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Orthogonal Families of Bicircular Quartics, Quadratic Differentials, and Edwards Normal Form

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Abstract: Orthogonal families of bicircular quartics are naturally viewed as pairs of singular foliations of $\hat{\mathbb{C}}$ by vertical and horizontal trajectories of a non-vanishing quadratic differential. Yet the identification of these trajectories with real quartics in $\mathbb{CP}^2$ is subtle. Here, we give an efficient, geometric argument in the course of updating the classical theory of confocal families in the modern language of quadratic differentials and the Edwards normal form for elliptic curves. In particular, we define a *parameterized Edwards transformation,* providing explicit birational equivalence between each curve in a confocal family and a fixed curve in normal form.

Keywords: Edwards normal form; quadratic differential; confocal family; bicircular quartic; foci; elliptic curve

MSC: 01A55; 14H52; 30F30; 33E05; 51N35; 37F75

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1. Introduction

The *geometric representation* of elliptic functions by algebraic plane curves was an important theme in nineteenth century mathematics. Out of a number of threads woven into this theme, we consider two types of elliptic curve parameterization, whose relationship to each other is not obvious. We spell out the connection between the two using quadratic differentials, the birational equivalence of curves, and certain bicircular quartics viewed as a generalization of the Edwards normal form.

For the elementary sine function $x = \sin t$, the two types of parameterization are almost equally well known:

(A) Since $x = \sin t$ satisfies the ODE $(\frac{dx}{dt})^2 = g(x) = 1 - x^2$, the functions x and $y = x'$ together parameterize the circle $y^2 = g(x)$.

(B) The complex function $z = \sin(s + it)$ parameterizes ellipses $s \mapsto \sin(s + it_0)$ and hyperbolas $t \mapsto \sin(s_0 + it)$ in $\mathbb{C} \simeq \mathbb{R}^2$ with common foci at the critical values of sine—i.e., the zeros of $g(z)$, $z = \pm 1$.

The most obvious difference between A and B is this: while A parameterizes a single curve, B parameterizes *curve families*. In fact, the ellipses and hyperbolas in B evidently form an *orthogonal system*. That they are also *confocal* is part of what deserves further geometric explanation, though it is often presented as a computational fact: one applies trig identities to $x + iy = \sin(s + it)$ to eliminate the parameter s or t and obtain quadric curves $f(x, y) = 0$, from which the foci $(x, y) = (\pm 1, 0)$ may be read.

Turning to elliptic functions, the story of A is by far the more famous nowadays: "Replace $g(x) = 1 - x^2$ in the above ODE by a cubic or quartic $g(x)$ with distinct roots, use x and $y = x'$ to parameterize the elliptic curve $y^2 = g(x), \ldots$". Such an idea (if not yet the modern notion of *elliptic curve*) would have been implicit in early work on the theory of elliptic functions, but according to Stillwell [1], the idea was first explicitly discussed by Clebsch in 1864 [2].

The corresponding curve parameterizations B appear in the 1860 paper of F. H. Siebeck [3]: "On a family of curves of the fourth degree which are related to elliptic functions". Here, among other things, Siebeck demonstrates that the Jacobi elliptic sine and cosine functions $\operatorname{sn} z = \operatorname{sn}(z, k)$ and $\operatorname{cn} z = \operatorname{cn}(z, k)$ map horizontal and vertical lines to confocal orthogonal systems of quartic plane curves with four (real) foci at the critical values of the elliptic function. Siebeck's eye-catching discovery is illustrated in Figure 1, which is reproduced from his paper.

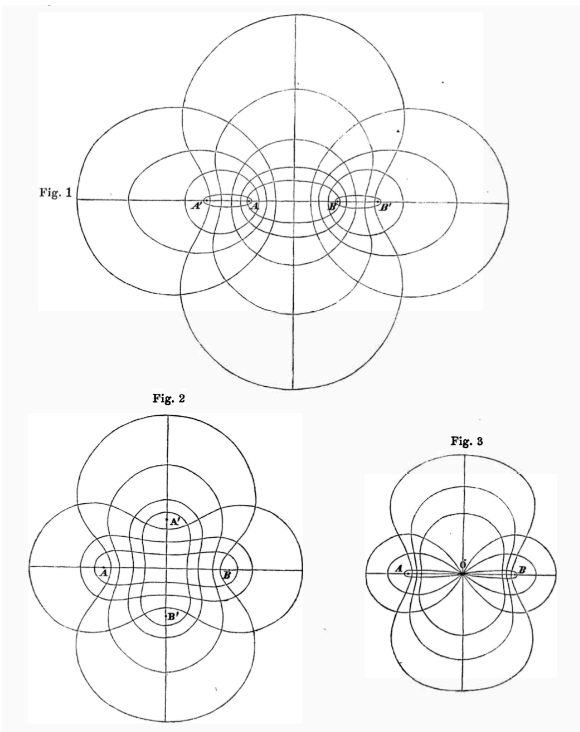

Figure 1. Figure from Siebeck's 1860 paper [3]: three types of confocal families of bicircular quartics.

Siebeck himself began with the analogy to Euclidean conics, yet he was unaware that his bicircular quartics actually represented *non-Euclidean conics*: In fact, all three graphics in Figure 1 may be viewed as confocal families of hyperbolic conics [4]. Although spherical conics had already been considered by Chasles (and subsequently Darboux), and there was also a more comprehensive treatment of non-Euclidean conics by 1882 [5], it might have been challenging even then to make a fully satisfactory connection to Siebeck's work. (Further comments and historical references are included in Appendix D, where we also briefly summarize the contents of [3], which is in German.)

A large part of the issue here is that there are several different natural contexts within which all such Euclidean and non-Euclidean conics may be viewed. The notion of *focus* itself takes on various—not always equivalent—meanings, depending on the setting. For this reason, we devote two sections to the topic. In Section 2, we have the audacity to give a new definition of focus; it suits the present purposes at least. In Appendix C, we provide the historical context and broader justification.

Yet there is no escaping the fact that the relationship $A \leftrightarrow B$ itself calls for descriptions of relevant objects in two settings at once. We first consider bicircular quartics as real curves in the complex projective plane $\mathbb{C}\mathrm{P}^2$—which is the setting of B (that of A being essentially the Riemann sphere $\hat{\mathbb{C}} \simeq \mathbb{C}\mathrm{P}^1$, where the ODE lives). The classical theory of bicircular quartics is a rich subject, and we require several sections (including appendices) to provide sufficient details (Sections 3 and 4, Appendices A, B and E).

This is the case even though Section 3 essentially short-circuits much of the theory; namely, a three-dimensional real projective space of bicircular quartics $P^3 \subset P^8$ suffices to represent all bicircular quartics, up to inversive equivalence (as explained in Appendix E). Within P^3, we eventually find all the relevant orthogonal families, including the three types in Figure 1 (as well as the Euclidean ellipses/hyperbolas—so Siebeck's *analogy* is really more than that).

A simple scaling puts any elliptic curve in P^3 in *standard position*—which fixes the set of four *mirrors* of inversion symmetry which any such curve is known to possess. (This is not Siebeck's normalization, which is imposed by the Jacobi elliptic functions, but is essentially the *generalized Edwards normal form* of Section 6.)

Standard position (SP) sets up the section on quadratic differentials Section 5, where we begin to establish the bridge $A \leftrightarrow B$. Each elliptic curve k in SP determines a *focal quadratic differential* $Q_{\{k\}}$, defined on $\hat{\mathbb{C}}$, with poles at its foci. Meanwhile, the *clinant quadratic differential* Q_k is defined on k as an elliptic curve. Despite being defined on different Riemann surfaces, Q_k and $Q_{\{k\}}$ are found to "agree" (along the real locus of k), by virtue of a key identity (Lemma 1). Now the point is that all curves in a given confocal family have the same focal quadratic differential $Q_{\{k\}}$, which effectively "glues together" all the Q_k's. But in $\hat{\mathbb{C}}$, the orthogonal family is simply the pair of foliations defined by the *horizontal* and *vertical trajectories* of $Q_{\{k\}}$.

Our use of Q_k may not be essential here, but it has a certain geometric appeal. In effect, we are introducing yet a third classical curve parameterization into the mix (see Remark 1): (C) Arc length parameterization. Namely, $Q_k = dx^2 + dy^2$ may be described as (an analytic continuation of) the usual arc length element along $k_{\mathbb{R}}$, and among its trajectories is the arc-length parameterization of $k_{\mathbb{R}}$ itself. Q_k is intimately related to our subject, already because it has singularities corresponding to the foci of k.

Remark 1. *Among the geometric representations of elliptic functions, C played perhaps the most central role historically [1]. It was the lemniscate integral that led Euler and Gauss to key insights about elliptic integrals and functions, and Abel to his subdivisibility result for the lemniscate, analogous to Gauss's result for the circle. The latter results involve ruler and compass constructibility and Fermat primes; some modern analogues involve origami constructibility and Pierpont primes [6–10]. In particular, the Kiepert trefoil admits such a result [11], and the related fact that this exceptional sextic curve has unit speed parameterization by Dixon elliptic functions [12] is explained in [13] using the clinant quadratic differential* Q_k.

We note that arc length parameterization is almost never defined by elliptic (or elementary) functions—precisely because the analytic continuation of the arc length parameterization of $k_{\mathbb{R}}$ develops branch points at singularities of Q_k [13,14]. A rare exception is provided by the circle (where C happens to coincide with A in the case of $\sin t$). Another is the Bernoulli lemniscate, which may be parameterized by unit speed via the *lemniscate elliptic sine* function, $\mathrm{sl}\, t = \mathrm{sn}(t, i)$.

The Bernoulli lemniscate leads into the subject of Sections 6 and 7. Each confocal family in Figure 1 contains a *Cassini oval*—the lemniscate being the Cassinian in the bottom (trinodal) family. Quartics in the Edwards normal form may be interpreted as Cassinians (simply by replacing the affine coordinates x, y in the form by *isotropic coordinates* $u = x + iy, v = x - iy$). The confocal families may be regarded as *variations* from Cassinian *core curves* (Section 7).

Such a variation is what provides the *generalized Edwards normal form*, where a key element of the Edwards theory carries over by means of a simple generalization (using again Lemma 1). Namely, where Edwards uses a birational transformation $\mathcal{E}$ to relate any ENF quartic to $z^2 = (a^2 - x^2)(1 - a^2x^2)$, a parameterized version of the same transformation, $\mathcal{E}_{p,q}$, relates all bicircular quartics in a given confocal family to (a minor modification of) the same special quartic. But the latter equation essentially stands in for the confocal quadratic differential $Q_{\{k\}}$, so $\mathcal{E}_{p,q}$ provides an explicit *algebraic integrability* result for the ODE defining trajectories of $Q_{\{k\}}$.

Further, although it is unsurprising that a confocal family is made up of equivalent elliptic curves—and there are several ways of seeing this—it follows from $\mathcal{E}_{p,q}$ that any two curves in the family are related by a rather simple birational transformation. After observations based on this fact, the discussion wraps up with a summary of the global geometric picture: P^3 is stratified via a discriminant Δk, whose highest dimensional strata are foliated by confocal families; roughly speaking, the foci fix the family, while the movable pair of singular foci provide the variation through equivalent elliptic curves. To conclude with a metaphor, if P^3 is a little galaxy, and foci are slowly drifting stars, the singular foci are asteroids in hyperbolic orbit.

2. Foci of Real Algebraic Plane Curves

If $f(x,y) = \sum c_{\mu\nu} x^\mu y^\nu$ is a polynomial with $c_{\mu\nu} \in \mathbb{R}$, then f determines a *real algebraic plane curve* as the set of solutions to $f(x,y) = 0$ in $\mathbb{R}^2$ or $\mathbb{C}^2$. This may be extended to complex projective space $\mathbb{C}P^2$ as the set of solutions to $F(x,y,z) = 0$, where $F(x,y,z) = z^d f(\frac{x}{z}, \frac{y}{z})$ is a homogeneous polynomial. Given a homogeneous real polynomial $F(x,y,z)$, we can of course recover $f(x,y) = F(x,y,1)$. Letting $\bar{f}(x,y) = \overline{f(\bar{x},\bar{y})} = \sum \overline{c_{\mu\nu}} x^\mu y^\nu$ denote the complex conjugation of coefficients, we have $\bar{f} = f$ as the reality condition for the curve.

We will also make use of *isotropic coordinates* $u = x + iy, v = x - iy, w = z$. In these coordinates, the curve is defined by the polynomial $K(u,v,w) := F(\frac{u+v}{2}, \frac{u-v}{2i}, w)$, or $k(u,v) = K(u,v,1)$. In these coordinates, reality is given by the condition $k(u,v) = \bar{k}(v,u)$.

In what follows, F and f will be used for real curves in ordinary (rectangular) coordinates (x,y,z), and k and K for real curves in isotropic coordinates (u,v,w).

The *circular points* $I = (1 : i : 0)$ and $J = (1 : -i : 0)$ in $\mathbb{C}P^2$ are the ideal points which belong to all circles $(x - x_0 z)^2 + (y - y_0 z)^2 = R^2 z^2$. A line $ax + by + cz = 0$ in $\mathbb{C}\mathrm{P}^2$ is *isotropic* if it contains exactly one of the two points I or J. An *isotropic tangent line* is a line through I or J tangent to the curve F.

Poncelet was the first to interpret foci of a conic as points of intersection of isotropic tangent lines [15,16]; Plücker used the same idea to define (real and complex) foci for curves of a higher degree [17]. Subsequently, classical texts considered *infinite foci* [18] and *singular foci* [19] but generally distinguished the latter from *ordinary foci*.

Coolidge simply excludes infinite and singular foci by definition [20]: *"Definition. A point of intersection of tangents to a curve from the two circular points at infinity, the points of contact being both finite, is called a 'focus' of the curve"*. He adds that it "would not be wise" to include singular foci, which fail to behave like ordinary foci under *inversion* (e.g., circle centers do not invert to circle centers).

But a singular focus may be *double, triple*, etc.; of these, only the double singular foci ought to be excluded. For instance, within confocal families of bicircular quartics sharing four foci, one finds Cassinians—which have two ordinary, and two triple (singular) foci. This only makes sense when the latter two points count as foci.

As just described, the needed definition might appear to be rather ad hoc—but it is not. In fact, we adopt an entirely different definition of the (real) *foci* of a curve, thus resolving all ambiguities in the classical notion (see Appendix C).

We use the fact that a $\mathbb{C}$-irreducible curve $f(x,y) = 0$ may be regarded as a compact Riemann surface $\mathcal{K} = \mathcal{K}_f$—extending the analytic submanifold structure on the regular set $f_{\mathrm{reg}} \subset \mathbb{C}^2$ by the addition of finitely many points. (This amounts to the *resolution of singularities* [21–23]; but for all curves arising here, elementary versions of the result suffice [24].)

Isotropic coordinates define *isotropic projections* $\pi_I, \pi_J : \mathbb{C}^2 \to \mathbb{C}$ by $\pi_I(u,v) = u$ and $\pi_J(u,v) = v$. These restrict to analytic functions on f_{reg}, and extend meromorphically to $\pi_I, \pi_J : \mathcal{K} \to \hat{\mathbb{C}} = \mathbb{C}\mathrm{P}^1$. In particular, we consider $\pi = \pi_I$ as a branched covering of Riemann surfaces. A *ramification point* $P \in \mathcal{K}$ is a point of higher multiplicity; a *branch* *point* $u_0 = \pi(P) \in \hat{\mathbb{C}}$ is the image of such a point (usage as in [24]). In other words, $|\pi^{-1}(u_0)| < \deg(\pi)$, where $|\pi^{-1}(u_0)|$ is the number of distinct points in the inverse image and $\deg(\pi) = \max_u |\pi^{-1}(u)|$ is the *degree* of the mapping $\pi : \mathcal{K} \to \hat{\mathbb{C}}$.

Definition 1. *A focus $u_0 \in \hat{\mathbb{C}}$ of a $\mathbb{C}$-irreducible real curve f is a branch point of extended isotropic projection $\pi : \mathcal{K} \to \hat{\mathbb{C}}$. If f is reducible, its focal set is the union* $\mathrm{foc}(f) = \cup_j \mathrm{foc}(f_j)$.

In particular, the natural place for real foci will not be the real plane, but $\hat{\mathbb{C}}$; all the more so for foci as poles of quadratic differentials (Section 5). In the meantime, this section presents some preliminaries, including a formula for the number of foci. The focal set itself is considered in Section 3.

Because I, J are swapped by the *real symmetry* of $\mathbb{C}\mathrm{P}^2$, $(x : y : z) \leftrightarrow (\bar{x} : \bar{y} : \bar{z})$, it will suffice to discuss mostly I. An isotropic line L through I has the equation $x + iy = u_0 z$ for some $u_0 \in \mathbb{C}$; using standard identifications $\mathbb{C} \simeq \mathbb{R}^2 \simeq \{(x : y : 1) : x, y \in \mathbb{R}\} \subset \mathbb{C}\mathrm{P}^2$, the complex number $u_0 = x_0 + iy_0$ is understood to represent the point $(x_0 : y_0 : 1)$ where L meets the *real plane*. (For heuristic purposes, we sometimes regard the ideal line $\overline{IJ} : z = 0$ as isotropic, but it does not represent a point in the real plane $\mathbb{R}\mathrm{P}^2$.)

In homogeneous isotropic coordinates $u = x + iy$, $v = x - iy$, $w = z$ on $\mathbb{C}\mathrm{P}^2$, the line L has the equation $u = u_0 w$, but we will mostly use the non-homogeneous equation $u = u_0$ for the affine line $L \setminus \{I\}$. Isotropic projection $\pi = \pi_I$ ("from I onto the real plane") sends each point of $L \setminus \{I\}$ to u_0.

The Riemann–Hurwitz formula for a branched covering $\pi : \mathcal{K} \to \mathcal{L}$ of compact Riemann surfaces of genus $g(\mathcal{K})$ and $g(\mathcal{L})$ is

$$g(\mathcal{K}) = 1 + \deg(\pi)(g(\mathcal{L}) - 1) + \frac{1}{2} \sum b_P(\pi).$$

Here, $b_P(\pi) := (\mathrm{mult}_P(\pi) - 1)$ denotes the *branching number* of π at P [24] (so $b_P(\pi) = 0$ for all but finitely many points, $b_P(\pi) = 1$ for simple ramification points, etc.).

The *total branching number* of π is denoted $\mathcal{B}(\pi) := \sum b_P(\pi)$. In the case of a meromorphic function, $\pi : \mathcal{K} \to \hat{\mathbb{C}}$, solving for $\mathcal{B}$ in the Riemann–Hurwitz formula with $g(\mathcal{L}) = g(\hat{\mathbb{C}}) = 0$ gives $\mathcal{B}(\pi) = 2(\deg(\pi) + g(\mathcal{K}) - 1)$.

Returning to algebraic curves, Definition 1 identifies the foci of $0 = f(x, y) = k(u, v)$ with branch points of $\pi : \mathcal{K} \to \hat{\mathbb{C}}$—projection from the circular point I. Then $\mathcal{B}(\pi)$ is the *number of foci* (counted with multiplicity), which the above formula may therefore be used to determine.

For this purpose, we consider the *multiplicity* $m = \mu_I(K)$ of I with respect to the corresponding projective curve K, e.g., K is *circular* when $m \geq 1$ and *bicircular* when $m \geq 2$. Then, the projection π has degree $\deg(\pi) = \max_u |\pi^{-1}(u)| = \deg(K) - \mu_I(K) = d - m$.

Putting $\deg(\pi) = \deg(K) - \mu_I(K)$ and $g(K) := g(\mathcal{K})$ into the formula for $\mathcal{B}(\pi)$ gives the number of foci of K:

$$|\mathrm{foci}(K)| = B = 2(\deg(K) - \mu_I(K) + g(K) - 1). \tag{1}$$

In our applications, $|\mathrm{foci}(K)|$ will in fact count *distinct* foci u_0 since these will always be *simple*: $b_P(\pi) = 0$, for all but one point in $\pi^{-1}(u_0)$, and for that point, $b_{P_0}(\pi) = 1$. (Note: the classical term *simple focus* has a different meaning as is explained in Appendix C.)

To begin with the main example, let K be a *bicircular quartic*: K is an irreducible real quartic with double points at I and J. In particular, suppose K has $\delta = 2$ nodes or $\kappa = 2$ cusps, and no other singularities. Then, by the Clebsch formula [21], the curve has genus $g(K) = \frac{1}{2}(d-1)(d-2) - \delta - \kappa = 1$. Thus, K is an elliptic curve. Since $\mu_I(K) = 2$, Equation (1) gives $|\mathrm{foci}(K)| = 2(g(K) + \deg(K) - \mu_I(K) - 1) = 2(1 + 4 - 2 - 1) = 4$. Thus, we have the following.

Proposition 1. *A bicircular quartic has four foci.*

Aside from the elliptic curves, we will also have occasion to consider rational bicircular quartics. For example, the Bernoulli lemniscate B is *trinodal* ($\delta = 3$), giving $g = 0$ and $|\text{foci}(B)| = 2(0+4-2-1) = 2$. In fact, B is *inverse* to a (rectangular) hyperbola H, which satisfies $|\text{foci}(H)| = 2(0+2-0-1) = 2$. The fact that $|\text{foci}(H)| = |\text{foci}(B)|$ is an instance of *inversive invariance*, to be discussed below.

3. Bicircular Quartics: Focal and Elliptic Discriminants

A bicircular quartic takes the form

$$F(x,y,z) = a(x^2+y^2)^2 + (bx+cy)(x^2+y^2)z + F_2(x,y,z)z^2 = 0,$$

with $F_2(x,y,z)$ homogeneous of degree 2. It depends on *nine* homogeneous coefficients $a, b, c, \dots \in \mathbb{R}$, and may thus be represented by a point in eight-dimensional projective space $\{a : b : c : \dots\} \in \mathrm{P}^8$.

However, every bicircular quartic is symmetric with respect to a pair of real, mutually orthogonal circles of inversion (or lines of reflection). For brevity, we refer to these as *mirrors*. In fact, the following result is classical (see, for example [19]):

Theorem 1. *A bicircular quartic (or circular cubic) is self-inverse with respect to each of four mutually orthogonal circles or lines, and the sixteen (real and complex) foci lie by fours on these four circles. At least two and at most three of these circles are real.*

The proof may be found in Appendix E.

Thus, for the purpose of studying confocal families up to inversive equivalence, it will suffice to consider such curves in *rectilinear position* (symmetric with respect to the x, y-axes):

$$F(x,y,z) = s(x^2+y^2)^2 + 2(px^2+qy^2)z^2 + rz^4 = 0. \tag{2}$$

In isotropic coordinates, $K(u,v,w) = F(x,y,z) = F(\frac{u+v}{2}, \frac{u-v}{2i}, w)$:

$$K(u,v,w) = su^2v^2 + \frac{p-q}{2}(u^2+v^2)w^2 + (p+q)uvw^2 + rw^4 = 0. \tag{3}$$

Thus, we may represent a bicircular quartic K as a point in the real projective space $\mathrm{P}^3 = \{(p : q : r : s)\}$ of rectilinear positions.

Now, we begin to analyze the foci of K. Like any bicircular quartic, K is quadratic in either of the first two isotropic coordinates u, v, which simply reflects the fact that isotropic projection $\pi : \mathcal{K} \to \mathbb{C}$ has degree *two*. In particular, the affine curve $k(u,v) = K(u,v,1) = 0$ may be expressed in the form $k = A(u)v^2 + B(u)v + C(u)$:

$$k(u,v) = (su^2 + \frac{p-q}{2})v^2 + (p+q)u\,v + (\frac{p-q}{2}u^2 + r). \tag{4}$$

A point $u_0 \in \mathbb{C}$ is *regular* if $k(u_0, v)$ has two distinct roots. Otherwise, u_0 is *exceptional*: the number of distinct solutions to the equation $k(u_0, v) = 0$ is 0 or 1 (or ∞ in case $k(u,v)$ has a linear component $u - u_0$). A focus $u_0 \in \mathbb{C}$ is exceptional, but the converse is false in general (as will be seen below).

To identify exceptional points, we consider the *v-discriminant* of k:

$$\Delta_v k = (\Delta_v k)(u) := B^2 - 4AC = 2(q-p)(su^4 + r) + 4(pq - rs)u^2, \tag{5}$$

We will also refer to $\Delta_v k$ as the *focal discriminant* (see Proposition 2). In the generic, *elliptic case,* $\pi : \mathcal{K} \to \hat{\mathbb{C}}$ is a branched double cover with *four* branch points, i.e., foci, and these are the roots of the quartic $\Delta_v k$.

To distinguish elliptic cases ($|\mathrm{foci}(K)| = 4$) from non-elliptic cases ($|\mathrm{foci}(K)| < 4$), we take the discriminant Δ_u of the quartic $\Delta_v k$:

$$\Delta K = \Delta k := \frac{1}{16384}\Delta_u(\Delta_v k) = rs(p-q)^2(p^2-rs)^2(q^2-rs)^2. \tag{6}$$

This *elliptic discriminant* gives a natural stratification $\mathrm{P}^3 = \cup \mathrm{P}^k_j$. (Note: we use "elliptic" to distinguish Δk from the usual curve discriminant—which vanishes for any singular quartic; in particular, for genus $g < 3$.) The *discriminant locus* $\mathcal{D} : \Delta K = 0$ consists of lower-dimensional strata defined via factors of ΔK. The 3-strata P^3_j—the *four* connected components of the complement $\mathrm{P}^3 \setminus \mathcal{D} = \cup_{j=0}^3 \mathrm{P}^3_j$—are foliated by hyperbolas representing confocal families of equivalent elliptic curves (see Section 4 and Appendix B). Several of the 2-strata P^2_j are likewise foliated by hyperbolas representing confocal families of rational (trinodal) quartics and conics, while others represent circle pairs (Appendix A). Finally, the circles/lines of reflection for the bicircular quartics in P^3 (Appendix E) belong to the 1/0-strata (albeit with $p = q = \pm i$ in one case).

To summarize the significance of the discriminants, we state a proposition (proved in Appendix A):

Proposition 2. *Let k be a quartic in rectilinear position (Equation (4)), with focal discriminant $\Delta_v k$ (Equation (5)) and elliptic discriminant Δk (Equation (6)).*
(a) $u_0 \in \mathbb{C}$ is a focus of k if and only if it is a simple root of $\Delta_v k$;
(b) k is an elliptic curve $\Leftrightarrow \Delta k \neq 0 \Leftrightarrow |\mathrm{foci}(K)| = 4$;
(c) k is rational-non-circular $\Leftrightarrow |\mathrm{foci}(K)| = 2$;
(d) k is circular or degenerate $\Leftrightarrow |\mathrm{foci}(K)| = 0$.

4. The Confocal Families of Bicircular Quartics $\mathcal{H}^r_\alpha$

In this section, we consider the confocal families of curves represented by points in the space P^3 of rectilinear positions. For this purpose, we replace the elliptic strata P^3_j by *reduced strata* $\hat{\mathrm{P}}^2_j$ obtained simply as follows. Since $s \neq 0$ and $r \neq 0$, we may scale coefficients using $s = 1$, then scale the affine curves $k(u,v) = 0$ (Equation (4)), via $(u,v) \mapsto (\lambda u, \lambda v)$ so that $r = \pm 1$. With this scaling, $f(x,y) = k(u,v)$ is said to be in *standard position*.

Remark 2. *Standard position has the following geometric significance. All bicircular quartics are known to be symmetric with respect to four mirrors—circles of inversion (Appendix E). While rectilinear position makes two of these mirrors the x and y axes, standard position fixes the remaining two: for $r = 1$, the (real/imaginary) circles $x^2 + y^2 = \pm 1$; for $r = -1$, the (complex) circles $x^2 + y^2 = \pm i$.*

The reduced strata $\hat{\mathrm{P}}^2_j$ are open subsets of one of the p,q-*planes* $\mathbb{R}^2_r := \{(p : q : r : 1)\}$, $r = \pm 1$. As will be seen presently, each $\hat{\mathrm{P}}^2_j$ is foliated by hyperbolas $\mathcal{H}_\alpha$ (Equation (10)); for a fixed value of a parameter α, which determines the four foci, $\mathcal{H}_\alpha$ represents a confocal family of curves. In fact, α is related to the j-invariant for K (Remark 3), and $\mathcal{H}_\alpha$ consists of equivalent elliptic curves (Section 6). $\mathcal{H}_\alpha$ itself is "parameterized" by the *squared singular focus* $\sigma^2 = \frac{q-p}{2}$ (the singular foci $\pm\sigma$ is discussed in Appendix A).

Thus, we express the standard position:

$$k(u,v) = (u^2 - \sigma^2)v^2 + (p+q)uv + (r - \sigma^2 u^2);\ \sigma^2 = \frac{q-p}{2}. \tag{7}$$

We "normalize" the focal discriminant $\Delta_v k$ (Equation (5)) by factoring out $4\sigma^2$. Noting that $p \neq q$ on $\hat{\mathrm{P}}^2_j$, we obtain the monic polynomial:

$$\delta(u) = \delta^r_\alpha(u) := \frac{\Delta_v k}{4\sigma^2} = (u^2 - b^2)(u^2 - \frac{r}{b^2}) = u^4 - 2\alpha u^2 + r. \tag{8}$$

Here, the foci (roots of $\delta(u)$) are denoted $\pm b, \pm\frac{\sqrt{r}}{b}$ and we introduce the *focal parameter* α:

$$\alpha := \frac{pq-r}{p-q} = \frac{1}{2}(b^2 + \frac{r}{b^2}). \tag{9}$$

The foci are distinct from the singular foci $\pm\sigma$ (except in the case of Cassinians, where $q = -p = \sigma^2$ and $\alpha = \frac{1}{2}(\sigma^2 + \frac{r}{\sigma^2})$). Finally, we note that these formulas are meaningful even when $r = 0$, and will thus include the trinodal quartics as a limiting case.

Remark 3. *The focal parameter α is related to the cross ratio of the four foci (or the "pencil of isotropic tangents"). Namely, $\lambda = [p_0, p_1, p_2, p_3] := \frac{(p_0-p_1)(p_2-p_3)}{(p_1-p_2)(p_3-p_0)}$, becmes $\lambda = [b, -\frac{\sqrt{r}}{b}, \frac{\sqrt{r}}{b}, -b] = \frac{\alpha}{2\sqrt{r}} + \frac{1}{2}$, i.e., $\alpha = \sqrt{r}(2\lambda - 1)$. Ultimately, this implies equivalence of confocal bicircular quartics, since λ may be shown to determine the j-invariant of k as an elliptic curve. But we do not require the j-invariant for this conclusion since we have more direct arguments in Section 6.*

We formally define *elliptic confocal families* $\mathcal{H}^r_\alpha \subset \mathrm{R}^2_r$ of a given type $r = \pm 1$ by fixing the focal parameter α, while allowing p, q to vary.

Definition 2. *For $r = 1$ and real $\alpha \neq \pm 1$, or for $r = -1, \alpha \in \mathbb{R}$, we define $\mathcal{H}^r_\alpha = \{K = (p : q : r : 1) \in \mathrm{P}^3 : p \neq q, \frac{pq-r}{p-q} = \alpha\}$.*

For a given $\alpha = \frac{pq-r}{p-q}$, we see that indeed $\mathcal{H} = \mathcal{H}^r_\alpha$ consists of points (p, q) on a rectangular hyperbola in the p, q-plane R^2_r:

$$\mathcal{H}\colon (p+\alpha)(q-\alpha) + \alpha^2 - r = 0. \tag{10}$$

For variable α, $\mathcal{H}$ in fact describes a pencil of hyperbolas, each of which represents a confocal family of elliptic curves.

In case $r = -1$, each quartic is one-circuited, and the mirrors are the x and y axes and the *complex* circles $x^2 + y^2 = \pm i$ (Appendix E). The left side of Figure 2 shows the orthogonal pair of foliations for the particular choice $\alpha = 0$.

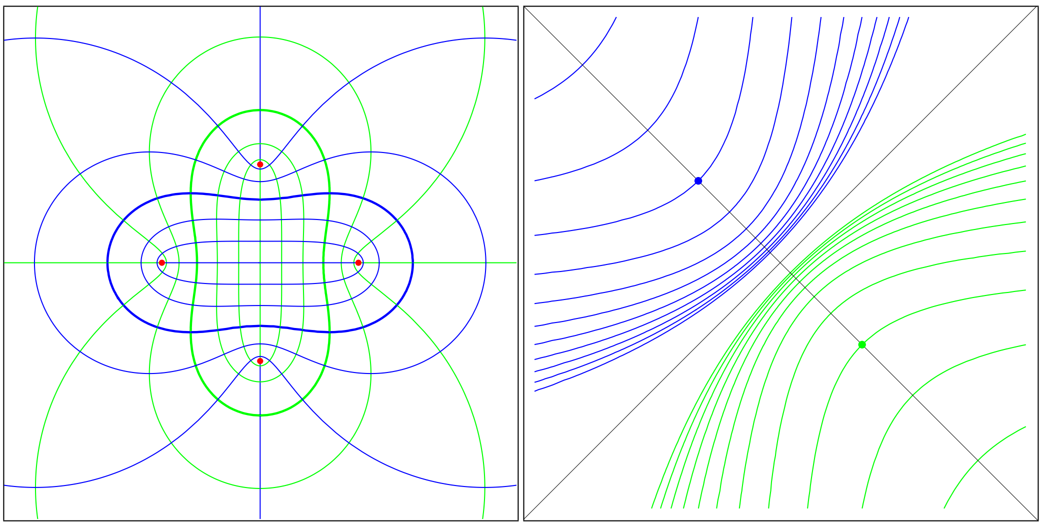

Figure 2. (**Left**): The orthogonal (green/blue) foliations of the confocal family $\mathcal{H}^r_\alpha$ with $\alpha = 0$, $r = -1$; the foci are the four *red* points. The (green/blue) *bold* curves are the two Cassinians in $\mathcal{H}^{-1}_0$. (**Right**): The p, q-plane for $r = -1$. The region $\hat{\mathrm{P}}^2_0 = \{p \neq q\}$ is filled by hyperbolas $\mathcal{H}_\alpha$ representing confocal families for each $\alpha \in \mathbb{R}$. The (green/blue) points and their branches represent the Cassinians and corresponding foliations of $\mathcal{H}^{-1}_0$.

On the right side of the figure, the complement of the diagonal in the p,q-plane is foliated by the hyperbolas $\mathcal{H}_\alpha, \alpha \in \mathbb{R}$. The upper branch of $\mathcal{H}_\alpha$ represents all the (blue) curves enclosing the foci $\pm b$; the lower branch represents (green) curves enclosing the foci $\pm ib$. Points on the anti-diagonal $p = -q$ represent *Cassinians*. These play a distinguished role in Sections 6 and 7; each represents the "half-way" curve in its family (see Remark 8).

Remark 4. *It is easy to explain why two curves in the confocal family pass through a general point $(x_0, y_0) \in \mathbb{R}^2$. (Orthogonality will follow from Theorem 2.) Consider the line in the p,q-plane:*

$$\mathcal{L}_{(x_0,y_0)} : f(x_0, y_0) = (x_0^2 + y_0^2)^2 + 2x_0^2 p + 2y_0^2 q + r = 0. \tag{11}$$

Since $\mathcal{L}_{(x_0,y_0)}$ is a line of negative slope (for $x_0, y_0 \neq 0$), and the hyperbola $\mathcal{H}$ has a positive slope, there is a point of intersection on each branch. (For points on the x or y axes, the horizontal/vertical lines $\mathcal{L}_{(x_0,y_0)}$ meet $\mathcal{H}$ in only one finite point, and determine just one irreducible quartic; the other quartic is either y^2z^2 or x^2z^2.)

Remark 5. *The confocal family plotted in the figure is atypical of the case $r = -1$ in one respect; it belongs to the exceptional case $\alpha = 0$, and consequently the foci $\pm 1, \pm i$ happen to be concyclic (with the harmonic cross ratio). The other values of α result in a rhombic focal set. We note that the case $r = 1$ also includes a confocal family with foci $\pm 1, \pm i$ (but its curves are two-circuited). Since the picture is more complicated to describe in detail for $r = 1$, we defer this discussion to Appendix B.*

5. Bicircular Quartics and Quadratic Differentials

Though the constructions of this section apply, in principle, to bicircular quartics in general, we usually assume a standard (or at least rectilinear) position. We define the *focal quadratic differential of $k(u,v)$*,

$$Q_{\{k\}} = Q_\alpha^r = \frac{du^2}{\delta(u)} = \frac{du^2}{u^4 - 2\alpha u^2 + r} = \frac{du^2}{(u^2 - b^2)(u^2 - \frac{r}{b^2})}, \tag{12}$$

a quadratic differential on $\hat{\mathbb{C}}$, whose horizontal/vertical trajectories are parameterized by solutions to the ODE: $(\frac{du}{dt})^2 = \pm\delta(u)$.

Since the poles of $Q_{\{k\}}$ are exactly the foci of the confocal family $\mathcal{H}_\alpha^r$, it is already plausible that such trajectories of Q_α^r represent the two confocal foliations defining $\mathcal{H}_\alpha^r$. Of course, we depend on the identification of the real plane $\mathbb{R}^2 = \{(x,y)\}$ with $\mathbb{C} = \{u = x + iy\}$ to relate the curves in the two settings. To be quite precise, we make a distinction between the confocal families $\mathcal{H}$ in $\mathbb{RP}^2$ and the corresponding *orthogonal families* $\mathcal{F} = \mathcal{F}_\alpha^r$ in $\hat{\mathbb{C}}$, which differ also in the following respect. In order for $\mathcal{F}$ to define a pair of foliations on the complement of the focal set in $\hat{\mathbb{C}}$, the leaves of $\mathcal{F}$ are understood to include the real mirrors of $\mathcal{H}$ (or arcs between foci on such the circles/lines). The notion of the orthogonal family is discussed in greater detail in Appendix B, where the case $r = 1$ more fully illustrates the difference between $\mathcal{H}$ and $\mathcal{F}$.

The divide between the two settings $\mathbb{RP}^2$ (or $\mathbb{CP}^2$) and $\hat{\mathbb{C}}$ is undoubtedly reflected in the difficulty of identifying curves in $\mathcal{H}$ with those in $\mathcal{F}$ by direct computation. For instance, one may integrate the ODE for trajectories $u(t) = x(t) + iy(t)$ of the quadratic differential Q_α^r in terms of elliptic functions, but the process of eliminating t to obtain quartics $f(x,y)$ (or $k(u,v)$) requires extensive computations involving various elliptic function identities (as in [3]). But since the ODE and the required implicit solution are both specified by polynomials—$\delta(u)$ and $k(u,v)$—one may naturally seek to bypass the elliptic functions altogether.

This is, in fact, not hard to do, with the help of quadratic differentials. For this purpose, we first introduce an auxiliary quadratic differential Q_k defined on a given real, irreducible curve $f(x,y) = k(u,v) = 0$. Heuristically, Q_k may be obtained via the analytic continuation of the real arc length element $ds^2 = dx^2 + dy^2 = du\,dv$ along the curve in $\mathbb{R}^2$ to the complex curve $\Gamma \subset \mathbb{C}^2$. Along a real arc $\gamma \subset \Gamma \cap \mathbb{R}^2$, Q_k is *positive*, i.e., γ is *horizontal*.

More formally, isotropic coordinates $u = x + iy$, $v = x - iy$ on k_{reg} extend to meromorphic functions on the Riemann surface $\mathcal{K}$ as in Section 2. The corresponding meromorphic differentials $du = (dx + idy)$, $dv = (dx - idy)$ satisfy the equation $dk = k_u du + k_v dv = 0$. Then, we may express Q_k, locally, in various ways:

$$Q_k = du dv = \frac{dv}{du} du^2 = -\frac{k_u}{k_v} du^2 = -\frac{k_v}{k_u} dv^2. \quad (13)$$

We call Q_k the *clinant quadratic differential of* k, for reasons to be explained below (see also [11,14,25] for geometric applications of Q_k).

In particular, one of u or v will be used as the local coordinate near a regular point of the curve, e.g., if $v = S(u)$ is one of the locally defined analytic functions satisfying $k(u, S(u)) = 0$, then $Q_k = S'(u)du^2$. Here, S denotes the *Schwarz function*, as defined by Davis [26]. Given an analytic arc $\gamma \subset \mathbb{C}$, this is the unique analytic function $w = S(z)$, defined in a suitable neighborhood of γ, satisfying $\bar{z} = S(z)$ along γ.

The *clinant* (or *inclination function*) along γ is the complex unit $S'(z) = \frac{d\bar{z}}{dz} = e^{-2i\theta}$, where θ measures the angle that the tangent to γ makes with the real axis. (This explains our name for $Q_k = S'(u)du^2$. The word *clinant* was defined as such by Davis [26] but appears to date back to Franklin [27], who explains the usage for the reciprocal, $e^{2i\theta}$. Franklin may have been familiar with Darboux's earlier use of the quantity $\frac{dx+idy}{dx-idy} = e^{2i\theta}$ [28].) Now observe that if $\gamma(\tau) = x + iy = u$ parameterizes a real arc of k, we may write $\gamma' = \rho e^{i\theta}$, and thus verify the required condition $(\gamma')^2 S'(\gamma) = \rho^2 > 0$ for γ to be a horizontal arc of $Q_k = S'(u)du^2$.

Now to relate the trajectories of Q_k and $Q_{\{k\}}$, we require the following lemma (which will also be used in Section 6):

Lemma 1. *Let $k(u, v)$ be a bicircular quartic (Equation (4)) and let $\Delta_v k$ be its focal discriminant (Equation (5)). Then the polynomial $(\frac{\partial k}{\partial v})^2 - \Delta_v k$ belongs to the ideal $(k) \subset \mathbb{R}[u, v]$. In fact, the following identity holds (introducing the "operator" $\Box_v$)*

$$\Box_v k(u, v) := (\frac{\partial k}{\partial v})^2 - \Delta_v k = 4(u^2 - \sigma^2) k(u, v), \quad (14)$$

where $\sigma^2 = \frac{q-p}{2s}$ is the squared singular focus of k. Interchanging the roles of u, v, the corresponding statement holds for $\Box_u k(u, v)$.

Proof. Equation (14) may be verified by a straightforward computation, which we omit. But the main consequence—that $\Box_u k$ and $\Box_v k$ belong to the ideal (k)—may be shown more simply, as follows. We note that the discriminant of a quadratic polynomial $P(x) = ax^2 + bx + c$ is the square of its derivative at a root x_0; that is, $\Delta P = b^2 - 4ac = (2ax_0 + b)^2 = (P'(x_0))^2$. This applies to $k(u, v)$, which can be regarded as a quadratic polynomial in either variable, u or v. Thus, along the curve $k = 0$, the corresponding partial derivatives and discriminants are related by $\Delta_v k(u) = b^2 - 4ac = k_v^2$, $\quad \Delta_u k(v) = B^2 - 4AC = k_u^2$. $\square$

The proof of the following theorem uses the lemma to compare the two quadratic differentials $Q_{\{k\}}$ and Q_k along the real locus of k:

Theorem 2. *A circuit of a bicircular quartic k in standard position is a horizontal or vertical trajectory of its focal quadratic differential $Q_{\{k\}} = \frac{du^2}{\delta(u)}$. Thus, the curves comprising the orthogonal family of algebraic curves $\mathcal{F}_\alpha^r$ are exactly the trajectories of $Q_\alpha^r = Q_{\{k\}}$.*

Proof. Combining Lemma 1 with $dk = k_u du + k_v dv = 0$, we conclude that the following separable ODE holds along k (see Remark 6):

$$\left(\frac{dv}{du}\right)^2 = \left(-\frac{k_u}{k_v}\right)^2 = \frac{\Delta_u k}{\Delta_v k} = \frac{\delta(v)}{\delta(u)}.$$

Here, $\Delta_u k = 4\sigma^2 \delta(v)$ and $\Delta_v k = 4\sigma^2 \delta(u)$ involve the same monic polynomial δ for reasons of symmetry. Namely, bicircular quartics in rectilinear position have the reflection symmetry $f(x, -y) = f(x, y) = \sum a_{rs} x^r y^{2s}$. In this case, $k(u, v)$ and $k(v, u)$ have identical, real coefficients, and consequently so do $\Delta_u k$ and $\Delta_v k$. (This symmetry is convenient, but our argument generalizes to all bicircular quartics.)

In particular, at a real point of the curve, we have $\delta(v) = \delta(\bar{u}) = \overline{\delta(u)}$, so the above equation gives $(\frac{dv}{du})^2 = (\frac{|\delta(u)|}{\delta(u)})^2$. Taking the square roots, we can write this as $\frac{dv}{du} = \frac{\pm\lambda}{\delta(u)}$, with $\lambda = |\delta(u)| > 0$. Since a real arc of k is an arc of its clinant quadratic differential $Q_k = \frac{dv}{du} du^2$, it follows that it is also a horizontal or vertical arc of $Q_{\{k\}} = du^2/\delta(u)$.

Next, we use this conclusion (the first claim) to prove its converse. Namely, for given r and α, we wish to show that all horizontal and vertical trajectories of the quadratic differential $Q = Q_\alpha^r$ correspond to curves in the orthogonal family $\mathcal{F} = \mathcal{F}_\alpha^r$.

So let $u_0 = x_0 + iy_0 \in \mathbb{C}$ be any point other than a pole of Q, and let $\gamma_\pm$ be the horizontal and vertical trajectories of Q through u_0. As discussed in Remark 4, there are also two distinct curves $k_\pm \in \mathcal{F}$ containing the point u_0. For most u_0, these are both irreducible bicircular quartics $f_\pm(x, y) = k_\pm(u, v)$. By the first claim, each agrees with one or the other trajectory $\gamma_\pm$. But $k_\pm$ cannot both agree with the same trajectory; so, γ_+ and γ_- must each correspond to one of the curves $k_\pm \in \mathcal{F}$. The exceptional trajectories of Q_α are easily identified with (arcs of) a line/circle of reflection. $\square$

Not only are all (horizontal/vertical) trajectories of $Q_\alpha = \frac{du^2}{\delta(u)}$ algebraic but Q_α will be seen to be *algebraically integrable*: the (elliptic) trajectories of Q_α may be produced directly from the elliptic curve $w^2 = \delta(u)$ via explicit birational transformation (Theorem 3).

Remark 6. *Separable ODEs of the form* $(\frac{dy}{dx})^2 = \frac{f(y)}{f(x)}$*, i.e.,* $\frac{dy}{\sqrt{f(y)}} = \pm\frac{dx}{\sqrt{f(x)}}$*, feature prominently in the classical literature. This is especially so in the case of quartics* $f(x) = ax^4 + bx^3 + cx^2 + dx + e$*, where the ODE is intimately related to the addition theorem for elliptic integrals [1,29,30]. See also [31], Ch. XIV, which begins with a method of algebraic integration of Lagrange. (This is not exactly algebraic integrability in the sense of Theorem 3.)*

6. Edwards Normal Form and Algebraic Integrability

In 2007, Harold Edwards [29] presented an alternative to the elliptic curve normal forms associated with Jacobi or Weierstrass elliptic functions. Any elliptic curve can be represented as a quartic in the *Edwards normal form* (ENF): $a^2(x^2y^2 + 1) = x^2 + y^2$, $a^5 \neq a$. One advantage of this normal form is that the addition $p + p' = p''$ on the elliptic curve may be expressed in a very symmetrical form: $x'' = \frac{1}{a}\frac{xy' + yx'}{1 + xyx'y'}$, $y'' = \frac{1}{a}\frac{yy' - xx'}{1 - xyx'y'}$ (see Theorem 3.1 in [29]).

Edwards notes the close resemblance to the original Euler–Gauss addition law for the special elliptic curve $x^2 + y^2 + x^2y^2 = 1$, which famously generalized the addition law on the circle $x^2 + y^2 = 1$—the addition formulas for $x = \sin t$ and $y = \cos t$. He also implicitly raises a historical question: "I have not been able to find [this addition law] in the literature. If it is not new, it is certainly not as well known as it deserves to be". We will not attempt to address this historical question, but we believe that the best answer is to be found in the investigations of bicircular quartics and their connections to elliptic functions and integrals (see Appendix D for references).

Without going too far afield here, we now proceed to interpret ENF quartics (in the real case) as Cassinians, then relate the Edwards theory more generally to the confocal families of bicircular quartics and their focal quadratic differentials. In fact, we will regard the confocal families as a kind of *generalized ENF*—obtained by "variation of Cassinians" (or in one case *Cayley Cassinians*) by a one-parameter family of birational equivalences.

To begin with the *ovals of Cassini*, we consider Equation (7) in the special case $q = -p = \sigma^2$ (so $\alpha = \frac{1}{2}(q + \frac{r}{q}) = \frac{1}{2}(\sigma^2 + \frac{r}{\sigma^2})$):

$$f := (x^2 + y^2)^2 - 2\sigma^2(x^2 - y^2) + r; \quad k := u^2v^2 - \sigma^2(u^2 + v^2) + r. \tag{15}$$

Observe that $k(u,v)$ is exactly ENF after substitutions $x \to u,\ y \to v, a \to 1/\sigma$—provided $r = 1$; here, we allow $r = -1$ as well. (Note: To put Cassinians with $r = -1$ *exactly* into ENF has certain advantages, but this would ultimately have a modified *standard position*—with awkward consequences for our discussion of the *non-concyclic* case, mirrors, Siebeck's result, etc.)

To develop his normal form, Edwards makes essential use of an auxiliary quartic curve which is birationally equivalent to ENF. In our notation for the Cassinian $k(u,v)$, the equivalent curve is $h(u,w) := w^2 - (\sigma^2 - u^2)(r - \sigma^2u^2)$—roughly, the Legendre normal form. The required birational equivalence has the simple form $(u,v) \overset{\mathcal{E}}{\longleftrightarrow} (u,w)$, and is given by: $w := v(u^2 - \sigma^2)$. In fact, substitution into $h(u,w)$ results in $k(u,v)$ (modulo an extra factor $(u^2 - \sigma^2)$, which can be neglected). Likewise, $v = w/(u^2 - \sigma^2)$ turns $k(u,v)$ into $h(u,w)$.

Where does this birational equivalence come from, and can it be generalized? In view of Lemma 1, the fact that $w = v(u^2 - \sigma^2) = \frac{1}{2}\frac{\partial k}{\partial v}$ is the key. This (and normalization by σ) yields the following generalization of the above equivalence $(u,v) \overset{\mathcal{E}}{\longleftrightarrow} (u,w)$ to bicircular quartics $k(u,v)$ in standard position (Equation (7)):

$$2\sigma w := \frac{\partial k}{\partial v} = 2(u^2 - \sigma^2)v + (p+q)u; \ v = \frac{2\sigma w - (p+q)u}{2(u^2 - \sigma^2)}, \tag{16}$$

where $\sigma^2 = \frac{q-p}{2}$. Writing Lemma 1 as $(\frac{\partial k}{\partial v})^2 - \Delta_v k = (\frac{\partial k}{\partial v})^2 - 4\sigma^2\delta(u) = 4(u^2 - \sigma^2)k$, we find that $4\sigma^2(w^2 - \delta(u)) = 4(u^2 - \sigma^2)k$, so the bicircular quartic k has the equivalent normal form $h(u,w) := w^2 - \delta(u)$:

$$h: \ w^2 = \delta(u) = (u^2 - b^2)(u^2 - \frac{r}{b^2}) = u^4 - 2\alpha u^2 + r. \tag{17}$$

The resulting "new curve" h is now independent of p, q—the reason we normalized via σ.

For $r = \pm 1$, let $\mathcal{E}_{p,q}$ denote the *parameterized Edwards transformation* defined by Equation (16). Trajectories of the quadratic differential $Q^r_\alpha = \frac{du^2}{\delta(u)}$ satisfy the ODE $(\frac{du}{dt})^2 = \delta(u)$, to which we associate the elliptic curve $h : w^2 = \delta(u)$. By applying $\mathcal{E}_{p,q}$ to the latter, for points (p,q) on the hyperbola $\mathcal{H}_\alpha$, all but finitely many trajectories of Q^r_α are obtained (the exceptions being linear or circular arcs). In fact, for a given *initial condition* $u_0 = x_0 + iy_0$, the pair of trajectories through u_0 are thus explicitly produced (the required p, q are given in terms of u_0 by the quadratic formula). This is what we mean by the following:

Theorem 3. *The ODE* $(\frac{du}{dt})^2 = \delta(u)$ *is algebraically integrable.*

7. Birational Equivalence of Confocal Curves

In this section, we consider how two curves in a confocal family are related to each other. Namely, if k and k' are two such bicircular quartics, we can use $\mathcal{E}_{p,q}$ and $\mathcal{E}_{p',q'}$ to relate both to h, and hence to each other: $k(u,v) \leftrightarrow h(u,w) \leftrightarrow k'(u,v)$. To be clear, the resulting equivalence $k \simeq k'$ is a *complex* birational equivalence of real curves.

More explicitly, if k and k' are represented by two points (p,q) and (p',q') on $\mathcal{H}_\alpha$, the two formulas of Equation (16) may be combined to prove the following.

Theorem 4. *If k and k' are confocal bicircular quartics in standard position, they are related by the following birational equivalence:*

$$\frac{u^2-\sigma'^2}{\sigma'}v' + \frac{p'+q'}{2\sigma'}u = \frac{u^2-\sigma^2}{\sigma}v + \frac{p+q}{2\sigma}u \tag{18}$$

$$\frac{u^2-\sigma'^2}{\sigma'^2}k'(u,v') = \frac{u^2-\sigma^2}{\sigma^2}k(u,v). \tag{19}$$

Proof. With $\sigma'^2 = \frac{q'-p'}{2}$, Equation (18) is obtained straightforwardly from Equation (16) as described above. Solving for

$$v' = \frac{\frac{\sigma'}{\sigma}(2(u^2-\sigma^2)v + (p+q)u) - (p'+q')u}{2(u^2-\sigma'^2)}$$

and inserting this expression into $k'(u,v')$ leads to the equation

$$\frac{u^2-\sigma'^2}{\sigma'^2}k'(u,v') - 2\alpha' u^2 = \frac{u^2-\sigma^2}{\sigma^2}k(u,v) - 2\alpha u^2.$$

Under the assumption $\alpha = \alpha' := \frac{p'q'-r}{p'-q'}$, this gives Equation (19), expressing the equivalence of the two curves $k'(u,v)$ and $k(u,v)$. □

Remark 7. *As the theorem illustrates, birational transformations are to binodal quartics as projective transformations are to nonsingular cubics. Namely, if two of the latter plane curves c, c' are equivalent to the elliptic curves ($j(c) = j(c')$), they are in fact projectively equivalent (though not necessarily real projectively equivalent, in the case of real curves [32]). But projective transformations do not suffice for binodal quartics, e.g., if k, k' are confocal bicircular quartics, no projective transformation preserves the shared isotropic tangents to k, k' while appropriately moving the isotropic tangents at circular points. The fact that birational transformations fill the gap is not surprising, given that the two classes of curves are themselves related by inversive equivalence (Appendix E), but this is quite awkward to exploit directly.*

Equations (18) and (19) may also be used, e.g., to transfer the group structure from Cassians (ENF) to bicircular quartics in standard position, or to show that such confocal $k(u,v)$ and $k'(u,v)$ are *rationally* equivalent, say, when $p, q, p', q' \in \mathbb{Q}$ and $(\frac{\sigma'}{\sigma})^2 = \frac{q'-p'}{q-p} = \frac{m^2}{n^2}$, for integers m, n.

The simplest case of the latter arises from reflection symmetry of the hyperbola $\mathcal{H}^r_\alpha$ across the anti-diagonal: $(p,q) \leftrightarrow (-q,-p)$. In this case, $\frac{\sigma'}{\sigma} = 1$, and Equation (18) reduces simply to $v' = v + \frac{(p+q)u}{u^2-\sigma^2}$. Introducing self-evident notation, Equation (19) becomes

$$k(u,v;p,q) = k(u, v + \frac{(p+q)u}{u^2-\sigma^2}; -q, -p) \tag{20}$$

The involution $k(u,v;p,q) \xleftrightarrow{\mathcal{R}} k(u,v;-q,-p)$ fixes Cassinians, and pairs the remaining curves in a confocal family with the same singular foci $\pm\sigma$. In fact, $\mathcal{R}$ corresponds to the *Cassinian reflection* as explained in the following remark.

Remark 8. *For an analytic arc $\gamma \subset \mathbb{C} = \{u = x+iy\}$ with Schwarz function $S(u)$, the Schwarz reflection in γ is the antiholomorphic involution $\mathcal{R}_\gamma(u) := \overline{S(u)}$ fixing γ [26], e.g., the Schwarz function of the circle $|u| = \rho$, is $S(u) = \rho^2/u$ and $\mathcal{R}_\gamma(u) := \rho^2/\bar{u}$ gives circle inversion.*

Turning now to Cassinians—say, one of those in Figure 2—such a curve γ determines a maximal ring domain [33]: D is foliated by closed trajectories of $Q = \frac{du^2}{\delta(u)}$, including γ, and is *maximal with respect to this property. In fact, γ is the core curve of D—it is the unique curve fixed by an antiholomorphic involution of D.*

To check this, we solve for v in Equation (15), obtaining the two-valued function $S(u) = \sqrt{\frac{\sigma^2 u^2 - r}{u^2 - \sigma^2}}$. The branch points of $S(u)$ are the foci $\pm\sigma, \sqrt{r}/\sigma$. These, together with $0, \infty$ lie on the boundary of D, and they correspond under reflection $\mathcal{R}_\gamma$ according to: $\pm\sigma \overset{\mathcal{R}_\gamma}{\longleftrightarrow} \infty$, $\pm\sqrt{r}/\sigma \overset{\mathcal{R}_\gamma}{\longleftrightarrow} 0$. Evidently, Schwarz reflection gives antiholomorphic involution $\mathcal{R}_\gamma : D \to D$ fixing γ. (In case $r = 1$, similar comments apply to the left and right halves of Figure A1.)

Now it turns out that $\mathcal{R}$, as defined after Equation (20), agrees with $\mathcal{R}_\gamma$ in the sense that it swaps the same pairs of curves foliating D. Yet this is rather curious; $\mathcal{R}_\gamma$ is a fixed antiholomorphic involution on D, while $\mathcal{R}$ is realized by complex birational transformations defining Riemann surface equivalences for each of the paired real curves.

In view of the pairing $\mathcal{R}$ of confocal curves with shared singular foci, it is reasonable to regard the squared singular focus $\sigma^2 = \frac{q-p}{2}$ as the "geometric parameter" for a confocal family. (Of course, p or q may be used to rationally parameterize $\mathcal{H}^r_\alpha$, but for most purposes, there does not appear to be a great advantage to doing so.)

Thus, the global geometric picture may be described roughly as follows. Returning to the projective space of bicircular quartics in rectilinear position, $\mathrm{P}^3 = \{(p : q : r : s)\}$, the complement of the discriminant locus ($\Delta k \neq 0$) is foliated by confocal families; the foci determine the confocal family, while the singular foci fix the position (essentially) on a given confocal family.

8. Conclusions

We considered real bicircular quartics in two different contexts: as algebraic curves in $\mathbb{C}\mathrm{P}^2$, and as members of confocal families of curves in $\hat{\mathbb{C}}$. The relationship between these two points of view was established by the use of quadratic differentials Q_k defined on each elliptic curve k in a confocal family $\{k\}$, and a quadratic differential $Q_{\{k\}}$ defined on $\hat{\mathbb{C}}$. A moduli space of bicircular quartics (w.r.t. inversive equivalence) was presented as a stratified real projective space P^3, whose points in the top-dimensional strata represent elliptic curves with four ordinary foci and two singular foci. A generic quartic k was essentially specified by its focal discriminant $\Delta_v k$, a quartic polynomial in a variable u whose roots are the foci of k, and a parameter σ^2, the squared singular focus, which parametrized the family of confocal quartics. We showed explicitly that curves in each family are all birationally equivalent via a parameterized Edwards transformation.

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Appendix A. The Space of Bicircular Quartics $\mathbf{P}^3 = \cup \mathbf{P}^k_j$

Here, we give further details regarding the stratification $\mathrm{P}^3 = \cup \mathrm{P}^k_j$ defined by the elliptic discriminant ΔK, and prove Proposition 2.

First, for K as in Equation (3), we read off the tangents to K at I from the lowest-order terms in the affine view $v = 1$:

$$K(u, 1, w) = su^2 + \frac{p-q}{2} w^2 + (p+q)uw^2 + \frac{p-q}{2} u^2 w^2 + rw^4.$$

Unless $s = 0$ or $p = q$, I is indeed a node, with distinct tangents

$$L_+L_- = su^2 + \frac{p-q}{2}w^2 = s(u - \sigma w)(u + \sigma w);\ \sigma = \sqrt{\frac{q-p}{2s}}.$$

Here, $\pm\sigma$ are the two *singular foci* of K. These are usually not foci (as noted in Section 2); exceptions will be discussed shortly. For future reference, we observe that $\pm\sigma$ are the roots of $A(u) = (su^2 + \frac{p-q}{2})$, the leading coefficient of $k(u, v)$ (Equation (4))—so $k(\pm\sigma, v)$ has at most one root.

In particular, when $\Delta K \neq 0$, K has nodes at I and J. In fact, there are no other singularities. It is useful to verify this directly by showing that the system $F_x = F_y = F_z = 0$ has no solution with $z = 1$. We scale coefficients so that $s = 1$:

$$F_x = 4x(x^2 + y^2 + p),\ F_y = 4y(x^2 + y^2 + q),\ F_z = 4(px^2 + qy^2 + r)$$

For these all to vanish, either: $xy \neq 0 \Rightarrow p = q$; $x = y = 0 \Rightarrow r = 0$; $x = 0, y \neq 0 \Rightarrow r = q^2$; or $y = 0, x \neq 0 \Rightarrow r = p^2$. In each case, a factor of ΔK vanishes. To conclude, $\Delta K \neq 0 \Rightarrow K$ is an elliptic curve.

Conversely, by considering the factors of ΔK, we find that $\Delta K = 0$ implies that K is non-elliptic. In particular, the 2-dimensional strata P^2_j—defined by exactly one distinct factor of ΔK vanishing—consist of rational or reducible curves satisfying one of the following descriptions (here, we normalize by $s = 1$, except when $s = 0$):

- $\underline{r = 0}$ Trinodal: $F = (x^2 + y^2)^2 + 2(px^2 + qy^2)$: nodes $I, J, O = (0, 0)$.
- $\underline{s = 0}$ Conics: $F = F_2z^2 = (2px^2 + 2qy^2 + rz^2)z^2$.
- $\underline{p = q}$ Concentric circles: $K = C_1C_2 = u^2v^2 + 2puv + r$, center O.
- $\underline{p^2 = r}$ Circle pairs: $K = C_+C_-;\ C_\pm := ((u \pm \sigma)(v \mp \sigma) + \frac{p+q}{2})$.
- $\underline{q^2 = r}$ Circle pairs: $K = C_+C_-;\ C_\pm := ((u \pm \sigma)(v \pm \sigma) + \frac{p+q}{2})$.

For the last two, the "centers" $\pm\sigma = \pm\sqrt{\frac{q-p}{2}}$ (singular foci) represent points on either the x-axis or the y-axis as symmetry requires. Of the five non-elliptic cases, we refer to the last three as *circular*, and to the first two as *rational-non-circular*.

Curves in the 1 and 0-strata (two or three factors vanish) play a special role. In case $r = 1$, $F = (x^2 + y^2 \pm 1)^2$, x^2z^2, y^2z^2 correspond to the four *mirrors* of reflection symmetry (Appendix E); in case $r = -1$, $F = (x^2 + y^2 \pm i)^2$, x^2z^2, y^2z^2 are the four mirrors (however, the first two require the imaginary coefficients $p = q = \pm i$).

Before turning to the proof of Proposition A1, we make a few additional observations. First, a curve K which is a product of circles/lines has no foci. Second, $u_0 = \infty$ is never a focus of any $K \in P^3$ ($\pi^{-1}(\infty) \subset \mathcal{K}$ consists of the "two points at J", when J is a node, and the two ideal points of a conic otherwise). A finite node $P_0 = (u_0, v_0)$ never gives a focus u_0; in particular, $u_0 = 0$ in the trinodal case is not a focus. In fact, $u_0 = 0$ is never a focus of any $K \in P^3$. The remaining issues addressed in the proof of the following result have mostly to do with singular foci and Cassinians.

Proposition A1. *Let k be a quartic in rectilinear position (Equation (4)), with focal discriminant $\Delta_v k$ (Equation (5)) and elliptic discriminant Δk (Equation (6)):*

(a) $u_0 \in \mathbb{C}$ is a focus of k if and only if it is a simple root of $\Delta_v k$;

(b) k is an elliptic curve $\Leftrightarrow \Delta k \neq 0 \Leftrightarrow |\text{foci}(K)| = 4$;

(c) k is a rational non-circular $\Leftrightarrow |\text{foci}(K)| = 2$;

(d) k is circular or degenerate $\Leftrightarrow |\text{foci}(K)| = 0$.

Proof. We already noted the first equivalence in (b). The second follows from Equation (1), and likewise for (c) and (d). For (a), there are two possibilities to consider: $B(u_0) = (p + q)u_0 \neq 0$, and $B(u_0) = 0$.

Case $B(u_0) \neq 0$ First, assume u_0 is a focus. Then, $A(u_0) \neq 0$; otherwise, $k = Bv + C$ has a simple root $v_0 = -C/B$, so u_0 cannot be a branch point of π. Thus, k is in fact quadratic in v. Since k cannot have *two* distinct roots, it must have a double root, so $(\Delta_v k)(u_0) = 0$.

To see that u_0 must be a *simple* root of $\Delta_v k$, we consider the different possibilities for K. All roots are simple in the elliptic case $\Delta K \neq 0$. Likewise, in the conic case $s = 0$, both roots are simple (see Equation (5)). For the trinodal case $r = 0$, the only non-simple root is $u = 0$ (again, see Equation (5)), but $B = (p+q)u_0 \neq 0$. Finally, the circular cases have no foci so need not be considered.

Conversely, let u_0 be a simple root of $\Delta_v k$. Then, $k(u_0, v)$ has a double root v_0. Consider the corresponding finite point $P_0 = (u_0, v_0)$. We know that P_0 cannot be the node $P_0 = O$ of a trinodal curve, which yields a double root of $\Delta_v k$. It cannot be a double point for one of the circular cases; in these cases, $\Delta_v k$ is a "square" (e.g., when $r = p^2$, $\Delta_v k = 2(q-p)(p+u^2)^2$). The only possibility is that P_0 is a regular point of K, but a ramification point of isotropic projection π—that is, a point of isotropic tangency. So u_0 is a focus.

Case $B(u_0) = 0$ We already dealt with $u_0 = 0$ (which only arises in cases $r = 0$ or $p = q$), so we need only consider the case $p = -q$. When $s = 0$, this is just the case of a rectangular hyperbola, and there is nothing new to consider.

When $r = 0$, K is a *Bernoulli lemniscate*. In this case, $\Delta_v k = 4qu^2(u^2 - q)$ has simple roots $u_0 = \pm\sqrt{q}$. Note that $A(u_0) = 0$, i.e., $u_0 = \pm\sigma$ is a singular focus. In fact, $A = B = 0$, $C = pu_0^2 \neq 0$, and $k(u_0, v) = C$ is a non-zero constant. (Geometrically, the isotropic line $u = u_0 w$ meets K *four* times at I, a biflecnode, and u_0 is a *triple focus*.) Thus, u_0 is a branch point of π, that is, a focus of K.

Finally, when $p = -q$ and $\Delta K \neq 0$, K is a Cassinian. In this case, there are two *triple foci* and two *ordinary foci*—all of which are foci and simple roots of $\Delta_v k$. The two types of foci correspond to the following two subcases: (i) $A = B = 0$, $C \neq 0$, which gives rise to the singular foci $u_0 = \pm\sqrt{q}$ (as for the lemniscate); (ii) $A \neq 0$, $B = C = 0$, which gives the ordinary foci satisfying $0 = C = pu^2 + r$, i.e., $u_0 = \pm\sqrt{-r/p}$. $\square$

Appendix B. Concyclic Confocal Families

Figure A1 shows the orthogonal pair of foliations $\mathcal{H}^r_\alpha$ in case $r = 1$ (for the particular choice $\alpha = 1 + \sqrt{2}$). The mirrors are the x and y axes and the *real/imaginary* circles $x^2 + y^2 = \pm 1$ (Appendix E). Each quartic is two-circuited; in one foliation (blue), the two circuits of a curve are swapped by reflection in the y-axis; in the other foliation (green), the two circuits are swapped by inversion in $x^2 + y^2 = 1$.

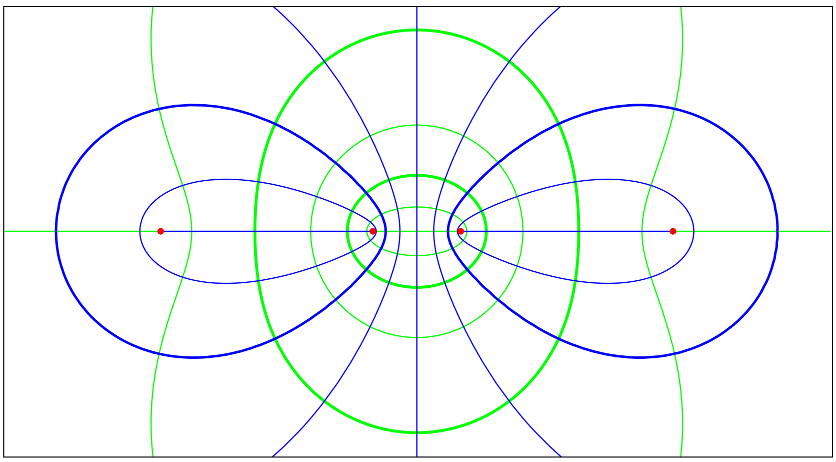

Figure A1. Confocal family of type $r = 1$ ($\alpha = 1 + \sqrt{2}$); the foci are the four *red* points. Each irreducible quartic in $\mathcal{H}^1_\alpha$ is two-circuited—e.g., see *bold* blue/green curves (the bold blue one is the unique Cassinian in $\mathcal{H}^1_\alpha$).

In this example, all foci are on the x-axis. But this is not the only possibility for a confocal family with $r = 1$. To determine all subcases, consider the normalized focal discriminant $\delta = \delta^r_\alpha$ (Equation (8)) and corresponding "normalized" elliptic discriminant:

$$\delta k := \frac{1}{256}\Delta\delta = \frac{r}{16}(b^2 - \frac{r}{b^2})^4 = r(\alpha^2 - r)^2 \tag{A1}$$

The curve k is elliptic unless $\delta k = 0$, i.e., $r = 0$ or $r = \alpha^2$.

Thus, in the present case $r = 1$, k is elliptic for $\alpha = \frac{pq-1}{p-q} \neq \pm 1$; that is, unless k belongs to one of the circular cases $p^2 = 1$ or $q^2 = 1$. In fact, the circular locus $\mathcal{C} : (p-q)(p^2-1)(q^2-1) = 0$ (where $\alpha = \pm 1$ or ∞) divides the p,q-plane into the following three reduced elliptic strata $\hat{\mathrm{P}}^2_j$ indicated in Figure A2:

$$\hat{\mathrm{P}}^2_1 : \alpha > 1 \text{ (blue)}; \quad \hat{\mathrm{P}}^2_2 : -1 < \alpha < 1 \text{ (yellow)}; \quad \hat{\mathrm{P}}^2_3 : \alpha < -1 \text{ (green)}.$$

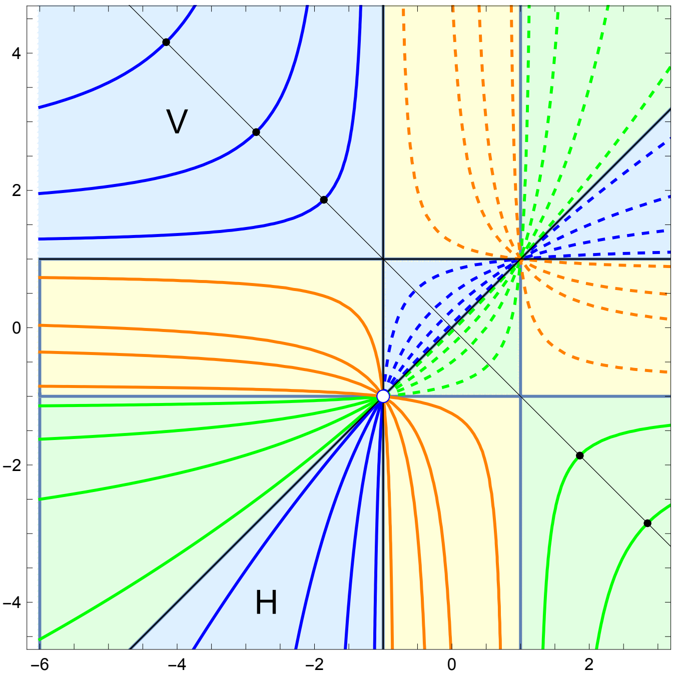

Figure A2. The p,q-plane R_1 is foliated by hyperbolas: $\mathcal{H}_\alpha$ represents a confocal family with concyclic foci on the x-axis (blue), y-axis (green), or unit circle (orange). H/V denote (blue) regions whose points represent horizontal/vertical trajectories. Dashed curves are explained in Remark A2.

We note that the three cases correspond to the four roots $\pm b, \pm 1/b$ of δ^1_α being *real* ($\hat{\mathrm{P}}^2_1$), *imaginary* ($\hat{\mathrm{P}}^2_3$), or of *unit modulus* ($\hat{\mathrm{P}}^2_2$)—see Figure A3, left. In particular, the foci are always concyclic; in fact, the three subcases are all inversively equivalent.

Remark A1. *We expand on the last comment. The two subcases $\alpha > 1$ and $\alpha < -1$ are related by $\frac{\pi}{2}$-rotation of the x,y-plane. The subcase $-1 < \alpha < 1$ is related to the first via the Cayley map $\mathrm{C}(u) = \frac{u-i}{u+i}$ (or the corresponding transformation on the x,y-plane). We note that $\mathrm{C}(u)$ is an order three map which cyclically permutes the real mirrors in case $r = 1$.*

Although this relationship implies that there is no essential difference from the other subcases, our descriptions have to be modified somewhat. For instance, note that the anti-diagonal $q = -p$ does not intersect any of the corresponding hyperbolas—shown in orange in Figure A2. In other words, there are no Cassinians in such S^1-concyclic confocal families. There are, however, curves which play essentially the same role. These curves—which are represented by points along the line $q = p + 2$ (i.e., $\sigma^2 = 1$)—are obtained from Cassinians via the Cayley map, so we call these curves Cayley Cassinians.

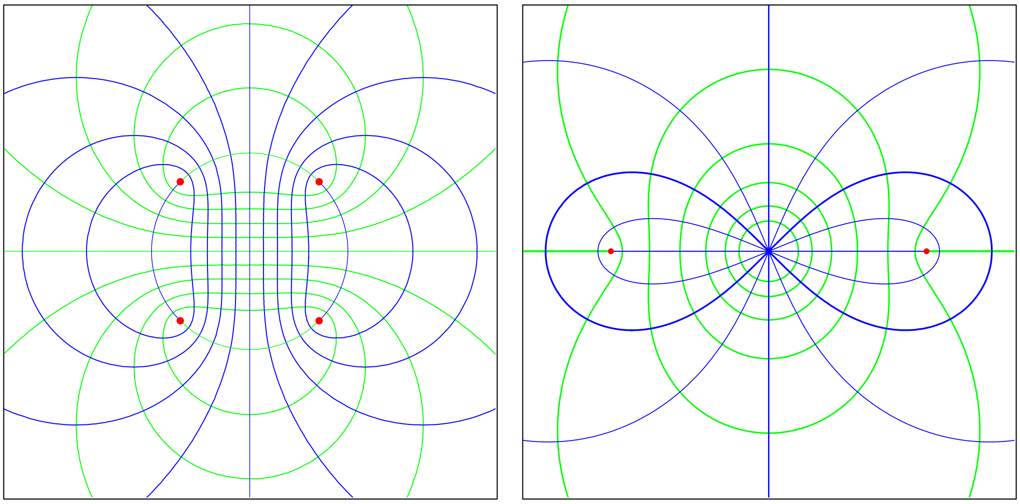

Figure A3. (**Left**): The confocal family $\mathcal{H}^r_\alpha = \mathcal{H}^1_0$; green/blue curves are horizontal/vertical trajectories of Q^r_α. (**Right**): Confocal family of type $r = 0$. The blue curves are Booth lemniscates; the *bold* one is the Bernoulli lemniscate (the unique Cassinian in the family). Green curves are S^1-inverted ellipses.

Returning to the subcase $\alpha > 1$, the anti-diagonal meets both branches of each hyperbola $\mathcal{H}_\alpha$—yet only one Cassinian appears in Figure A1. The apparent contradiction is resolved by the following remark, which explains one of the main differences between the two cases $r = \pm 1$, and between confocal vs. orthogonal families.

Remark A2. *Though a confocal family is understood here to consist of real curves, a given curve in the family may have an empty real locus in case $r = 1$. Note that $f(x, y) = (x^2 + y^2)^2 + 2(px^2 + qy^2) + 1 \geq 1$ when $p, q \geq 0$. In all other cases, f has a minimum either $f(\pm\sqrt{-p}, 0) = 1 - p^2$ or $f(0, \pm\sqrt{-q}, 0) = 1 - q^2$. Thus, f has no real points whenever $p, q \geq -1$. In Figure A2, solid portions of hyperbolas indicate curves in the foliation on $\hat{\mathbb{C}}$. In the blue region (where $\alpha > 1$), every point on an upper branch represents a curve in the orthogonal family $\mathcal{F}^1_\alpha$ (in fact, a vertical trajectory of Q^1_α). On the other hand, the dashed portions of certain hyperbola branches indicate "empty" curves, e.g., on the lower branch of a hyperbola in the blue region (where $\alpha > 1$), only the points up to $\bigcirc = (-1, -1)$ (indicated by a white dot) represent (horizontal) trajectories in the foliation $\mathcal{F}^1_\alpha$. Finally, $\bigcirc$ itself represents the "squared circle" $(x^2 + y^2 - 1)^2$ in $\mathcal{P}^3$, which has no foci, and does not belong to the confocal family $\mathcal{H}^1_\alpha$, as strictly defined. But as Figure A1 illustrates, the unit circle (like all real mirrors) clearly belongs to the orthogonal family $\mathcal{F}^1_\alpha$.*

For convenient reference, we list the types of confocal families (and their figures):

Type $r = 1$ (Figure A1; Figure A3, left)) The four roots $\pm b, \pm 1/b$ of δ^1_α are always concyclic. In each subcase $\hat{\mathrm{P}}^2_1, \hat{\mathrm{P}}^2_2, \hat{\mathrm{P}}^2_3$, K is an elliptic curve forming two smooth closed curves in $\mathbb{R}^2$. In fact, the three subcases are inversively equivalent.

Type $r = -1$ (Figure 2) For $b > 0$, $\alpha = \frac{1}{2}(b^2 - \frac{1}{b^2})$ takes all real values. The four roots $\pm b, \pm i/b$ of δ^r_α are *rhombic* (non-concyclic, except when $b = 1$, i.e., $\alpha = 0$). The bicircular quartic k is an elliptic curve whose real locus consists of one smooth closed curve in $\mathbb{R}^2$.

Type $r = 0$ (Figure A3, right) k is a trinodal quartic with a (real or imaginary) pair of foci $u_0 = \pm b$. If $pq < 0$, the double point at the origin is a node and k is a *Booth lemniscate*. When $pq > 0$, it is an acnode, and the rest of k appears as an *oval* in $\mathbb{R}^2$. The case $r = 0$ may be regarded as the transitional case between $r = 1$ and $r = -1$.

Type $s = 0$ k is a conic ($K = w^4 k(u/w, v/w) \in \mathrm{P}^3$) with a (real or imaginary) pair of foci; the ellipses ($pq > 0$) and hyperbolas ($pq < 0$) are related to the curves of type $r = 0$ by inversion in the unit circle.

Appendix C. Remarks on the Classical Notion of Focus

Aside from providing geometrical intuition, our foray into the classical heuristics of foci is meant to point out how confusions arise, and how they are resolved by Definition 1.

Focal properties of conics have been known since antiquity, but it was Kepler who introduced the Latin word *focus*. Poncelet interpreted a conic's foci as points of intersection of the isotropic tangent lines through I with those through J. Plücker then used the latter to *define* (real and complex) foci for curves of higher degree (see [16], p.160).

Here, we are mostly concerned with *real foci* ($\mathbb{R}$-foci), a term which distinguishes such real points $P \in \mathbb{R}^2$ from the general $\mathbb{C}$-foci of Plücker's definition, and from the *foci* $u_0 \in \hat{\mathbb{C}}$ of Definition 1.

In the case of an ellipse or hyperbola, we note that there are two isotropic tangents from each point I, J. The tangents from I are paired with those from J via complex conjugation (as is the case for any real curve); consequently, two of the four $\mathbb{C}$-foci are real foci.

More generally, the number m of tangent lines to a curve F from a general point P is the *class* of F—which can be shown to be independent of P, and is in fact the degree of the dual curve F^*. For a curve with δ nodes and κ cusps (and no more complicated singularities), one of Plücker's equations is $m(K) = n(n-1) - 2\delta - 3\kappa$.

Letting $P = I$, one accordingly expects m^2 $\mathbb{C}$-foci and m $\mathbb{R}$-foci. Indeed, this is usually the case for *non-circular* curves. Thus, for example, an ellipse or hyperbola, which has class $m = 2$, has two real foci. (A parabola would appear to be an exception, but according to Definition 1, it has a second focus at $u_0 = \infty$.)

Now consider a circle that has a regular point at I. If P is a nearby point, there are two tangents from P since $m = 2$. If P is allowed to approach I, these will merge into one tangent at I (which may be viewed as contributing *two* to the class m). This would presumably explain why the circle has just one real focus, namely, its center.

But a circle's center is a *singular focus*—the classical term for a focus resulting from a tangent at I—often viewed as quite different from an *ordinary* (or *simple*) real focus. (According to Definition 1, a circle has no foci; on the unit circle $uv = 1$, e.g., π is evidently 1-1.)

Next, consider a bicircular quartic with a pair of nodes at the circular points. With $\delta = 2$, $\kappa = 0$, Plücker's equation gives $m = 8$. Of course, such curves do not have so many real foci. In fact, there can only be *six* distinct tangents through I, given that each of the two tangents at the node I counts (at least) *twice*. Thus, if we discard the resulting pair of singular foci, we would generally expect to obtain four real foci, as in Proposition 1.

But simply *discarding the singular foci* is not quite correct. This can be seen by considering the special bicircular quartics known as *Cassinians* (or *ovals of Cassini*, after the astronomer who proposed such curves as candidates for planetary orbits [1]). Cassinians belong again to the binodal case ($\delta = 2$ or 3), but happen to be *biflecnodal* [18,34]; that is, each of the two tangents at I meets its branch *three* times. Such a tangent also meets the other branch once, thus accounting for all *four* intersections of the tangent with the curve. Whereas the tangent to a circle at I results in a *double focus* (classical term), the two singular foci of Cassinians are *triple foci*. Further, in view of the fact that a Cassinian has class $m = 8$, such a curve accordingly has only two ordinary real foci. (A similar anomaly arises for *Cartesians*, which are bicircular quartics with $\kappa = 2$; these have triple foci by virtue of their cusps at I, along with three ordinary foci.)

Yet Cassinians are plainly seen to occur in confocal families of bicircular quartics sharing *four* foci. Thus, in contrast with Proposition 1, the classical enumeration of foci would appear to be full of caveats: *A bicircular quartic K has four foci, provided the two singular foci are discarded—except when K is a Cassinian (or Cartesian), in which case the singular foci should not be discarded!*

In other words, *triple foci count, even though double foci do not*. But from the standpoint of Definition 1, this is easily explained: What ties the triple and ordinary foci of Cassinians together—and sets both apart from the double foci—is that they are branch points of isotropic projection. (In fact, they are both simple branch points.) Likewise, quadruple foci, etc., are *foci* (though such higher order branch points of π_I do not occur in our examples).

Appendix D. Siebeck's (Other) Theorem

Note that the identity of Siebeck appears to be a bit uncertain. He is likely the author of the Ph.D. thesis listed as F. H. Siebeck, Universität Breslau 1845, "On conic surfaces for any circumscribed surface". In the important reference [35], the author of [36] is listed as J. Siebeck—mistakenly, we believe. Regardless, Siebeck seems to be mostly known for *Siebeck's theorem* [36], which characterizes the zeros of $f(z) = \sum_1^3 \frac{m_j}{z-z_j}$ as the foci of a related conic (see [37] for the hyperbolic case). But his result on quartics and elliptic functions also deserves to be well known—hence, this brief appendix on his paper [3].

Figure 1 itself delivers on the promise of an earlier paper [38], "On the graphical representation of imaginary functions". For this reason, and because of the satisfying parallel to *sine* and confocal conics, it is the kind of thing one might expect to see browsing textbooks "at random". But so far, we have found just [39], which shows Siebeck's first graphic.

In fact, it is hard to tell from the literature how much attention Siebeck's paper received at the time. What does seem clear is that Siebeck's work should not be viewed as an isolated contribution, and the general theme (B) would have seemed much more mainstream over one hundred years ago than it is today.

Darboux studied the confocal systems of Cartesians and *cycliques* (or bicircular quartics) [28,40–44], in the context of the theory of projective algebraic plane curves, triply orthogonal systems of surfaces and spherical conics. (Note: though the connection between bicircular quartics and spherical conics was made in Siebeck's time, the intrinsic hyperbolic setting for conics seems to have received much less attention—see, however, [4,45–47].) Darboux and others investigating such topics were building on the geometric contributions of Poncelet, Plücker [17], Chasles [48], and Kummer [49], to name a few. Over several decades, a number of authors [27,50–57] also explored the intimate connections between such curves and parallel developments in complex function theory—the Jacobi and Weierstrass elliptic functions, their double periodicity and addition theorems. (See [58] for historical background and many additional references.)

Returning to the Siebeck paper itself, we note that direct verifications of the curve parameterizations (along the lines discussed in Section 1) involve elliptic function identities; the computations are much more complicated than in the case of Euclidean conics. In the course of developing a manageable approach to such computations, it seems that Siebeck also discovered several of the main properties of such curves. He did so prior to the closely related work of other authors—e.g., Darboux (on *cycliques*) and Casey (on *bicircular quartics*)—and without the benefit of somewhat more systematic treatments of foci, including his own [36].

Although Plücker's general notion of foci was well established by 1860, Siebeck's provisional treatment of foci for the curves in question probably reflected the state of understanding at the time. Siebeck considers three different features of his confocal families to establish the connection with accepted notions of foci in projective geometry. Namely, he finds analogues of the *string* and *reflection* properties of conics, and also the orthogonal pair of foliations with singularities at the presumed foci.

Although the first paper [38] suggests that Siebeck's starting point was the parameterization by elliptic functions $\operatorname{sn} z = \operatorname{sn}(z,k)$ and $\operatorname{cn} z = \operatorname{cn}(z,k)$, he begins [3] with an analogue of the string property to define the class of curves as follows. Fix a pair of points A, B, and let a general point P on a curve satisfy an equation of the form

$$mS^2 + nD^2 = 4a^2, \tag{A2}$$

where $S = AP + PB$ and $D = AP - PB$ denote the sum and difference of distances from P to A and B, and m and n are real numbers.

By purely algebraic means, Siebeck discovers the underlying geometric symmetries of Equation (A2) to show that there are actually *eight* special points with respect to which similar equations for the same curves may be written down. (Thus, he locates half of the *sixteen* complex foci of his curve families.) He then determines that exactly *four* of the latter points are real, and finds two qualitatively distinct *types* of such curve families, depending on the pattern of the quadruple of points. Ultimately, the two cases turn out to correspond to the ramification points of the elliptic functions $\operatorname{sn} z$ and $\operatorname{cn} z$.

Figure 1 illustrates the main qualitative difference between the orthogonal systems parameterized by $\operatorname{sn} z$ and $\operatorname{cn} z$. In the case of $\operatorname{sn} z$ (upper curves in the figure), the four foci A, B, A', B' are *concyclic* (in fact, collinear). For the elliptic cosine $\operatorname{cn} z$ (shown on the lower left), the foci are *non-concyclic* (and lie symmetrically on an orthogonal pair of axes). Finally, there is an intermediate case (lower right), in which the elliptic function becomes elementary as a pair of foci in either of the previous two cases collide.

Next, Siebeck shows that the family of curves breaks up into two orthogonal subfamilies, concluding: "The points A, B, A', B' are therefore foci in the usual sense". He does not elaborate on "the usual sense", but it seems likely that he was familiar with *Kummer's theorem*.

The striking heuristic argument of Kummer, based on Plücker's definition of foci, goes as follows. Assume a family of curves $f(x, y; a)$ such that exactly two such curves $f(x, y, a_\pm)$ meet orthogonally at a given general point (x_0, y_0). The latter condition seems evident in Figure 1, and is exactly what Siebeck verifies algebraically (seemingly for real planar points (x_0, y_0)). The orthogonality condition may be written $f_x(x, y; a_+) f_x(x, y; a_-) + f_y(x, y; a_+) f_y(x, y; a_-) = 0$. Kummer apparently assumes analytic dependence on a, so the latter equation holds also at complex points—which must therefore include the "invisible" points of intersection of arbitrarily nearby curves in the planar foliations by $f(x, y; a)$. Therefore, in the limit where a pair of such curves coincide, $a_\pm = a$ and the earlier equation becomes $f_x(x, y; a)^2 + f_y(x, y; a)^2 = 0$. This may be thought of as the condition of "self-orthogonality" of the vector $(v_1, v_2) = (f_x(x, y; a), f_y(x, y; a))$ with respect to the complex-linear extension of the Euclidean metric on the plane. In other words, (v_1, v_2) is an *isotropic* (or *null*) vector, i.e., it has an *imaginary slope* $v_2/v_1 = \pm i$. This occurs because the family $f(x, y, a_\pm)$ has an *envelope* belonging to a set of isotropic lines. These are in fact isotropic tangents to each curve in the family $f(x, y, a)$, which is therefore confocal.

The third focal property which Siebeck establishes for $f(x, y; a)$ is a generalization of the focal property of conics. Given a point P on one of the curves $f(x, y; a)$, let θ be its angular coordinate, and let $\theta_1, \ldots, \theta_4$ be the angular coordinates formed by the vectors from each of the four foci to P. Examining each of the two types of systems $f(x, y, a)$, Siebeck verifies that in both cases, the following relationship holds among the five angles:

$$\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) = \tan(2\theta) \tag{A3}$$

In other words, up to the addition of $\frac{\pi}{2}$, the equation $\theta = \frac{1}{2}(\theta_1 + \theta_2 + \theta_3 + \theta_4)$ holds. Further, Siebeck argues, as two of the four foci tend to infinity, that f tends to a conic, and the latter equation becomes $\theta = \frac{1}{2}(\theta_1 + \theta_2)$—which is a way of expressing the familiar reflection property of the ellipse.

We remark that the analogy to the classical reflection property becomes much closer when Siebeck's quartic curves are interpreted as non-Euclidean conics [4,59]. We note also that Equation (A3) can be immediately read off from the quadratic differential $Q = \frac{du^2}{\delta(u)}$ defining the family of curves as trajectories (Section 5).

Appendix E. Circles of Inversion for Circular Cubics

In this appendix, we prove Theorem 1 on the existence of four orthogonal mirrors. A mirror C is given by an equation:

$$C(x, y) = c_0(x^2 + y^2) - 2c_1 x - 2c_2 y + c_3. \tag{A4}$$

For $c_0 \neq 0$, we may consider the center c and squared radius κ,

$$c = (\frac{c_1}{c_0}, \frac{c_2}{c_0}), \quad \kappa = \rho^2 := \frac{c_1^2 + c_2^2 - c_0 c_3}{c_0^2}. \tag{A5}$$

If the coefficients c_i are real, then C may be *real* (real center and real radius), or *imaginary* (real center and imaginary radius). Otherwise, C is *complex*, in which case there is a complex conjugate mirror $\bar{C}$. Point circles $\kappa = 0$ are excluded.

For a mirror with a center at the origin and squared radius κ, inversion is given by $\iota_\kappa(x, y) = (\frac{\kappa x}{x^2+y^2}, \frac{\kappa y}{x^2+y^2})$—essentially, the quadratic transformation $(x : y : z) \mapsto (\kappa xz : \kappa yz : (x - iy)(x + iy))$ in $\mathbb{C}P^2$. For $c \neq (0, 0)$, inversion is obtained by conjugating such ι_κ by translation. The meaning of *orthogonal* for non-real mirrors is discussed below.

We first show that a bicircular quartic can be put in rectilinear position; for this, it suffices to find an orthogonal pair of real circles of inversion since these become perpendicular axes after inversion about one of the two finite points of intersection. Further, since the inverse of a bicircular quartic with respect to a general point on the curve is a circular cubic, the problem reduces to finding such a pair of circles for a given circular cubic. We now describe the elementary construction for this purpose, outlined in [19]; see Figure A4.

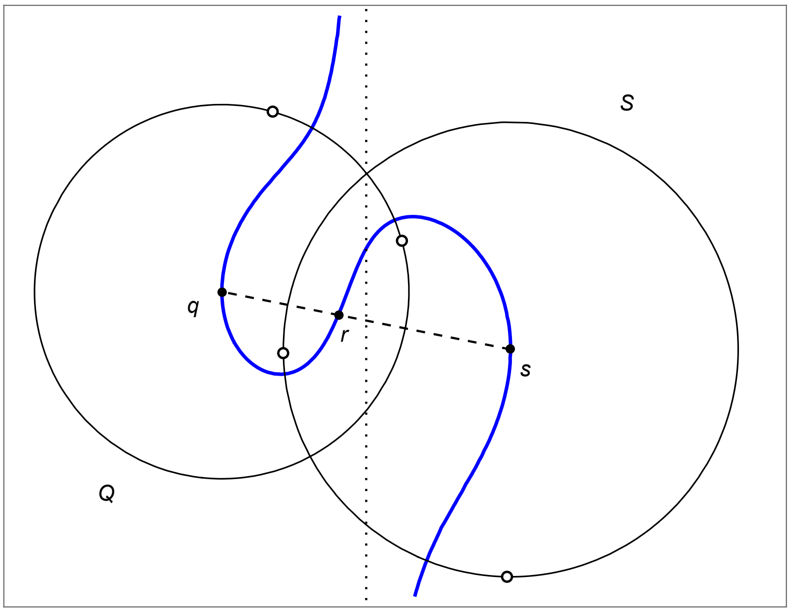

Figure A4. A one-circuited circular cubic and its two real circles of inversion Q, S (centers q, s); real foci $\circ$ (two on each circle); asymptote; collinear points qrs.

Let $f(x, y)$ be a circular cubic, assumed to be real and nonsingular. We may also assume that the unique real ideal point p is not a flex (which will be the case, e.g., if a bicircular quartic is inverted about a generic point). By rotation, we may assume that f has cubic term $x(x^2 + y^2)$, so $p = (0 : 1 : 0)$. There are four tangents to f from p (aside from the tangent at p itself), namely, the vertical lines $x = x_j$, where $x_j \in \mathbb{C}$ is one of the four roots of the discriminant $\Delta_y f(x, y)$. (f being of class six, four tangents from p, was expected.) When the point of tangency is translated to the origin, the cubic has the form $f = x(x^2 + y^2) + ax^2 + 2hxy + by^2 + \kappa x$. Then, one easily verifies that f is symmetric with respect to inversion in the circle $x^2 + y^2 = \kappa$. Here, the "radius" $\sqrt{\kappa} \in \mathbb{C}$ is non-zero since f is nonsingular. To be precise, f is preserved by the inversion ι_κ in the sense that $f(x, y) = \frac{1}{\kappa^2}(x^2 + y^2)^2 f(\frac{\kappa x}{x^2+y^2}, \frac{ky}{x^2+y^2})$ (the extra linear components $(x^2 + y^2)^2$ being "superfluous").

Translating the center of the circle $x^2 + y^2 = \kappa$ back to the point of tangency of the original curve gives the corresponding mirror C_j. One can do the same for each of the four roots x_j to obtain four mirrors $C_1, \ldots, C_4$.

We want to find real mirrors; for this, it suffices to show that at least two of the points x_j are real. We recall that all nonsingular real cubics may be divided into two classes: those whose real locus consists of the odd circuit alone, and those which also have an *even circuit*

f_e (a simple closed curve in $\mathbb{R}^2$). The key is to examine the mirrors with centers on the *odd circuit* f_o (unbounded real connected component) of f.

Figure A4 shows a cubic of the former type, together with the vertical asymptote $x = x_0$ (the tangent line to f at p); the argument works just the same for two-circuited cubics.

Since p is not a flex, f_o meets $x = x_0$ in a finite real point and extends to the left and right of this line. In fact, f_o has a left-most point q and a right-most point s, which are points of vertical tangency. Thus, q and s are real centers of inversion as constructed above. But we still need to verify that the corresponding mirrors have real radii $\sqrt{\kappa}$, and that they meet orthogonally.

We denote these two mirrors $Q = S_1$ and $S = S_2$. Let $L = \overline{qs}$ be the line joining their centers, and let r be the point on f_o obtained as the third intersection of L with f. (L cannot intersect f_e since it would have to meet it twice.) Viewing f_o as a topological circle containing p, the four points have cyclic order (p, q, r, s) on f_o. Regarding ι_Q, by restriction, as a homeomorphism of the circle f_o, note that ι_Q swaps p with q and swaps r with s. Thus, ι_Q has real fixed points "between" each pair. But the fixed points of ι_Q belong to Q, which must therefore be real. Likewise, ι_S swaps points according to $(p, q, r, s) \xmapsto{\iota_S} (s, r, q, p)$, and S is real.

For heuristics, it is expedient at this point to identify the composition of circle inversions $T := \iota_Q \iota_S$ with a Möbius transformation of $\mathbb{C} \simeq \mathbb{R}^2$. In this sense, T must be *elliptic*—with a pair of real fixed points $Q \cap S$—otherwise, T would have infinite order. In fact, $(p, q, r, s) \xmapsto{T} (r, s, p, q)$, so $T^2 = Id$. In other words, inversions ι_Q and ι_S commute, which can only happen if Q and S meet perpendicularly. Thus, a circular cubic has an orthogonal pair of mirrors as required for the rectilinear position.

Some additional remarks may be in order. Suppose C is any mirror of a circular cubic f and ℓ is one of the four isotropic tangents through the circular point I. If K is the finite intersection point of C with ℓ, then $\ell = \overline{IK}$ inverts to the isotropic line $\iota_C \overline{IK} = \overline{JK}$. This line must also be a tangent since ι_C takes f to itself. Thus, K is among the sixteen (complex) foci of f, four of which must lie on C. The same argument applies to each mirror; thus, we established the last part of the classical result.

Next, we comment on the remaining mirrors S_3, S_4. In the one-circuited case, these are a complex conjugate pair of mirrors $S_3 = T, S_4 = \bar{T}$ determined by the two non-real centers on f_o. In the two-circuited case, the centers of S_3, S_4 are real, these being the left- and right-most points on f_e; it turns out that one mirror T is real and the other U is imaginary. Evidently, T is orthogonal to Q and S, and consequently all four mirrors are mutually self-inverse.

When a non-real mirror is involved, the notion of *orthogonality* requires a definition. First, let C, C' be real circles, with centers c, c' and squared radius κ, κ'. If C' and C meet perpendicularly in $\mathbb{R}^2$, the Pythagorean theorem gives $|c - c'|^2 = \kappa + \kappa'$. Let C, as in Equation (A4), be represented by the vector $C = (c_0, c_1, c_2, c_3)$, and likewise for C'. One finds, using Equation (A5), that perpendicularity is equivalent to orthogonality with respect to the bilinear form

$$\langle C, C' \rangle := 2(c_1 c'_1 + c_2 c'_2) - c'_0 c_3 - c_0 c'_3.$$

We note the relation to *Möbius circle geometry* [60]. If a plane $a_1 x_1 + a_2 x_2 + a_3 x_3 = a_4$ and sphere $x_1^2 + x_2^2 + x_3^2 = 1$ in $\mathbb{R}^3$ intersect in a circle A, there corresponds a planar circle/line C as above via stereographic projection $(x_1, x_2, x_3) = (\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1})$. It's coefficients are: $c_0 = a_3 - a_4, c_1 = -a_1, c_2 = -a_2, c_3 = -a_3 - a_4$. Then a short computation yields the Minkowski inner product on vectors A, A' representing spherical circles: $\langle C, C' \rangle = 2(a_1 a'_1 + a_2 a'_2 + a_3 a'_3 - a_4 a'_4) =: 2\langle A, A' \rangle$. In particular, note that $\langle C, C \rangle = 2c_0^2 \kappa > 0$ for real circles, which explains why there can be at most three mutual orthogonal real mirrors.

Proceeding to the general case, orthogonality may be defined by the same equation $\langle C, C' \rangle = 0$ by complex bilinear extension. We note some formal consequences of this definition. First, no orthogonal pair can be imaginary. For if C and C' are two such circles, we can assume $c_0 = c'_0 = 1$, for simplicity, and obtain

$$0 \leq (c_1 - c'_1)^2 + (c_2 - c'_2)^2 = c_1^2 + c_2^2 + {c'}_1^2 + {c'}_2^2 - 2(c_1 c'_1 + c_2 c'_2)$$
$$= c_1^2 + {c'}_1^2 + c_2^2 + {c'}_2^2 - c_3 - c'_3 = \kappa + \kappa' < 0.$$

This leaves three possibilities for a set of mutually orthogonal mirrors:
(1) Three real mirrors and one imaginary mirror.
(2) Two real mirrors and two complex conjugate mirrors.
(3) Two pairs of two complex conjugate mirrors.
However, the third type cannot exist. Suppose $C = (c_0, c_1, c_2, c_3) = v + iw$ is the vector representation of a complex circle, where v and w are circles with real coefficients. The condition that C be orthogonal to $\bar{C}$ is $v \cdot v + w \cdot w = 0$. Note that this implies one of v and w is a real circle and the other is imaginary.

Now assume there are two pairs of complex mirrors $C_1, \bar{C}_1, C_2, \bar{C}_2$ that are mutually orthogonal. Taking the inner product of $C_1 = v_1 + iw_1$ with $C_2 = v_2 + iw_2$, we have

$$v_1 \cdot v_2 - w_1 \cdot w_2 = 0, \quad v_1 \cdot w_2 + w_1 \cdot v_2 = 0.$$

Comparing this with the equation $C_1 \cdot \bar{C}_2 = 0$, we conclude that the real and imaginary part of each is orthogonal to both the real and the imaginary part of the other. But either v_i or w_i is an imaginary circle, and imaginary circles cannot be orthogonal to each other.

To conclude this section, we list the mirrors and complex foci of bicircular quartics F in standard position. First, the x and y axes are a pair of real mirrors. Further, one easily verifies that the circles $C_\pm = x^2 + y^2 \mp \sqrt{r}$ are mirrors using ι_κ with $\kappa = \mp\sqrt{r}$. (All mirrors can also be "derived" from the elliptic discriminant; each is found by setting three factors of ΔK equal to zero.) Thus, in case $r = 1$, there is a real circle C_+ and an imaginary circle C_-. In case $r = -1$, there is a pair of complex conjugate mirrors with squared radius $\kappa = \pm i$. The apparent departure from the cubic case is rather striking: the two circular mirrors are "concentric", with center $(0, 0)$ not on the curve. But there is no contradiction to any of the above general claims about systems of mirrors; circle centers, being double foci, do not respect inversive symmetry.

We list the sixteen complex foci for case $s = r = 1, b > 1$:

$$\underline{y = 0} : (\pm b, 0), (\pm\frac{1}{b}, 0); \quad \underline{x = 0} : (0, \pm ib), (0, \pm\frac{i}{b});$$

$$\underline{x^2 + y^2 = 1} : (\pm\frac{b^2 + 1}{2b}, \pm i\frac{b^2 - 1}{2b});$$
$$\underline{x^2 + y^2 = -1} : (\pm\frac{b^2 - 1}{2b}, \pm i\frac{b^2 + 1}{2b}).$$

Likewise, the sixteen complex foci for $s = 1, r = -1, b > 0$ are

$$\underline{y = 0} : (\pm b, 0), (\pm\frac{i}{b}, 0); \quad \underline{x = 0} : (0, \pm\frac{1}{b}), (0, \pm ib);$$

$$\underline{x^2 + y^2 = i} : (\pm\frac{b^2 + i}{2b}, \pm\frac{ib^2 + 1}{2b});$$
$$\underline{x^2 + y^2 = -i} : (\pm\frac{b^2 - i}{2b}, \pm\frac{ib^2 - 1}{2b}).$$

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