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Triangular Norm-Based Elements on Bounded Lattices

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Abstract: In this study, we introduce the notion of the T-irreducible element as a generalization of the notion of the meet-irreducible element in complete lattices. We derive some related properties of these elements and T-prime elements. We prove that T-irreducible elements and T-prime elements are preserved under the isomorphism that is generated by the same t-norm. We discuss the relationship between the sets of T-prime elements and co-atoms under some conditions. We illustrate this discussion with some examples. We also give some characterizations for the sets of T-irreducible elements and T-prime elements on the direct product of lattices. Then, we show that Theorem 2 given by Karaçal and Sağıroğlu is false by giving some counterexamples. We present a necessary and sufficient condition for the mentioned theorem to be correct.

Keywords: lattice; triangular norm; irreducible; prime; direct product

MSC: 03B52; 06B05

1. Introduction

Decompositions into "irreducible parts" play a central role in many structure theories and often the background is a purely order-theoretical one. The concepts of prime, coprime, join-irreducible, meet-irreducible, atom, and co-atom work well, and these types of elements can be used for the characterization of lattices of certain types. For example, a lattice is distributive if and only if every join-irreducible element is coprime. Similarly, a lattice is meet-pseudocomplemented if and only if each atom is coprime. Furthermore, lattices are decomposed into sublattices by these elements in order that properties of the sublattice generally help to derive properties of the original lattice.

Triangular norms (briefly t-norms) are a special class of aggregation operators that are used in a wide range of fields of application: applied mathematics (e.g., probability, statistics, decision theory) and computer sciences (e.g., artificial intelligence, operations research), as well as many applied fields (economics and finance, pattern recognition and image processing, data fusion, multicriteria decision aid, automated reasoning, etc.) [1]. T-norms have become especially popular as models for fuzzy sets intersection. They act as an extended operator of meet that is used as a conjunction in fuzzy logic [2]. They are also applied in probabilistic metric spaces, many-valued logic, non-additive measures and integrals, etc. [3]. Many authors have studied with t-norms on bounded lattices [4–9]. Yılmaz and Kazancı have presented a method for generating t-norms on a finite distributive lattice by means of $\vee$-irreducible elements [10]. Karaçal and Sağıroğlu have introduced the concepts of the T-prime element and T-prime radical element and have studied their properties on complete lattices [11].

This paper is organized as follows. In Section 2, we recall some preliminaries regarding the lattice theory and triangular norms. In Section 3, we define the concept of the T-irreducible element and investigate some algebraic properties of the sets of T-irreducible elements and T-prime elements. We also give some illustrative examples and characterizations of the sets of T-irreducible elements and T-prime elements on direct product of lattices. In Section 4, we point out that Theorem 2 in [11] is not true in general. We give some appropriate counterexamples. We suggest a necessary and sufficient condition that makes Theorem 2 [11] correct.

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Citation: Yılmaz, Ş.; İşçi, R. Triangular Norm-Based Elements on Bounded Lattices. *Axioms* **2024**, *13*, 23. https://doi.org/10.3390/axioms13010023

Academic Editor: Florin Felix Nichita

Received: 11 October 2023
Revised: 10 December 2023
Accepted: 12 December 2023
Published: 29 December 2023

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2. Preliminaries

In this section, we recall some definitions and previous results for the sake of convenience; see [4,12–14].

A lattice is a partially ordered set L where any two elements have a greatest lower bound (also called meet or infimum) denoted by $x \wedge y$, and a least upper bound (also called join or supremum) denoted by $x \vee y$. A bounded lattice is a lattice L in which there exist two elements $0, 1$ such that $0 \leq x \leq 1$ for all $x \in L$. Here, 0 is said to be the least element and 1 is said to be the greatest element. The elements x, y in a lattice L are comparable if $x \leq y$ or $y \leq x$. Otherwise, x and y are incomparable; in notation, $x \parallel y$. A chain is a lattice in which there are no pairs of incomparable elements. Throughout the paper, L denotes a bounded lattice with at least two elements, unless otherwise stated.

Definition 1. *An element x of L is called a co-atom if x is a maximal element of the set $L - \{1\}$. We denote the set of all co-atoms in L by $C(L)$.*

Definition 2. *The length of a finite chain $x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_{n-1} \leq x_n$ in L is defined to be $n - 1$. The length $\ell(L)$ of L is defined as the supremum of the lengths of the chains in L. If $\ell(L)$ is finite, then L is said to be of finite length.*

Definition 3. *Let $(L_1, \leq_1, 0_1, 1_1)$ and $(L_2, \leq_2, 0_2, 1_2)$ be two bounded lattices. The direct product of L_1 and L_2 is the bounded lattice $(L_1 \times L_2, \leq, (0_1, 0_2), (1_1, 1_2))$ with partial order relation $\leq$ defined by $(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq_1 x_2$ in L_1 and $y_1 \leq_2 y_2$ in L_2.*

Definition 4. *A triangular norm (briefly t-norm) is a binary operation on L that is monotone, commutative, associative, and has the neutral element 1.*

Definition 5. *Let T_1, T_2 be two t-norms on L. T_1 is called smaller than T_2 or T_2 is called greater than T_1 if $T_1(x, y) \leq T_2(x, y)$ for all $x, y \in L$. In this case, we write $T_1 \leq T_2$.*

Example 1. *The following are the smallest and the greatest t-norm on L, respectively:*

$$T_W(x, y) = \begin{cases} y, & x = 1 \\ x, & y = 1 \\ 0, & \text{otherwise.} \end{cases}$$

$$T_M(x, y) = x \wedge y.$$

Proposition 1. *Let T_1 and T_2 be t-norms on bounded lattices L_1 and L_2, respectively. Then, the direct product $T_1 \times T_2$ of T_1 and T_2 defined by*

$$(T_1 \times T_2)((x_1, y_1), (x_2, y_2)) = (T_1(x_1, x_2), T_2(y_1, y_2))$$

is a t-norm on the product lattice $L_1 \times L_2$.

Definition 6. *Let L_1 and L_2 be two lattices and $\varphi : L_1 \to L_2$. Then, φ is called a lattice homomorphism if and only if, for all $x, y \in L_1$, $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$ and $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$. If φ is bijective, then φ is called a lattice isomomorphism.*

Definition 7 ([11])**.** *Let L be a complete lattice and T be a t-norm on L. A $p \in L \backslash \{1\}$ is called a $T-$ prime element if and only if it satisfies $T(x, y) \leq p \Rightarrow x \leq p$ or $y \leq p$ for all $x, y \in L$.*

3. Irreducibles and Primes Based on a T-Norm

In this section, our goal is to present a proper definition for the concepts of the T-irreducible element and T-prime elements in bounded lattices and study their algebraic properties. We investigate some features of the sets of T-irreducible elements and T-

prime elements. We show that lattice isomorphism, which is generated by a t-norm T, preserves T-irreducible elements and T-prime elements. We investigate the relationship between the sets of T-prime elements and co-atoms under some conditions. We give some characterizations for the sets of T-irreducible elements and T-prime elements on product lattices. We present some related properties of these sets. Throughout the section, we illustrate our points with some examples.

Definition 8. *Let T be a t-norm on L and $p \in L\backslash\{1\}$.*

(i) *The element p is called $T-$ prime if and only if it satisfies $T(a,b) \leq p \Rightarrow a \leq p$ or $b \leq p$ for all $a, b \in L$.*

(ii) *The element p is called $T-$ irreducible if and only if it satisfies $T(a,b) = p \Rightarrow a = p$ or $b = p$ for all $a, b \in L$.*

The sets of all T-prime elements and all T-irreducible elements of L are denoted by $T_p(L)$ and $T_i(L)$, respectively.

Proposition 2. *Let T be a t-norm on L. Then, $T_p(L) \subseteq T_i(L)$.*

Proof. Let $x \in T_p(L)$ and $x = T(a,b)$, for $a, b \in L$. Then, $T(a,b) \leq x$. Since x is T-prime, we have $a \leq x$ or $b \leq x$. We also know that $x \leq a \wedge b$. So, $x \leq a$ and $x \leq b$. Therefore, $x = a$ or $x = b$. Thus, $x \in T_i(L)$. □

Example 2. *Let $L_1 = \{0,1\}$ and $L_2 = \{0, \frac{1}{2}, 1\}$. Then, the function T on $L_1 \times L_2$ defined by*

$$T(x,y) = T(y,x) = \begin{cases} x, & y = (1,1) \\ (0,1), & x = y = (0,1) \\ (1,0), & x, y \in \left\{(1,0), (1,\frac{1}{2})\right\} \\ (0,0), & \text{otherwise.} \end{cases}$$

is a t-norm on $L_1 \times L_2$ [15]. Then, we obtain that

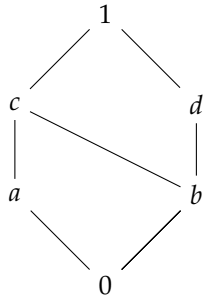
$$T_i(L) = \left\{(0,\frac{1}{2}), (0,1), (1,\frac{1}{2})\right\},$$

$$T_p(L) = \left\{(0,1), (1,\frac{1}{2})\right\}.$$

Proposition 3. *Let T_1, T_2 be t-norms on L and $T_1 \leq T_2$. Then, $T_{1p}(L) \subseteq T_{2p}(L)$.*

Proof. Let $x \in T_{1p}(L)$ and $T_{2p}(a,b) \leq x$ for $a, b \in L$. Then, $T_{1p}(a,b) \leq x$. Since x is T_1-prime, $a \leq x$ or $b \leq x$. Hence, $x \in T_{2p}(L)$. □

Example 3. *Let $L = \{0, a, b, c, d, 1\}$ be the lattice given with the following diagram:*



Consider the t-norms T_1, T_2 on L defined by

$$T_1(x,y) = \begin{cases} x \wedge y, & x = 1 \text{ or } y = 1 \\ x \wedge y \wedge d, & \text{otherwise.} \end{cases}, \quad T_2(x,y) = \begin{cases} d, & x = d, y = d \\ x \wedge y, & x = 1 \text{ or } y = 1 \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $T_2 \leq T_1$. We have that $T_{1_p}(L) = \{a, c\}$, $T_{2_p}(L) = \{c\}$. So, we obtain that $T_{2_p}(L) \subseteq T_{1_p}(L)$.

Remark 1. *Since $T_D \leq T \leq T_M$ for any t-norm T on L, then $T_{Dp}(L) \subseteq T_p(L) \subseteq T_{Mp}(L)$. One can wonder whether or not $T_{1i}(L) \subseteq T_{2i}(L)$ holds, where T_1 and T_2 are two t-norms on L and $T_1 \leq T_2$. We give a negative answer to this question using a counterexample as follows:*

Example 4. *Let $L = [0,1]$. Consider the t-norms T_D, T_M, and T_P, where T_P is given by $T_P(x,y) = x.y$ for all x, y in L. We know that $T_D \leq T_P \leq T_M$. We obtain that $T_{Di}(L) = L - \{0,1\}, T_{Mi}(L) = L - \{1\}$ and $T_{Pi}(L) = \{0\}$. We see that it satisfies $T_{Pi}(L) \subseteq T_{Mi}(L)$ but it holds neither $T_{Di}(L) \subseteq T_{Pi}(L)$ nor $T_{Pi}(L) \subseteq T_{Di}(L)$.*

Let L_1 and L_2 be bounded lattices, $\varphi : L_1 \longrightarrow L_2$ be a lattice isomorphism, and T be a t-norm on L_1. Then, it is straightforward that T_φ defined by $T_\varphi(a,b) = \varphi\big(T\big(\varphi^{-1}(a), \varphi^{-1}(b)\big)\big)$ is a t-norm on L_2.

Theorem 1. *Let L_1 and L_2 be bounded lattices, $\varphi : L_1 \longrightarrow L_2$ be a lattice isomorphism, and T be a t-norm on L_1. Then, the following statements hold:*

(i) If $p \in T_p(L_1)$, then $\varphi(p) \in T_{\varphi_p}(L_2)$.

(ii) If $p \in T_i(L_1)$, then $\varphi(p) \in T_{\varphi_i}(L_2)$.

Proof. (i) Let $a, b \in L_2$ and $T_\varphi(a,b) \leq \varphi(p)$. Then, we have $\varphi\big(T\big(\varphi^{-1}(a), \varphi^{-1}(b)\big)\big) \leq \varphi(p)$. Since φ is a lattice isomorphism, we obtain that $T\big(\varphi^{-1}(a), \varphi^{-1}(b)\big) \leq p$. Because $p \in T_p(L_1)$, we have $\varphi^{-1}(a) \leq p$ or $\varphi^{-1}(b) \leq p$. Thus, $a \leq \varphi(p)$ or $b \leq \varphi(p)$ and, hence, $\varphi(p) \in T_{\varphi_p}(L_2)$.

(ii) Let $a, b \in L_2$ and $T_\varphi(a,b) = \varphi(p)$. Therefore, $\varphi\big(T\big(\varphi^{-1}(a), \varphi^{-1}(b)\big)\big) = \varphi(p)$. Since φ is one-to-one, then $T\big(\varphi^{-1}(a), \varphi^{-1}(b)\big) = p$. Because $p \in T_i(L_1)$, $\varphi^{-1}(a) = p$ or $\varphi^{-1}(b) = p$. Hence, $a = \varphi(p)$ or $b = \varphi(p)$. As a consequence, $\varphi(p) \in T_{\varphi_i}(L_2)$. □

Proposition 4. *Consider the t-norm $T = T_D$ on L. If $C(L) = \emptyset$, then it follows that $T_p(L) = \emptyset$.*

Proof. Suppose that $T_p(L) \neq \emptyset$. Then, there exists an element x in $T_p(L)$. Since $C(L) = \emptyset$, then $x \notin C(L)$ and hence there exists y in L such that $x < y < 1$. We obtain that $T(y,y) = 0 \leq x$ but $y \not\leq x$. Therefore, we obtain that $x \notin T_p(L)$, which is a contradiction. So, $T_p(L) = \emptyset$. □

Lemma 1. *Let L be of finite length and $|C(L)| = 1$. If we denote c as the single co-atom, then $x \leq c$ for any $x \in L \backslash \{1\}$.*

Proof. Let $C(L) = \{c\}$ and $x \neq 1$ in L. Since c is a co-atom, then $c \leq c \vee x$ and thus $c \vee x = c$ or $c \vee x = 1$. Let us denote $X = \{1 \neq x \in L : c \vee x = 1\}$. Assume that X is nonempty. Since L is of finite length, every chain in L is finite and so it has an upper bound. Thus, by Zorn 's lemma, X has a maximal element. Let m be a maximal element of X. Then, for each $t \in L \backslash X$ and $m < t$, we obtain that $t \vee x \geq m \vee x = 1$. If $t \neq 1$, then $t \in X$, which contradicts that m is a maximal element of X. Then, $t = 1$ and hence m is a co-atom in L. Since $m \in X, m \vee c = 1$, we obtain that $m \parallel c$. So, $m \neq c$, which contradicts that $|C(L)| = 1$. So, we obtain that $X = \emptyset$. Then, it is required that $x \vee c = c$, and it is equivalent to $x \leq c$. □

Theorem 2. *Let L be of finite length. Consider the t-norm $T = T_D$ on L. If $|C(L)| = 1$, then $T_p(L) = C(L)$.*

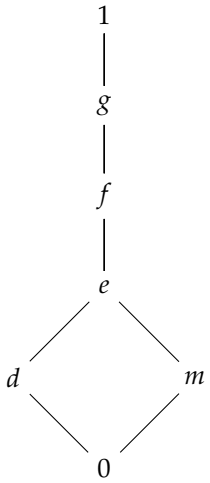
Proof.

(i) If $\ell(L) = 1$, then we have that $L = \{0, 1\}$ and $C(L) = \{0\}$. It is easily seen that $0 \in T_p(L)$. Therefore, $T_p(L) = C(L)$.

(ii) If $\ell(L) \geq 2$, then the set $L \setminus (C(L) \bigcup \{1\})$ is nonempty, and let x be its element. If we denote c as the only element of $C(L)$, from Lemma 1 we have that $x < c$. Since $T(c, c) = 0 \leq x$ but $c \not\leq x$, then $x \notin T_p(L)$. To show that $c \in T_p(L)$, it follows from the fact that $0 \neq T(a, b) \leq c$ for the case $a = 1$ or $b = 1$ and, in such case, the second element is less than or equal to c, and it trivially satisfies the T-prime condition. The case $T(a, b) = 0$ and $a = 1$ or $b = 1$ is similar. Finally, if $T(a, b) = 0$ and $a, b < 1$, then $a, b < c$, which also satisfies the T-prime condition.

□

Example 5. *Let $L = \{0, 1, g, f, e, d, m\}$ be the lattice with the following diagram.*



Consider the t-norm $T = T_D$ on L. It is easily seen that $C(L) = \{g\}$ and $T_p(L) = \{g\}$. Consequently, we have that $C(L) = T_p(L)$.

Proposition 5. *Consider the t-norm $T = T_D$ on L. If $|C(L)| \geq 2$, then $T_p(L) = \emptyset$.*

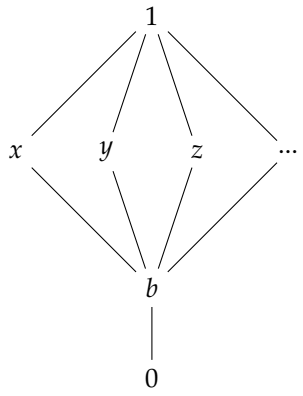
Proof. Let $x \in L \setminus \{1\}$.

(i) Let $x \in L \setminus C(L)$. Then, for $y \in C(L)$, we obtain that $T(y, y) = 0 \leq x$, but $y \not\leq x$. Hence, $x \notin T_p(L)$.

(ii) Let $x \in C(L)$. Since $|C(L)| \geq 2$, then there exist $y \in C(L), y \neq x$. Then, $T(y, y) = 0 \leq x$, but $y \not\leq x$ because $x \parallel y$. Thus, $x \notin T_p(L)$.

Consequently, $T_p(L) = \emptyset$. □

Example 6. *Let $L = \{0, b, x, y, z \dots\}$ be the lattice given by the following diagram:*



It is easily seen that $C(L) = \{x, y, z \dots\}$. Consider the t-norm $T = T_D$ on L. Then, $T(y, z) = 0 \leq x$ but $y \not\leq x$ and $z \not\leq x$. So, $x \notin T_p(L)$. Similarly, we get that $0, x, y, .. \notin T_p(L)$. As a result, we obtain that $T_p(L) = \emptyset$.

The following example shows that the fact that $T_p(L) = \emptyset$ does not imply that $T = T_D$.

Example 7. *Consider the lattice $L = [0, 1]$ and the t-norm $T(x, y) = T_L(x, y) = \max(x + y - 1, 0)$ on L. Note that $C(L) = \emptyset$ and hence $|C(L)| = 0$.*

(i) *Let $T(x, y) = 0$. Then, $x + y - 1 \leq 0$ and hence $x + y \leq 1$. $T\left(\frac{1}{2}, \frac{1}{2}\right) = 0$ but $\frac{1}{2} \not\leq 0$. Hence, $0 \notin T_p(L)$.*

(ii) *Let $T(x, y) \leq \frac{1}{2}$. Then, $\max(x + y - 1, 0) \leq \frac{1}{2}$.*

- $\max(x + y - 1, 0) = 0 \Rightarrow x \leq \frac{1}{2}$ *or* $y \leq \frac{1}{2}$.
- $\max(x + y - 1, 0) = x + y - 1 \leq \frac{1}{2} \Rightarrow x + y \leq \frac{3}{2}$. *Since* $T(\frac{3}{4}, \frac{3}{4}) = \frac{1}{2}$ *but* $\frac{3}{4} \not\leq \frac{1}{2}$, *we have that* $\frac{1}{2} \notin T_p(L)$.

(iii) *Let $T(x, y) = \max(x + y - 1, 0) \leq p$, $p \in (0, \frac{1}{2})$.*

- $\max(x + y - 1, 0) = 0 \Rightarrow x + y \leq 1$. *Since* $T\left(\frac{1}{2}, \frac{1}{2}\right) = 0 \leq p$ *but* $\frac{1}{2} \not\leq p$, *then* $p \notin T_p(L)$.
- $\max(x + y - 1, 0) = x + y - 1 \leq p \Rightarrow x + y \leq p + 1$. *We obtain that* $T\left(\frac{p+1}{2}, \frac{p+1}{2}\right) = p \leq p$ *but* $\frac{p+1}{2} \not\leq p$. *If* $\frac{p+1}{2} \not\leq p$, *then we obtain that* $1 \leq p$, *which is a contradiction. Therefore,* $p \notin T_p(L)$.

(iv) *Let $T(x, y) = \max(x + y - 1, 0) \leq q, q \in (\frac{1}{2}, 1)$.*

- $\max(x + y - 1, 0) = 0 \Rightarrow x + y \leq 1 \Rightarrow x \leq \frac{1}{2}$ *or* $y \leq \frac{1}{2}$.
- $\max(x + y - 1, 0) = x + y - 1 \Rightarrow 0 \leq x + y - 1 \leq q \Rightarrow 1 \leq x + y \leq q + 1$. $T\left(\frac{q+1}{2}, \frac{q+1}{2}\right) = q \leq q$. *But,* $q < 1 \Rightarrow 2q < q + 1 \Rightarrow q < \frac{q+1}{2}$. *So,* $\frac{q+1}{2} \not\leq q$.

Therefore, $q \notin T_p(L)$.

So, $T_p(L) = \emptyset$. By the following, we characterize the sets of T-irreducible elements and T-prime elements on product lattices.

Theorem 3. *Let $(L_1, \leq, 0_{L_1}, 1_{L_1})$ and $(L_2, \leq, 0_{L_2}, 1_{L_2})$ be two bounded lattices and T_1, T_2 be two t-norms on L_1, L_2, respectively. Then, $(T_1 \times T_2)_i(L_1 \times L_2) = \left(T_{1_i}(L_1) \times \{1_{L_2}\}\right) \cup \left(\{1_{L_1}\} \times T_{2_i}(L_2)\right)$.*

Proof. Suppose that $(p, q) \in \left(T_{1_i}(L_1) \times \{1_{L_2}\}\right) \cup \left(\{1_{L_1}\} \times T_{2_i}(L_2)\right)$. Hence, $(p, q) \in T_{1_i}(L_1) \times \{1_{L_2}\}$ or $(p, q) \in \{1_{L_1}\} \times T_{2_i}(L_2)$. Therefore, $(p \in T_{1_i}(L_1)$ and $q = 1_{L_2})$ or $(p = 1_{L_1}$ and $q \in T_{2_i}(L_2))$.

Let $(T_1 \times T_2)((x, y), (a, b)) = (p, q)$. In this case, we have that

(i) $(T_1 \times T_2)((x, y), (a, b)) = (p, 1_{L_2})$ and $p \in T_{1_i}(L_1)$

or

(ii) $(T_1 \times T_2)((x,y),(a,b)) = (1_{L_1}, q)$ and $q \in T_{2_i}(L_2)$.

If $(T_1 \times T_2)((x,y),(a,b)) = (p, 1_{L_2})$ and $p \in T_{1_i}(L_1)$, then $(T_1(x,a), T_2(y,b)) = (p,\ 1_{L_2})$. Hence $T_1(x,a) = p$ and $T_2(y,b) = 1_{L_2}$. Since $p \in T_{1_i}(L_1)$, we obtain that ($x = p$ or $a = p$) and ($y = b = 1_{L_2}$). Then, $(p, 1_{L_2}) = (x,y)$ or $(p, 1_{L_2}) = (a,b)$. Hence, $(p,\ 1_{L_2}) \in (T_1 \times T_2)_i(L_1 \times L_2)$. Similarly, if $(T_1 \times T_2)((x,y),(a,b)) = (1_{L_1}, q)$ and $q \in T_{2_i}(L_2)$, then we obtain that $(1_{L_1}, q) \in (T_1 \times T_2)_i(L_1 \times L_2)$. Therefore,

$$(T_{1_i}(L_1) \times \{1_{L_2}\}) \cup (\{1_{L_1}\} \times T_{2_i}(L_2)) \subseteq (T_1 \times T_2)_i(L_1 \times L_2). \tag{1}$$

Assume that $(p,q) \in (T_1 \times T_2)_i(L_1 \times L_2)$ and $q \notin T_{2_i}(L_2), p \notin T_{1_i}(L_2)$. Then, $q = 1_{L_2}$ or $\exists y, b \in L_2 :\ T_2(y,b) = q$ and $y \neq q, b \neq q$. Similarly, we have $p = 1_{L_1}$ or $\exists x, a \in L_1 :$ $T_1(x,a) = p$ and $x \neq p, a \neq p$. Let $p \neq 1_{L_1}$ and $q \neq 1_{l_2}$.

$$(p,q) = (T_1(x,a), T_2(y,b)) = (T_1 \times T_2)((x,y),(a,b))$$

$$\Rightarrow (p,q) = (x,y) or (p,q) = (a,b)$$

$$\Rightarrow (p = x, q = y)\ or\ (p = a, q = b).$$

That is a contradiction. Therefore, $p \in 1_{L_1} \vee q = 1_{L_2}$.

If $p = 1_{L_1}$, then $(T_1 \times T_2)((x,y),(a,b)) = (1_{L_1}, q)$ and hence $(T_1(x,a), T_2(y,b)) = (1_{L_1}, q)$. So, we have $T_1(x,a) = 1_{L_1}$ and $T_2(y,b) = q$. Thus, $x = a = 1_{L_1}$ and also $y = q$ or $b = q$. Therefore, $q \in T_{2_i}(L_2)$.

Similarly, if $q = 1_{L_2}$, then it follows that $p \in T_{1_i}(L_1)$. Hence, $(p,q) \in T_{1_i}(L_1) \times \{1_{L_2}\} \cup \{1_{L_1}\} \times T_{2_i}(L_2)$. Therefore,

$$(T_1 \times T_2)_i(L_1 \times L_2) \subseteq (T_{1_i}(L_1) \times \{1_{L_2}\}) \cup (\{1_{L_1}\} \times T_{2_i}(L_2)). \tag{2}$$

Then, it follows from (1) and (2) that

$$(T_1 \times T_2)_i(L_1 \times L_2) = (T_{1_i}(L_1) \times \{1_{L_2}\}) \cup (\{1_{L_1}\} \times T_{2_i}(L_2)).$$

□

Theorem 4. *Let* $(L_1, \leq, 0_{L_1}, 1_{L_1})$ *and* $(L_2, \leq, 0_{L_2}, 1_{L_2})$ *be two bounded lattices and* T_1, T_2 *be two t-norms on* L_1, L_2*, respectively. Then,* $(T_1 \times T_2)_p(L_1 \times L_2) = \left(T_{1_p}(L_1) \times \{1_{L_2}\}\right) \cup \left(\{1_{L_1}\} \times T_{2_p}(L_2)\right)$.

Proof. Assume that $(p,q) \in \left(T_{1_p}(L_1) \times \{1_{L_2}\}\right) \cup \left(1_{L_1} \times T_{2_p}(L_2)\right)$. Then,

$(p,q) \in T_{1_p}(L_1) \times \{1_{L_2}\}$ or $(p,q) \in \{1_{L_1}\} \times T_{2_p}(L_2)$. So, ($p \in T_{1_p}(L_1)$ and $q = 1_{L_2}$) or ($p = 1_{L_1}$ and $q \in T_{2_p}(L_2)$).

Let $(T_1 \times T_2)((x,y),(a,b)) \leq (p,q)$. In this case, we have that

(i) $(T_1 \times T_2)((x,y),(a,b)) \leq (p, 1_{L_2})$ and $p \in T_{1_p}(L_1)$

or

(ii) $(T_1 \times T_2)((x,y),(a,b)) = (1_{L_1}, q)$ and $q \in T_{2_p}(L_2)$.

If $(T_1 \times T_2)((x,y),(a,b)) \leq (p, 1_{L_2}), p \in T_{1_p}(L_1)$, then $T_1(x,a) \leq p$ and $T_2(y,b) \leq 1_{L_2}$. Since $p \in T_{1_p}(L_1)$, then $x \leq p\ or\ a \leq p$. Therefore, $(x,y) \leq (p, 1_{L_2})$ or $(a,b) \leq (p, 1_{L_2})$. Consequently, $(p, 1_{L_2}) \in (T_1 \times T_2)_p(L_1 \times L_2)$.

Similarly, if $(T_1 \times T_2)((x,y),(a,b)) = (1_{L_1}, q)$ and $q \in T_{2_p}(L_2)$, we obtain that $(1_{L_1}, q) \in (T_1 \times T_2)_p(L_1 \times L_2)$.

Therefore, we obtain that

$$\left(T_{1_p}(L_1) \times \{1_{L_2}\}\right) \cup \left(\{1_{L_1}\} \times T_{2_p}(L_2)\right) \subseteq (T_1 \times T_2)_p(L_1 \times L_2). \tag{3}$$

Conversely, let $(p,q) \in (T_1 \times T_2)_p(L_1 \times L_2)$ and $p \notin T_{1_p}(L_1), q \notin T_{2_p}(L_2)$. In this case, we obtain that $p = 1_{L_1}$ or there exist x, a in L_1 such that $T_1(x,a) \leq p$ and $x \not\leq p, a \not\leq p$. Similarly, we have that $q = 1_{L_2}$ or there exist y, b in L_2 such that $T_2(y,b) \leq q$ and $y \not\leq q, b \not\leq q$.

Assume that $p \neq 1_{L_1}$ and $q \neq 1_{L_2}$. Then $(p,q) \geq (T_1(x,a), T_2(y,b)) = (T_1 \times T_2)((x,y),(a,b))$. Because $(p,q) \in (T_1 \times T_2)_p(L_1 \times L_2)$, we obtain that $(p,q) \geq (x,y)$ or $(p,q) \geq (a,b)$. So, we obtain that $p \geq x, q \geq y$ or $p \geq a, q \geq b$, which is a contradiction. Consequently, it holds that $p = 1_{L_1}$ or $q = 1_{L_2}$.

(i) If $p = 1_{L_1}$, then $(T_1 \times T_2)((x,y),(a,b)) \leq (1_{L_1}, q)$. Thus, $(T_1(x,a), T_2(y,b)) \leq (1_{L_1}, q)$. So, we have $T_1(x,a) \leq 1_{L_1}$ and $T_2(y,b) \leq q$. Thus, $y \leq q$ or $b \leq q$. Hence, $q \in T_{2_p}(L_2)$.

(ii) Let $q = 1_{L_2}$. Similarly, it is proved that $p \in T_{1_p}(L_1)$.

As a consequence, we have that

$$(T_1 \times T_2)_p(L_1 \times L_2) \subseteq \Big(T_{1_p}(L_1) \times \{1_{L_2}\}\Big) \cup \Big(\{1_{L_1}\} \times T_{2_p}(L_2)\Big). \tag{4}$$

Therefore, it follows from (3) and (4) that

$$(T_1 \times T_2)_p(L_1 \times L_2) = \Big(T_{1_p}(L_1) \times \{1_{L_2}\}\Big) \cup \Big(\{1_{L_1}\} \times T_{2_p}(L_2)\Big).$$

□

Corollary 1. *Let $(L_1, \leq, 0_{L_1}, 1_{L_1})$ and $(L_2, \leq, 0_{L_2}, 1_{L_2})$ be two bounded lattices and T_1, T_2 be two t-norms on L_1, L_2, respectively. If $T_{1_p}(L_1) = T_{1_i}(L_1)$ and $T_{2_p}(L_2) = T_{2_i}(L_2)$, then $(T_1 \times T_2)_p(L_1 \times L_2) = (T_1 \times T_2)_i(L_1 \times L_2)$.*

Proof. This is straightforward from Theorems 3 and 4. □

Corollary 2. *Let $(L_1, \leq, 0_{L_1}, 1_{L_1})$ and $(L_2, \leq, 0_{L_2}, 1_{L_2})$ be two bounded lattices and T_1, T_2 be two t-norms on L_1, L_2, respectively. Then,*

$$\big[T_{1_i}(L_1) \times T_{2_i}(L_2)\big] \cap (T_1 \times T_2)_i(L_1 \times L_2) = \emptyset.$$

Proof. If $T_{1_i}(L_1) \times T_{2_i}(L_2) = \emptyset$ or $(T_1 \times T_2)_i(L_1 \times L_2) = \emptyset$, then the proof is obvious. Assume that $T_{1_i}(L_1) \times T_{2_i}(L_2) \neq \emptyset$ and $(T_1 \times T_2)_i(L_1 \times L_2) \neq \emptyset$.

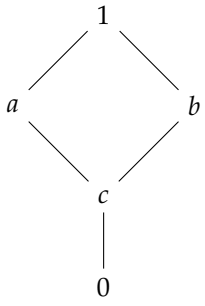
Suppose that $\big[T_{1_i}(L_1) \times T_{2_i}(L_2)\big] \cap (T_1 \times T_2)_i(L_1 \times L_2) \neq \emptyset$ and (p,q) is its element. Then, $(p,q) \in T_{1_i}(L_1) \times T_{2_i}(L_2)$ and $(p,q) \in (T_1 \times T_2)_i(L_1 \times L_2)$. So, we obtain that ($p \in T_{1_i}(L_1)$ and $q \in T_{2_i}(L_2)$) and also we have that ($p = 1_{L_1}$ and $q \in T_{2_i}(L_2)$) or ($p \in T_{1_i}(L_1)$ and $q = 1_{L_2}$). That is a contradiction. Therefore, $\big[T_{1_i}(L_1) \times T_{2_i}(L_2)\big] \cap (T_1 \times T_2)_i(L_1 \times L_2) = \emptyset$. □

Corollary 3. *Let $(L_1, \leq, 0_{L_1}, 1_{L_1})$ and $(L_2, \leq, 0_{L_2}, 1_{L_2})$ be two bounded lattices and T_1, T_2 be two t-norms on L_1, L_2, respectively. Then,*

$$\Big[T_{1_p}(L_1) \times T_{2_p}(L_2)\Big] \cap \Big[(T_1 \times T_2)_p(L_1 \times L_2)\Big] = \emptyset.$$

Proof. The proof follows similarly to the one given in Corollary 2. □

Example 8. *Let $L_1 = \{1, a, b, c, 0\}$ be the lattice given by following Hasse diagram and consider the lattice $L_2 = \{1, g, f, e, d, m, 0\}$ given in Example 5.*



Let us consider the t-norms by the following

$$T_1(x,y) = T_M(x,y),$$

$$T_2(x,y) = \begin{cases} e, & x = f \text{ and } y = f \\ x \wedge y, & \text{otherwise.} \end{cases}$$

on L_1, L_2*, respectively.*

It is easily seen that $T_{1i}(L_1) = \{a, b, 0\} = T_{1p}(L_1)$ *and* $T_{2_i}(L_2) = \{d, m, g, f\} = T_{2_p}(L_2)$. *Hence,*

$$(T_1 \times T_2)_i(L_1 \times L_2) = (T_1 \times T_2)_p(L_1 \times L_2)$$
$$= \{(1,g),(1,f),(1,d),(1,m),(a,u),(b,u),(0,u)\}.$$

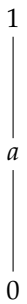
On the other hand,

$$T_{1i}(L_1) \times T_{2_i}(L_2) = T_{1p}(L_1) \times T_{2_p}(L_2)$$
$$= \{(a,d),(a,m),(a,g),(a,f),(b,d),(b,m),(b,g),(b,f),(0,d),(0,m),(0,g),(0,f)\}.$$

Thus,

$$\left[T_{1_i}(L_1) \times T_{2_i}(L_2)\right] \cap \left[(T_1 \times T_2)_i(L_1 \times L_2)\right] = \emptyset,$$
$$\left[T_{1_p}(L_1) \times T_{2_p}(L_2)\right] \cap \left[(T_1 \times T_2)_p(L_1 \times L_2)\right] = \emptyset.$$

Example 9. *Let* $L_1 = \{1, a, 0\}$ *be a chain given by the following diagram and consider the t-norm* $T_1 = T_M$ *on* L_1*:*



Assume that $L_2 = \{u, x, y, z, t, w\}$ *is a chain given by following diagram and consider the t-norm* $T_2 = T_D$ *on* L_2*:*



It is easy to see that

$$T_{1_i}(L_1) = \{a, 0\} = T_{1_p}(L_1),$$
$$T_{2_i}(L_2) = \{x, y, z, t\}, T_{2_p}(L_2) = \{x\}.$$

So,

$$T_{1_i}(L_1) \times T_{2_i}(L_2) = \{(a,x),(a,y),(a,z),(a,t),(0,x),(0,y),(0,z),(0,t)\},$$
$$T_{1_p}(L_1) \times T_{2_p}(L_2) = \{(a,x),(0,x)\}.$$

Moreover,

$$(T_1 \times T_2)_i(L_1 \times L_2) = \{(1,x),(1,y),(1,z),(1,t),(a,u),(0,u)\},$$
$$(T_1 \times T_2)_p(L_1 \times L_2) = \{(1,x),(a,u),(0,u)\}.$$

4. The Lattice $L_R(T)$

Karaçal and Sağıroğlu [11] have constructed a lattice by using T-prime elements in a complete lattice L and they have suggested a new t-norm on the constructed lattice using Theorem 2 [11]. In this section, we show that Theorem 2 in [11] is false by giving some counterexamples. We present a necessary and sufficient condition for the mentioned theorem to be correct.

Definition 9 ([11]). *Let T be a t-norm on a complete lattice L and $x \in L$. If the set $\{p \in T_p(L) \mid p \geq x\}$ is nonempty, then the infimum of this set is called the $T-$ prime radical of x denoted by $R_T(x)$. Since the infimum of the empty set is equal to 1, in the case where $\{p \in T_p(L) \mid p \geq x\} = \emptyset$, then $R_T(x) = 1$. x is called a $T-$ prime radical element if $x = R_T(x)$. The set of all T-prime radical elements in L is denoted by $L_R(T)$. Clearly, $1 \in L_R(T)$.*

Proposition 6 ([11]). *Let T be a t-norm on a complete lattice L. Then, $R_T(T(a,b)) = R_T(a) \wedge R_T(b)$ for every $a, b \in L$.*

Proposition 7 ([11]). *The set $L_R(T)$ is a complete lattice.*

Proposition 8. *Let T_1, T_2 be t-norms on a complete lattice L and $T_1 \leq T_2$. Then, $L_R(T_1)$ is a $\wedge$-subsemilattice of $L_R(T_2)$.*

Proof. First, we show that $L_R(T_1) \subseteq L_R(T_2)$. Let $x \in L_R(T_1)$ be arbitrary. Then,

$$x = \bigwedge \left\{ p \in T_{1p}(L) \;\middle|\; p \geq x \right\} \geq \bigwedge \left\{ p \in T_{2p}(L) \;\middle|\; p \geq x \right\}$$

On the other side, since $x \leq \bigwedge \left\{ p \in T_{2_p}(L) \;\middle|\; p \geq x \right\}$, then $x = \bigwedge \left\{ p \in T_{2_p}(L) \;\middle|\; p \geq x \right\}$. This shows that $x \in L_R(T_2)$. Therefore, $L_R(T_1) \subseteq L_R(T_2)$.

From Proposition 7, we know that $L_R(T_1)$ and $L_R(T_2)$ are complete lattices ($\wedge$-subsemilattices of L (Proposition 5, Remark 2, [11])). The fact that infimums on L, L_1 and L_2 coincide implies that $x \wedge_{L_R(T_1)} y = x \wedge_L y = x \wedge_{L_R(T_2)} y$ for any $x, y \in L_R(T_1)$. It proves that $L_R(T_1)$ is a $\wedge$-subsemilattice of $L_R(T_2)$. □

The notation $T \downarrow L_R(T)$ denotes the restriction of T to $L_R(T)$.

Theorem 5 (Theorem 2, [11])**.** *$T \downarrow L_R(T)$ is an infinitely $\cup$-distributive t-norm on $L_R(T)$. Authors in the proof of (Theorem 2, [11]) claimed the following:*

"... We have that $R_T(T(a,b)) = R_T(a) \wedge R_T(b)$ for every $a, b \in L$ by Proposition 6. Thus for $a, b \in L_R(T)$, $(T(a,b) \in L_R(T)$...."

But, for any $a, b \in L_R(T)$, we obtain that $R_T(T(a,b)) = R_T(a) \wedge R_T(b) = a \wedge b$. Thus, unless $T = T_M$, this need not imply $R_T(T(a,b)) = T(a,b)$. Now, we give the following counterexamples to show that Theorem 4.5 is really false.

Example 10. *Consider the lattice given in Example 5 and define a t-norm T on L as follows [11]:*

$$T(x,y) = \begin{cases} e, & x = f \text{ and } y = f \\ x \wedge y, & \text{otherwise.} \end{cases}$$

Then, $T_p(L) = \{g, f, d, m\}$, $L_R(T) = \{1, g, f, d, m, 0\}$. Since $(T \downarrow L_R(T))(f,f) = T(f,f) = e \notin L_R(T)$, we have that T is not closed on $L_R(T)$. So, it is not a t-norm on $L_R(T)$.

Example 11. *Consider the lattice given in Example 3. Then,*

$$T(x,y) = T_d(x,y) = \begin{cases} T_D(x,y), & (x,y) \neq (d,d) \\ d, & (x,y) = (d,d). \end{cases}$$

is a t-norm on L.

It is easily seen that $T_p(L) = \{c\}$ and $L_R(T) = \{c, 1\}$.

$(T \downarrow L_R(T))(c,c) = T(c,c) = 0 \notin L_R(T)$. Hence, $T \downarrow L_R(T)$ is not a t-norm on $L_R(T)$.

By the following, we provide a characterization for the statement in Theorem 5 (Theorem 2, [11]) to be true.

Theorem 6. *$T \downarrow L_R(T)$ on $L_R(T)$ is a t-norm if and only if $T = \wedge$.*

Proof. **$\Leftarrow$:** Since $L_R(T)$ is a complete lattice, by Proposition 7, for any $a, b \in L_R(T)$, we have that $a \wedge b \in L_R(T)$ and $1 \in L_R(T)$. Thus, if $T = \wedge$, then, for $a, b \in L_R(T)$, $T(a,b) = a \wedge b \in L_R(T)$. Since T is associative, commutative, and monotone in L, it is clear that $T \downarrow L_R(T)$ satisfies these properties in $L_R(T)$. Finally, using $1 \in L_R(T)$, it holds that $T \downarrow L_R(T)$ is a t-norm on $L_R(T)$.

$\Rightarrow$: Assume that $T \downarrow L_R(T)$ is a t-norm on $L_R(T)$. Thus, for $a, b \in L_R(T)$, we have that $T(a,b) \in L_R(T)$. Then, we obtain that $R_T(T(a,b)) = T(a,b)$. So, it follows that $R_T(a) \wedge R_T(b) = T(a,b)$. Therefore, $a \wedge b = T(a,b)$ and hence $T = \wedge$. □

5. Conclusions

We have defined the notion of the T-irreducible element as an extension of the notion of the meet-irreducible element in bounded lattices and have studied its properties. We have investigated the relationship between T-irreducible elements and T-prime elements. We have characterized the sets of T-irreducible elements and T-prime elements in product lattices. We have also presented a correction to a theorem in [11]. It should be interesting to further explore the conditions (depending on the lattice or the t-norm) under which

T-irreducible elements and T-prime elements coincide. For further work, T-irreducible elements may be used to construct new lattices and/or new t-norms on lattices.

Author Contributions: Conceptualization, Ş.Y.; methodology, Ş.Y.; software, Ş.Y.; validation, Ş.Y. and R.İ.; formal analysis, Ş.Y. and R.İ.; investigation, Ş.Y. and R.İ.; resources, Ş.Y. and R.İ.; data curation, Ş.Y.; writing—original draft preparation, Ş.Y.; writing—review and editing, Ş.Y.; visualization, Ş.Y. and R.İ.; supervision, Ş.Y.; project administration, Ş.Y.; funding acquisition, Ş.Y. and R.İ. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: There is no dataset related to this manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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