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Almost Boyd-Wong Type Contractions under Binary Relations with Applications to Boundary Value Problems

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Abstract: This article is devoted to investigating the fixed point theorems for a new contractivity contraction, which combines the idea involved in Boyd-Wong contractions, strict almost contractions and relational contractions. Our results improve and expand existing fixed point theorems of literature. In process, we deduce a metrical fixed point theorem for strict almost Boyd-Wong contractions. To demonstrate the credibility of our results, we present a number of a few examples. Applying our findings, we find a unique solution to a particular periodic boundary value problem.

Keywords: $\mathcal{G}$-closed binary relation; fixed points; strict almost contractions

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1. Introduction

Banach contraction principle (abbreviation: BCP) is most significant and classical tool in nonlinear functional analysis. Besides guaranteeing the prevalence of unique fixed point, the BCP also provides a constructive method that approximates the fixed point. Due to its simplicity, the BCP was made attractive from the application aspects. In this direction, many authors utilized contraction mappings to prove the existence of solutions of boundary value problems (abbreviated as: BVP), integral equations and matrix equations etc. Various generalizations of this interesting result have been heavily investigated the branch of research; however the readers are advised to study some recent work contained in [1–4].

One of the generalizations of BCP that have attracted much attention during the last half-century was due to Boyd and Wong [5]. Indeed, Boyd and Wong [5] improved the contraction condition by replacing Lipschitz constant $k \in (0,1)$ with a control function belonging to the following family:

$$\Phi = \Big\{\varphi : [0,\infty) \to [0,\infty) : \varphi(a) < a, \text{ for all } a > 0 \text{ and } \limsup_{s \to a^+} \varphi(s) < a, \text{ for all } a > 0\Big\}.$$

Theorem 1 ([5])**.** *Assume that for some $\varphi \in \Phi$, a self-function $\mathcal{G}$ on a complete metric space $(\mathbf{P}, \rho)$ satisfies*

$$\rho(\mathcal{G}p, \mathcal{G}q) \leq \varphi(\rho(p,q)), \quad \textit{for all } p, q \in \mathbf{P}.$$

Then $\mathcal{G}$ possesses a unique fixed point.

Such contractivity condition is called nonlinear contraction or φ-contraction. Under the restriction $\varphi(a) = k \cdot a$, $0 < k < 1$, φ-contraction reduces to usual contraction and Theorem 1 reduces to the BCP.

In 2004, Berinde [6] introduced yet a new generalization of BCP, often called "almost contraction".

Definition 1 ([6,7])**.** *A self-function $\mathcal{G}$ on a metric space $(\mathbf{P}, \rho)$ is referred to as an almost contraction if there exists $k \in (0,1)$ and there exists $L \in [0, \infty)$ satisfying*

$$\rho(\mathcal{G}p, \mathcal{G}q) \leq k\rho(p,q) + L\rho(p, \mathcal{G}q), \quad \textit{for all } p, q \in \mathbf{P}.$$

By symmetric property of ρ, the above condition is identical to:

$$\rho(\mathcal{G}p, \mathcal{G}q) \leq k\rho(p,q) + L\rho(q, \mathcal{G}p), \quad \text{for all } p, q \in \mathbf{P}.$$

Theorem 2 ([6])**.** *An almost contraction self-function on a complete metric space admits a fixed point.*

The idea of almost contraction has been developed by various researchers, e.g., see [8–13]. An almost contraction remains weakly Picard operator so that it need not admit a unique fixed but sequence of Picard iteration converges to a fixed point of underlying mapping. To obtain a uniqueness theorem, Babu et al. [8] defined slightly stronger class of almost contraction condition.

Definition 2 ([8])**.** *A self-function $\mathcal{G}$ on a metric space $(\mathbf{P}, \rho)$ is named as strict almost contraction if there exists $k \in (0,1)$ and there exists $L \in [0, \infty)$ satisfying*

$$\rho(\mathcal{G}p, \mathcal{G}q) \leq k\rho(p,q) + L\min\{\rho(p, \mathcal{G}p), \rho(q, \mathcal{G}q), \rho(p, \mathcal{G}q), \rho(q, \mathcal{G}p)\}, \quad \textit{for all } p, q \in \mathbf{P}.$$

Clearly, a strict almost contraction is an almost contraction; but not conversely as shown by Example 2.6 [8].

Theorem 3 ([8])**.** *A strict almost contraction self-function on a complete metric space offers a unique fixed point.*

In recent times, an attractive research direction of metric fixed point theory is to demonstrate the fixed point results in relational metric space. Such results require that the contraction for just comparative elements (with respect to underlying binary relation) be satisfied. As of now, relational contractions are still weaker than to usual contractions. This trend is initiated by Alam and Imdad [14], wherein the authors obtained a variant of BCP in the structure of relational metric space. Since than, various results in this direction have been established. To cite some of them, we refer [15–27] besides others.

In this paper, we subsume two contractivity conditions mentioned as earlier (i.e., φ-contraction and strict almost contraction). We undertake the relation-preserving variant of this newly obtained contraction and adopted it for demonstrating the existence and uniqueness of fixed points in the structure of relational metric space. Our existence result assumes the underlying relation to be $\mathcal{G}$-closed and locally $\mathcal{G}$-transitive. However, uniqueness result requires to impose an additional hypothesis (i.e., $\mathcal{S}$-directedness) on certain subset of ambient space. Several examples are delivered, which attest to the credibility of our findings.

As was already indicated, a weaker contraction condition is implemented compared to what is found in the most recent research. Due to such limitations, the results demonstrated here can be applied in certain types of boundary value problems, nonlinear matrix equations and nonlinear integral equations, wherein classical fixed point theorems cannot be applied. For the sake of limitation, we adopt an application of our results to a BVP satisfying certain additional hypotheses, which shows the validation of our results.

2. Preliminaries

In what follows, the set of: natural numbers, whole numbers and real numbers will be denoted by $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{R}$, respectively. Recall that a subset of $\mathbf{P}^2$ is said to a binary relation (or, a relation) on the set $\mathbf{P}$.

Let us assume that $\mathbf{P}$ is the given set, $\mathcal{G} : \mathbf{P} \to \mathbf{P}$ is a mapping, $\mathcal{S}$ is a relation on $\mathbf{P}$ and ρ remains a metric on $\mathbf{P}$.

Definition 3 ([14])**.** *The points $p, q \in \mathbf{P}$ are called $\mathcal{S}$-comparative if $(p, q) \in \mathcal{S}$ or $(q, p) \in \mathcal{S}$. We denote such a pair by $[p, q] \in \mathcal{S}$.*

Definition 4 ([28])**.** *The relation $\mathcal{S}^{-1} := \{(p, q) \in \mathbf{P}^2 : (q, p) \in \mathcal{S}\}$ is called inverse of $\mathcal{S}$. Also, $\mathcal{S}^s := \mathcal{S} \cup \mathcal{S}^{-1}$ defines a symmetric relation on $\mathbf{P}$, often called symmetric closure of $\mathcal{S}$.*

Remark 1 ([14])**.** $(p, q) \in \mathcal{S}^s \Longleftrightarrow [p, q] \in \mathcal{S}$.

Definition 5 ([29])**.** *For a subset $\mathbf{Q} \subseteq \mathbf{P}$, the set*

$$\mathcal{S}|_{\mathbf{Q}} := \mathcal{S} \cap \mathbf{Q}^2,$$

a relation on $\mathbf{Q}$, is termed as the restriction of $\mathcal{S}$ on $\mathbf{Q}$.

Definition 6 ([14])**.** *$\mathcal{S}$ is referred to as $\mathcal{G}$-closed if for every pair $p, q \in \mathbf{P}$ verifying $(p, q) \in \mathcal{S}$, one has*

$$(\mathcal{G}p, \mathcal{G}q) \in \mathcal{S}.$$

Definition 7 ([14])**.** *A sequence $\{p_n\} \subset \mathbf{P}$ satisfying $(p_n, p_{n+1}) \in \mathcal{S}$, for all $n \in \mathbb{N}$, is termed as $\mathcal{S}$-preserving.*

Definition 8 ([15])**.** *The metric space $(\mathbf{P}, \rho)$ is called $\mathcal{S}$-complete if each $\mathcal{S}$-preserving Cauchy sequence in $\mathbf{P}$ converges.*

Definition 9 ([15])**.** *The mapping $\mathcal{G}$ is referred to as $\mathcal{S}$-continuous at $p \in \mathbf{P}$ if for every $\mathcal{S}$-preserving sequence $\{p_n\} \subset \mathbf{P}$ with $p_n \xrightarrow{\rho} p$,*

$$\mathcal{G}(p_n) \xrightarrow{\rho} \mathcal{G}(p).$$

A mapping, which remains $\mathcal{S}$-continuous function at every point, is called $\mathcal{S}$-continuous.

Definition 10 ([14])**.** *$\mathcal{S}$ is termed as ρ-self-closed if each $\mathcal{S}$-preserving convergent sequence in $\mathbf{P}$ contains a subsequence, within which each term is $\mathcal{S}$-comparative to the limit of sequence.*

Definition 11 ([30])**.** *A subset $\mathbf{Q} \subseteq \mathbf{P}$ is termed as $\mathcal{S}$-directed if for every pair $p, q \in \mathbf{Q}$, $\exists\, m \in \mathbf{P}$ satisfying $(p, m) \in \mathcal{S}$ and $(q, m) \in \mathcal{S}$.*

Definition 12 ([16])**.** *$\mathcal{S}$ is referred as locally $\mathcal{G}$-transitive if for each $\mathcal{S}$-preserving sequence $\{p_n\} \subset \mathcal{G}(\mathbf{P})$ (with range $\mathbf{E} = \{p_n : n \in \mathbb{N}\}$), $\mathcal{S}|_{\mathbf{E}}$ remains transitive.*

Proposition 1 ([16])**.** *If $\mathcal{S}$ is $\mathcal{G}$-closed, then $\mathcal{S}$ is $\mathcal{G}^n$-closed, for each $n \in \mathbb{N}_0$.*

Definition 13 ([31])**.** *A sequence $\{p_n\} \subset \mathbf{P}$ is called semi-Cauchy if for all $n \in \mathbb{N}$, it satisfies*

$$\lim_{n \to \infty} \rho(p_n, p_{n+1}) = 0.$$

Clearly, every Cauchy sequence is semi-Cauchy but not conversely.

Lemma 1 ([32])**.** *If $\{p_n\}$ remains a sequence in a metric space $(\mathbf{P}, \rho)$, which is not a Cauchy, then there exists $\epsilon_0 > 0$ and there exists subsequences $\{p_{n_k}\}$ and $\{p_{l_k}\}$ of $\{p_n\}$ verifying*

(i) $k \leq l_k < n_k$ *for all* $k \in \mathbb{N}$,
(ii) $\rho(p_{l_k}, p_{n_k}) > \epsilon_0$ *for all* $k \in \mathbb{N}$,

(iii) $\rho(p_{l_k}, p_{n_{k-1}}) \leq \epsilon_0$ *for all* $k \in \mathbb{N}$.

Moreover, if $\{p_n\}$ *is semi-Cauchy then*

(iv) $\lim\limits_{k\to\infty} \rho(p_{l_k}, p_{n_k}) = \epsilon_0$,

(v) $\lim\limits_{k\to\infty} \rho(p_{l_k}, p_{n_k+1}) = \epsilon_0$,

(vi) $\lim\limits_{k\to\infty} \rho(p_{l_k+1}, p_{n_k}) = \epsilon_0$,

(vii) $\lim\limits_{k\to\infty} \rho(p_{l_k+1}, p_{n_k+1}) = \epsilon_0$.

Making use of symmetric property of metric ρ, we can conclude the following:

Proposition 2. *Given any* $\varphi \in \Phi$ *and a constant* $L \geq 0$, *the following conditions are equivalent:*

(I) $\rho(\mathcal{G}p, \mathcal{G}q) \leq \varphi(\rho(p,q)) + L\min\{\rho(p,\mathcal{G}p), \rho(q,\mathcal{G}q), \rho(p,\mathcal{G}q), \rho(q,\mathcal{G}p)\}$,
for all $p, q \in \mathbf{P}$ *with* $(p,q) \in \mathcal{S}$.

(II) $\rho(\mathcal{G}p, \mathcal{G}q) \leq \varphi(\rho(p,q)) + L\min\{\rho(p,\mathcal{G}p), \rho(q,\mathcal{G}q), \rho(p,\mathcal{G}q), \rho(q,\mathcal{G}p)\}$,
for all $p, q \in \mathbf{P}$ *with* $[p,q] \in \mathcal{S}$.

3. Main Results

Firstly, we present the following result on the existence of fixed point for relational strict almost φ-contraction.

Theorem 4. *Suppose that* $(\mathbf{P}, \rho)$ *is a metric space,* $\mathcal{S}$ *is a relation on* $\mathbf{P}$ *while* $\mathcal{G} : \mathbf{P} \to \mathbf{P}$ *remains a mapping. Also, suppose the following assumptions are contented:*

(i) $(\mathbf{P}, \rho)$ *is* $\mathcal{S}$*-complete,*

(ii) *there exists* $p_0 \in \mathbf{P}$ *satisfying* $(p_0, \mathcal{G}p_0) \in \mathcal{S}$,

(iii) $\mathcal{S}$ *is locally* $\mathcal{G}$*-transitive and* $\mathcal{G}$*-closed,*

(iv) $\mathbf{P}$ *is* $\mathcal{S}$*-continuous, or* $\mathcal{S}$ *is* ρ*-self-closed,*

(v) *there exists* $\varphi \in \Phi$ *and* $L \geq 0$ *verifying*

$$\rho(\mathcal{G}p, \mathcal{G}q) \leq \varphi(\rho(p,q)) + L\min\{\rho(p,\mathcal{G}p), \rho(q,\mathcal{G}q), \rho(p,\mathcal{G}q), \rho(q,\mathcal{G}p)\},$$
$$\text{for all } p, q \in \mathbf{P} \text{ with } (p,q) \in \mathcal{S}.$$

Then, $\mathcal{G}$ *admits a fixed point.*

Proof. We'll prove the outcome in several steps:

Step-1. We'll construct Picard sequence $\{p_n\} \subset \mathbf{P}$ with initial point $p_0 \in \mathbf{P}$ as follows:

$$p_n := \mathcal{G}^n(p_0) = \mathcal{G}(p_{n-1}), \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

Step-2. We'll that $\{p_n\}$ is an $\mathcal{S}$-preserving sequence. By assumption (ii), $\mathcal{G}$-closedness of $\mathcal{S}$ and Proposition 1, we derive

$$(\mathcal{G}^n p_0, \mathcal{G}^{n+1} p_0) \in \mathcal{S},$$

which due to availability of (1) reduces to

$$(p_n, p_{n+1}) \in \mathcal{S}, \quad \text{for all } n \in \mathbb{N}_0. \tag{2}$$

Step-3. We'll show that $\{p_n\}$ is semi-Cauchy, i.e., $\lim\limits_{n\to\infty} \rho(p_n, p_{n+1}) = 0$.

Let us denote $\rho_n := \rho(p_n, p_{n+1})$. If $\rho_{n_0} = \rho(p_{n_0}, p_{n_0+1}) = 0$ for some $n_0 \in \mathbb{N}_0$, then in lieu of (1), one has $\mathcal{G}(p_{n_0}) = p_{n_0}$. Thus, p_{n_0} is a fixed point of $\mathcal{G}$ and hence we are done.

In case $\rho_n > 0$, for all $n \in \mathbb{N}_0$, employing assumption (v), (1) and (2), we get

$$\begin{aligned} \rho_n &= \rho(p_n, p_{n+1}) = \rho(\mathcal{G}p_{n-1}, \mathcal{G}p_n) \\ &\leq \varphi(\rho(p_{n-1}, p_n)) + L\min\{\rho(p_{n-1}, p_n), \rho(p_n, p_{n+1}), \rho(p_{n-1}, p_{n+1}), 0\}, \end{aligned}$$

so that

$$\rho_n \leq \varphi(\rho_{n-1}) \quad \text{for all } n \in \mathbb{N}_0. \tag{3}$$

Employing the property of φ in (3), we derive

$$\rho_n \leq \varphi(\rho_{n-1}) < \rho_{n-1}, \quad \text{for all } n \in \mathbb{N},$$

i.e., $\{\rho_n\}$ is a monotonically decreasing sequence of positive reals. Further, $\{\rho_n\}$ remains bounded below by '0'. Consequently, there exists $l \geq 0$ such that

$$\lim_{n\to\infty} \rho_n = l. \tag{4}$$

Now, we assert that $l = 0$. Quite the contrary, if $l > 0$ then letting upper limit in (3) and using (4) and the property of Φ, we find

$$l = \limsup_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \varphi(\rho_{n-1}) = \limsup_{\rho_n\to l^+} \varphi(\rho_{n-1}) < l,$$

which is a contradiction so that $l = 0$. Thus, we have

$$\lim_{n\to\infty} \rho_n = 0. \tag{5}$$

Step-4. We'll show that $\{p_n\}$ is a Cauchy sequence. If $\{p_n\}$ is not Cauchy, then by Lemma 1, $\exists\, \epsilon_0 > 0$ and there exists subsequences $\{p_{n_k}\}$ and $\{p_{l_k}\}$ of $\{p_n\}$ satisfying

$$k \leq l_k < n_k,\ \rho(p_{l_k}, p_{n_k}) > \epsilon_0 \geq \rho(p_{l_k}, p_{n_{k-1}}), \quad \text{for all } k \in \mathbb{N}.$$

Denote $\delta_k := \rho(p_{l_k}, p_{n_k})$. As $\{p_n\}$ is $\mathcal{S}$-preserving (due to (2)) and $\{p_n\} \subset \mathcal{G}(\mathbf{P})$ (due to (1)), using locally $\mathcal{G}$-transitivity of $\mathcal{S}$, we find $(p_{l_k}, p_{n_k}) \in \mathcal{S}$. Therefore, by using the contractivity condition (v), we obtain

$$\begin{aligned} \rho(p_{l_k+1}, p_{n_k+1}) &= \rho(\mathcal{G}p_{l_k}, \mathcal{G}p_{n_k}) \\ &\leq \varphi(\rho(p_{l_k}, p_{n_k})) + L\min\{\rho(p_{l_k}, \mathcal{G}p_{l_k}), \rho(p_{n_k}, \mathcal{G}p_{n_k}), \rho(p_{l_k}, \mathcal{G}p_{n_k}), \rho(p_{n_k}, \mathcal{G}p_{l_k})\} \end{aligned}$$

so that

$$\rho(p_{l_k+1}, p_{n_k+1}) \leq \varphi(\delta_k) + L\min\{\rho_{l_k}, \rho_{n_k}, \rho(p_{l_k}, p_{n_k+1}), \rho(p_{n_k}, p_{l_k+1})\}. \tag{6}$$

Letting upper limit in (6) and making use of Lemma 1 and the property of Φ, we find

$$\epsilon_0 = \limsup_{k\to\infty} \rho(p_{l_k+1}, p_{n_k+1}) \leq \limsup_{k\to\infty} \varphi(\delta_k) + L\min\{0, 0, \epsilon_0, \epsilon_0\} = \limsup_{s\to\epsilon_0^+} \varphi(s) < \epsilon_0,$$

which arises a contradiction. Thus, $\{p_n\}$ remains Cauchy.

Since the sequence $\{p_n\}$ is an $\mathcal{S}$-preserving and Cauchy, therefore by assumption (i), there exists $\bar{p} \in \mathbf{P}$ verifying $p_n \xrightarrow{\rho} \bar{p}$.

Step-5. We'll show that $\bar{p}$ is the fixed point of $\mathcal{G}$ by using the assumption (iv). Suppose that the mapping $\mathcal{G}$ is $\mathcal{S}$-continuous. As $\{p_n\}$ remains $\mathcal{S}$-preserving verifying $p_n \xrightarrow{\rho} \bar{p}$, $\mathcal{S}$-continuity of $\mathcal{G}$ yields that $p_{n+1} = \mathcal{G}(p_n) \xrightarrow{\rho} \mathcal{G}(\bar{p})$. Owing to the uniqueness property of convergence limit, we get $\mathcal{G}(\bar{p}) = \bar{p}$.

If $\mathcal{S}$ is ρ-self-closed, then $\{p_n\}$ contains a subsequence $\{p_{n_k}\}$ verifying $[p_{n_k}, \bar{p}] \in \mathcal{S}$, for all $k \in \mathbb{N}$. Using assumption (v), Proposition 2 and $[p_{n_k}, \bar{p}] \in \mathcal{S}$, we obtain

$$\begin{aligned} \rho(p_{n_k+1}, \mathcal{G}\bar{p}) &= \rho(\mathcal{G}p_{n_k}, \mathcal{G}\bar{p}) \\ &\leq \varphi(\rho(p_{n_k}, \bar{p})) + L\min\{\rho(p_{n_k}, p_{n_k+1}), 0, \rho(p_{n_k}, \bar{p}), \rho(\bar{p}, p_{n_k+1})\} \\ &= \varphi(\rho(p_{n_k}, \bar{p})). \end{aligned}$$

We claim that

$$\rho(p_{n_k+1}, \mathcal{G}\bar{p}) \leq \rho(p_{n_k}, \bar{p}), \quad \text{for all } k \in \mathbb{N}. \tag{7}$$

If $\rho(p_{n_{k_0}}, \bar{p}) = 0$ for some $k_0 \in \mathbb{N}$, then we find $\rho(\mathcal{G}p_{n_{k_0}}, \mathcal{G}\bar{p}) = 0$ so that $\rho(p_{n_{k_0}+1}, \mathcal{G}\bar{p}) = 0$ and hence (7) occurs for such $k_0 \in \mathbb{N}$. In either case, we have $\rho(p_{n_k}, \bar{p}) > 0$ for all $k \in \mathbb{N}$. By the definition of Φ, we get $\rho(p_{n_k+1}, \mathcal{G}\bar{p}) \leq \varphi(\rho(p_{n_k}, \bar{p})) < \rho(p_{n_k}, \bar{p})$ for all $k \in \mathbb{N}$. Thus (7) occurs for any $k \in \mathbb{N}$. Putting limit of (7) and utilizing $p_{n_k} \xrightarrow{\rho} \bar{p}$, we derive $p_{n_k+1} \xrightarrow{\rho} \mathcal{G}(\bar{p})$. Due to uniqueness property of limit, we find $\mathcal{G}(\bar{p}) = \bar{p}$ so that $\bar{p}$ remains a fixed point of $\mathcal{G}$. □

Next, we present the following uniqueness result.

Theorem 5. *Along with the hypotheses of Theorem 4, if $\mathcal{G}(\mathbf{P})$ is $\mathcal{S}$-directed, then $\mathcal{G}$ possesses a unique fixed point.*

Proof. In view of Theorem 4, choose $\bar{p}, \bar{q} \in \mathbf{P}$ verifying

$$\mathcal{G}(\bar{p}) = \bar{p} \text{ and } \mathcal{G}(\bar{q}) = \bar{q}. \tag{8}$$

As $\bar{p}, \bar{q} \in \mathcal{G}(\mathbf{P})$, by our hypothesis, there exists $m \in \mathbf{P}$ satisfying

$$(\bar{p}, m) \in \mathcal{S} \quad \text{and} \quad (\bar{q}, m) \in \mathcal{S}. \tag{9}$$

Denote $\varrho_n := \rho(\bar{p}, \mathcal{G}^n m)$. Using (8), (9) and assumption (v), one obtains

$$\begin{aligned} \varrho_n = \rho(\bar{p}, \mathcal{G}^n m) &= \rho(\mathcal{G}\bar{p}, \mathcal{G}(\mathcal{G}^{n-1}m)) \\ &\leq \varphi(\rho(\bar{p}, \mathcal{G}^{n-1}m)) + L\min\{0, \rho(\mathcal{G}^{n-1}m, \mathcal{G}^n m), \rho(\bar{p}, \mathcal{G}^n m), \rho(\mathcal{G}^{n-1}m, \bar{p})\} \\ &= \varphi(\varrho_{n-1}) \end{aligned}$$

so that

$$\varrho_n \leq \varphi(\varrho_{n-1}). \tag{10}$$

If for some $n_0 \in \mathbb{N}$, $\varrho_{n_0} = 0$, then we have $\varrho_{n_0} \leq \varrho_{n_0-1}$. Otherwise in case $\varrho_n > 0$, for all $n \in \mathbb{N}$, using the definition of Φ, (10) reduces to $\varrho_n < \varrho_{n-1}$. Hence, in both cases, we have

$$\varrho_n \leq \varrho_{n-1}.$$

Using the arguments similar to Theorem 4, above inequality gives rise to

$$\lim_{n\to\infty} \varrho_n = \lim_{n\to\infty} \rho(\bar{p}, \mathcal{G}^n m) = 0. \tag{11}$$

Similarly, one can find

$$\lim_{n\to\infty} \rho(\bar{q}, \mathcal{G}^n m) = 0. \tag{12}$$

By using (11), (12) and the triangular inequality, one has

$$\rho(\bar{p}, \bar{q}) = \rho(\bar{p}, \mathcal{G}^n m) + \rho(\mathcal{G}^n m, \bar{q}) \to 0, \quad \text{as} \quad n \to \infty.$$

This asserts that $\bar{p} = \bar{q}$. Therefore, $\mathcal{G}$ possesses a unique fixed point. □

4. Examples and Consequences

Intending to illustrate Theorems 4 and 5, consider the following examples.

Example 1. *Take* $\mathbf{P} = [0, \infty)$ *with usual metric. Let* $\mathcal{G} : \mathbf{P} \to \mathbf{P}$ *be a mapping defined by* $\mathcal{G}(p) = \frac{p}{p+1}$*. Consider* $\mathcal{S} := \{(p,q) \in \mathbf{P}^2 : p - q > 0\}$*. Then* $(\mathbf{P}, \rho)$ *is* $\mathcal{S}$*-complete and* $\mathcal{G}$ *is* $\mathcal{S}$*-continuous. Also,* $\mathcal{S}$ *is locally* $\mathcal{G}$*-transitive and* $\mathcal{G}$*-closed binary relation on* $\mathbf{P}$*. Define the auxiliary function* $\varphi : [0,\infty) \to [0,\infty)$ *by* $\varphi(a) = \frac{a}{a+1}$ *and choose* $L \geq 0$ *arbitrarily. Then for all* $(p,q) \in \mathcal{S}$*, we have*

$$\begin{aligned}
\rho(\mathcal{G}p, \mathcal{G}q) &= \left| \frac{p}{p+1} - \frac{q}{q+1} \right| = \left| \frac{p-q}{1+p+q+pq} \right| \\
&\leq \frac{p-q}{1+(p-q)} = \frac{\rho(p,q)}{1+\rho(p,q)} \\
&\leq \phi(\rho(p,q)) + L\min\{\rho(p,\mathcal{G}p), \rho(q,\mathcal{G}q), \rho(p,\mathcal{G}q), \rho(q,\mathcal{G}p)\}.
\end{aligned}$$

Therefore, the assumption (v) *of Theorem 4 is satisfied. Similarly, rest of the conditions of Theorem 4 and Theorem 5 hold. Consequently,* $\mathcal{G}$ *admits a unique fixed point* $\bar{p} = 0$*.*

Example 2. *Take* $\mathbf{P} = [0,1]$ *with usual metric. Let* $\mathcal{G} : \mathbf{P} \to \mathbf{P}$ *be a mapping defined by*

$$\mathcal{G}(p) = \begin{cases} p^2, & \text{if } 0 \leq p < 1/4 \\ 0, & \text{if } 1/4 \leq p \leq 1. \end{cases}$$

Consider $\mathcal{S} := \leq$*. Clearly,* $(\mathbf{P}, \rho)$ *is* $\mathcal{S}$*-complete. Also,* $\mathcal{S}$ *is locally* $\mathcal{G}$*-transitive and* $\mathcal{G}$*-closed binary relation on* $\mathbf{P}$*. Here,* $\mathcal{G}$ *is not* $\mathcal{S}$*-continuous. But* $\mathcal{S}$ *is* ρ*-self-closed. Also,* $\mathcal{G}$ *satisfies the contractivity condition* (v) *for the auxiliary function* $\varphi(a) = a/2$ *and for the constant* $L = 1$*. Similarly, rest of the conditions of Theorem 4 and Theorem 5 hold. Consequently,* $\mathcal{G}$ *admits a unique fixed point* $\bar{p} = 0$*.*

Example 3. *Take* $\mathbf{P} = (0,4]$ *with usual metric. Let* $\mathcal{G} : \mathbf{P} \to \mathbf{P}$ *be a mapping defined by*

$$\mathcal{G}(p) = \begin{cases} 1, & \text{if } 0 \leq p < 3 \\ 4, & \text{if } 3 \leq p \leq 4. \end{cases}$$

Define a relation $\mathcal{S} := \{(p,q) : 1 \leq p \leq q \leq 2 \text{ or } 3 \leq p \leq q \leq 4\}$ *on* $\mathbf{P}$*. Clearly,* $(\mathbf{P}, \rho)$ *is* $\mathcal{S}$*-complete and* $\mathcal{G}$ *is* $\mathcal{S}$*-continuous. Also,* $\mathcal{S}$ *is locally* $\mathcal{G}$*-transitive and* $\mathcal{G}$*-closed binary relation on* $\mathbf{P}$*. Here,* $\mathcal{G}$ *satisfies the contractivity condition* (v) *for the auxiliary function*

$$\varphi(a) = \begin{cases} 1/2 & \text{if } 0 \leq a \leq 1 \\ a - 1/2 & \text{if } a > 1 \end{cases}$$

and for the constant $L \geq 1$*. Similarly, rest of the conditions of Theorem 4 hold. Consequently,* $\mathcal{G}$ *admits a fixed point.*

Note that $\mathcal{G}(\mathbf{P})$ *is not* $\mathcal{S}$*-directed as there is no element in* $\mathbf{P}$ *which remains simultaneously* $\mathcal{S}$*-comparative with* 1 *and* 4*. Thus far Theorem 5 is not applicable to present example. Indeed,* $\mathcal{G}$ *possesses two fixed points, (*$p = 1$ *and* $q = 4$*).*

Now, making use of our results, we'll obtain some well known fixed point theorems from a review of current research.. Under the restriction $\mathcal{S} = \mathbf{P}^2$, the universal relation, Theorem 5 deduces the following metrical fixed point theorem under strict almost Boyd-Wong contraction.

Corollary 1. *Suppose that $(\mathbf{P}, \rho)$ is a complete metric space and $\mathcal{G} : \mathbf{P} \to \mathbf{P}$ is a function. If $\exists$ $\varphi \in \Phi$ and $L \geq 0$ verifying*

$$\rho(\mathcal{G}p, \mathcal{G}q) \leq \varphi(\rho(p,q)) + L \min\{\rho(p, \mathcal{G}p), \rho(q, \mathcal{G}q), \rho(p, \mathcal{G}q), \rho(q, \mathcal{G}p)\}, \text{ for all } p, q \in \mathbf{P},$$

then $\mathcal{G}$ admits a unique fixed point.

Corollary 1 further reduces to Theorem 1 for $L = 0$, while it reduces to Theorem 3 for $\varphi(a) = ka,\ k \in (0,1)$.

Particularly for $L = 0$, Theorem 5 deduces the following result of Alam and Imdad [16].

Corollary 2. *[16] Suppose that $(\mathbf{P}, \rho)$ is metric space, $\mathcal{S}$ is a relation on $\mathbf{P}$ while $\mathcal{G} : \mathbf{P} \to \mathbf{P}$ remains a mapping. Also,*

(i) *$(\mathbf{P}, \rho)$ is $\mathcal{S}$-complete,*
(ii) *there exists $p_0 \in \mathbf{P}$ satisfying $(p_0, \mathcal{G}p_0) \in \mathcal{S}$,*
(iii) *$\mathcal{S}$ is $\mathcal{G}$-closed,*
(iv) *$\mathbf{P}$ is $\mathcal{S}$-continuous, or $\mathcal{S}$ is ρ-self-closed,*
(v) *there exists $\varphi \in \Phi$ verifying*

$$\rho(\mathcal{G}p, \mathcal{G}q) \leq \varphi(\rho(p,q)), \qquad \text{for all } p, q \in \mathbf{P} \text{ with } (p,q) \in \mathcal{S}.$$

Then, $\mathcal{G}$ admits a fixed point. Moreover, if $\mathcal{G}(\mathbf{P})$ is $\mathcal{S}$-directed, then $\mathcal{G}$ possesses a unique fixed point.

Taking $\mathcal{S} = \preceq$, a partial order in Corollary 2, we get Theorem 9 of Kutbi et al. [33], which sharpens and enriches several existing results, viz., Theorem 2.1 of Wu and Liu [34], Theorem 5 of Kutbi et al. [35], Theorem 10 of Karapinar et al. [36] and Theorem 1.2 of Karapinar and Roldán-López-de-Hierro [37].

Example 2 can not be covered by Corollary 2 for if we take $p = 1/5$ and $q = 1/4$, then the inequality

$$\rho(\mathcal{G}p, \mathcal{G}q) \leq \varphi(\rho(p,q))$$

is never satisfied. This substantiates the utility and novelty of Theorem 5 over Corollary 2.

5. Applications to Boundary Value Problems

Let us consider the following BVP:

$$\begin{cases} v'(r) = \Psi(r, v(r)), & r \in [0,c] \\ v(0) = v(c) \end{cases} \tag{13}$$

wherein $\Psi : [0,c] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In the follow-up, by Θ, we'll denote the class of continuous and monotonically increasing functions $\theta : [0,\infty) \to [0,\infty)$ verifying $\theta(a) < a,$ for all $a > 0$. Obviously, $\Theta \subset \Phi$.

As usual, the collection of real valued continuous (continuously differentiable) functions on the interval $[0,c]$ will be denoted by $\mathcal{C}[0,c]$ ($\mathcal{C}'[0,c]$).

Following [38], we say that $\tilde{v} \in \mathcal{C}'[0,c]$ is a lower solution of (13) if

$$\begin{cases} \tilde{v}'(r) \leq \Psi(r, \tilde{v}(r)), & r \in [0,c] \\ \tilde{v}(0) \leq \tilde{v}(c). \end{cases}$$

Our main result of this section runs as under:

Theorem 6. *Along with the Problem (13), if there exists $\tau > 0$, and $\theta \in \Theta$ satisfying*

$$0 \leq [\Psi(r,b) + \tau b] - [\Psi(r,a) + \tau a] \leq \tau\theta(b-a), \quad \textit{for all } r \in [0,c] \textit{ and for all } a,b \in \mathbb{R} \textit{ with } a \leq b, \tag{14}$$

then the Problem (13) possesses a unique solution provided it has a lower solution.

Proof. Rewrite (13) as

$$\begin{cases} v'(r) + \tau v(r) = \Psi(r, v(r)) + \tau v(r), & \text{for all } r \in [0,c] \\ v(0) = v(c) \end{cases}$$

which remains equivalent to the Fredholm integral equation:

$$v(r) = \int_0^c M(r,\xi)[\Psi(\xi, v(\xi)) + \tau v(\xi)]d\xi. \tag{15}$$

Here $M(r,\xi)$ is the Green function, defined by

$$M(r,\xi) = \begin{cases} \frac{e^{\tau(c+\xi-r)}}{e^{\tau c}-1}, & 0 \leq \xi < r \leq c \\ \frac{e^{\tau(\xi-r)}}{e^{\tau c}-1}, & 0 \leq r < \xi \leq c. \end{cases}$$

Denote $\mathbf{P} := \mathcal{C}[0,c]$. Consider the function $\mathcal{G} : \mathbf{P} \to \mathbf{P}$ defined by

$$(\mathcal{G}v)(r) = \int_0^c M(r,\xi)[\Psi(\xi, v(\xi)) + \tau v(\xi)]d\xi, \quad \text{for all } r \in [0,c]. \tag{16}$$

Define a relation $\mathcal{S}$ on $\mathbf{P}$ by

$$\mathcal{S} = \{(u,v) \in \mathbf{P} \times \mathbf{P} : u(r) \leq v(r), \text{ for all } r \in [0,c]\}. \tag{17}$$

In lieu of one of the hypothesis, let $\tilde{v} \in \mathcal{C}'[0,c]$ be a lower solution of (13). Now, we shall show that $(\tilde{v}, \mathcal{G}\tilde{v}) \in \mathcal{S}$. We have

$$\tilde{v}'(r) + \tau\tilde{v}(r) \leq \Psi(r, \tilde{v}(r)) + \tau\tilde{v}(r), \quad \text{for all } r \in [0,c].$$

By multiplying to both of the sides with $e^{\tau r}$, we obtain

$$(\tilde{v}(r)e^{\tau r})' \leq [\Psi(r, \tilde{v}(r)) + \tau\tilde{v}(r)]e^{\tau r}, \quad \text{for all } r \in [0,c],$$

thereby yielding

$$\tilde{v}(r)e^{\tau r} \leq \tilde{v}(0) + \int_0^r [\Psi(\xi, \tilde{v}(\xi)) + \tau\tilde{v}(\xi)]e^{\tau\xi}d\xi, \quad \text{for all } r \in [0,c]. \tag{18}$$

Using the fact $\tilde{v}(0) \leq \tilde{v}(c)$, we find

$$\tilde{v}(0)e^{\tau c} \leq \tilde{v}(c)e^{\tau c} \leq \tilde{v}(0) + \int_0^c [\Psi(\xi, \tilde{v}(\xi)) + \tau\tilde{v}(\xi)]e^{\tau\xi}d\xi$$

so that

$$\tilde{v}(0) \leq \int_0^c \frac{e^{\tau\xi}}{e^{\tau c}-1}[\Psi(\xi, \tilde{v}(\xi)) + \tau\tilde{v}(\xi)]d\xi. \tag{19}$$

Employing (18) and (19), we find

$$\begin{aligned} \tilde{v}(r)e^{\tau r} &\leq \int_0^c \frac{e^{\tau\xi}}{e^{\tau c}-1}[\Psi(\xi, \tilde{v}(\xi)) + \tau\tilde{v}(\xi)]d\xi + \int_0^r e^{\tau\xi}[\Psi(\xi, \tilde{v}(\xi)) + \tau\tilde{v}(\xi)]d\xi \\ &= \int_0^r \frac{e^{\tau(c+\xi)}}{e^{\tau c}-1}[\Psi(\xi, \tilde{v}(\xi)) + \tau\tilde{v}(\xi)]d\xi + \int_r^c \frac{e^{\tau\xi}}{e^{\tau c}-1}[\Psi(\xi, \tilde{v}(\xi)) + \tau\tilde{v}(\xi)]d\xi, \end{aligned}$$

implying thereby

$$\begin{aligned} \tilde{v}(r) &\leq \int_0^r \frac{e^{\tau(c+\xi-r)}}{e^{\tau c}-1}[\Psi(\xi,\tilde{v}(\xi))+\tau\tilde{v}(\xi)]d\xi + \int_r^c \frac{e^{\tau(\xi-r)}}{e^{\tau c}-1}[\Psi(\xi,\tilde{v}(\xi))+\tau\tilde{v}(\xi)]d\xi \\ &= \int_0^c M(r,\xi)[\Psi(\xi,\tilde{v}(\xi))+\tau\tilde{v}(\xi)]d\xi \\ &= (\mathcal{G}\tilde{v})(r), \quad \text{for all } r\in[0,c], \end{aligned}$$

so that $(\tilde{v},\mathcal{G}\tilde{v})\in\mathcal{S}$.

Next, we shall verify that $\mathcal{S}$ is $\mathcal{G}$-closed. Choose $u,v\in\mathbf{P}$ such that $(u,v)\in\mathcal{S}$. Making use of (14), we find

$$\Psi(r,u(r))+\tau u(r)\leq\Psi(r,v(r))+\tau v(r), \quad \text{for all } r\in[0,c]. \tag{20}$$

By (16), (20) and $M(r,\xi)>0$, for all $r,\xi\in[0,c]$, we obtain

$$\begin{aligned} (\mathcal{G}u)(r) &= \int_0^c M(r,\xi)[\Psi(\xi,u(\xi))+\tau u(\xi)]d\xi \\ &\leq \int_0^c M(r,\xi)[\Psi(\xi,v(\xi))+\tau v(\xi)]d\xi \\ &= (\mathcal{G}v)(r), \quad \text{for all } r\in[0,c], \end{aligned}$$

which in view of (17) yields that $(\mathcal{G}u,\mathcal{G}v)\in\mathcal{S}$ and hence the conclusion is immediate.

Now, equip a metric ρ on $\mathbf{P}$ as

$$\rho(u,v)=\sup_{r\in[0,c]}|u(r)-v(r)|, \quad \forall\, u,v\in\mathbf{P}. \tag{21}$$

Clearly, the metric space $(\mathbf{P},\rho)$ is $\mathcal{S}$-complete. To verify the contraction condition, take $u,v\in\mathbf{P}$ such that $(u,v)\in\mathcal{S}$. Making use of (14), (16) and (21), we find

$$\begin{aligned} \rho(\mathcal{G}u,\mathcal{G}v) &= \sup_{r\in[0,c]}|(\mathcal{G}u)(r)-(\mathcal{G}v)(r)| = \sup_{r\in[0,c]}\big((\mathcal{G}v)(r)-(\mathcal{G}u)(r)\big) \\ &\leq \sup_{r\in[0,c]}\int_0^c M(r,\xi)[\Psi(\xi,v(\xi))+\tau v(\xi)-\Psi(\xi,u(\xi))-\tau u(\xi)]d\xi \\ &\leq \sup_{r\in[0,c]}\int_0^c M(r,\xi)\tau\theta(v(\xi)-u(\xi))d\xi. \end{aligned} \tag{22}$$

Since, we have $0\leq v(\xi)-u(\xi)\leq\rho(u,v)$, therefore monotonicity of θ provides that

$$\theta(v(\xi)-u(\xi))\leq\theta(\rho(u,v)).$$

Using above inequality, (22) reduces to

$$\begin{aligned} \rho(\mathcal{G}u,\mathcal{G}v) &\leq \tau\theta(\rho(u,v))\sup_{r\in[0,c]}\int_0^c M(r,\xi)d\xi \\ &= \tau\theta(\rho(u,v))\sup_{r\in[0,c]}\frac{1}{e^{\tau c}-1}\left[\frac{1}{\tau}e^{\tau(c+\xi-r)}\Big|_0^r+\frac{1}{\tau}e^{\tau(\xi-r)}\Big|_r^c\right] \\ &= \tau\theta(\rho(u,v))\frac{1}{\tau(e^{\tau c}-1)}(e^{\tau c}-1) \\ &= \theta(\rho(u,v)) \end{aligned}$$

implying thereby

$$\rho(\mathcal{G}u,\mathcal{G}v)\leq\theta(\rho(u,v))+L\min\{\rho(u,\mathcal{G}u),\rho(v,\mathcal{G}v),\rho(u,\mathcal{G}v),\rho(v,\mathcal{G}u)\},$$
$$\text{for all } u,v\in\mathbf{P} \text{ satisfying } (u,v)\in\mathcal{S}$$

where $L\geq 0$ is arbitrary.

Let $\{v_n\}\subset\mathbf{P}$ be an $\mathcal{S}$-preserving sequence that converges to $v\in\mathbf{P}$ implying thereby $v_n(r)\leq v(r)$, for all $n\in\mathbb{N}$ and for all $r\in[0,c]$. By (17), we have $(v_n,v)\in\mathcal{S}$, for all $n\in\mathbb{N}$. Thus, $\mathcal{S}$ is ρ-self-closed. Therefore, the assumptions (i)-(v) of Theorem 4 holds and so $\mathcal{G}$ admits a fixed point.

Take arbitrary $u,v\in\mathbf{P}$ so that $\mathcal{G}(u),\mathcal{G}(u)\in\mathcal{G}(\mathbf{P})$. Set $w:=\max\{\mathcal{G}u,\mathcal{G}v\}$ implying thereby $(\mathcal{G}u,\omega)\in\mathcal{S}$ and $(\mathcal{G}v,\omega)\in\mathcal{S}$. This shows that the set $\mathcal{G}(\mathbf{P})$ is $\mathcal{S}$-directed. Consequently, using Theorem 5, $\mathcal{G}$ possesses a unique fixed point, which leads to the desired unique solution of (13). □

6. Conclusions

In this manuscript, we have investigated the fixed point results via a locally $\mathcal{G}$-transitive relation under a strict almost φ-contraction in the sense of Boyd and Wong [5]. We also deduced a corresponding result in abstract metric space, which generalizes the main results of Boyd and Wong [5] (i.e., Theorem 1) and Babu et al. [8] (i.e., Theorem 3). On the other hand, for a partial order relation, Theorems 4 and 5 reduce to the enriched versions of several existing results. This substantiates the utility of our results in comparison to other known findings in the literature. To demonstrate our findings, we constructed three examples. Examples 1 and 2 illustrate Theorem 5 which respectively verifies two distinct alternating assumptions (firstly, $\mathbf{P}$ is $\mathcal{S}$-continuous; secondly, $\mathcal{S}$ is ρ-self-closed). On the other hand, Example 3 satisfies the hypotheses of only existence result (i.e., Theorem 4) and fails to be uniqueness.

As a future work, one can prove the analogues of Theorems 4 and 5 for locally finitely $\mathcal{G}$-transitive relation under strict almost φ-contraction following the results of Alam et al. [17]. On applying of our findings, we established the existence and uniqueness theorem for BVP when a lower solution exists. Analogously, one can prove similar result in the presence of an upper solution.

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