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Fisher Information, Asymptotic Behavior, and Applications for Generalized Order Statistics and Their Concomitants Based on the Sarmanov Family

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Abstract: In this paper, the Fisher information (FI), relevant to m-generalized order statistics (m-GOSs) and their concomitants of the shape-parameter of the Sarmanov family of bivariate distributions, is investigated. In addition, we study the concomitants of m-GOSs from this family. Furthermore, we look at how those concomitants were distributed collectively. The FI contained in the scale and shape parameters of the exponential and power function distributions, respectively, in concomitants of m-GOSs is obtained. A study of the asymptotic behavior of the concomitants of ordinary order statistics is also provided. Some versatile applications for this study are offered. As a final step, we examined a bivariate real-world data set for illustrative purposes.

Keywords: Sarmanov family; concomitants; generalized order statistics; Fisher information; asymptotic behavior

MSC: 62B10; 62G30

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1. Introduction

The creation of bivariate distributions with specified marginals is one of the pivotal problems in statistical theory and its applications to modeling bivariate data. This is because marginal distributions are typically used as a piece of prior knowledge when modeling bivariate data. Since the creation of the Farlie–Gumbel–Morgenstern (FGM) family in the 1960s, numerous researchers have created and examined a variety of generalizations about it to enhance the correlation between its components (see [1–3]). One of the most adaptable and robust extensions of the traditional FGM family of bivariate distribution functions (DFs) is the Sarmanov family, denoted by SAR(.), which was suggested and used by Sarmanov [4] to describe hydrological phenomena. Recently, the superiority of this family over all known extensions of the FGM family has been revealed in a series of studies, namely, those of [5–8]. The DF and probability density function (PDF) of SAR(α) are given, respectively, by

$$F_{T,Z}(t,z)=F_T(t)F_Z(z)\left[1+3\alpha\overline{F}_T(t)\overline{F}_Z(z)+5\alpha^2(2F_T(t)-1)(2F_Z(z)-1)\overline{F}_T(t)\overline{F}_Z(z)\right]$$

and

$$\begin{aligned} f_{T,Z}(t,z) &= f_T(t)f_Z(z)[1+3\alpha(2F_T(t)-1)(2F_Z(z)-1) \\ &+ \frac{5}{4}\alpha^2(3(2F_T(t)-1)^2-1)(3(2F_Z(z)-1)^2-1)\Big],\ |\alpha|\le\frac{\sqrt{7}}{5}, \end{aligned} \quad (1)$$

where the survival function $\overline{F}(.) = 1 - F(.)$. In situations when the marginals are uniform, the correlation coefficient is α. Due to this, the correlation coefficient ρ for this family's lowest and highest values are -0.529 and 0.529, respectively (see [9]).

Kamps [10] introduced the idea of GOSs as a unifying model for ascendingly ordered random variables (RVs). Burkschat et al. [11] introduced the idea of dual GOSs (DGOSs) to provide a common approach to descendingly ordered RVs such as reversed ordinary order statistics (OOSs) and lower records models. The subclasses m-GOSs and m-DGOSs of GOSs and DGOSs, respectively, contain the most significant models of ordered RVs, including OOSs, lower and higher record values, $k-$records, sequential order statistics (SOSs), and progressive type II censoring with a constant scheme. Burkschatet et al. [11] demonstrated that there is a direct connection between DGOSs and GOSs (cf. Theorem 3.3). As a result, every outcome of the DGOS model has an equivalent outcome to the GOS model. This is why we only take into account the m-GOS model. Let $F(.)$ be any continuous DF of your choosing. Then the RVs $T(1,n,m,k) \leq T(2,n,m,k) \leq \ldots \leq T(n,n,m,k)$ $(k > 0,\ m \geq -1)$ are said to be m-GOSs if their joint PDF (JPDF) is given by

$$f^{(m,k)}_{1,2,\ldots,n:n}(t_1,t_2,\ldots,t_n) = \left(\prod_{j=1}^{n}\gamma_j\right)\left(\prod_{j=1}^{n-1}\overline{F}^{m}(t_j)f(t_j)\right)\overline{F}^{k-1}(t_n)f(t_n),$$

where $F^{-1}(1) \geq t_n \geq \ldots \geq t_1 \geq F^{-1}(0)$ and $\gamma_j = k + (n-j)(m+1) > 0, j = 1,2,\ldots,n$ (note that $\gamma_n = k$). The marginal PDF of the rth m-GOS, $1 \leq r \leq n$, is given by (cf. [10])

$$f_{T(r,n,m,k)}(t) = \frac{C_{r-1}}{(r-1)!}(\overline{F}(t))^{\gamma_r - 1}f(t)g_m^{r-1}(F(t)). \tag{2}$$

Moreover, the JPDF of the rth and sth, $1 \leq r < s \leq n$, m-GOSs is given by

$$\begin{aligned} f_{T(r,n,m,k),T(s,n,m,k)}(t,z) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!}\overline{F}^{m}(t)f(t)g_m^{r-1}(F(t)) \\ &\times\ [h_m(F(z)) - h_m(F(t))]^{s-r-1}\overline{F}^{\gamma_s - 1}(z)f(z),\ z \geq t, \end{aligned} \tag{3}$$

where $C_{r-1} = \prod_{i=1}^{r}\gamma_i, r = 1,2,\ldots,n, g_m(t) = h_m(t) - h_m(0), t \in [0,1)$ and $h_m(t) = \frac{-1}{(m+1)}(1-t)^{m+1}$ if $m \neq -1$, while $h_{-1}(t) = -\log(1-t)$.

Due to its applicability in selection processes and prediction problems, the study of concomitants has recently experienced a resurgence in interest. The concept of the concomitants of OOSs dates back to David [12], while Yang [13] outlined the general theory of concomitants of OOSs. An excellent review of the concomitants of OOSs can be found in [14]. Compared to the concomitants of OOSs, the concomitants of GOSs and DGOSs have not been extensively studied. A few authors, including [15–21] have investigated this topic.

Suppose $(T_i, Z_i), i = 1,2,\ldots,n$, is a random sample from a bivariate DF $F_{T,Z}(t,z)$. The m-GOSs of the Z-variate that are connected with the m-GOSs for the T sample are referred to as concomitants and are denoted by the symbols $Z_{[1,n,m,k]}, Z_{[2,n,m,k]}, \ldots, Z_{[n,n,m,k]}$.

The PDF of the concomitant $Z_{[r,n,m,k]}$ of the rth m-GOS $T(r,n,m,k), 1 \leq r \leq n$, is given by (see [15,16])

$$f_{[r,n,m,k]}(z) = \int_{-\infty}^{\infty} f_{Z|T}(z|t)f_{T(r,n,m,k)}(t)dt, \tag{4}$$

where $f_{T(r,n,m,k)}(t)$ is defined by (2) and $f_{Z|T}(z|t)$ is the conditional PDF of Z given T. Additionally, the JPDF of the concomitants $Z_{[r,n,m,k]}$ and $Z_{[s,n,m,k]}$, $1 \leq r < s \leq n$, is given by (see [15,16])

$$f_{[r,s,n,m,k]}(z_1,z_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{t_2} f_{Z|T}(z_1|t_1)f_{Z|T}(z_2|t_2)f_{T(r,n,m,k),T(s,n,m,k)}(t_1,t_2)dt_1dt_2. \tag{5}$$

An important property of the FI is that it measures how much information can be gleaned from a sample of data produced by a probability distribution regarding unknown parameters. There is a direct link between the FI and the concepts of efficiency and sufficiency. It follows that a sufficient statistic for a family of probability distributions contains all the information about the unknown parameter in the sample, and hence, the sample from which the statistics are generated provides no additional information. Using the asymptotic variance of an asymptotically efficient estimator, we can determine the Cramér–Rao lower limit, which assumes an unbiased estimator of an unknown parameter. According to Cramér–Rao, any unbiased estimator must have at least the same variance as the inverse of the FI. If a sample is large enough, we can use that information to set bounds on the variance of one particular estimate of the unknown parameter as well as an approximate sampling distribution.

Consider an RV T that has a PDF $f(t;\lambda)$, where $\lambda \in \Lambda$ is an unknown parameter with a parameter space Λ. Under certain regularity conditions (cf. [22,23]), the FI of the real parameter $\lambda \in \Lambda$, contained in the RV T, is defined as

$$I_\lambda(T;\lambda) := \mathrm{E}\left(\frac{\partial \log f(T;\lambda)}{\partial \lambda}\right)^2.$$

We also assume that appropriate regularity conditions are satisfied (cf. [24]), that permits the representation

$$I_\lambda(T;\lambda) = -\mathrm{E}\left(\frac{\partial^2 \log f(T;\lambda)}{\partial \lambda^2}\right).$$

The rest of this paper is organized as follows. In Section 2, the FI relevant to m-GOSs and their concomitants of the shape parameter of the Sarmanov family of bivariate distributions is derived. In Section 3, we investigate the concomitant $Z_{[r,n,m,k]}$, $1 \le r \le n$, based on SAR(α) with general marginals. The FI of the shape parameter of concomitants of SOSs and record values based on SAR(α) with power function distribution marginal are obtained in Section 4. Furthermore, the FI of the scale parameter of concomitants of SOSs and record values based on SAR(α) with exponential distribution marginal are obtained in Section 5. In Section 6, the joint DF of the bivariate concomitants of m-GOSs based on SAR(α) is derived. Moreover, in Section 7, we study the asymptotic behavior of the concomitants of the OOSs; as a result, we suggest a new method for estimating the shape parameter α and a simple fitting test for SAR(α). In Section 8, a reliability modeling application of the paper's findings is shown. We examine a bivariate real-world data set for illustrative purposes in Section 9. Finally, Section 10 includes the conclusion of the study.

2. FI of α Based on the Copula of SAR(α)

Let T and Z be uniformly distributed RVs over $(0,1)$, written $T, Z \sim U(0,1)$, and let them be jointly distributed as the Sarmanov family (1). We obtain the Sarmanov copula with the following equation:

$$f_{T,Z}(t,z;\alpha) = 1 + 3\alpha(2t-1)(2z-1) + \frac{5}{4}\alpha^2(3(2t-1)^2-1)(3(2z-1)^2-1),\ 0 \le t,z \le 1,\ |\alpha| \le \frac{\sqrt{7}}{5}. \quad (6)$$

The shape parameter α is the only unknown parameter in the copula (6). In this section, $f_{T,Z}(t,z)$ (together with any corresponding PDFs of concomitants or m-GOSs) is more conveniently written $f_{T,Z}(t,z,\alpha)$. The JPDF of $(T_{r,n,m,k},\ Z_{[r,n,m,k]})$, based on (2), (4) and (6), is given by

$$f_{[r,n,m,k]}(t,z;\alpha) = \frac{C_{r-1}}{(r-1)!} f_{T,Z}(t,z;\alpha)(1-t)^{\gamma_r-1} g_m^{r-1}(t). \quad (7)$$

Remark 1. *The set*

$$\Omega = \{\alpha : \mid 3\alpha\varphi(t)\varphi(z) + \frac{5}{4}\alpha^2(3\varphi^2(t)-1)(3\varphi^2(z)-1) \mid < 1,\ \forall\, 0 \le t,z \le 1\},\ \varphi(t) = 2t-1,$$

is taken into consideration before formulating Theorem 1 about the FI in $(T_{r,n,m,k}, Z_{[r,n,m,k]})$. *From this point forward, we exclusively discuss* $\alpha \in \Omega \cap \Upsilon$, *where* $\Upsilon = \{\alpha : \mid \alpha \mid \le \frac{\sqrt{7}}{5}\}$. *Since* $\alpha = 0 \in \Omega \cap \Upsilon$, *the set* $\Omega \cap \Upsilon$ *is not empty.*

Theorem 1. *Let T and* $Z \sim U(0,1)$ *with JPDF (6). Furthermore, let* $1 \le r \le n$ *and* $\alpha \in \Omega \cap \Upsilon$. *Then, under some regular conditions, the FI in* $(T_{r,n,m,k}, Z_{[r,n,m,k]})$ *of* α *is given by*

$$I_\alpha(T_{r,n,m,k}, Z_{[r,n,m,k]}; \alpha) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}(-1)^i(\alpha)^i(\frac{1}{2})^j[J_1+J_2+J_3], \tag{8}$$

where

$$\begin{aligned} J_1 &= (3)^{i-j+2}(\frac{5\alpha}{2})^j \sum_{l=0}^{j}\sum_{f=0}^{i-j+2l+2}\binom{j}{l}\binom{i-j+2l+2}{f}(-3)^l(-2)^f\prod_{h=1}^{r}\frac{\gamma_h}{\gamma_h+f} \\ &\times \sum_{t=0}^{j}\sum_{p=0}^{i-j+2t+2}\binom{j}{t}\binom{i-j+2t+2}{p}\frac{(-3)^t(-2)^p}{1+p}, \end{aligned}$$

$$\begin{aligned} J_2 &= (3)^{i-j}(\frac{5\alpha}{2})^{j+2} \sum_{v=0}^{j+2}\sum_{u=0}^{i-j+2v}\binom{j+2}{v}\binom{i-j+2v}{u}(-3)^v(-2)^u\prod_{h=1}^{r}\frac{\gamma_h}{\gamma_h+u} \\ &\times \sum_{c=0}^{j+2}\sum_{z=0}^{i-j+2c}\binom{j+2}{c}\binom{i-j+2c}{z}\frac{(-3)^c(-2)^z}{1+z}, \end{aligned}$$

and

$$\begin{aligned} J_3 &= (3)^{i-j+1}(5\alpha)^{j+1}(\frac{1}{2})^j \sum_{a=0}^{j+1}\sum_{b=0}^{i-j+2a+1}\binom{j+1}{a}\binom{i-j+2a+1}{b}(-3)^a(-2)^b\prod_{h=1}^{r}\frac{\gamma_h}{\gamma_h+b} \\ &\times \sum_{d=0}^{j+1}\sum_{g=0}^{i-j+2d+1}\binom{j+1}{d}\binom{i-j+2d+1}{g}\frac{(-3)^d(-2)^g}{1+g}. \end{aligned}$$

Proof. Using the Sarmanov copula (6) and (7), the FI of α is given by

$$\begin{aligned} I_\alpha(T_{r,n,m,k}, Z_{[r,n,m,k]}; \alpha) &= -\mathrm{E}\left(\frac{\partial^2 \log f_{[r,n,m,k]}(T_{r,n,m,k}, Z_{[r,n,m,k]}; \alpha)}{\partial\alpha^2}\right) \\ &= C_{r,n}\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^i\binom{i}{j}(\alpha)^i(\frac{1}{2})^j \\ &\times \int_0^1\int_0^1\left(3\varphi(t)\varphi(z) + \frac{5}{2}\alpha(3\varphi^2(t)-1)(3\varphi^2(z)-1)\right)^2(3\varphi(t)\varphi(z))^{i-j} \\ &\times \left(\frac{5}{2}\alpha(3\varphi^2(t)-1)(3\varphi^2(z)-1)\right)^j(1-t)^{\gamma_r-1}\left(\frac{1-(1-t)^{m+1}}{m+1}\right)^{r-1}dtdz \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^i\binom{i}{j}(\alpha)^i(\frac{1}{2})^j[J_1+J_2+J_3], \end{aligned} \tag{9}$$

where

$$J_1 = \frac{C_{r-1}}{(r-1)!}\int_0^1\int_0^1 (3\varphi(t)\varphi(z))^{i-j+2}(\frac{5}{2}\alpha(3\varphi^2(t)-1)(3\varphi^2(z)-1))^j$$

$$\times(1-t)^{\gamma_r-1}\left(\frac{1-(1-t)^{m+1}}{m+1}\right)^{r-1}dtdz$$

$$= (3)^{i-j+2}(\frac{5\alpha}{2})^j\sum_{l=0}^{j}\sum_{f=0}^{i-j+2l+2}\binom{j}{l}\binom{i-j+2l+2}{f}(-3)^l(-2)^f\prod_{h=1}^{r}\frac{\gamma_h}{\gamma_h+f}$$

$$\sum_{t=0}^{j}\sum_{p=0}^{i-j+2t+2}\binom{j}{t}\binom{i-j+2t+2}{p}\frac{(-3)^t(-2)^p}{1+p}, \tag{10}$$

$$J_2 = \frac{C_{r-1}}{(r-1)!}\int_0^1\int_0^1 (3\varphi(t)\varphi(z))^{i-j}(\frac{5}{2}\alpha(3\varphi^2(t)-1)(3\varphi^2(z)-1))^{j+2}$$

$$\times(1-t)^{\gamma_r-1}\left(\frac{1-(1-t)^{m+1}}{m+1}\right)^{r-1}dtdz$$

$$= (3)^{i-j}(\frac{5\alpha}{2})^{j+2}\sum_{v=0}^{j+2}\sum_{u=0}^{i-j+2v}\binom{j+2}{v}\binom{i-j+2v}{u}(-3)^v(-2)^u\prod_{h=1}^{r}\frac{\gamma_h}{\gamma_h+u}$$

$$\sum_{c=0}^{j+2}\sum_{z=0}^{i-j+2c}\binom{j+2}{c}\binom{i-j+2c}{z}\frac{(-3)^c(-2)^z}{1+z}, \tag{11}$$

and

$$J_3 = \frac{C_{r-1}}{(r-1)!}\int_0^1\int_0^1 (3\varphi(t)\varphi(z))^{i-j+1}(5\alpha(3\varphi^2(t)-1)(3\varphi^2(z)-1))^{j+1}(\frac{1}{2})^j$$

$$\times(1-t)^{\gamma_r-1}\left(\frac{1-(1-t)^{m+1}}{m+1}\right)^{r-1}dtdz$$

$$= (3)^{i-j+1}(5\alpha)^{j+1}(\frac{1}{2})^j\sum_{a=0}^{j+1}\sum_{b=0}^{i-j+2a+1}\binom{j+1}{a}\binom{i-j+2a+1}{b}(-3)^a(-2)^b\prod_{h=1}^{r}\frac{\gamma_h}{\gamma_h+b}$$

$$\sum_{d=0}^{j+1}\sum_{g=0}^{i-j+2d+1}\binom{j+1}{d}\binom{i-j+2d+1}{g}\frac{(-3)^d(-2)^g}{1+g}. \tag{12}$$

Combining (9)–(12), we obtain (8). This completes the proof. □

Record case: By using the proof of Theorem 1 and letting $m=-1, k=1$, we obtain the record values T_n and its concomitant $Z_{[n]}$. Then we have

$$j_1 = \frac{1}{\Gamma(n)}\int_0^1\int_0^1 (3\varphi(t)\varphi(z))^{i-j+2}(\frac{5}{2}\alpha(3\varphi^2(t)-1)(3\varphi^2(z)-1))^j(-\log(1-t))^{n-1}dtdz$$

$$= (3)^{i-j+2}(\frac{5\alpha}{2})^j\sum_{l=0}^{j}\sum_{f=0}^{i-j+2l+2}\binom{j}{l}\binom{i-j+2l+2}{f}\frac{(-3)^l(-2)^f}{(f+1)^n}$$

$$\sum_{t=0}^{j}\sum_{p=0}^{i-j+2t+2}\binom{j}{t}\binom{i-j+2t+2}{p}\frac{(-3)^t(-2)^p}{1+p}, \tag{13}$$

$$j_2 = \frac{1}{\Gamma(n)}\int_0^1\int_0^1 (3\varphi(t)\varphi(z))^{i-j}(\frac{5}{2}\alpha(3\varphi^2(t)-1)(3\varphi^2(z)-1))^{j+2}(-\log(1-t))^{n-1}dtdz$$

$$= (3)^{i-j}(\frac{5\alpha}{2})^{j+2}\sum_{v=0}^{j+2}\sum_{u=0}^{i-j+2v}\binom{j+2}{v}\binom{i-j+2v}{u}\frac{(-3)^v(-2)^u}{(u+1)^n}$$
$$\sum_{c=0}^{j+2}\sum_{z=0}^{i-j+2c}\binom{j+2}{c}\binom{i-j+2c}{z}\frac{(-3)^c(-2)^z}{1+z}, \tag{14}$$

and

$$j_3 = \frac{1}{\Gamma(n)}\int_0^1\int_0^1(3\varphi(t)\varphi(z))^{i-j+1}(5\alpha(3\varphi^2(t)-1)(3\varphi^2(z)-1))^{j+1}(\frac{1}{2})^j(-\log(1-t))^{n-1}dtdz$$
$$= (3)^{i-j+1}(5\alpha)^{j+1}(\frac{1}{2})^j\sum_{a=0}^{j+1}\sum_{b=0}^{i-j+2a+1}\binom{j+1}{a}\binom{i-j+2a+1}{b}\frac{(-3)^a(-2)^b}{(b+1)^n}$$
$$\sum_{d=0}^{j+1}\sum_{g=0}^{i-j+2d+1}\binom{j+1}{d}\binom{i-j+2d+1}{g}\frac{(-3)^d(-2)^g}{1+g}. \tag{15}$$

Tables 1 and 2 display the FI of the parameter α of SOSs (i.e., $k = m = 1$) and record values, respectively, as a function of n, r, and α, for $\alpha = \mp 0.2, \mp 0.15, \mp 0.1, \mp 0.05$, where $\alpha \in \Omega \cap \Upsilon$. The entries are computed using the relations (10)–(12) and (13)–(15). Tables 1 and 2, as well as Figures 1–4, can be used to extrapolate the following intriguing characteristics:

- For $n > 1$, the value of $I_\alpha(T_{r,n,1,1}, Z_{[r,n,1,1]}; \alpha)$ decreases when the difference between r and the sample size n decreases.
- For fixed n and r, the value of $I_\alpha(T_{r,n,1,1}, Z_{[r,n,1,1]}; \alpha)$ is equal to $I_\alpha(T_{r,n,1,1}, Z_{[r,n,1,1]}; -\alpha)$.
- The value of $I_\alpha(T_n, Z_{[n]}; \alpha)$ increases when n increases, and the value of $I_\alpha(T_n, Z_{[n]}; \alpha)$ stabilizes nearly at $n = 18$ (increases by a very small amount with rising n and the increase disappears with about three decimal places or more).
- For fixed n, we have $I_\alpha(T_n, Z_{[n]}; \alpha) = I_\alpha(T_n, Z_{[n]}; -\alpha)$.
- The value of FI increases as $|\alpha|$ increases.

Table 1. FI for $(T_{r,n,1,1}, Z_{[r,n,1,1]})$ of the parameter α.

n	r	α = 0.05	α = −0.05	α = 0.1	α = −0.1	α = 0.15	α = −0.15	α = 0.2	α = −0.2
1	1	1.008	1.008	1.033	1.033	1.077	1.077	1.144	1.144
5	1	2.026	2.026	2.052	2.052	2.100	2.100	2.176	2.176
5	2	1.193	1.193	1.214	1.214	1.250	1.250	1.304	1.304
5	3	0.588	0.588	0.613	0.613	0.657	0.657	0.721	0.721
5	4	0.376	0.376	0.405	0.405	0.454	0.454	0.526	0.526
5	5	1.146	1.146	1.170	1.170	1.213	1.213	1.279	1.279
9	1	2.413	2.413	2.445	2.445	2.504	2.504	2.598	2.598
9	2	1.855	1.855	1.876	1.876	1.913	1.913	1.972	1.972
9	3	1.350	1.350	1.368	1.368	1.399	1.399	1.445	1.445
9	4	0.910	0.910	0.930	0.930	0.964	0.964	1.015	1.015
9	5	0.553	0.553	0.578	0.578	0.620	0.620	0.683	0.683
9	6	0.309	0.309	0.338	0.338	0.389	0.389	0.463	0.463
9	7	0.233	0.233	0.265	0.265	0.318	0.318	0.397	0.397
9	8	0.452	0.452	0.480	0.480	0.527	0.527	0.596	0.596
9	9	1.413	1.413	1.436	1.436	1.478	1.478	1.541	1.541
15	1	2.637	2.637	2.674	2.674	2.745	2.745	2.858	2.858

Table 1. *Cont.*

n	r	α = 0.05	α = −0.05	α = 0.1	α = −0.1	α = 0.15	α = −0.15	α = 0.2	α = −0.2
15	2	2.274	2.274	2.301	2.301	2.350	2.350	2.427	2.427
15	3	1.928	1.928	1.948	1.948	1.983	1.983	2.038	2.038
15	4	1.601	1.601	1.617	1.617	1.646	1.646	1.690	1.690
15	5	1.295	1.295	1.311	1.311	1.338	1.338	1.379	1.379
15	6	1.014	1.014	1.031	1.031	1.061	1.061	1.105	1.105
15	7	0.760	0.760	0.780	0.780	0.815	0.815	0.867	0.867
15	8	0.539	0.539	0.563	0.563	0.604	0.604	0.666	0.666
15	9	0.357	0.357	0.385	0.385	0.433	0.433	0.504	0.504
15	10	0.224	0.224	0.255	0.255	0.309	0.309	0.388	0.388
15	11	0.155	0.155	0.188	0.188	0.245	0.245	0.328	0.328
15	12	0.1731	0.1731	0.206	0.206	0.262	0.262	0.344	0.344
15	13	0.322	0.322	0.352	0.352	0.402	0.402	0.475	0.475
15	14	0.702	0.702	0.725	0.725	0.764	0.764	0.823	0.823
15	15	1.661	1.661	1.683	1.683	1.724	1.724	1.787	1.787

Table 2. FI for $(T_n, Z_{[n]})$ of the parameter α.

n	α = 0.05	α = −0.05	α = 0.1	α = −0.1	α = 0.15	α = −0.15	α = 0.2	α = −0.2
1	1.008	1.008	1.033	1.033	1.077	1.077	1.142	1.142
2	1.342	1.342	1.368	1.368	1.414	1.414	1.483	1.483
3	1.954	1.954	1.983	1.983	2.038	2.038	2.119	2.119
4	2.409	2.409	2.444	2.444	2.511	2.511	2.610	2.610
5	2.687	2.687	2.728	2.728	2.807	2.807	2.922	2.922
6	2.843	2.843	2.888	2.888	2.977	2.977	3.103	3.103
7	2.927	2.927	2.975	2.975	3.069	3.069	3.203	3.203
8	2.970	2.970	3.020	3.020	3.118	3.118	3.257	3.257
9	2.993	2.993	3.044	3.044	3.143	3.143	3.285	3.285
10	3.004	3.004	3.056	3.056	3.156	3.156	3.299	3.299
11	3.010	3.010	3.062	3.062	3.163	3.163	3.306	3.306
12	3.013	3.013	3.065	3.065	3.166	3.166	3.310	3.310
13	3.014	3.014	3.066	3.066	3.168	3.168	3.312	3.312
14	3.015	3.015	3.067	3.067	3.169	3.169	3.313	3.313
15	3.01549	3.01549	3.06759	3.06759	3.16942	3.16942	3.31306	3.31306
16	3.01568	3.01568	3.06778	3.06778	3.16963	3.16963	3.3133	3.3133
17	3.01577	3.01577	3.06788	3.06788	3.16974	3.16974	3.31342	3.31342
18	3.01582	3.01582	3.06793	3.06793	3.16979	3.16979	3.31347	3.31347
19	3.01584	3.01584	3.06795	3.06795	3.16982	3.16982	3.3135	3.3135
20	3.01585	3.01585	3.06797	3.06797	3.16983	3.16983	3.31352	3.31352
21	3.01586	3.01586	3.06797	3.06797	3.16984	3.16984	3.31353	3.31353
22	3.01586	3.01586	3.06798	3.06798	3.16984	3.16984	3.31353	3.31353
23	3.01586	3.01586	3.06798	3.06798	3.16984	3.16984	3.31353	3.31353
24	3.01586	3.01586	3.06798	3.06798	3.16985	3.16985	3.31353	3.31353
25	3.01587	3.01587	3.06798	3.06798	3.16985	3.16985	3.31353	3.31353
26	3.01587	3.01587	3.06798	3.06798	3.16985	3.16985	3.31353	3.31353

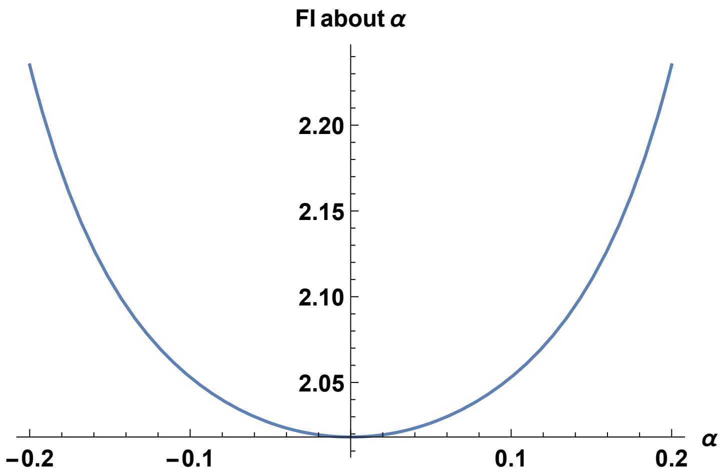

Figure 1. FI in $(T_{r,n,1,1}, Z_{[r,n,1,1]})$ of the parameter α at $r = 1$ and $n = 5$.

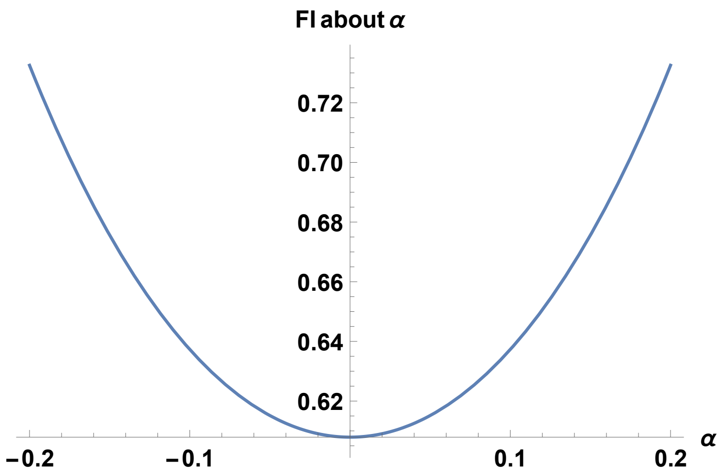

Figure 2. FI in $(T_{r,n,1,1}, Z_{[r,n,1,1]})$ of the parameter α at $r = 10$ and $n = 20$.

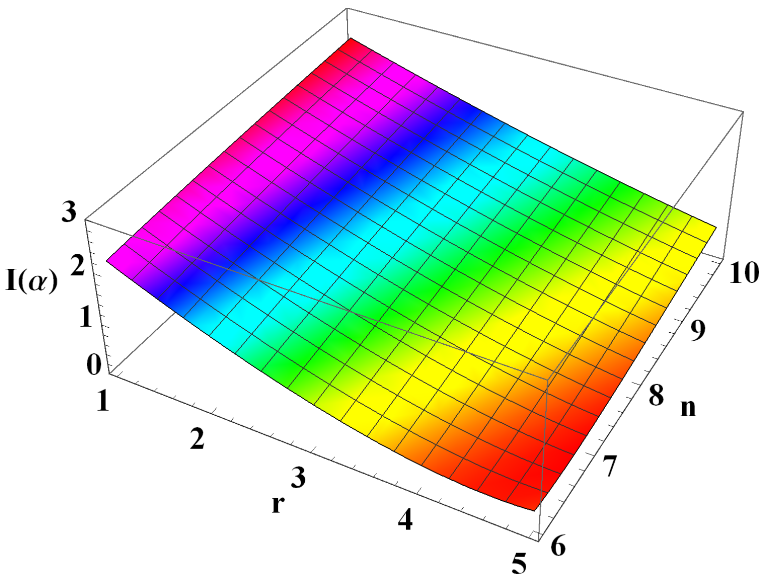

Figure 3. FI in $(T_{r,n,1,1}, Z_{[r,n,1,1]})$ of the parameter α at $\alpha = 0.1$.

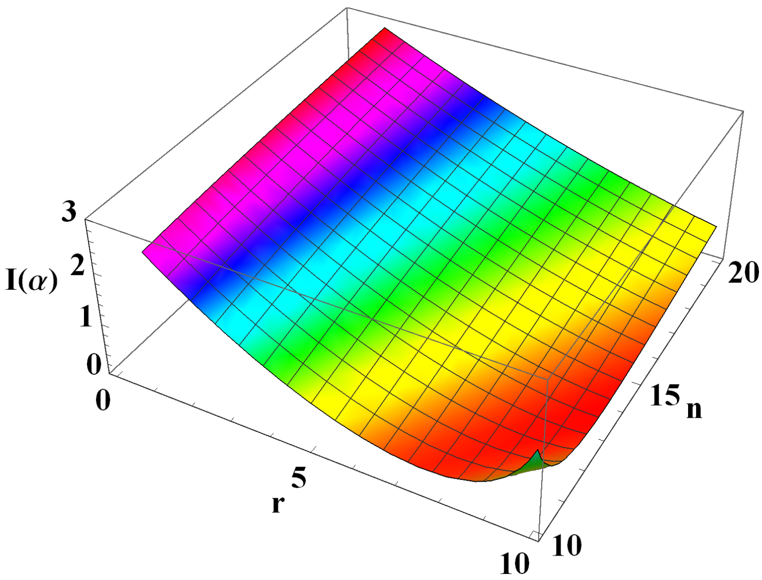

Figure 4. FI in $(T_{r,n,1,1}, Z_{[r,n,1,1]})$ of the parameter α at $\alpha = -0.2$.

3. Distributional Properties of Concomitants of m-GOSs Based on SAR(α)

In this section, the marginal DF, moment-generating function (MGF), and moments of concomitants of m-GOSs for SAR(α) are obtained. Moreover, the joint DF of the bivariate concomitants of m-GOSs based on SAR(α) (defined by (1)) is derived.

3.1. Marginal Distribution of Concomitants of m-GOSs

Based on SAR(α), the following theorem provides a suitable representation for the PDF $f_{[r,n,m,k]}(z)$.

Theorem 2. *Let $m \neq -1$, $V_i \sim F_Z^{i+1}$, $i = 1, 2$. Then*

$$f_{[r,n,m,k]}(z) = \left(1 - 3X_{r,n:1}^{(m,k)} + \frac{5}{2}X_{r,n:2}^{(m,k)}\right) f_Z(z) + \left(3X_{r,n:1}^{(m,k)} - \frac{15}{2}X_{r,n:2}^{(m,k)}\right) f_{V_1}(z) + 5X_{r,n:2}^{(m,k)} f_{V_2}(z), \tag{16}$$

where $X_{r,n:1}^{(m,k)} = \alpha(2I_1^{(m,k)} - 1)$, $X_{r,n:2}^{(m,k)} = \alpha^2\left(12\left(I_2^{(m,k)} - I_1^{(m,k)}\right) + 2\right)$, *and* $I_p^{(m,k)} = \prod_{i=1}^{r} \frac{\gamma_i}{\gamma_i + p}$, $p \in \Re^+$.

Proof. First, for every $p \in \Re^+$, consider the integration

$$\begin{aligned} I_p &= \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} \overline{F}_T^p(t) \overline{F}_T^{\gamma_r - 1}(t) f_T(t) g_m^{r-1}(F_T(t)) dt \\ &= \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} \overline{F}_T^{\gamma_r + p - 1}(t) \left(\frac{1}{m+1}[1 - \overline{F}_T{}^{m+1}(t)]\right)^{r-1} f_T(t) dt. \end{aligned}$$

Taking the transformation $w = \frac{1}{m+1}[1 - \overline{F}_T^{m+1}(t)]$, we obtain

$$I_p = \frac{C_{r-1}}{(r-1)!} \int_0^{\frac{1}{m+1}} w^{r-1} [1 - (m+1)w]^{\frac{\gamma_r + p}{m+1} - 1} dw.$$

Furthermore, by using the transformation $u = (m+1)w$, we obtain

$$
\begin{aligned}
I_p &= \frac{C_{r-1}}{(r-1)!(m+1)^r}\int_0^1 u^{r-1}[1-u]^{\frac{\gamma_r+p}{m+1}-1}du = \frac{C_{r-1}}{(r-1)!(m+1)^r}\beta\left(r,\frac{\gamma_r+p}{m+1}\right)\\
&= \frac{C_{r-1}}{(r-1)!(m+1)^r}\frac{\Gamma(r)\Gamma(\frac{\gamma_r+p}{m+1})}{\Gamma(r+\frac{\gamma_r+p}{m+1})} = \frac{C_{r-1}}{(m+1)^r}\frac{\Gamma(\frac{\gamma_r+p}{m+1})}{(r-1+\frac{\gamma_r+p}{m+1})(r-2+\frac{\gamma_r+p}{m+1})\ldots.(\frac{\gamma_r+p}{m+1})\Gamma(\frac{\gamma_r+p}{m+1})}\\
&= \frac{\gamma_1\gamma_2\ldots.\gamma_r}{(\gamma_1+p)(\gamma_2+p)\ldots.(\gamma_r+p)} = \prod_{i=1}^{r}\frac{\gamma_i}{\gamma_i+p}.
\end{aligned}
$$

Thus

$$
I_p = \frac{C_{r-1}}{(r-1)!}\int_{-\infty}^{\infty}\overline{F}_T^p(t)\overline{F}_T^{\gamma_r-1}(t)f_T(t)g_m^{r-1}(F_T(t))dt = \prod_{i=1}^{r}\frac{\gamma_i}{\gamma_i+p} = I_p^{(m,k)}. \tag{17}
$$

Now, by using (2) and (4), we obtain

$$
\begin{aligned}
f_{[r,n,m,k]}(z) &= \int_{-\infty}^{\infty} f_Z(z)[1+3\alpha(1-2F_T(t))(1-2F_Z(z))+\frac{5}{4}\alpha^2(3(1-2F_T(t))^2-1)\\
&\times\ (3(1-2F_Z(z))^2-1)]\frac{C_{r-1}}{(r-1)!}\overline{F}_T{}^{\gamma_r-1}(t)f_T(t)g_m^{r-1}(F_T(t))dt\\
&= \frac{C_{r-1}}{(r-1)!}\int_{-\infty}^{\infty}[f_Z(z)+3\alpha(2\overline{F}_T(t)-1)(f_{V_1}(z)-f_Z(z))\\
&+\ \frac{5}{4}\alpha^2(3(2\overline{F}_T(t)-1)^2-1)(4f_{V_2}(z)-6f_{V_1}(z)+2f_Z(z))]\overline{F}_T^{\gamma_r-1}(t)f_T(t)g_m^{r-1}(F_T(t))dt.
\end{aligned}
$$

Then, by considering the definition of $X_{r,n:i}^{(m,k)}, i=1,2,$ and incorporating (17), with $p=1,2$, in the above integrations, the required result directly follows. This completes the proof. □

Remark 2. *Husseiny et al. [8] dealt with the issue $m=-1$ and $k=1$, which involves the case of record values.*

Remark 3. *By putting $m=0$ and $k=1$ (which leads to $\gamma_r=n-r+1$), we obtain*

$$
I_p^{(0,1)} = \prod_{i=1}^{r}\frac{\gamma_i}{\gamma_i+p} = \frac{\beta(n-r+1+p,r)}{\beta(r,n-r+1)},\ p=1,2.
$$

Therefore, relation (16) provides the PDF of the concomitant of the rth OOS, $Z_{[r,n,0,1]}$, based on SAR(α), which was revealed by Barakat et al. [6].

Example 1. *Consider the r-out-of-n system, where the distribution of the remaining components' life spans may shift following each component failure. This system, which is known in the literature as an SOS model, is an m-GOS model with $m=k=1$ and $\gamma_i=2(n-i)+1,\ i=1,2,\ldots,n-1$ (cf. [10,25]). Now, consider the case that the generalized exponential (GE) DF, $F(t)=\left(1-e^{-\theta t}\right)^{\lambda}$, $t;\lambda,\theta>0$, as potential marginals of $SAR(\alpha)$, denoted by SAR-GE($\theta_1,\lambda_1;\theta_2,\lambda_2$). With a little basic algebra, we can demonstrate that the PDF of the concomitant of the rth SOS, $Z_{[r,n,1,1]}$, for SAR-GE($\theta_1,\lambda_1;\theta_2,\lambda_2$), is given by*

$$
\begin{aligned}
f_{[r,n,1,1]}(z) &= \lambda_2\theta_2(1-e^{-\theta_2 z})^{\lambda_2-1}e^{-\theta_2 z}\Big\{(1-3X_{r,n:1}^{(1,1)}+\frac{5}{2}X_{r,n:2}^{(1,1)})+2(3X_{r,n:1}^{(1,1)}-\frac{15}{2}X_{r,n:2}^{(1,1)})(1-e^{-\theta_2 z})^{\lambda_2}\\
&+\ 15X_{r,n:2}^{(1,1)}(1-e^{-\theta_2 z})^{2\lambda_2}\Big\},
\end{aligned}
$$

where $X^{(1,1)}_{r,n:1} = \alpha(2I^{(1,1)}_1 - 1)$, $X^{(1,1)}_{r,n:2} = \alpha^2\left(12\left(I^{(1,1)}_2 - I^{(1,1)}_1\right) + 2\right)$, *and* $I^{(1,1)}_p = \frac{\beta(r,n-r+\frac{1+p}{2})}{\beta(r,n-r+\frac{1}{2})}$, $p = 1,2$.

3.2. The MGF and Moments of Concomitants

In this subsection, we derive the MGF and moments of $Z_{[r,n,m,k]}$ based on SAR(α) for any arbitrary marginal. Relying on (16), the MGF of $Z_{[r,n,m,k]}$ based on SAR(α) is given by

$$M_{[r,n,m,k]}(b) = \left(1 - 3X^{(m,k)}_{r,n:1} + \frac{5}{2}X^{(m,k)}_{r,n:2}\right)M_Z(b) + \left(3X^{(m,k)}_{r,n:1} - \frac{15}{2}X^{(m,k)}_{r,n:2}\right)M_{V_1}(b) + 5X^{(m,k)}_{r,n:2}M_{V_2}(b),$$

where $M_Z(b), M_{V_1}(b)$, and $M_{V_2}(b)$ are the MGFs of the RVs Z, V_1, and V_2, respectively. Thus, using (16), the ℓth moment of $Z_{[r,n,m,k]}$ based on SAR(α) is given by

$$\mu^{(\ell)}_{[r,n,m,k]} = (1 - 3X^{(m,k)}_{r,n:1} + \tfrac{5}{2}X^{(m,k)}_{r,n:2})\mu^{(\ell)}_Z + (3X^{(m,k)}_{r,n:1} - \tfrac{15}{2}X^{(m,k)}_{r,n:2})\mu^{(\ell)}_{V_1} + 5X^{(m,k)}_{r,n:2}\mu^{(\ell)}_{V_2}, \quad (18)$$

where $\mu^{(\ell)}_Z = E[Z^\ell]$, $\mu^{(\ell)}_{V_1} = E[{V_1}^\ell]$, and $\mu^{(\ell)}_{V_2} = E[{V_2}^\ell]$. In general, if $h(.)$ is a measurable function (i.e., an RV), then (16) yields

$$\mathrm{E}[h(Z_{[r,n,m,k]})] = \left(1 - 3X^{(m,k)}_{r,n:1} + \frac{5}{2}X^{(m,k)}_{r,n:2}\right)\mathrm{E}[h(Z)] + \left(3X^{(m,k)}_{r,n:1} - \frac{15}{2}X^{(m,k)}_{r,n:2}\right)\mathrm{E}[h(V_1)] + 5X^{(m,k)}_{r,n:2}\mathrm{E}[h(V_2)],$$

provided the expectations exist. Thus, we obtain the following general recurrence relation:

$$\begin{aligned}
\mathrm{E}[h(Z_{[r,n,m,k]})] \quad &- \quad \mathrm{E}[h(Z_{[r-1,n,m,k]})] = -3\left(X^{(m,k)}_{r,n:1} - X^{(m,k)}_{r-1,n:1}\right)(\mathrm{E}[h(Z)] - \mathrm{E}[h(V_1)]) \\
&+ \quad \frac{5}{2}\left(X^{(m,k)}_{r,n:2} - X^{(m,k)}_{r-1,n:2}\right)(\mathrm{E}[h(Z)] - 3\mathrm{E}[h(V_1)] + 2\mathrm{E}[h(V_2)]) \\
&= \quad \frac{30\alpha^2}{\gamma_r}\left[\prod_{i=1}^{r}\frac{\gamma_i}{\gamma_i+1} - 2\prod_{i=1}^{r}\frac{\gamma_i}{\gamma_i+2}\right](\mathrm{E}[h(Z)] - 3\mathrm{E}[h(V_1)] + 2\mathrm{E}[h(V_2)]) \\
&+ \quad \frac{6\alpha}{\gamma_r}\left[\prod_{i=1}^{r}\frac{\gamma_i}{\gamma_i+1}\right](\mathrm{E}[h(Z)] - \mathrm{E}[h(V_1)]).
\end{aligned}$$

4. FI of the Shape Parameter of Power Function Distribution Marginal

In this section, the FI of the shape parameter of concomitants of SOSs and record values based on SAR(α) with power function distribution marginal are obtained. Moreover, numerical studies are conducted to study the behavior of the FI of the shape parameter in each model.

I—SOSs case: Let $f_Z(z) = cz^{c-1}$, $c \geq 10$, $0 \leq z \leq 1$. By using Example 1, we obtain the marginal PDF of the concomitant $Z_{[r,n,1,1]}$ based on the power function distribution as:

$$f_{[r,n,1,1]}(z;\alpha,c) = \left(1 - 3X^{(1,1)}_{r,n:1} + \frac{5}{2}X^{(1,1)}_{r,n:2}\right)cz^{c-1} + \left(3X^{(1,1)}_{r,n:1} - \frac{15}{2}X^{(1,1)}_{r,n:2}\right)(2cz^{2c-1}) + 5X^{(1,1)}_{r,n:2}(3cz^{3c-1}).$$

The last equation, after some algebra, can be rewritten as follows:

$$f_{[r,n,1,1]}(z;\alpha,c) = (cz^{c-1})A_1 + (2cz^{2c-1})A_2 + (3cz^{3c-1})A_3,$$

where $A_1 = \left(1 - 3X^{(1,1)}_{r,n:1} + \frac{5}{2}X^{(1,1)}_{r,n:2}\right)$, $A_2 = \left(3X^{(1,1)}_{r,n:1} - \frac{15}{2}X^{(1,1)}_{r,n:2}\right)$, and $A_3 = 5X^{(1,1)}_{r,n:2}$. Therefore,

$$\frac{\partial \log f_{[r,n,1,1]}(z;\alpha,c)}{\partial c} = \frac{1}{c} + \log z + \frac{A_2z^c \log z + 2A_3z^{2c}\log z}{A_1 + A_2z^c + A_3z^{2c}},$$

which implies

$$\frac{\partial^2 \log f_{[r,n,1,1]}(z;\alpha,c)}{\partial c^2} = \frac{-1}{c^2} + \frac{2A_2 z^c (\log z)^2 + 12A_3 z^{2c} (\log z)^2}{A_1 + 2A_2 z^c + 3A_3 z^{2c}} - \left(\frac{2A_2 z^c \log z + 6A_3 z^{2c} \log z}{A_1 + 2A_2 z^c + 3A_3 z^{2c}}\right)^2. \tag{19}$$

Thus, (19) yields

$$I_c(Z_{[r,n,1,1]};\alpha,c) = -\int_0^1 \frac{\partial^2 \log f_{[r,n,1,1]}(z;\alpha,c)}{\partial c^2} f_{[r,n,1,1]}(z;\alpha,c)dz = l_1 + l_2 + l_3,$$

where

$$l_1 = \frac{1}{c^2}\int_0^1 cz^{c-1}\left(A_1 + 2A_2 z^c + 3A_3 z^{2c}\right)dz = \frac{1}{c^2},$$

$$l_2 = -\int_0^1 cz^{c-1}\left(2A_2 z^c (\log z)^2 + 12A_3 z^{2c} (\log z)^2\right)dz = -\frac{(9A_2 + 16A_3)}{18c^2},$$

and

$$l_3 = \int_0^1 cz^{c-1} \frac{(2A_2 z^c \log z + 6A_3 z^{2c} \log z)^2}{A_1 + 2A_2 z^c + 3A_3 z^{2c}} dz.$$

Therefore, we obtain $I_c(Z_{[r,n,1,1]};\alpha,c) = \frac{1}{c^2}\left(1 - \frac{(9A_2+16A_3)}{18}\right) + l_3$, where l_3 can be evaluated using MATHEMATICA.

II—Record case: Let $f_Z(z) = cz^{c-1}$, $c \geq 1$, $0 \leq z \leq 1$. Using Theorem 2.2 of Husseiny et al. [8], we obtain the marginal PDF of the concomitant $Z_{[n]}$ based on the power function distribution as:

$$f_{[n]}(z;\alpha,c) = \left(1 - 3X^{(\alpha)}_{n:1} + \frac{5}{2}X^{(\alpha)}_{n:2}\right)cz^{c-1} + \left(3X^{(\alpha)}_{n:1} - \frac{15}{2}X^{(\alpha)}_{n:2}\right)(2cz^{2c-1}) + 5X^{(\alpha)}_{n:2}(3cz^{3c-1}),$$

where $X^{(\alpha)}_{n:1} = \alpha(1 - 2^{-(n-1)})$ and $X^{(\alpha)}_{n:2} = \alpha^2(12(3^{-n} - 2^{-n}) + 2)$. The last equation, after some algebra, can be rewritten as follows:

$$f_{[n]}(z;\alpha,c) = (cz^{c-1})A'_1 + (2cz^{2c-1})A'_2 + (3cz^{3c-1})A'_3,$$

where $A'_1 = \left(1 - 3X^{(\alpha)}_{n:1} + \frac{5}{2}X^{(\alpha)}_{n:2}\right)$, $A'_2 = \left(3X^{(\alpha)}_{n:1} - \frac{15}{2}X^{(\alpha)}_{n:2}\right)$, and $A'_3 = 5X^{(\alpha)}_{n:2}$. By using the same proof steps, we obtain

$$I_c(Z_{[n]};\alpha,c) = \frac{1}{c^2}\left(1 - \frac{(9A'_2 + 16A'_3)}{18}\right) + l'_3,$$

where $l'_3 = \int_0^1 cz^{c-1}\frac{(2A'_2 z^c \log z + 6A'_3 z^{2c} \log z)^2}{A'_1 + 2A'_2 z^c + 3A'_3 z^{2c}}dz$ can be evaluated with MATHEMATICA.

Tables 3 and 4 display the FI of the shape parameter c of SOSs and record values, respectively, as a function of n, r, and α. The entries were computed using MATHEMATICA ver. 12. The following interesting features can be extracted from Tables 3 and 4:

- For $n > 1$, the value of $I_c(Z_{[r,n,1,1]};\alpha,c)$ increases when the difference between the rank r and the sample size n decreases, and the value of $I_c(Z_{[r,n,1,1]};-\alpha,c)$ decreases when the difference between the rank r and the sample size n decreases.
- The value of $I_c(Z_{[n]};\alpha,c)$ decreases when n increases, and the value of $I_c(Z_{[n]};-\alpha,c)$ increases when n increases.
- The value of $I_c(Z_{[n]};\alpha,c)$ stabilizes nearly at $n = 15$ (increases by a very small amount with rising n and the increase disappears with about three decimal places or more).
- The value of the FI $I_c(Z_{[n]};\alpha,c)$ decreases as c increases. This fact can be easily checked theoretically and also using MATHEMATICA.

Table 3. FI for $Z_{[r,n,1,1]}$ for $c = 2$.

n	r	$\alpha = 0.2$	$\alpha = -0.2$	$\alpha = 0.3$	$\alpha = -0.3$	$\alpha = 0.4$	$\alpha = -0.4$	$\alpha = 0.52$	$\alpha = -0.52$
1	1	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
3	1	0.207	0.309	0.191	0.346	0.178	0.391	0.169	0.457
3	2	0.236	0.273	0.234	0.287	0.236	0.304	0.249	0.328
3	3	0.282	0.226	0.301	0.218	0.321	0.212	0.349	0.210
5	1	0.198	0.322	0.178	0.372	0.165	0.436	0.168	0.543
5	2	0.215	0.301	0.203	0.332	0.196	0.368	0.195	0.417
5	3	0.233	0.279	0.231	0.297	0.237	0.319	0.260	0.351
5	4	0.256	0.254	0.264	0.261	0.277	0.273	0.305	0.301
5	5	0.294	0.218	0.321	0.207	0.350	0.199	0.391	0.195
7	1	0.195	0.329	0.173	0.385	0.161	0.460	0.183	0.596
7	2	0.206	0.313	0.190	0.353	0.178	0.402	0.171	0.474
7	3	0.218	0.297	0.209	0.326	0.206	0.359	0.215	0.404
7	4	0.232	0.281	0.230	0.301	0.237	0.325	0.266	0.361
7	5	0.247	0.264	0.253	0.277	0.267	0.295	0.302	0.330
7	6	0.267	0.244	0.281	0.247	0.299	0.259	0.330	0.289
7	7	0.301	0.213	0.332	0.200	0.368	0.191	0.418	0.185
15	1	0.189	0.338	0.167	0.404	0.162	0.499	0.242	0.699
15	2	0.194	0.330	0.173	0.386	0.161	0.463	0.185	0.601
15	3	0.191	0.322	0.180	0.371	0.167	0.432	0.167	0.532
15	4	0.205	0.315	0.189	0.356	0.178	0.407	0.172	0.481
15	5	0.211	0.307	0.198	0.343	0.191	0.385	0.191	0.443
15	6	0.217	0.210	0.209	0.331	0.206	0.366	0.219	0.414
15	7	0.223	0.292	0.219	0.319	0.223	0.349	0.249	0.393
15	8	0.230	0.285	0.229	0.308	0.239	0.335	0.278	0.376
15	9	0.237	0.277	0.240	0.297	0.254	0.321	0.301	0.363
15	10	0.245	0.270	0.251	0.286	0.268	0.309	0.317	0.351
15	11	0.253	0.261	0.262	0.274	0.282	0.295	0.330	0.340
15	12	0.262	0.251	0.275	0.260	0.296	0.279	0.340	0.326
15	13	0.273	0.240	0.290	0.244	0.313	0.258	0.353	0.302
15	14	0.288	0.226	0.312	0.222	0.339	0.227	0.378	0.252
15	15	0.314	0.205	0.355	0.188	0.405	0.175	0.480	0.168

Table 4. FI for $Z_{[n]}$ for $c = 2$.

n	$\alpha = 0.2$	$\alpha = -0.2$	$\alpha = 0.3$	$\alpha = -0.3$	$\alpha = 0.4$	$\alpha = -0.4$	$\alpha = 0.52$	$\alpha = -0.52$
1	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
2	0.217	0.292	0.203	0.318	0.190	0.346	0.178	0.386
3	0.201	0.317	0.181	0.362	0.166	0.419	0.163	0.513
4	0.193	0.331	0.171	0.389	0.160	0.471	0.197	0.624
5	0.189	0.338	0.166	0.405	0.162	0.503	0.253	0.712
6	0.187	0.342	0.164	0.414	0.166	0.522	0.306	0.773
7	0.186	0.344	0.163	0.418	0.168	0.532	0.346	0.813
8	0.185037	0.344692	0.162958	0.420282	0.169475	0.537309	0.373865	0.837843
9	0.184766	0.345181	0.162747	0.421449	0.170267	0.540086	0.390449	0.85174
10	0.18463	0.345427	0.162643	0.422039	0.170683	0.541498	0.399759	0.859283
11	0.184562	0.345549	0.162591	0.422335	0.170898	0.542212	0.404733	0.86324
12	0.184527	0.345611	0.162565	0.422483	0.171006	0.54257	0.407312	0.865272
13	0.18451	0.345642	0.162552	0.422558	0.171061	0.542751	0.408628	0.866304
14	0.184501	0.345657	0.162545	0.422595	0.171089	0.542841	0.409293	0.866824

Table 4. *Cont.*

n	$\alpha = 0.2$	$\alpha = -0.2$	$\alpha = 0.3$	$\alpha = -0.3$	$\alpha = 0.4$	$\alpha = -0.4$	$\alpha = 0.52$	$\alpha = -0.52$
15	0.184497	0.345665	0.162542	0.422614	0.171103	0.542886	0.409627	0.867085
16	0.184495	0.345668	0.162541	0.422623	0.17111	0.542909	0.409795	0.867216
17	0.184494	0.34567	0.16254	0.422628	0.171113	0.54292	0.409879	0.867281
18	0.184493	0.345671	0.162539	0.42263	0.171115	0.542926	0.409921	0.867314
19	0.184493	0.345672	0.162539	0.422631	0.171116	0.542928	0.409942	0.86733
20	0.184493	0.345672	0.162539	0.422632	0.171116	0.54293	0.409953	0.867339
21	0.184493	0.345672	0.162539	0.422632	0.171116	0.542931	0.409958	0.867343
22	0.184493	0.345672	0.162539	0.422632	0.171116	0.542931	0.409961	0.867345
23	0.184493	0.345672	0.162539	0.422632	0.171116	0.542931	0.409962	0.867346
24	0.184493	0.345672	0.162539	0.422632	0.171116	0.542931	0.409963	0.867346
25	0.184493	0.345672	0.162539	0.422632	0.171116	0.542931	0.409963	0.867347

5. FI of the Scale Parameter of Exponential Distribution Marginal

In this section, the FI of the scale parameter of concomitants of SOSs and record values based on SAR(α) with exponential distribution marginal are obtained. Moreover, numerical studies are conducted to study the behavior of the FI of the scale parameter in each model.

I—SOSs case: Let $f_Z(z) = \frac{1}{\theta}\exp(\frac{-z}{\theta})$, $z, \theta > 0$. We obtain the marginal PDF of the concomitant $Z_{[r,n,1,1]}$ based on the exponential distribution as:

$$f_{[r,n,1,1]}(z;\alpha,\theta) = \frac{1}{\theta}\exp(\frac{-z}{\theta}) + 3X^{(1,1)}_{r,n:1}\left(\frac{2}{\theta}\exp(\frac{-z}{\theta})\left(1-\exp(\frac{-z}{\theta})\right) - \frac{1}{\theta}\exp(\frac{-z}{\theta})\right)$$

$$+\frac{5}{2}X^{(1,1)}_{r,n:2}\left(\frac{1}{\theta}\exp(\frac{-z}{\theta}) - \frac{6}{\theta}\exp(\frac{-z}{\theta})\left(1-\exp(\frac{-z}{\theta})\right) + \frac{6}{\theta}\exp(\frac{-z}{\theta})(1-\exp(\frac{-z}{\theta}))^2\right).$$

The last equation, after some algebra, can be rewritten as follows:

$$f_{[r,n,1,1]}(z;\alpha,\theta) = \frac{1}{\theta}\exp(\frac{-z}{\theta})\left(D_1 + D_2\exp(\frac{-z}{\theta}) + D_3\exp(\frac{-2z}{\theta})\right).$$

where $D_1 = \left(1 + 3X^{(1,1)}_{r,n:1} + \frac{5}{2}X^{(1,1)}_{r,n:2}\right)$, $D_2 = -\left(6X^{(1,1)}_{r,n:1} + 15X^{(1,1)}_{r,n:2}\right)$, and $D_3 = \left(15X^{(1,1)}_{r,n:2}\right)$. Therefore, using the proof in Subsection 4.1 in Barakat et al. [6], we obtain

$$I_\theta(Z_{[r,n,1,1]};\alpha,\theta) = \int_0^\infty \left(\frac{\partial \log f_{[r,n,1,1]}(z;\alpha,\theta)}{\partial\theta}\right)^2 f_{[r,n,1,1]}(z;\alpha,\theta)\,dz = \frac{1}{\theta^2}\sum_{i=1}^{10} K_i,$$

where $K_1 = 4\left(2D_1 + \frac{D_2}{4} + \frac{2D_3}{27}\right)$, $K_2 = 1$, $K_3 = D_1^2\int_0^\infty \frac{z^2e^{-z}}{D_1+D_2e^{-z}+D_3e^{-2z}}\,dz$, $K_4 = D_3^2\int_0^\infty \frac{z^2e^{-5z}}{D_1+D_2e^{-z}+D_3e^{-2z}}\,dz$, $K_5 = -4\left(D_1 + \frac{D_2}{4} + \frac{D_3}{9}\right)$, $K_6 = -8D_1$, $K_7 = \frac{8D_3}{27}$, $K_8 = 2D_1$, $K_9 = \frac{-2D_3}{9}$, and $K_{10} = -2D_1D_3\int_0^\infty \frac{z^2e^{-3z}}{D_1+D_2e^{-z}+D_3e^{-2z}}\,dz$. Therefore, we obtain

$$I_\theta(Z_{[r,n,1,1]};\alpha,\theta) = \frac{1}{\theta^2}\left(1 - 2D_1 - \frac{2D_3}{27} + K_3 + K_4 + K_{10}\right), \quad (20)$$

where K_3, K_4, and K_{10} can be evaluated with MATHEMATICA.

II—Record case: Let $f_Z(z) = \frac{1}{\theta}\exp(\frac{-z}{\theta})$, $z, \theta > 0$. By using Theorem 2.2 of Husseiny et al. [8], we obtain the marginal PDF of the concomitant $Z_{[n]}$ based on the exponential function distribution as:

$$f_{[n]}(z;\alpha,\theta) = \frac{1}{\theta}\exp(\frac{-z}{\theta}) + 3X^{(\alpha)}_{n:1}\left(\frac{2}{\theta}\exp(\frac{-z}{\theta})\left(1-\exp(\frac{-z}{\theta})\right) - \frac{1}{\theta}\exp(\frac{-z}{\theta})\right)$$

$$+\frac{5}{2}X_{n:2}^{(\alpha)}\left(\frac{1}{\theta}\exp(\frac{-z}{\theta})-\frac{6}{\theta}\exp(\frac{-z}{\theta})\left(1-\exp(\frac{-z}{\theta})\right)+\frac{6}{\theta}\exp(\frac{-z}{\theta})(1-\exp(\frac{-z}{\theta}))^2\right).$$

The last equation, after some algebra, can be rewritten as follows:

$$f_{[n]}(z;\alpha,c)=\frac{D_1'}{\theta}\exp(\frac{-z}{\theta})+\frac{2D_2'}{\theta}\exp(\frac{-z}{\theta})\left(1-\exp(\frac{-z}{\theta})\right)+\frac{3D_3'}{\theta}\exp(\frac{-z}{\theta})\left(1-\exp(\frac{-z}{\theta})\right)^2$$

where $D_1'=\left(1+3X_{n:1}^{(\alpha)}+\frac{5}{2}X_{n:2}^{(\alpha)}\right)$, $D_2'=-\left(6X_{n:1}^{(\alpha)}+15X_{n:2}^{(\alpha)}\right)$, and $D_3'=\left(15X_{n:2}^{(\alpha)}\right)$. By using the same proof steps, we obtain

$$I_\theta(Z_{[n]};\alpha,\theta)=\frac{1}{\theta^2}\left(1-2D_1'-\frac{2D_3'}{27}+K_3'+K_4'+K_{10}'\right),$$

where

$$K_3'=(D_1')^2\int_0^\infty\frac{z^2e^{-z}}{D_1'+D_2'e^{-z}+D_3'e^{-2z}}dz,$$

$$K_4'=(D_3')^2\int_0^\infty\frac{z^2e^{-5z}}{D_1'+D_2'e^{-z}+D_3'e^{-2z}}dz,$$

and

$$K_{10}'=-2D_1'D_3'\int_0^\infty\frac{z^2e^{-3z}}{D_1'+D_2'e^{-z}+D_3'e^{-2z}}dz.$$

The integrations K_3', K_4', and K_{10}' can be evaluated with MATHEMATICA.

Remark 4. *When the value of θ increases, the value of FI of $E(Z)$ decreases.*

The FI $I_\theta(Z_{[r,n,1,1]};\alpha,\theta)$ can be computed using (20) and MATHEMATICA. Tables 5 and 6 provide the values of $I_\theta(Z_{[r,n,1,1]};\alpha,\theta)$ and $I_\theta(Z_{[n]};\alpha,\theta)$ for $\theta=0.5$ and $\theta=1$. From Tables 5 and 6, the following properties can be extracted:

- For $n>1$, the value of $I_\theta(Z_{[r,n,1,1]};\alpha,\theta)$ decreases when the difference between the rank r and the sample size n decreases, and the value of $I_\theta(Z_{[r,n,1,1]};-\alpha,\theta)$ increases when the difference between the rank r and the sample size n decreases.
- The value of $I_\theta(Z_{[n]};\alpha,\theta)$ increases when n increases, and the value of $I_\theta(Z_{[n]};-\alpha,\theta)$ decreases when n increases.
- The value of $I_\theta(Z_{[n]};\alpha,\theta)$ is often stabilized nearly at $n=17$.
- The value of FI $I_\theta(Z_{[n]};\alpha,\theta)$ decreases as θ increases. This fact can be easily checked theoretically and also using MATHEMATICA.

Remark 5. *A fairly high fluctuation of the FI with n is shown in Tables 4 and 6. Rhe FI fluctuates between increasing and decreasing with n. Since there is no sample size n, rather than the rank (or order) of the particular record value, this conclusion is appropriate and not at all unusual for the record value model. At $n=i,j$, where $i>j$, for instance, it is possible that FI in Z_i (or in $Z_{[i]}$) will be less than, bigger than, or equal to FI in Z_j (or in $Z_{[j]}$). For other examples of this kind of fluctuation in record values, see Figure 1 in Amini and Ahmadi [26].*

Remark 6. *The code used for producing Tables 1 and 6 is given in Appendix A for the reader as an example.*

Table 5. FI in $Z_{[r,n,1,1]}$ for exponential distribution.

				$\theta = 0.5$								$\theta = 1$			
n	**r**	$\alpha = 0.2$	$\alpha = -0.2$	$\alpha = 0.3$	$\alpha = -0.3$	$\alpha = 0.4$	$\alpha = -0.4$	**n**	**r**	$\alpha = 0.2$	$\alpha = -0.2$	$\alpha = 0.3$	$\alpha = -0.3$	$\alpha = 0.4$	$\alpha = -0.4$
1	1	4	4	4	4	4	4	1	1	1	1	1	1	1	1
3	1	4.942	3.315	5.544	3.052	6.260	2.846	3	1	1.236	0.829	1.386	0.763	1.565	0.711
3	2	4.363	3.777	4.593	3.740	4.862	3.782	3	2	1.091	0.944	1.148	0.935	1.215	0.946
3	3	3.618	4.512	3.484	4.812	3.395	5.143	3	3	0.904	1.128	0.871	1.203	0.849	1.286
5	1	5.162	3.175	5.957	2.855	6.980	2.634	5	1	1.290	0.794	1.490	0.714	1.745	0.659
5	2	4.811	3.432	5.311	3.246	5.881	3.135	5	2	1.203	0.858	1.328	0.811	1.470	0.784
5	3	4.458	3.729	4.751	3.700	5.098	3.789	5	3	1.114	0.932	1.188	0.925	1.274	0.947
5	4	4.060	4.096	4.172	4.223	4.375	4.438	5	4	1.015	1.024	1.043	1.056	1.094	1.109
5	5	3.487	4.704	3.306	5.130	3.183	5.607	5	5	0.8717	1.176	0.827	1.282	0.796	1.402
7	1	5.261	3.112	6.158	2.772	7.365	2.580	7	1	1.315	0.778	1.540	0.693	1.841	0.645
7	2	5.005	3.292	5.653	3.033	6.429	2.842	7	2	1.251	0.823	1.413	0.758	1.607	0.711
7	3	4.754	3.490	5.218	3.345	5.742	3.299	7	3	1.189	0.872	1.305	0.836	1.436	0.825
7	4	4.501	3.708	4.824	3.685	5.206	3.799	7	4	1.125	0.927	1.206	0.921	1.301	0.950
7	5	4.229	3.959	4.429	4.049	4.724	4.272	7	5	1.057	0.990	1.107	1.0123	1.181	1.068
7	6	3.902	4.276	3.958	4.489	4.138	4.780	7	6	0.975	1.069	0.990	1.122	1.034	1.195
7	7	3.413	4.814	3.202	5.316	3.051	5.890	7	7	0.8533	1.204	0.801	1.329	0.763	1.472
15	1	5.401	3.027	6.459	2.671	7.989	2.590	15	1	1.350	0.757	1.615	0.668	1.997	0.648
15	2	5.275	3.108	6.183	2.768	7.407	2.581	15	2	1.319	0.777	1.546	0.692	1.852	0.645
15	3	5.152	3.193	5.932	2.888	6.918	2.674	15	3	1.288	0.798	1.483	0.722	1.730	0.668
15	4	5.032	3.282	5.701	3.024	6.504	2.841	15	4	1.258	0.820	1.425	0.756	1.626	0.710
15	5	4.914	3.375	5.488	3.174	6.152	3.057	15	5	1.229	0.844	1.372	0.793	1.538	0.764
15	6	4.797	3.472	5.290	3.332	5.851	3.304	15	6	1.199	0.868	1.323	0.833	1.463	0.826
15	7	4.680	3.574	5.103	3.496	5.589	3.562	15	7	1.170	0.893	1.276	0.874	1.397	0.890
15	8	4.561	3.680	4.924	3.665	5.356	3.818	15	8	1.140	0.920	1.231	0.916	1.339	0.955
15	9	4.439	3.793	4.747	3.837	5.143	4.063	15	9	1.101	0.948	1.187	0.959	1.286	1.016
15	10	4.312	3.914	4.568	4.014	4.937	4.295	15	10	1.078	0.978	1.142	1.003	1.234	1.074
15	11	4.175	4.046	4.378	4.191	4.721	4.516	15	11	1.0438	1.011	1.095	1.050	1.180	1.129
15	12	4.023	4.195	4.165	4.404	4.469	4.744	15	12	1.006	1.0488	1.0412	1.101	1.117	1.186
15	13	3.845	4.373	3.905	4.648	4.135	5.0131	15	13	0.961	1.093	0.976	1.162	1.034	1.253
15	14	3.620	4.609	3.556	4.989	3.626	5.426	15	14	0.905	1.152	0.889	1.247	0.907	1.357
15	15	3.277	5.0165	3.005	5.676	2.799	6.474	15	15	0.819	1.254	0.751	1.419	0.691	1.619

Table 6. FI in $Z_{[n]}$ for exponential distribution.

			$\theta = 0.5$							$\theta = 1$			
n	$\alpha = 0.2$	$\alpha = -0.2$	$\alpha = 0.3$	$\alpha = -0.3$	$\alpha = 0.4$	$\alpha = -0.4$	n	$\alpha = 0.2$	$\alpha = -0.2$	$\alpha = 0.3$	$\alpha = -0.3$	$\alpha = 0.4$	$\alpha = -0.4$
1	4	4	4	4	4	4	1	1	1	1	1	1	1
2	4.677	3.464	5.082	3.241	5.542	3.045	2	1.169	0.866	1.270	0.810	1.386	0.761
3	5.072	3.212	5.793	2.894	6.709	2.655	3	1.268	0.803	1.448	0.724	1.677	0.664
4	5.291	3.084	6.230	2.732	7.529	2.566	4	1.323	0.771	1.557	0.683	1.882	0.641
5	5.408	3.019	6.480	2.661	8.047	2.599	5	1.352	0.755	1.620	0.665	2.012	0.650
6	5.469	2.986	6.616	2.629	8.347	2.650	6	1.367	0.747	1.654	0.657	2.087	0.663
7	5.499	2.969	6.688	2.614	8.511	2.681	7	1.375	0.742	1.672	0.654	2.128	0.672
8	5.515	2.961	6.725	2.607	8.597	2.712	8	1.379	0.740	1.681	0.652	2.149	0.678
9	5.523	2.956	6.743	2.604	8.641	2.724	9	1.381	0.739	1.686	0.651	2.160	0.681
10	5.527	2.954	6.753	2.602	8.664	2.730	10	1.382	0.739	1.688	0.651	2.166	0.683
11	5.529	2.953	6.757	2.601	8.675	2.734	11	1.382	0.738	1.689	0.650	2.169	0.684
12	5.530	2.952	6.751	2.601	8.681	2.736	12	1.382	0.738	1.690	0.650	2.170	0.684
13	5.53027	2.95216	6.76093	2.60083	8.68401	2.73698	13	1.38257	0.73804	1.69023	0.650208	2.171	0.684245
14	5.53051	2.95202	6.76152	2.60073	8.68545	2.73742	14	1.38263	0.738005	1.69038	0.650182	2.17136	0.684355
15	5.53063	2.95195	6.76182	2.60067	8.68618	2.73764	15	1.38266	0.737988	1.69046	0.650169	2.17154	0.68441
16	5.5307	2.95192	6.76197	2.60065	8.68654	2.73775	16	1.38267	0.737979	1.69049	0.650162	2.17163	0.684438
17	5.53073	2.9519	6.76204	2.60064	8.68672	2.73781	17	1.38268	0.737975	1.69051	0.650159	2.17168	0.684452
18	5.53074	2.95189	6.76208	2.60063	8.68681	2.73784	18	1.38269	0.737973	1.69052	0.650157	2.1717	0.684459
19	5.53075	2.95189	6.7621	2.60063	8.68685	2.73785	19	1.38269	0.737972	1.69053	0.650156	2.17171	0.684462
20	5.53075	2.95189	6.76211	2.60062	8.68688	2.73786	20	1.38269	0.737971	1.69053	0.650156	2.17172	0.684464
21	5.53076	2.95188	6.76211	2.60062	8.68689	2.73786	21	1.38269	0.737971	1.69053	0.650156	2.17172	0.684465
22	5.53076	2.95188	6.76212	2.60062	8.68689	2.73786	22	1.38269	0.737971	1.69053	0.650156	2.17172	0.684465
23	5.53076	2.95188	6.76212	2.60062	8.68689	2.73786	23	1.38269	0.737971	1.69053	0.650156	2.17172	0.684466
24	5.53076	2.95188	6.76212	2.60062	8.68689	2.73786	24	1.38269	0.737971	1.69053	0.650156	2.17172	0.684466
25	5.53076	2.95188	6.76212	2.60062	8.68689	2.73786	25	1.38269	0.737971	1.69053	0.650156	2.17172	0.684466
26	5.53076	2.95188	6.76212	2.60062	8.68689	2.73786	26	1.38269	0.737971	1.69053	0.650156	2.17172	0.684466
27	5.53076	2.95188	6.76212	2.60062	8.68689	2.73786	27	1.38269	0.737971	1.69053	0.650156	2.17172	0.684466
28	5.53076	2.95188	6.76212	2.60062	8.68689	2.73786	28	1.38269	0.737971	1.69053	0.650156	2.17172	0.684466
29	5.53076	2.95188	6.76212	2.60062	8.68689	2.73786	29	1.38269	0.737971	1.69053	0.650156	2.17172	0.684466
30	5.53076	2.95188	6.76212	2.60062	8.68689	2.73786	30	1.38269	0.737971	1.69053	0.650156	2.17173	0.684466

6. Joint Distribution of Concomitants of m-GOSs Based on SAR(α)

The following theorem gives the JPDF $f_{[r,s,n,m,k]}(z_1, z_2)$ (defined by (5)) of the concomitants $Z_{[r,n,m,k]}$ and $Z_{[s,n,m,k]}$, $r < s$, based on SAR(α).

Theorem 3. *Let $m \neq -1$, $V_i \sim F_Z^{i+1}$, $i = 1, 2$. Then,*

$$\begin{aligned}
f_{[r,s,n,m,k]}(z_1, z_2) = {} & f_Z(z_1) f_Z(z_2) + \left(3X_{r,n:1}^{(m,k)} - \frac{5}{2} X_{r,n:2}^{(m,k)}\right) f_Z(z_2)(f_{V_1}(z_1) - f_Z(z_1)) + \left(3X_{s,n:1}^{(m,k)}\right. \\
& \left. - \frac{5}{2} X_{s,n:2}^{(m,k)}\right) f_Z(z_1)(f_{V_1}(z_2) - f_Z(z_2)) + 5X_{r,n:2}^{(m,k)} f_Z(z_2)(f_{V_2}(z_1) - f_{V_1}(z_1)) \\
& + 5X_{s,n:2}^{(m,k)} f_Z(z_1)(f_{V_2}(z_2) - f_{V_1}(z_2)) + \left(9X_{r,s,n:1}^{(m,k)} - \frac{15}{2} X_{r,s,n:2}^{(m,k)} - \frac{15}{2} X_{r,s,n:3}^{(m,k)}\right. \\
& \left. + \frac{25}{4} X_{r,s,n:4}^{(m,k)}\right)(f_{V_1}(z_1) - f_Z(z_1))(f_{V_1}(z_2) - f_Z(z_2)) + \left(15X_{r,s,n:2}^{(m,k)} - \frac{25}{2} X_{r,s,n:4}^{(m,k)}\right) \\
& \times (f_{V_2}(z_1) - f_{V_1}(z_1))(f_{V_1}(z_2) - f_Z(z_2)) + \left(15X_{r,s,n:3}^{(m,k)} - \frac{25}{2} X_{r,s,n:4}^{(m,k)}\right)(f_{V_1}(z_1) \\
& - f_Z(z_1))(f_{V_2}(z_2) - f_{V_1}(z_2)) + 25X_{r,s,n:4}^{(m,k)}(f_{V_2}(z_1) - f_{V_1}(z_1))(f_{V_2}(z_2) - f_{V_1}(z_2)),
\end{aligned}$$

where $X_{s,n:1}^{(m,k)}$ and $X_{s,n:2}^{(m,k)}$ are defined by replacing r with s in $X_{r,n:1}^{(m,k)}$ and $X_{r,n:2}^{(m,k)}$, respectively,

$$X_{r,s,n:1}^{(m,k)} = \alpha^2 \left(\left(4I_{1,1}^{(m,k)} - 1\right) + X_{r,n:1}^{(m,k)} + X_{s,n:1}^{(m,k)}\right),$$

$$X_{r,s,n:2}^{(m,k)} = \alpha^3 \left(12(I_{2,0}^{(m,k)} - 2I_{2,1}^{(m,k)}) + 10(2I_{1,1}^{(m,k)} - I_{1,0}^{(m,k)}) + X_{s,n:1}^{(m,k)} + X_{r,s,n:1}^{(m,k)}\right),$$

$$X_{r,s,n:3}^{(m,k)} = \alpha^3 \left(12(I_{0,2}^{(m,k)} - 2I_{1,2}^{(m,k)}) + 10(2I_{1,1}^{(m,k)} - I_{0,1}^{(m,k)}) + X_{r,n:1}^{(m,k)} + X_{r,s,n:1}^{(m,k)}\right),$$

$$X_{r,s,n:4}^{(m,k)} = \alpha^4 \left(48(3I_{2,2}^{(m,k)} - I_{1,2}^{(m,k)} - I_{2,1}^{(m,k)} + I_{1,1}^{(m,k)}) + 2(X_{r,s,n:2}^{(m,k)} + X_{r,s,n:3}^{(m,k)}) - 4(X_{r,n:1}^{(m,k)} + X_{s,n:1}^{(m,k)} - 1)\right),$$

and

$$I_{p,q}^{(m,k)} = \prod_{i=1}^{r} \frac{\gamma_i}{\gamma_i + p + q} \prod_{i=r+1}^{s} \frac{\gamma_i}{\gamma_i + q}, \ p, q \in \Re.$$

Proof. Using (3) and (5), we obtain

$$\begin{aligned}
f_{[r,s,n,m,k]}(z_1, z_2) = {} & \int_{-\infty}^{\infty} \int_{-\infty}^{t_2} f_Z(z_1)[1 + 3\alpha(1 - 2F_T(t_1))(1 - 2F_Z(z_1)) + \frac{5}{4}\alpha^2(3(1 - 2F_T(t_1))^2 - 1) \\
\times \ & (3(1 - 2F_Z(z_1))^2 - 1)] f_Z(z_2)[1 + 3\alpha(1 - 2F_T(t_2))(1 - 2F_Z(z_2)) \\
+ \ & \frac{5}{4}\alpha^2(3(1 - 2F_T(t_2))^2 - 1)(3(1 - 2F_Z(z_2))^2 - 1)] \frac{C_{s-1}}{(r-1)!(s-r-1)!} \overline{F}_T^m(t_1) \\
\times \ & \left(\frac{1 - \overline{F}_T^{m+1}(t_1)}{m+1}\right)^{r-1} \left(\frac{\overline{F}_T^{m+1}(t_1) - \overline{F}_T^{m+1}(t_2)}{m+1}\right)^{s-r-1} \overline{F}_T^{\gamma_s - 1}(t_2) f_T(t_1) f_T(t_2) dt_1 dt_2.
\end{aligned}$$

By using the relations $f_{V_1} = 2f_Z F_Z$ and $f_{V_2} = 3f_Z F_Z^2$ and carrying out some algebra, we obtain

$$
\begin{aligned}
f_{[r,s,n,m,k]}(z_1,z_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{t_2} [f_Z(z_1)f_Z(z_2) + 3\alpha f_Z(z_1)f_Z(z_2)(1-2F_T(t_2))(1-2F_Z(z_2)) \\
&+ \frac{5}{4}\alpha^2 f_Z(z_1)f_Z(z_2)(3(1-2F_T(t_2))^2-1)(3(1-2F_Z(z_2))^2-1) \\
&+ 3\alpha f_Z(z_1)f_Z(z_2)(1-2F_T(t_1))(1-2F_Z(z_1)) + 9\alpha^2 f_Z(z_1)f_Z(z_2)(1-2F_T(t_1)) \\
&\times (1-2F_Z(z_1))(1-2F_T(t_2))(1-2F_Z(z_2)) + \frac{15}{4}\alpha^3 f_Z(z_1)f_Z(z_2)(1-2F_T(t_1)) \\
&\times (1-2F_Z(z_1))(3(1-2F_T(t_2))^2-1)(3(1-2F_Z(z_2))^2-1) + \frac{5}{4}\alpha^2 f_Z(z_1)f_Z(z_2) \\
&\times (3(1-2F_T(t_1))^2-1)(3(1-2F_Z(z_1))^2-1) + \frac{15}{4}\alpha^3 f_Z(z_1)f_Z(z_2) \\
&\times (3(1-2F_T(t_1))^2-1)(3(1-2F_Z(z_1))^2-1)(1-2F_T(t_2))(1-2F_Z(z_2)) \\
&+ \frac{25}{4}\alpha^4 f_Z(z_1)f_Z(z_2)(3(1-2F_T(t_1))^2-1)(3(1-2F_Z(z_1))^2-1) \\
&\times (3(1-2F_T(t_2))^2-1)(3(1-2F_Z(z_2))^2-1)]\frac{C_{s-1}}{(r-1)!(s-r-1)!}\overline{F}_T^m(t_1) \\
&\times \left(\frac{1-\overline{F}_T^{m+1}(t_1)}{m+1}\right)^{r-1}\left(\frac{\overline{F}_T^{m+1}(t_1)-\overline{F}_T^{m+1}(t_2)}{m+1}\right)^{s-r-1}\overline{F}_T^{\gamma_s-1}(t_2)f_T(t_1)f_T(t_2)dt_1dt_2.
\end{aligned}
$$

However, using algebra with $p=1, q=0$ for $t=1$ and $p=0, q=1$, for $t=2$ and utilizing Lemma 1 [16], we obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{t_2} \alpha(2\overline{F}_T(t_b)-1)\frac{C_{s-1}}{(r-1)!(s-r-1)!}\overline{F}_T^m(t_1)\left(\frac{1-\overline{F}_T^{m+1}(t_1)}{m+1}\right)^{r-1} \\
&\times\left(\frac{\overline{F}_T^{m+1}(t_1)-\overline{F}_T^{m+1}(t_2)}{m+1}\right)^{s-r-1}\overline{F}_T^{\gamma_s-1}(t_2)f_T(t_1)f_T(t_2)dt_1dt_2 \\
&= \begin{cases} \alpha\left(2I_{1,0}^{(m,k)}-1\right) = \alpha\left(2\prod_{i=1}^{r}\frac{\gamma_i}{\gamma_i+1}-1\right) = \alpha\left(2I_1^{(m,k)}-1\right) = X_{r,n:1}^{(m,k)}, & b=1, \\ \alpha\left(2I_{0,1}^{(m,k)}-1\right) = \alpha\left(2\prod_{i=1}^{s}\frac{\gamma_i}{\gamma_i+1}-1\right) = X_{s,n:1}^{(m,k)}, & b=2. \end{cases}
\end{aligned}
$$

Finally, in the same way, we can obtain $X_{r,n:2}^{(m,k)}$, $X_{s,n:2}^{(m,k)}$, $X_{r,s,n:1}^{(m,k)}$, $X_{r,s,n:2}^{(m,k)}$, $X_{r,s,n:3}^{(m,k)}$, and $X_{r,s,n:4}^{(m,k)}$, which completes the proof. □

Remark 7. *Husseiny et al. [8] dealt with the issue $m=-1$ and $k=1$, which involves the case of record values.*

Theorem 3 has the immediate result that the joint MGF of the concomitants $Z_{[r,n,m,k]}$ and $Z_{[s,n,m,k]}$, $r<s$, based on SAR(α) is provided by

$$
\begin{aligned}
M_{[r,s,n,m,k]}(b_1,b_2) &= M_Z(b_1)M_Z(b_2) + \left(3X_{r,n:1}^{(m,k)} - \frac{5}{2}X_{r,n:2}^{(m,k)}\right)M_Z(b_2)(M_{V_1}(b_1)-M_Z(b_1)) + \Big(3X_{s,n:1}^{(m,k)} \\
&- \frac{5}{2}X_{s,n:2}^{(m,k)}\Big)M_Z(b_1)(M_{V_1}(b_2)-M_Z(b_2)) + 5X_{r,n:2}^{(m,k)}M_Z(b_2)(M_{V_2}(b_1)-M_{V_1}(b_1)) \\
&+ 5X_{s,n:2}^{(m,k)}M_Z(b_1)(M_{V_2}(b_2)-M_{V_1}(b_2)) + \left(9X_{r,s,n:1}^{(m,k)} - \frac{15}{2}X_{r,s,n:2}^{(m,k)} - \frac{15}{2}X_{r,s,n:3}^{(m,k)} + \frac{25}{4}X_{r,s,n:4}^{(m,k)}\right) \\
&\times (M_{V_1}(b_1)-M_Z(b_1))(M_{V_1}(b_2)-M_Z(b_2)) + \left(15X_{r,s,n:2}^{(m,k)} - \frac{25}{2}X_{r,s,n:4}^{(m,k)}\right)(M_{V_2}(b_1)
\end{aligned}
$$

$$- M_{V_1}(b_1))(M_{V_1}(b_2) - M_Z(b_2)) + \left(15X^{(m,k)}_{r,s,n:3} - \frac{25}{2}X^{(m,k)}_{r,s,n:4}\right)(M_{V_1}(b_1) - M_Z(b_1))$$
$$\times (M_{V_2}(b_2) - M_{V_1}(b_2)) + 25X^{(m,k)}_{r,s,n:4}(M_{V_2}(b_1) - M_{V_1}(b_1))(M_{V_2}(b_2) - M_{V_1}(b_2)). \tag{21}$$

Remark 8. *When $m = 0$ and $k = 1$ (i.e., for the OOSs model), it is simple to verify*

$$X^{(0,1)}_{r,s,n:1} = \alpha^2\left(\left(4I^{(0,1)}_{1,1} - 1\right) + X^{(0,1)}_{r,n:1} + X^{(0,1)}_{s,n:1}\right),$$

$$X^{(0,1)}_{r,s,n:2} = \alpha^3\left(12(I^{(0,1)}_{2,0} - 2I^{(0,1)}_{2,1}) + 10(2I^{(0,1)}_{1,1} - I^{(0,1)}_{1,0}) + X^{(0,1)}_{s,n:1} + X^{(0,1)}_{r,s,n:1}\right),$$

$$X^{(0,1)}_{r,s,n:3} = \alpha^3\left(12(I^{(0,1)}_{0,2} - 2I^{(0,1)}_{1,2}) + 10(2I^{(0,1)}_{1,1} - I^{(0,1)}_{0,1}) + X^{(0,1)}_{r,n:1} + X^{(0,1)}_{r,s,n:1}\right),$$

$$X^{(0,1)}_{r,s,n:4} = \alpha^4\left(48(3I^{(0,1)}_{2,2} - I^{(0,1)}_{1,2} - I^{(0,1)}_{2,1} + I^{(0,1)}_{1,1}) + 2(X^{(0,1)}_{r,s,n:2} + X^{(0,1)}_{r,s,n:3}) - 4(X^{(0,1)}_{r,n:1} + X^{(0,1)}_{s,n:1} - 1)\right),$$

and

$$I^{(0,1)}_{p,q} = \frac{\beta(r, n-r+p+q+1)\beta(s-r, n-s+q+1)}{\beta(r, s-r, n-s+1)},\ p,q \in \Re^+.$$

Moreover, $X^{(0,1)}_{s,n:1}$ and $X^{(0,1)}_{s,n:2}$ are defined by replacing r with s in $X^{(0,1)}_{r,n:1}$ and $X^{(0,1)}_{r,n:2}$, respectively (see Remark 3).

The product moment $\mathrm{E}[Z_{[r,n,m,k]}Z_{[s,n,m,k]}] = \mu_{[r,s,n,m,k]}$ is obtained directly from (21) by

$$\mu_{[r,s,n,m,k]} = \left[3\left(X^{(m,k)}_{r,n:1} + X^{(m,k)}_{s,n:1}\right) - \frac{5}{2}\left(X^{(m,k)}_{r,n:2} + X^{(m,k)}_{s,n:2}\right)\right]\mu_Z(\mu_{V_1} - \mu_Z) + 25X^{(m,k)}_{r,s,n:4}(\mu_{V_2} - \mu_{V_1})^2$$
$$+ \frac{5}{2}\left(X^{(m,k)}_{r,n:2} + X^{(m,k)}_{s,n:2}\right)\mu_Z(\mu_{V_2} - \mu_{V_1}) + \left[15\left(X^{(m,k)}_{r,s,n:2} + X^{(m,k)}_{r,s,n:3}\right) - 25X^{(m,k)}_{r,s,n:4}\right](\mu_{V_1} - \mu_Z)$$
$$\times (\mu_{V_2} - \mu_{V_1}) + \left(9X^{(m,k)}_{r,s,n:1} - \frac{15}{2}X^{(m,k)}_{r,s,n:2} - \frac{15}{2}X^{(m,k)}_{r,s,n:3} + \frac{25}{4}X^{(m,k)}_{r,s,n:4}\right)(\mu_{V_1} - \mu_Z)^2 + \mu_Z^2.$$

7. Asymptotic Behavior of the Concomitants for the OOSs Model

It is generally known that extremes among the Ts may coincide with extremes among the Zs, but not always (cf. [27]). Due to this circumstance, some researchers became curious about the rank $\mathcal{R}_{[r:n]} = \sum_{j=1}^{n} \mathrm{I}(Z_{[r:n]} - Z_j)$ of $Z_{[r:n]} := Z_{[r,n,0,1]}$, where $\mathrm{I}(t) = 1$ if $t \geq 0$ and $\mathrm{I}(t) = 0$ if $t < 0$. The distribution of $\mathcal{R}_{[r:n]}$ was obtained by David et al. [28]. Barakat and El-Shandidy [29] gave a new representation of the DF and expected value of $\mathcal{R}_{[r:n]}$. Namely, for all $r, s = 2, 3, \ldots, n-1$, we have

$$A_{r:n}(s) = P(\mathcal{R}_{[r:n]} = s) = n[\mathrm{E}(\mathcal{C}(W_{r:n-1}, Y_{s:n-1})) - \mathrm{E}(\mathcal{C}(W_{r-1:n-1}, Y_{s:n-1}))$$
$$- \mathrm{E}(\mathcal{C}(W_{r:n-1}, Y_{s-1:n-1})) + \mathrm{E}(\mathcal{C}(W_{r-1:n-1}, Y_{s-1:n-1}))], \tag{22}$$

where $\mathcal{C}(.,.)$ is the copula of the bivariate DF $F_{T,Z}(t,z)$, i.e., $\mathcal{C}(w,y) = wy[1 + 3\alpha(1-w)(1-y) + 5\alpha^2(1-2w)(1-2y)(1-w)(1-y)]$. Moreover, $W_{j:n} = F_T(T_{j:n})$ and $Y_{j:n} = F_Z(Z_{j:n})$ are the jth uniform OOS with expectation $\mathrm{E}(W_{j:n}) = \mathrm{E}(Y_{j:n}) = \frac{j}{n+1}$. The representation (22) enables us to use the δ method (with one-step Taylor approximation) to compute an approximate formula for the DF $A_{r:n}(s)$, by

$$
\begin{aligned}
A_{r:n}(s) \quad &\sim \quad n\Big[\mathcal{C}(\frac{r}{n},\frac{s}{n}) - \mathcal{C}(\frac{r-1}{n},\frac{s}{n}) - \mathcal{C}(\frac{r}{n},\frac{s-1}{n}) + \mathcal{C}(\frac{r-1}{n},\frac{s-1}{n})\Big] \\
&= \quad \frac{1}{n^2}[1 + \frac{3\alpha}{n^2}((n+2)(rs-s+n) - r(n+1)^2+1) + \frac{5\alpha^2}{n^4}(-2nrs(rs+5nrs+4r^2-19r \\
&- \quad 15ns+18s-14n-24+3n^2+r) - 4s(rs^2-2r^2s-5rs+7s+4r^2+r-5) \\
&+ \quad 2nr(n^2r-nr-3n^2-8n+4r^2-7-3r) + 4r(2r^2-r-1) + 2ns(3ns-3n^2-13n \\
&+ \quad 10s-15) + n(n^3+6n^2+12+13n) + 12s^2-12s+4)].
\end{aligned}
$$

Let $A_n, a_n > 0$, and $B_n, b_n \in \Re$ be any appropriate normalizing constants. Theorem 5.5.1 in [27] (see also its extension Theorem 1.3.1 in [30]) suggests that the limiting distribution of the normalized rth concomitant $A_n^{-1}(Z_{[r:n]} - B_n)$ will generally depend on the conditional distribution of the normalized RV $A_n^{-1}(Z - B_n)$ given the normalized RV $a_n^{-1}(T - b_n)$ and the limit DF of $a_n^{-1}(T_{r:n} - b_n)$. Interestingly, for the Sarmanov family, this limit does not depend on the limit type of $a_n^{-1}(T_{r:n} - b_n)$. Moreover, we can choose $A_n = 1$ and $B_n = 0$, as suggested by the next Theorem 4.

Before we build Theorem 4, we need to review the extreme value theorem (EVT) and the local uniform convergence that it is related to. Let $F_T(t)$ be an absolutely continuous DF with the PDF $f_T(t) = F_T'(t)$. Moreover, assume that the EVT is satisfied, i.e., there exist sequences of constants $a_n > 0$ and $b_n \in \Re$ such that

$$
F_T^n(a_n t + b_n) \xrightarrow[n]{w} G_1(t), \tag{23}
$$

where $G_1(t)$ is one of the possible three max-stable DFs (see [27]) and " $\xrightarrow[n]{w}$ " denotes weak convergence as $n \to \infty$. Now, assume that the weak convergence (23) implies density convergence, i.e., $\mathcal{F}_n(t) := n a_n F_T^{n-1}(a_n t + b_n) f_T(a_n t + b_n) \to G_1'(t)$ as $n \to \infty$. It is known that the local uniform convergence of $\mathcal{F}_n(t)$ to $G_1'(t)$ is equivalent to the appropriate one of the following mutually exclusive von Mises conditions (see [31]) :

(V1) For some $\alpha > 0$, $\lim_{t\to\infty} \frac{tF_T'(t)}{\overline{F}_T(t)} = \alpha$.

(V2) For some $\alpha > 0$, $\lim_{t\uparrow x_o} \frac{(x_o - t)F_T'(t)}{\overline{F}_T(t)} = \alpha$, where $x_o < \infty$ is the right end point of $F_T(t)$.

(V3) Let F_T be absolutely continuous in a left neighborhood of $X_0 \le \infty$,

$$
\lim_{t\uparrow x_o} \frac{F_T'(t) \int_0^{x_o} \overline{F}_T(t)\,dt}{\overline{F}_T^2(t)} = 1.
$$

We require the following lemma, which was extended by David [32] and is owed to Galambos [27].

Lemma 1. *Let $F_T(t)$ satisfy one of the von Mises V1-V3 conditions and assume that (23) is satisfied for some sequences of constants $a_n > 0$ and $b_n \in \Re$. Further, suppose there exist constants $A_n > 0$ and B_n such that*

$$
F_{Z|T}(A_n z + B_n | T = a_n t + b_n) \xrightarrow[n]{w} \Psi_{Z|T}(z|t), \tag{24}
$$

uniformly for all z and t. Then,

$$
F_{[n-r+1:n]}(A_n Z + B_n) \xrightarrow[n]{w} \int_{-\infty}^{\infty} \Psi_{Z|T}(z|t) dG_r(t), \tag{25}
$$

where G_r is the DF of the rth lower record value from the extreme value DF G.

Theorem 4. *We suppose $F_T(t)$ is absolutely continuous with density $f_T(t) = F'_T(t)$. Moreover, assume that the sequences of constants $a_n > 0$ and $b_n \in \Re$ are such that $F^n_T(a_n t + b_n)$ weakly converges to a max-stable law $G_1(t)$. Assume that $F_T(t)$ satisfies one of the the von Mises conditions V1-V3. Then, for any fixed r with respect to n, $1 \leq r < n$, and arbitrary marginal F_Z, we obtain*

$$\begin{aligned} F_{[n-r+1:n]}(z) &:= P(Z_{[n-r+1:n]} \leq a_n z + b_n) \xrightarrow[n]{w} F_Z(z)\Big(1 + 3\alpha\overline{F}_Z(z) + 5\alpha^2(2F^2_Z(z) - 3F_Z(z) + 1)\Big) \\ &:= H^{(S)}(z), \end{aligned} \tag{26}$$

where the superscript object (S) denotes that the limit conditional DF $H^{(S)}$ is related to Sarmanov family.

Proof. Under the conditions of the theorem and in view of Lemma 1, if there exist constants $A_n > 0$ and B_n such that (24) is satisfied uniformly for all z and t, then we obtain (25). Now, since $F^n_T(a_n t + b_n) \xrightarrow[n]{w} G_1(t)$, where $G_1(t)$ is a non-degenerate DF, we have $n\overline{F}_T(a_n t + b_n)$ converging to $-\log G_1(t)$. Thus, $\overline{F}_T(a_n t + b_n) \to 0$ as $n \to \infty$ for all $t > \inf\{t : G_1(t) > 0\}$. Therefore, after some direct algebra (see also relation (2.4) in Barakat et al. [6]), we obtain

$$F_{Z|T}(A_n z + B_n | T = a_n t + b_n) \xrightarrow[n]{w} F_Z(z)\Big(1 + 3\alpha\overline{F}_Z(z) + 5\alpha^2(2F^2_Z(z) - 3F_Z(z) + 1)\Big) := H^{(S)}(z),$$

where $A_n = 1$ and $B_n = 0$. Thus, bearing in mind that $H^{(S)}(z)(:= \Psi_{Z|T}(z|t))$ does not depend on t and $\int_{-\infty}^{\infty} dG_r(t) = 1$, we obtain (26). This completes the proof. □

Theorem 4 reveals an interesting fact that all the concomitants of the upper extremes for large n constitute a sequence of i.i.d. RVs $Z_{[n-i+1]} := Z^*_i$, $i = 1, 2, \ldots, r$. This fact suggests an easy method to estimate the shape parameter α when the two marginals of SAR(α) are known and the DF $F_T(t)$ satisfies the condition given in Theorem 4. This suggested method relies on the ordinary least squares method of estimation for complete samples, which was originally proposed by Swain et al. [33]. Several authors used the least squares method for estimating the parameters from different distributions, among them [34,35]. The method is based on minimizing the function

$$\mathcal{L}(\alpha \mid \overline{z}^*) := \sum_{i=1}^{r}\left(H^{(S)}(z^*_{i:r}) - \frac{i}{r+1}\right)^2,$$

with respect to the unknown parameter α, where $\overline{z}^* := (z^*_{1:r}, z^*_{2:r}, \ldots, z^*_{r:r})$ is an observed ordered sample (i.e., $Z^*_{1:r} \leq Z^*_{2:r} \leq \ldots \leq Z^*_{r:r}$ are the OOSs of the random sample $Z^*_1, Z^*_2, \ldots, Z^*_r$).

Theorem 4 provides us with a simple fitting test to the model SAR(α) of a given bivariate data for a large n. Namely, when the parameter α and the two marginals F_T and F_Z are known, we can apply the Kolmogorov test (say) to check whether the DF $H^{(S)}(z)$ fits the data $z^*_1, z^*_2, \ldots, z^*_r$. This roughly corresponds to the fitting test for SAR(α) to the original bivariate data $(t_1, z_1), (t_2, z_2), \ldots, (t_n, z_n)$ when n is sufficiently large.

Remark 9. *It is important to note that different extensions of FGM can yield findings that are comparable to Theorem 4, and as a result, the suggested approach for estimating the shape parameter and the fitting test can be used in this situation. Consider, for instance, the type proposed by Huang and Kotz [36] (see also [37]), whose DF is provided by*

$$F_{T,Z}(t,z) = F_T(t)F_Z(z)\Big[1 + \lambda(1 - F^p_T(t))(1 - F^p_Z(z))\Big],\ p \geq 1,\ -p^{-2} \leq \lambda \leq p^{-1}.$$

Under the conditions of Theorem 4 (concerning the marginal F_T) and on proceeding as we have in Theorem 4, we can verify that

$$F_{[n-r+1:n]}(z) = P(Z_{[n-r+1:n]} \leq a_n z + b_n) \xrightarrow[n]{w} F_Z(z)(1 - \lambda p(1 - F^p_Z(z))) := H^{(K)}(z),$$

where the superscript object (K) denotes that the limit conditional DF $H^{(K)}$ is related to the type proposed by Huang and Kotz [36].

8. An Application in the Reliability Theory

We start this section by looking at the GE DF $F_T(t) = \left(1 - e^{-\theta t}\right)^{\lambda}$, $t; \lambda, \theta > 0$ (denoted by $GE(\theta; \lambda)$). Gupta and Kundu [38] showed that the ℓth moment of T is given by

$$\mu_T^{(\ell)} = \frac{\lambda \ell!}{\theta^{\ell}} \sum_{i=0}^{\varphi(\lambda-1)} \frac{(-1)^i}{(i+1)^{\ell+1}} A(\lambda-1, i),$$

where $A(\lambda-1, i) = \binom{\lambda-1}{i}$ and $\varphi(t) = \infty$ if t is non-integer and $\varphi(t) = t$ if t is integer. Moreover, the mean, variance, and MGF of $GE(\theta; \lambda)$ are given, respectively, by

$$\mu_T = \mathrm{E}(T) = \frac{B(\lambda)}{\theta}, \mathrm{Var}(T) = \sigma_T^2 = \frac{C(\lambda)}{\theta^2}, \text{ and } M_T(b) = \lambda \beta(\lambda, 1 - \frac{b}{\theta}), \tag{27}$$

where $B(\lambda) = \Psi(\lambda+1) - \Psi(1)$, $C(\lambda) = \Psi'(1) - \Psi'(\lambda+1)$, $\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, and $\Psi(.)$ is the digamma function, while $\Psi'(.)$ is its derivation (the trigamma function).

Using Theorem 3, the PDF of $Z_{[r,n,m,k]}$ based on SAR-GE$(\theta_1, \lambda_1; \theta_2, \lambda_2)$ is given by

$$\begin{aligned} f_{[r,n,m,k]}(z) &= \lambda_2 \theta_2 (1 - e^{-\theta_2 z})^{\lambda_2 - 1} e^{-\theta_2 z} \{ (1 - 3X_{r,n:1}^{(m,k)} + \frac{5}{2} X_{r,n:2}^{(m,k)}) + 2(3X_{r,n:1}^{(m,k)} \\ &- \frac{15}{2} X_{r,n:2}^{(m,k)})(1 - e^{-\theta_2 z})^{\lambda_2} + 15X_{r,n:2}^{(m,k)}(1 - e^{-\theta_2 z})^{2\lambda_2} \}. \end{aligned}$$

Moreover, by putting $\ell = 1$ in (18) and using the relation (27), we obtain the mean of $Z_{[r,n,m,k]}$ based on SAR-GE$(\theta_1, \lambda_1; \theta_2, \lambda_2)$ by

$$\mu_{[r,n,m,k]} = \frac{1}{\theta_2} \left[\left(1 - 3X_{r,n:1}^{(m,k)} + \frac{5}{2} X_{r,n:2}^{(m,k)} \right) B(\lambda_2) + \left(3X_{r,n:1}^{(m,k)} - \frac{15}{2} X_{r,n:2}^{(m,k)} \right) B(2\lambda_2) + 5X_{r,n:2}^{(m,k)} B(3\lambda_2) \right]. \tag{28}$$

Remark 10. *The mean of the concomitant of the rth OOS, based on SAR-GE$(\theta_1, \lambda_1; \theta_2, \lambda_2)$, i.e., by putting $m = 0$ and $k = 1$ in (28), is given by*

$$\mu_{[r,n,0,1]} = \frac{1}{\theta_2} \left[\left(1 - 3X_{r,n:1}^{(0,1)} + \frac{5}{2} X_{r,n:2}^{(0,1)} \right) B(\lambda_2) + \left(3X_{r,n:1}^{(0,1)} - \frac{15}{2} X_{r,n:2}^{(0,1)} \right) B(2\lambda_2) + 5X_{r,n:2}^{(0,1)} B(3\lambda_2) \right].$$

Eryilmaz [39] studied the FGM using exponential marginals in order to assess its reliability. Here, some of Eryilmaz's findings are expanded to the SAR-GE$(\theta_1, \lambda_1; \theta_2, \lambda_2)$. Let $T_i \sim$ GE$(\theta_1; \lambda_1)$ and $Z_i \sim$ GE$(\theta_2; \lambda_2)$ represent the ith component's lifetime and its lifetime utility, respectively, $i = 1, \ldots, n$. The definition of total utility for n components is the RV $\sum_{i=1}^{n} Z_i$. Aside from that, $\sum_{i=1}^{n} Z_i - Z_{[1:n]}$ gives the system's residual performance following the first failure. Even though the components are the same, depending on their location or who utilizes them, they may contribute differently or be more or less helpful to the operation of the complete system. A positive correlation exists between a component's lifetime and utility. Such a dependence can be modeled by SAR-GE$(\theta_1, \lambda_1; \theta_2, \lambda_2)$. The residual performance after time b is defined by the process (cf. [39])

$$S(b) = \sum_{i=N(b)+1}^{n} Z_{[i:n]}, \; b > 0,$$

where the process $N(b)$ denotes the number of failures up to time b, i.e., $P(N(b) = r) = \binom{n}{r} F_T^r(b) \, (1 - F_T(b))^{n-r}$, $r = 0, 1, \ldots, n$, with $P(N(b) = 0) = 1$. It is obvious that an engineer may benefit from understanding the mean value of $S(b)$ at numerous stages, including design and preventative maintenance. Using Proposition 1 of Eryilmaz [39] and after some algebra, we can show that

$$\begin{aligned} \mathrm{E}(S(b)) &= \frac{n}{\theta_2}[B(\lambda_2)(1-F_T(b)) + 3\alpha D(2\lambda_2)(F_{U_1}(b) - F_T(b)) \\ &+ \frac{5}{2}\alpha^2(2B(3\lambda_2) - 3B(2\lambda_2) + B(\lambda_2))(4F_{U_2}(b) - 6F_{U_1}(b) + 2F_T(b))], \end{aligned}$$

where $U_1 \sim \mathrm{GE}(\theta_1; 2\lambda_1)$ and $U_2 \sim \mathrm{GE}(\theta_1; 3\lambda_1)$.

On the other hand, when exactly $\mathcal{N}$ working components are present at a given time, it is helpful to understand the mean residual performance of the system. We take into account the conditional mean residual performance as defined by $\psi_{\mathcal{N}}(b) = E(S(b) = j|M(b) = n - N(b) = \mathcal{N})$, where $M(b)$ is the number of working components at time b. Now, using Theorem 1 of Eryilmaz [39], we obtain, after some algebra,

$$\begin{aligned} \psi_{\mathcal{N}}(b) &= \frac{\mathcal{N}\mathrm{E}(S(b))}{n(1-F_T(b))} = \frac{\mathcal{N}}{\theta_2}[B(\lambda_2) + 3\alpha D(2\lambda_2)\frac{(F_{U_1}(b) - F_T(b))}{1-F_T(b)} \\ &+ \frac{5}{2}\alpha^2(2B(3\lambda_2) - 3B(2\lambda_2) + B(\lambda_2))\frac{(4F_{U_2}(b) - 6F_{U_1}(b) + 2F_T(b))}{1-F_T(b)}]. \end{aligned}$$

By applying L'Hospital's rule, we obtain

$$\lim_{b\to\infty} \psi_{\mathcal{N}}(b) = \frac{\mathcal{N}}{\theta_2}\Big[B(\lambda_2) + 3\alpha D(2\lambda_2) + 5\alpha^2(2B(3\lambda_2) - 3B(2\lambda_2) + B(\lambda_2))\Big] = \lim_{b\to\infty} \mathrm{E}(Z|T=b),$$

where

$$\begin{aligned} \mathrm{E}(Z|T=b) &= \frac{1}{\theta_2}\{B(\lambda_2) + 3\alpha D(2\lambda_2)(2F_T(t) - 1) + \frac{5}{4}\alpha^2\Big[3(2F_T(t)-1)^2 - 1\Big] \\ &\times [4B(3\lambda_2) - 6B(2\lambda_2) + B(\lambda_2)]\}. \end{aligned}$$

9. Application to Diabetic Nephropathy Data

This section examines the FI of $\mathrm{E}(Z)$ of the exponential distribution through investigations of real-world data. In data on diabetic nephropathy, where there is a weak correlation between the two RVs, we review medical information. The data set covers the time period from January 2012 to August 2013 and was acquired from the database at Dr. Path Lal's lab. These data, which contain the average diabetes duration for 132 individuals with type 2 diabetic nephropathy throughout various periods, was processed by Grover et al. [40]. The RVs T and Z represent the mean serum creatinine levels (SrCr) and the mean duration of diabetes, respectively. This data set, which presents information from 19 patients, can be found in [41,42]. We fit the exponential distribution to Z and T, with the scale parameters θ_1 and θ_2, separately. The maximum likelihood (ML) estimates of the scale parameters are $(\hat{\theta}_1, \hat{\theta}_2) = (21.441, 2.10733)$. Moreover, the ML shape parameter α is $\hat{\alpha} = -0.52915$. Table 7 examines the FI of $\mathrm{E}(Z)$ for the concomitants $Z_{[r:19]}$, $r = 1, 2, 9, 10, 18, 19$, i.e., the concomitants of lower-extreme OOSs, upper-extreme OOSs, and central values. From Table 7, we see that the FI decreases with increasing r, and its maximum occurs at the minimum OOS.

Table 7. The FI in $Z_{[r:n]}$.

r	$I_\theta(Z_{[r:19]}; -0.52915)$
1	0.600
2	0.495
9	0.3124
10	0.305
18	0.153
19	0.197

10. Concluding Remarks and Future Work

In this paper, we studied the distributions of concomitants of m-GOSs based on the Sarmanov family of bivariate distributions with general marginals. Some distributional properties of these concomitants were revealed and studied, such as moments and MGF. In addition, we derived the joint DF of the bivariate concomitants of m-GOSs based on SAR(α). Moreover, FI relevant to m-GOSs and their concomitants of the shape parameter of SAR(α) was derived. Some calculations were conducted to obtain more information about the properties of FI of the parameter α for SOSs and record values models as special cases. One of the most useful of these revealed properties is the symmetry about the parameter α, i.e., $I_{\alpha}(T_{r,n,1,1}, Z_{[r,n,1,1]}; \alpha) = I_{\alpha}(T_{r,n,1,1}, Z_{[r,n,1,1]}; -\alpha)$. This property is also satisfied for record values.

The FI regarding the shape parameter of concomitants of SOSs and record values based on SAR(α) with power function distribution marginal were derived. We also obtained the FI of the scale parameter of concomitants of SOSs and record values based on SAR(α) with exponential distribution marginal. After performing numerical studies, we noticed that the value of $I_c(Z_{[n]}; \alpha, c)$ remained constant starting from $n > 15$ for the power function distribution marginal and the value of $I_{\theta}(Z_{[n]}; \alpha, \theta)$ was often constant starting at $n > 17$ for the exponential distribution marginal. The FI measures the information that is contained in the available samples about the unknown parameters. Therefore, the study of the effects of the sample sizes n and r on the FI (Tables 1–6) provides a very useful tool for selecting a suitable censoring sample. This is because the study sheds light on the places of the ordered sample where the information about the parameters is concentrated.

By analyzing the asymptotic behavior of the concomitants of OOSs, we proposed a new method for estimating the shape parameter α and a simple fitting test for SAR(α). The findings of the paper were applied to a reliability modeling application. A bivariate real-world data set was also examined for illustrative purposes.

Concomitants of GOSs are crucial in numerous selection processes. One such example is the employment of GOS concomitants in sampling methods like the ranking set sample, double sampling, and others. By examining the information content of various information measures, these sample designs may contribute something worthwhile to the body of knowledge. To conduct such a study, concomitants of GOSs can be created from other bivariate families, such as FGM and Sarmanov.

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Appendix A

Some code in MATHEMATICA

Code for Table 1:

$\alpha = 0.1;$

$n = 1;$
$r = 1;$
$\sum_{i=0}^{11} \sum_{j=0}^{i} (-1)^i Binomial[i,j] \alpha^i 0.5^j ((3)^{i-j+2} (\frac{5\alpha}{2})^j \sum_{l=0}^{j} Binomial[j,l](-3)^l \sum_{f=0}^{i-j+2l+2}$
$Binomial[i-j+2l+2,f](-2)^f \frac{Beta(r,n-r+\frac{f+1}{2})}{Beta(r,n-r+\frac{1}{2})} \sum_{t=0}^{j} Binomial[j,t](-3)^t \sum_{p=0}^{i-j+2t+2}$
$Binomial[i-j+2t+2,p] \frac{(-2)^p}{(p+1)} + (3)^{i-j} (\frac{5\alpha}{2})^{j+2} \sum_{v=0}^{j+2} Binomial[j+2,v](-3)^v \sum_{u=0}^{i-j+2v}$
$Binomial[i-j+2v,u](-2)^u \frac{Beta(r,n-r+\frac{u+1}{2})}{Beta(r,n-r+\frac{1}{2})} \sum_{c=0}^{j+2} Binomial[j+2,c](-3)^c \sum_{z=0}^{2c+i-j}$
$Binomial[2c+i-j,z] \frac{(-2)^z}{(z+1)} + (3)^{i-j+1} (5\alpha)^{j+1} (\frac{1}{2})^j \sum_{a=0}^{j+1} Binomial[j+1,a](-3)^a \sum_{b=0}^{2a+i-j+1}$
$Binomial[2a+i-j+1,b](-2)^b \frac{Beta(r,n-r+\frac{b+1}{2})}{B(r,n-r+\frac{1}{2})} \sum_{d=0}^{j+1} Binomial[j+1,d](-3)^d \sum_{g=0}^{2d+i-j+1}$
$Binomial[2d+i-j+1,g] \frac{(-2)^g}{(g+1)})$

Code for Table 6:

$n = 2;$
$\alpha = 0.2;$
$\theta = 0.5;$
$T = 1 + 3\alpha(1 - 2^{-(n-1)}) + \frac{5\alpha^2}{2}(12(3^{-n} - 2^{-n}) + 2);$
$U = -6\alpha(1 - 2^{-(n-1)}) - 15\alpha^2(12(3^{-n} - 2^{-n}) + 2);$
$V = 15\alpha^2(12(3^{-n} - 2^{-n}) + 2);$
$R = T^2 Integrate[(w^2/(T + Ue^{-w} + Ve^{-2w}))e^{-w}, \{w, 0, \infty\}];$
$X = V^2 Integrate[(w^2/(T + Ue^{-w} + Ve^{-2w}))e^{-5w}, \{w, 0, \infty\}];$
$L = -2TV Integrate[(w^2 e^{-3w})/(T + Ue^{-w} + Ve^{-2w}), \{w, 0, \infty\}];$
$(1 - 2T - (2/27)V + R + X + L)/\theta^2$

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