then $q(\overline{\mathcal{U}}) = \mathcal{U}_M$, $q(0) = b$ and $E(q) = \emptyset$.

Lemma 2 ([1]). Let $\mathcal{U}_{r_0} = \{z : |z| < r_0\}$, with $0 < r_0 < 1$. Suppose q given in (14) and $p(z) = b + b_n z^n + b_{n+1} z^{n+1} + \dots$ is analytic in \mathcal{U} with $n \geq 2$ and $p(z) \neq b$. If there exist points $z_0 = r_0 e^{i\theta_0} \in \mathcal{U}_M$ and $w_0 \in \partial\mathcal{U}$ such that $p(z_0) = q(w_0)$, $p(\mathcal{U}_{r_0}) \subset \mathcal{U}_M$ and

$$|z p'(z)| |M + \bar{b}e^{i\theta}|^2 \leq nM[M^2 - |b|^2] \quad (15)$$

when $z \in \overline{\mathcal{U}}_{r_0}$ and $\theta \in [0, 2\pi]$, then

$$z_0 p'(z_0) = n q(w_0) \frac{|q(w_0) - b|^2}{|q(w_0)|^2 - |b|^2},$$

$$\Re \left(\frac{z_0 p''(z_0)}{p'(z_0)} + 1 \right) \geq n \left(\frac{|q(w_0) - b|^2}{|q(w_0)|^2 - |b|^2} \right), \quad \text{and}$$

$$\Re \left(\frac{z_0^2 p'''(z_0)}{p'(z_0)} \right) \geq 6n^2 \frac{[|q(w_0) - b|^2]^2}{[|q(w_0)|^2 - |b|^2]^2}.$$

Our main objective in this article is to systematically investigate several potentially useful results that are based upon third-order strong differential subordinations and their applications in geometric function theory of complex analysis. Our results give interesting new properties and, together with other papers that appeared in recent years, could emphasize the perspective of the importance of third-order strong differential subordination theory and the generalized Srivastava–Attiya operator.

The organization of this article is as follows. In Section 2 below, we derive the notion of third-order strong differential subordination, some definitions and the interesting main results. We consider some suitable classes of admissible functions and investigate several third-order strong differential subordination properties of multivalent functions involving the Srivastava–Attiya operator defined by (2) in Section 3. Some corollaries and consequences of our main results are also presented in Sections 2 and 3. Finally, in the last Section 4, some potential directions for related further research are presented.

2. Main Results

Unless indicated otherwise, we assume throughout the sequel that $p \geq 2$, $z \in \mathcal{U}$ and $\zeta \in \overline{\mathcal{U}}$. We establish the third-order strong differential subordinations theorem. In this connection, we state the following definition.

Definition 3. Suppose $\Pi \in \mathbb{C}$ and $q \in \mathcal{Q}$. The class of admissible functions $\Xi_n[\Pi, q]$ consists of those functions

$$\Xi : \mathbb{C}^4 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$$

that fulfill the following admissibility condition:

$$\Xi(r_1, s_1, t_1, u_1; z, \zeta) \notin \Pi \quad (16)$$

whenever $r_1 = q(\zeta)$, $s_1 = n\zeta q'(\zeta)$,

$$\Re\left(\frac{t_1}{s_1} + 1\right) \geq n \left[\Re \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right]$$

and

$$\Re\left(\frac{u_1}{s_1}\right) \geq n^2 \left[\Re \frac{\zeta^2 q'''(\zeta)}{q'(\zeta)} \right],$$

for $\zeta \in \partial\mathcal{U} \setminus E(q)$.

Here, $\Xi_1[\Pi, q]$ is denoted as $\Xi[\Pi, q]$. We refer to two special subcases of this definition. If $\Xi : \mathbb{C}^3 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$, then (16) becomes $\Xi(r_1, s_1, t_1; z, \zeta) \notin \Pi$ when $r_1 = q(\zeta)$, $s_1 = n\zeta q'(\zeta)$ and

$$\Re\left(\frac{t_1}{s_1} + 1\right) \geq n \left[\Re \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right], \quad \text{for } \zeta \in \partial\mathcal{U} \setminus E(q).$$

If $\Xi : \mathbb{C}^2 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$, then (16) becomes $\Xi(q(\zeta), n\zeta q'(\zeta); z, \zeta) \notin \Pi$ when $\zeta \in \partial\mathcal{U} \setminus E(q)$. We also deduce from Definition 3 the inclusion relations $\Xi_n[\Pi', q] \subset \Xi_n[\Pi, q]$ if $\Pi' \subset \Pi$.

The following theorem is a key result in the notion of third-order strong differential subordination.

Theorem 1. Consider $p \in \mathcal{H}[b, n]$ and $q \in \mathcal{Q}(b)$ fulfills

$$\Re \frac{\zeta q''(\zeta)}{q'(\zeta)} \geq 0 \quad \text{and} \quad \left| \frac{z p'(z)}{q'(\zeta)} \right| \leq n, \quad (17)$$

where $\zeta \in \partial\mathcal{U} \setminus E(q)$. If Π is a set in \mathbb{C} , $\Xi \in \Xi_n[\Pi, q]$ and

$$\Xi(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z, \zeta) \in \Pi, \quad (18)$$

then

$$p(z) \prec q(z).$$

Proof. If we assume that $p \not\prec q$, then there exist points $z_0 = r_0 e^{i\theta_0} \in \mathcal{U}$ and $\zeta_0 \in \partial\mathcal{U} \setminus E(q)$ such that $p(z_0) = q(\zeta_0)$ and $p(\overline{\mathcal{U}}_{r_0}) \subset q(\mathcal{U})$. From (17), we see that (9) and (10) of Lemma 1 are satisfied when $z \in \mathcal{U}$ and $\zeta \in \partial\mathcal{U} \setminus E(q)$. The conditions of that lemma are satisfied;

we conclude that (11)–(13) also follow. Using these conditions with $r_1 = p(z_0)$, $s_1 = z_0 p'(z_0)$, $t_1 = z_0^2 p''(z_0)$, $u_1 = z_0^3 p'''(z_0)$ and $z = z_0$ in Definition 3 leads to

$$\Xi(p(z_0), z_0 p'(z_0), z_0^2 p''(z_0), z_0^3 p'''(z_0); z, \zeta) \notin \Pi,$$

which contradicts (18); thus, we have

$$p(z) \prec q(z).$$

□

In Theorem 1, inequalities (17) and (18) are the most necessary for solving third-order differential subordination. If third-order terms in (18) are missing, then they are not required to satisfy (17).

The next result is a special case where the behavior of q on $\partial\mathcal{U}$ is not known in Theorem 1.

Corollary 1. Suppose q is univalent in \mathcal{U} , $q(0) = b$ and set $q_\rho(z) \equiv q(\rho z)$ for $\rho \in (0, 1)$. Consider that $p \in \mathcal{H}[b, n]$ and q_ρ fulfill

$$\Re \frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)} \geq 0 \quad \text{and} \quad \left| \frac{z p'(z)}{q_\rho'(\xi)} \right| \leq n,$$

when $\xi \in \partial\mathcal{U} \setminus E(q)$. If Π is a set in \mathbb{C} and $\Xi \in \Xi_n[\Pi, q_\rho]$, then

$$\Xi(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z, \zeta) \subset \Pi$$

implies

$$p(z) \prec q(z).$$

Proof. Given q_ρ is univalent in $\partial\mathcal{U}$, and hence $E(q_\rho) = \emptyset$ and $q_\rho \in \mathcal{Q}(b)$. Since the class $\Xi_n[\Pi, q_\rho]$ is an admissible functions and from Theorem 1 we obtain $p \prec q_\rho$. Since $q_\rho \prec q$, here we conclude that $p \prec q$. □

In Definition 3, there are no specific conditions on Π . When $\Pi \neq \mathbb{C}$ is a simply connected domain and there is a conformal mapping h of \mathcal{U} onto Π , we denote the class $\Xi_n[h(\mathcal{U}), q]$ by $\Xi_n[h, q]$. The next two results are directly from Theorem 1 and Corollary 1.

Theorem 2. Consider $p \in \mathcal{H}[b, n]$ and $q \in \mathcal{Q}(b)$ and that they fulfill

$$\Re \frac{\xi q''(\xi)}{q'(\xi)} \geq 0 \quad \text{and} \quad \left| \frac{z p'(z)}{q'(\xi)} \right| \leq n,$$

where $\xi \in \partial\mathcal{U} \setminus E(q)$. If $\Xi \in \Xi_n[h, q]$ and $\Xi(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z, \zeta)$ is analytic in \mathcal{U} , then

$$\Xi(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z, \zeta) \prec h(z)$$

implies

$$p(z) \prec q(z).$$

Corollary 2. Suppose q is univalent in \mathcal{U} , with $q(0) = b$, and set $q_\rho(z) \equiv q(\rho z)$ for $\rho \in (0, 1)$. Consider that $p \in \mathcal{H}[b, n]$ and q_ρ fulfill

$$\Re \frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)} \geq 0 \quad \text{and} \quad \left| \frac{z p'(z)}{q_\rho'(\xi)} \right| \leq n$$

where $\xi \in \partial\mathcal{U} \setminus E(q)$. If $\Xi \in \Xi_n[h, q_p]$ and $\Xi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z, \xi)$ is analytic in \mathcal{U} , then

$$\Xi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z, \xi) \prec\prec h(z)$$

implies

$$p(z) \prec q(z).$$

We next specify the connection between the best dominant of a strong differential subordination and the solution of a corresponding differential equation.

Theorem 3. Consider $p \in \mathcal{H}[b, n]$, $\Xi : \mathbb{C}^4 \times \mathcal{U} \times \overline{\mathcal{U}} \rightarrow \mathbb{C}$ and that

$$\Xi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z, \xi)$$

is analytic in \mathcal{U} . Suppose h is univalent in \mathcal{U} and the differential equation

$$\Xi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z, \xi) = h(z) \quad (19)$$

has a solution $q \in \mathcal{Q}(b)$ and

$$\Re \frac{\xi q''(\xi)}{q'(\xi)} \geq 0 \quad \text{and} \quad \left| \frac{zp'(z)}{q'(\xi)} \right| \leq n,$$

where $\xi \in \partial\mathcal{U} \setminus E(q)$. If $\Xi \in \Xi_n[h, q]$, then

$$\Xi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z, \xi) \prec\prec h(z) \quad (20)$$

implies that

$$p(z) \prec q(z)$$

and q is the best dominant.

Proof. From Theorem 1, we have that q is a dominant of (20). Again, q fulfills (19) and it is a solution of (20). Thus, q will be dominated by all dominants of (20). Therefore, q is the best dominant. \square

We further pursue the family of admissible functions and theorems, when $q(\mathcal{U})$ is a disc. Since q is given by (14), the class denoted by $\Xi_n[\Pi, M, b]$. When $\Pi = \Delta$, the class denoted by $\Xi_n[M, b]$. Since $q(w) = Me^{i\theta}$ with $0 \leq \theta \leq 2\pi$ when $|w| = 1$, from Lemma 2 we derived the following.

Definition 4. Consider q to be given by (14), $n \geq 2$, and Π is a set in \mathbb{C} . For $\theta \in [0, 2\pi]$, the class $\Xi_n[\Pi, M, b]$ which consists of those functions

$$\Xi : \mathbb{C}^4 \times \mathcal{U} \times \overline{\mathcal{U}} \rightarrow \mathbb{C}$$

that fulfill the following admissibility condition

$$\Xi(r_1, s_1, t_1, u_1; z, \xi) \notin \Pi$$

whenever $r_1 = Me^{i\theta}$, $s_1 = nM \frac{|M - \bar{b}e^{i\theta}|^2}{M^2 - |b|^2} e^{i\theta}$

$$\begin{aligned} \Re \frac{t_1}{s_1} + 1 &\geq n \frac{|M - \bar{b}e^{i\theta}|^2}{M^2 - |b|^2} \quad \text{and} \\ \Re \frac{u_1}{s_1} &\geq 6n^2 \Re \frac{[\bar{b}M - |b|^2]^2}{[M^2 - |b|^2]^2}, \\ &\text{for } z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}. \end{aligned} \quad (21)$$

When $b = 0$, $0 \leq \theta \leq 2\pi$, we see from (21) that $\Xi_n[\Pi, M, 0]$ consists of those functions

$$\Xi : \mathbb{C}^4 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$$

that fulfill

$$\Xi(Me^{i\theta}, nMe^{i\theta}, L, N; z, \zeta) \notin \Pi$$

when

$$\Re(Le^{-i\theta}) \geq (n^2 - n)M \quad \text{and} \quad \Re(Ne^{-i\theta}) \geq 0. \quad (22)$$

The following result is the immediate consequence.

Theorem 4. Consider that the q given in (14) and $p \in \mathcal{H}[b, n]$ satisfy

$$|zp'(z)| |M + \bar{b}e^{i\theta}|^2 \leq Mn [M^2 - |b|^2],$$

where $z \in \mathcal{U}$ and $0 \leq \theta \leq 2\pi$. If $\Xi \in \Xi_n[\Pi, M, b]$, then

$$\Xi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z, \zeta) \subset \Pi$$

implies

$$p(z) \prec q(z).$$

Next, we obtain the following corollary when $b = 0$ in Theorem 4.

Corollary 3. Consider that $q(w) = Mw$ and $p \in \mathcal{H}[0, n]$ fulfill

$$|zp'(z)| \leq Mn$$

when $z \in \mathcal{U}$. If Π is a set in \mathbb{C} and $\Xi \in \Xi_n[\Pi, M, 0]$ as characterized by (22), then

$$\Xi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z, \zeta) \subset \Pi$$

implies

$$p(z) \prec Mz.$$

In this particular case, Theorem 4 becomes

Theorem 5. Consider that the q given in (14) and $p \in \mathcal{H}[b, n]$ satisfy (17). If Π is a set in \mathbb{C} and (i) $\Xi \in \Xi_n[\Pi, M, b]$, then

$$\Xi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z, \zeta) \subset \Pi \implies |p(z)| < M.$$

(ii) If $\Xi \in \Xi_n[M, b]$, then

$$|\Xi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z, \zeta)| < M \implies |p(z)| < M.$$

3. Applications with the Operator

By using the operator $\mathcal{I}_{p,\delta}^\lambda$, we establish the family of admissible functions to discuss the strong subordination properties.

Definition 5. Suppose Π is a set in \mathbb{C} and $q \in \mathcal{Q}$. The family of admissible functions $\Theta_I[\Pi, q]$ consists of functions

$$\Theta : \mathbb{C}^4 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$$

fulfilling the admissibility

$$\Theta(\alpha, \beta, \gamma, \eta; z, \zeta) \notin \Pi$$

$$\text{when } \alpha = q(\zeta), \beta = \frac{k\zeta q'(\zeta) + \delta q(\zeta)}{p + \delta},$$

$$\Re \left(\frac{(p + \delta)^2 \gamma - \delta^2 \alpha}{(p + \delta)\beta - \delta \alpha} - 2\delta \right) \geq k \left[\Re \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right]$$

and

$$\Re \left(\frac{(p + \delta)^2 (\eta(p + \delta) - 3\gamma(1 + \delta)) + (3 + 2\delta)\delta^2 \alpha}{(p + \delta)\beta - \delta \alpha} + 2 + 3(2 + \delta)\delta \right) \geq k^2 \left[\Re \frac{\zeta^2 q'''(\zeta)}{q'(\zeta)} \right],$$

for $\zeta \in \partial \mathcal{U} \setminus E(q)$ and $k \geq p$.

Theorem 6. Consider $\mathcal{I}_{p,\delta}^\lambda f(z) \in \mathcal{H}[0, p]$ with $p \geq 2$, $q \in \mathcal{Q}(0)$ and that they satisfy

$$\Re \frac{\zeta q''(\zeta)}{q'(\zeta)} \geq 0 \quad \text{and} \quad \left| \frac{z(\mathcal{I}_{p,\delta}^\lambda f(z))'}{q'(\zeta)} \right| \leq k, \quad (23)$$

when $\zeta \in \partial \mathcal{U} \setminus E(q)$ and $k \geq p$. If Π is a set in \mathbb{C} , $\Theta \in \Theta_I[\Pi, q]$ and $f(z) \in \mathcal{A}(p)$ satisfies

$$\Theta(\mathcal{I}_{p,\delta}^\lambda f(z), \mathcal{I}_{p,\delta}^{\lambda-1} f(z), \mathcal{I}_{p,\delta}^{\lambda-2} f(z), \mathcal{I}_{p,\delta}^{\lambda-3} f(z); z, \zeta) \subset \Pi, \quad (24)$$

then

$$\mathcal{I}_{p,\delta}^\lambda f(z) \prec q(z).$$

Proof. Let

$$g(z) := \mathcal{I}_{p,\delta}^\lambda f(z). \quad (25)$$

Differentiating (25) with respect to z , and using the identity (3), we obtain

$$\mathcal{I}_{p,\delta}^{\lambda-1} f(z) = \frac{zg'(z) + \delta g(z)}{p + \delta}. \quad (26)$$

Again, by differentiating (26), we have

$$\mathcal{I}_{p,\delta}^{\lambda-2} f(z) = \frac{z^2 g''(z) + (1 + 2\delta)zg'(z) + \delta^2 g(z)}{(p + \delta)^2}. \quad (27)$$

Further computations show that

$$\mathcal{I}_{p,\delta}^{\lambda-3} f(z) = \frac{z^3 g'''(z) + 3(1 + \delta)z^2 g''(z) + (1 + 3\delta + 3\delta^2)zg'(z) + \delta^3 g(z)}{(p + \delta)^3}. \quad (28)$$

Set the transformations from \mathbb{C}^4 to \mathbb{C} by

$$\begin{aligned}\alpha &= r_1, \quad \beta = \frac{s_1 + \delta r_1}{p + \delta}, \quad \gamma = \frac{t_1 + (1 + 2\delta)s_1 + \delta^2 r_1}{(p + \delta)^2}, \\ \eta &= \frac{u_1 + 3(1 + \delta)t_1 + (1 + 3\delta + 3\delta^2)s_1 + \delta^3 r_1}{(p + \delta)^3}.\end{aligned}\quad (29)$$

Let

$$\begin{aligned}\Xi(r_1, s_1, t_1, u_1; z, \zeta) &= \Theta(\alpha, \beta, \gamma, \eta; z, \zeta) \\ &= \Theta\left(r_1, \frac{s_1 + \delta r_1}{p + \delta}, \frac{t_1 + (1 + 2\delta)s_1 + \delta^2 r_1}{(p + \delta)^2}, \frac{u_1 + 3(1 + \delta)t_1 + (1 + 3\delta + 3\delta^2)s_1 + \delta^3 r_1}{(p + \delta)^3}; z, \zeta\right).\end{aligned}\quad (30)$$

Using Equations (25)–(28), and from (30), we obtain

$$\Xi(g(z), zg'(z), z^2 g''(z), z^3 g'''(z); z, \zeta) = \Theta(\mathcal{I}_{p,\delta}^\lambda f(z), \mathcal{I}_{p,\delta}^{\lambda-1} f(z), \mathcal{I}_{p,\delta}^{\lambda-2} f(z), \mathcal{I}_{p,\delta}^{\lambda-3} f(z); z, \zeta).$$

Therefore, the inclusion (24) leads to

$$\Xi(g(z), zg'(z), z^2 g''(z), z^3 g'''(z); z, \zeta) \in \Pi.$$

Now,

$$\frac{t_1}{s_1} + 1 = \frac{(p + \delta)^2 \gamma - \delta^2 \alpha}{(p + \delta) \beta - \delta \alpha} - 2\delta$$

and

$$\frac{u_1}{s_1} = \frac{(p + \delta)^2 (\eta(p + \delta) - 3\gamma(1 + \delta)) + (3 + 2\delta)\delta^2 \alpha}{(p + \delta) \beta - \delta \alpha} + 2 + 3(2 + \delta)\delta.$$

Hence, the admissibility condition in Definition 5 for $\Theta \in \Theta_I[\Pi, q]$ is equivalent to Definition 3. Thus, by use of (23) and applying Theorem 1, we obtain

$$g(z) \prec q(z)$$

or

$$\mathcal{I}_{p,\delta}^\lambda f(z) \prec q(z).$$

□

The hypothesis of Theorem 6 requires that the behavior of q on the boundary is not known.

Corollary 4. Consider q to be univalent in \mathcal{U} , with $q(0) = 0$, and set $q_\rho(z) \equiv q(\rho z)$ for $\rho \in (0, 1)$. Let $\mathcal{I}_{p,\delta}^\lambda f(z) \in \mathcal{H}[0, p]$ for $p \geq 2$ and let $\mathcal{I}_{p,\delta}^\lambda f(z)$ and q_ρ satisfy (23). If Π is a set in \mathbb{C} and $\Theta \in \Theta_I[\Pi, q_\rho]$ and $f(z) \in \mathcal{A}(p)$ fulfill

$$\Theta(\mathcal{I}_{p,\delta}^\lambda f(z), \mathcal{I}_{p,\delta}^{\lambda-1} f(z), \mathcal{I}_{p,\delta}^{\lambda-2} f(z), \mathcal{I}_{p,\delta}^{\lambda-3} f(z); z, \zeta) \subset \Pi,$$

then

$$\mathcal{I}_{p,\delta}^\lambda f(z) \prec q(z).$$

Proof. Proof of the corollary is an immediate consequence of using Theorem 6, and we obtain

$$\mathcal{I}_{p,\delta}^\lambda f(z) \prec q_\rho(z).$$

Since $q_\rho \prec q$, we conclude that

$$\mathcal{I}_{p,\delta}^\lambda f(z) \prec q(z).$$

□

In Definition 5, there are no special conditions on Π . When $\Pi \neq \mathbb{C}$, then there is some conformal mapping h of \mathcal{U} onto Π . Let it be denoted by $\Theta_I[h, q]$. We then obtain the results that are an immediate consequence of Theorem 6 and Corollary 4.

Theorem 7. Consider that $\mathcal{I}_{p,\delta}^\lambda f(z) \in \mathcal{H}[0, p]$ with $p \geq 2$ and $q \in \mathcal{Q}(0)$ and that they satisfy (23). If Π is a set in \mathbb{C} , $\Theta \in \Theta_I[\Pi, q]$, $f(z) \in \mathcal{A}(p)$ and

$$\Theta(\mathcal{I}_{p,\delta}^\lambda f(z), \mathcal{I}_{p,\delta}^{\lambda-1} f(z), \mathcal{I}_{p,\delta}^{\lambda-2} f(z), \mathcal{I}_{p,\delta}^{\lambda-3} f(z); z, \zeta)$$

is analytic in \mathcal{U} , then

$$\Theta(\mathcal{I}_{p,\delta}^\lambda f(z), \mathcal{I}_{p,\delta}^{\lambda-1} f(z), \mathcal{I}_{p,\delta}^{\lambda-2} f(z), \mathcal{I}_{p,\delta}^{\lambda-3} f(z); z, \zeta) \prec\prec h(z)$$

implies

$$\mathcal{I}_{p,\delta}^\lambda f(z) \prec q(z).$$

Corollary 5. Consider q to be univalent in \mathcal{U} , with $q(0) = 0$, and set $q_\rho(z) \equiv q(\rho z)$ for $\rho \in (0, 1)$. Let $\mathcal{I}_{p,\delta}^\lambda f(z) \in \mathcal{H}[0, p]$ for $p \geq 2$ and let $\mathcal{I}_{p,\delta}^\lambda f(z)$ and q_ρ satisfy (23). If Π is a set in \mathbb{C} , $\Theta \in \Theta_I[\Pi, q_\rho]$, $f(z) \in \mathcal{A}(p)$ and

$$\Theta(\mathcal{I}_{p,\delta}^\lambda f(z), \mathcal{I}_{p,\delta}^{\lambda-1} f(z), \mathcal{I}_{p,\delta}^{\lambda-2} f(z), \mathcal{I}_{p,\delta}^{\lambda-3} f(z); z, \zeta)$$

is analytic in \mathcal{U} , then

$$\Theta(\mathcal{I}_{p,\delta}^\lambda f(z), \mathcal{I}_{p,\delta}^{\lambda-1} f(z), \mathcal{I}_{p,\delta}^{\lambda-2} f(z), \mathcal{I}_{p,\delta}^{\lambda-3} f(z); z, \zeta) \prec\prec h(z)$$

implies

$$\mathcal{I}_{p,\delta}^\lambda f(z) \prec q(z).$$

We next indicate the connection between the best dominant and the solution of a strong differential subordination.

Theorem 8. Consider that $\mathcal{I}_{p,\delta}^\lambda f(z) \in \mathcal{H}[0, p]$ with $p \geq 2$, $\Theta : \mathbb{C}^4 \times \mathcal{U} \times \overline{\mathcal{U}} \rightarrow \mathbb{C}$ and that

$$\Theta(\mathcal{I}_{p,\delta}^\lambda f(z), \mathcal{I}_{p,\delta}^{\lambda-1} f(z), \mathcal{I}_{p,\delta}^{\lambda-2} f(z), \mathcal{I}_{p,\delta}^{\lambda-3} f(z); z, \zeta)$$

is analytic in \mathcal{U} . Suppose h is univalent in \mathcal{U} and $q \in \mathcal{Q}(0)$ is a solution of the following differential equation

$$\Theta(\mathcal{I}_{p,\delta}^\lambda f(z), \mathcal{I}_{p,\delta}^{\lambda-1} f(z), \mathcal{I}_{p,\delta}^{\lambda-2} f(z), \mathcal{I}_{p,\delta}^{\lambda-3} f(z); z, \zeta) = h(z) \quad (31)$$

and satisfies (23). If Π is a set in \mathbb{C} , $\Theta \in \Theta_I[h, q]$ and $f(z) \in \mathcal{A}(p)$ fulfills

$$\Theta(\mathcal{I}_{p,\delta}^\lambda f(z), \mathcal{I}_{p,\delta}^{\lambda-1} f(z), \mathcal{I}_{p,\delta}^{\lambda-2} f(z), \mathcal{I}_{p,\delta}^{\lambda-3} f(z); z, \zeta) \prec\prec h(z), \quad (32)$$

then

$$\mathcal{I}_{p,\delta}^\lambda f(z) \prec q(z)$$

and q is the best dominant.

Proof. From Theorem 6, we conclude that q is a dominant of (32). Since q satisfies (31), q is a solution of (32). Thus, q is dominated by all dominants of (32). Therefore, q is the best dominant. \square

Our next outcomes are for the specialized case of q being a disc, where q is given by (14) and the class $\Theta_I[\Pi, M, b]$. Also, we denote the class $\Theta_I[M, b]$, when $\Pi = \Delta$. And

$q(w) = Me^{i\theta}$ with $0 \leq \theta \leq 2\pi$ when $|w| = 1$. Notably, the case $q(z) = Mz$, $M > 0$ denotes the admissible functions class $\Theta_I[\Pi, M]$.

Definition 6. If Π is a set in \mathbb{C} , $M > 0$ and $p \geq 2$. The admissible functions class $\Theta_I[\Pi, M]$ consists of those functions

$$\Theta : \mathbb{C}^4 \times \mathcal{U} \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$$

such that

$$\Theta \left(Me^{i\theta}, \frac{k+\delta}{p+\delta} Me^{i\theta}, \frac{L + ((1+2\delta)k + \delta^2) Me^{i\theta}}{(p+\delta)^2}, \frac{N + 3(1+\delta)L + ((1+3\delta+3\delta^2)k + \delta^3) Me^{i\theta}}{(p+\delta)^2}; z, \zeta \right) \notin \Pi$$

whenever

$$\Re Le^{-i\theta} \geq (k^2 - k)M, \quad \Re Ne^{-i\theta} \geq 0 \\ \text{for } 0 \leq \theta \leq 2\pi \text{ and } k \geq p.$$

Corollary 6. Consider $q(z) = Mz$ and $\mathcal{I}_{p,\delta}^\lambda f(z) \in \mathcal{H}[0, p]$ with $p \geq 2$ to satisfy

$$|z(\mathcal{I}_{p,\delta}^\lambda f(z))'| \leq Mk,$$

when $z \in \mathcal{U}$ and $k \geq p$. If $\Theta \in \Theta_I[\Pi, M]$, $f(z) \in \mathcal{A}(p)$ satisfies

$$\Theta(\mathcal{I}_{p,\delta}^\lambda f(z), \mathcal{I}_{p,\delta}^{\lambda-1} f(z), \mathcal{I}_{p,\delta}^{\lambda-2} f(z), \mathcal{I}_{p,\delta}^{\lambda-3} f(z); z, \zeta) \subset \Pi,$$

then

$$\mathcal{I}_{p,\delta}^\lambda f(z) \prec q(z).$$

Corollary 7. Consider $q(z) = Mz$ and $\mathcal{I}_{p,\delta}^\lambda f(z) \in \mathcal{H}[0, p]$ with $p \geq 2$. If Π is a set in \mathbb{C} and (i) $\Theta \in \Theta_I[\Pi, M]$, $f(z) \in \mathcal{A}(p)$ satisfies

$$\Theta(\mathcal{I}_{p,\delta}^\lambda f(z), \mathcal{I}_{p,\delta}^{\lambda-1} f(z), \mathcal{I}_{p,\delta}^{\lambda-2} f(z), \mathcal{I}_{p,\delta}^{\lambda-3} f(z); z, \zeta) \subset \Pi \implies |p(z)| < M.$$

(ii) If $f(z) \in \mathcal{A}(p)$ and $\Theta \in \Theta_I[M]$, it satisfies

$$|\Theta(\mathcal{I}_{p,\delta}^\lambda f(z), \mathcal{I}_{p,\delta}^{\lambda-1} f(z), \mathcal{I}_{p,\delta}^{\lambda-2} f(z), \mathcal{I}_{p,\delta}^{\lambda-3} f(z); z, \zeta)| < M \implies |p(z)| < M.$$

4. Conclusions

This paper is intended to propose a new line of investigation for third-order strong differential subordination theories using some specific classes of admissible functions. In each theorem, the dominant and the best dominant, respectively, are established, replacing the functions considered as the dominant and the best dominant from the theorems with remarkable functions and using the properties which produce interesting corollaries. Using the operator, strong subordination results are obtained. The third-order strong differential subordination outcomes such as those here may serve as inspiration for future research on this subject, and in the theory of differential subordinations and superordinations of the third and higher orders as well. Here, we only used and explored the third-order strong differential subordinations.

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