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Bogdanov–Takens Bifurcation Analysis of a Learning-Process Model

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Abstract: In this paper, as a complement to the works by Monterio and Notargiacomo, we analyze the dynamical behavior of a learning-process model in a case where the system admits a unique interior degenerate equilibrium. Meanwhile, we acquire the sufficient condition for the cusp of codimension 2 and verify that the system undergoes Bogdanov–Takens bifurcation around the cusp. Finally, we give a numerical simulation to support the theoretical results.

Keywords: learning-process; cusp; universal unfolding; Bogdanov–Takens bifurcation

1. Introduction

With the development of human society, educational issues [1–5] are gradually attracting people’s attention and are not ignored. Doubts [6] are a universal phenomena during the process of learning knowledge. The influence and significance of doubt in experience of acquiring knowledge are focused on by many scholars [7–9]. As time goes by, there exists a dynamic process of mutual transformation between understanding and doubt. That is, understanding can be transformed into doubt, and doubt can also be transformed into understanding [10]. Therefore, Monterio and Notargiacomo [11] proposed that the learning process as the interplay between understanding and doubt can be studied by formulating and analyzing a dynamical system written in term of differential equations. In order to better study this, the entirety of knowledge is divided into two parts: one part is already understood, and the other is still doubted. The first “understanding–doubt” model was put forward as follows.

$$\begin{aligned}\dot{U} &= a\{U(U-1)(\alpha-U) - fUD\}\{1 - (U+D)\}, \\ \dot{D} &= b\{D(\beta-D) + gUD\}\{1 - (U+D)\},\end{aligned}\quad (1)$$

U and D describe the level of understanding and doubt with time t during the learning process, respectively. $0 < \alpha < 1$ and $0 < \beta < 1$ stand for the minimum background required to learn about a subject and the maximum level of doubt that learner can have about a subject that they did not learn about it, respectively. a and b represent the speed of the learning process. The parameters f and g describe the interaction between $U(t)$ and $D(t)$ and $fg > 0$. At the standing point of physical meaning, the term $1 - (U + D)$ restricts the dynamics to the right triangle domain given by $0 \leq U \leq 1$, $0 \leq D \leq 1$ and $0 \leq U + D \leq 1$. Authors have studied the stability of the boundary equilibria and provided some numerical simulations to verify the theoretical results.

Liu, Ding and Chen performed deep studies of System (1) in [12]. They gave a complete analysis of the qualitative properties of the interior equilibria and a singular line segment. They also investigated some local bifurcation, including transcritical, pitchfork, and Hopf bifurcation [13,14] on isolated equilibrium, and transcritical bifurcation without parameters on non-isolated equilibrium.



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However, dynamical behavior of the learning-process model in a case where the system admits a unique interior equilibrium can be studied further. Meanwhile, bifurcation of a higher codimension, which is the Bogdanov–Takens bifurcation of codimension 2, is worth analyzing since, compared to bifurcation of codimension 1, bifurcation of codimension 2 demonstrates more realistic dynamic behavior between understanding and doubt in the learning process.

Compared to the works by Liu, Ding and Chen in [12] we will give the analysis of Bogdanov–Takens bifurcation [15–17] of codimension 2 when System (1) admits a unique interior equilibrium (cusp).

This paper is arranged as follows. We obtain the sufficient conditions for the existence of the cusp [18] of codimension 2 in Section 2. In Section 3, we prove that System (1) undergoes Bogdanov–Takens bifurcation of codimension 2 and the theoretical results are verified by numerical simulation. A brief ends this paper in Section 4.

2. The Existence of the Cusp of Codimension 2

For simplicity, we make a time rescaling $d\tau = a dt$, and System (1) becomes

$$\begin{aligned} \dot{U} &= \{U(U - 1)(\alpha - U) - fUD\}\{1 - (U + D)\}, \\ \dot{D} &= r\{D(\beta - D) + gUD\}\{1 - (U + D)\}, \end{aligned} \tag{2}$$

where $r = \frac{b}{a}$.

For physical meaning, we only consider the dynamic behavior of System (2) in the closure $\Omega := \{(U, D) : 0 \leq U \leq 1, 0 \leq D \leq 1, 0 \leq U + D \leq 1\}$ for all possibilities of $(\alpha, \beta, f, g, r) \in \Sigma := \{0 < \alpha < 1, 0 < \beta < 1, fg > 0, r > 0\}$.

The equilibria of System (2) are determined by the following equation:

$$\begin{aligned} \{U(U - 1)(\alpha - U) - fUD\}\{1 - (U + D)\} &= 0, \\ r\{D(\beta - D) + gUD\}\{1 - (U + D)\} &= 0. \end{aligned} \tag{3}$$

There exists a singular line segment

$$L := \{(U, D) : U + D = 1, 0 \leq U \leq 1, 0 \leq D \leq 1\},$$

and three boundary equilibria $E_0(0, 0), E_1(0, \beta), E_2(\alpha, 0)$. The interior equilibria of System (2) are determined by the following equation:

$$\begin{aligned} (U - 1)(a - U) - fD &= 0, \\ \beta - D + gU &= 0. \end{aligned} \tag{4}$$

Substituting $D = gU + \beta$ into Equation (4), we obtain

$$F(U) = U^2 + (fg - \alpha - 1)U + f\beta + \alpha$$

and

$$F'(U) = 2U + (fg - \alpha - 1)U.$$

From $F(U) = 0$, we have

$$\alpha = \frac{Ufg + U^2 + \beta f - U}{U - 1}. \tag{5}$$

The Jacobian matrix of System (2) at any equilibrium $E(U, D)$ of System (2) takes the form

$$J_E = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where

$$\begin{aligned}
 a &= 4U^3 + (3D - 3\alpha - 6)U^2 + (2fD + 2\alpha + (2 + 2\alpha)(1 - D))U \\
 &\quad - (fD + \alpha)(1 - D), \\
 b &= U^3 + (f - 1 - \alpha)U^2 + (f(D - 1) + \alpha + fD)U, \\
 c &= (r - rg)D^2 + (rg(1 - U) - r(gU + \beta))D, \\
 d &= 3rD^2 - (2r(gU + \beta) + 2r(1 - U))D + r(gU + \beta)(1 - U),
 \end{aligned}$$

and $Det(J_E) = ad - bc$, $Tr(J_E) = a + d$.

Substituting (5) into $Det(J_E)$ and $F'(U)$, we can rewrite $Det(J_E)$ and $F'(U)$ as

$$Det(J_E) = r(Ug + \beta)(Ug + U + \beta - 1)^2UF'(U)$$

and

$$F'(U) = \frac{U^2 - \beta f - fg - 2U + 1}{U - 1}.$$

Therefore, from $F(0) > 0$ and $F(1) > 0$, we consider quadratic function $F(U)$ to have double zeros $U = U_*$ if $F'(U_*) = 0$, which means that System (2) admits a degenerate equilibrium $E_*(U_*, \beta + gU_*)$. Furthermore, if $Tr(J_{E_*}) = 0$ and $Det(J_{E_*}) = 0$, then we have

$$\begin{aligned}
 \alpha &= fg + 2U_* - 1 =: \alpha_1, \\
 \beta &= \frac{(U_*^2 - fg - 2U_* + 1)}{f} =: \beta_1, \\
 r &= \frac{gU_*f^2}{(fgU_* + U_*^2 - fg - 2U_* + 1)} =: r_1.
 \end{aligned}$$

Therefore, System (2) admits a degenerate equilibrium $E_*(U_*, \beta_1 + gU_*)$.

In order to ensure $0 < \alpha_1 < 1$, $0 < \beta_1 < 1$ and $r_1 > 0$, the parameters f , g and U_* should satisfy $\max\left\{\frac{1}{2} - \frac{1}{2}fg, 1 - \sqrt{fg + f}\right\} < U_* < \min\left\{1 - \frac{1}{2}fg, 1 - \sqrt{fg}\right\}$, $g > 0$, $f > 0$ and $0 < fg < 1$.

Additionally, equilibrium $E_*(U_*, \beta_1 + gU_*)$ should lie on the right triangle domain Ω , so we have $1 - fg - f < U_* < 1$. On the other hand, $1 - \sqrt{fg + f} < 1 - fg - f$, and we need to ensure $1 - \sqrt{fg} > 1 - fg - f$, thus $f > \frac{g}{(g+1)^2}$ holds.

Summing up the above, we have the following theorem:

Theorem 1. Based on $g > 0$, $f > \frac{g}{(g+1)^2}$, $0 < fg < 1$ and

$$\max\left\{\frac{1}{2} - \frac{1}{2}fg, 1 - fg - f\right\} < U_* < \min\left\{1 - \frac{1}{2}fg, 1 - \sqrt{fg}\right\},$$

there exists a unique degenerate equilibrium $E_*(U_*, \beta_1 + gU_*)$ in system (2).

Combining above results with Theorem 2.1 in [12], we have the following theorem:

Theorem 2. Depending on condition for existence of equilibrium E_* , boundary equilibria of system (2) have the following qualitative properties:

- (i) The origin $E_0: (0, 0)$ is a saddle;
- (ii) Equilibrium $E_1: (0, \beta_1)$ is a stable node;
- (iii) Equilibrium $E_2: (\alpha_1, 0)$ is an unstable node.

In the following, we determine type of equilibrium E_* .

Theorem 3. From condition for existence of degenerate equilibrium E_* and

$$\frac{fg(fg + U_*^2 - 1) + 2U_*(U_* - 1)^2}{fg + U_* - 1} \neq 0,$$

equilibrium E_* is a cusp of codimension 2.

Proof. Making a transformation $(x, y) = (U - U_*, D - \beta_1 - gU_*)$, then system (2) becomes

$$\begin{aligned} \frac{dx}{dt} &= a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + O(|(x, y)|^3), \\ \frac{dy}{dt} &= b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + O(|(x, y)|^3), \end{aligned} \tag{6}$$

where

$$\begin{aligned} a_{10} &= -gU_*(U_* - 1)(fg + f + U_* - 1), \\ a_{01} &= U_*(U_* - 1)(fg + f + U_* - 1), \\ a_{20} &= \frac{U_*^3 + (f-2)U_*^2 + (-f^2g^2 - 2f^2g + fg - f + 1)U_* + f^2g^2 + f^2g - fg}{f}, \\ a_{11} &= -fg + 2U_*f + U_*^2 - f - 2U_* + 1, \\ a_{02} &= U_*f, \\ b_{10} &= -(U_* - 1)(fg + f + U_* - 1)g^2U_*, \\ b_{01} &= gU_*(U_* - 1)(fg + f + U_* - 1), \\ b_{20} &= -fg^2U_*, \\ b_{11} &= -\frac{(2fg^2 + 2gU_* - 2g - U_* + 1)gU_*f}{fg + U_* - 1}, \\ b_{02} &= \frac{(2fg + f + 2U_* - 2)gU_*f}{fg + U_* - 1}. \end{aligned}$$

Since $a_{01} \neq 0$, we perform linear transformation $x = X, y = -\frac{a_{10}}{a_{01}}X + \frac{1}{a_{01}}Y$, then System (6) becomes

$$\begin{aligned} \frac{dX}{dt} &= Y + c_{20}X^2 + c_{11}XY + c_{02}Y^2 + O(|(X, Y)|^3), \\ \frac{dY}{dt} &= d_{20}X^2 + d_{11}XY + d_{02}Y^2 + O(|(X, Y)|^3), \end{aligned} \tag{7}$$

where

$$\begin{aligned} c_{20} &= \frac{U_*(U_* - 1)(fg + f + U_* - 1)}{f}, \\ c_{11} &= \frac{2fgU_* - fg + 2U_*f + U_*^2 - f - 2U_* + 1}{U_*(U_* - 1)(fg + f + U_* - 1)}, \\ c_{02} &= \frac{f}{U_*(U_* - 1)^2(fg + f + U_* - 1)^2}, \\ d_{20} &= -\frac{U_*^2(U_* - 1)^2(fg + f + U_* - 1)^2g}{f}, \\ d_{11} &= \frac{g(fg + f + U_* - 1)(fg - U_*^2 + 2U_* - 1)}{fg + U_* - 1}, \\ d_{02} &= \frac{fg}{(U_* - 1)(fg + U_* - 1)}. \end{aligned}$$

By Remark 1 of Section 2.13 in [19], we obtain an equivalent system of System (7) in the small neighborhood of $(0, 0)$ as follows

$$\begin{aligned} \frac{dX}{dt} &= Y + O(|(X, Y)|^3), \\ \frac{dY}{dt} &= \bar{d}_{20}X^2 + \bar{d}_{11}XY + O(|(X, Y)|^3), \end{aligned} \tag{8}$$

where

$$\begin{aligned} \bar{d}_{20} &= d_{20} = -\frac{U_*^2(U_* - 1)^2(fg + f + U_* - 1)^2g}{f} < 0, \\ \bar{d}_{11} &= 2c_{20} + d_{11} = \frac{(fg + f + U_* - 1)(fg(fg + U_*^2 - 1) + 2U_*(U_* - 1)^2)}{(fg + U_* - 1)f}. \end{aligned}$$

If

$$\frac{fg(fg + U_*^2 - 1) + 2U_*(U_* - 1)^2}{fg + U_* - 1} \neq 0,$$

then $\bar{d}_{20}\bar{d}_{11} \neq 0$, which means that degenerate equilibrium $E_*(U_*, gU_* + \beta_1)$ is a cusp of codimension 2. \square

Setting $\alpha = \frac{3}{25}, \beta = \frac{13}{60}, f = \frac{3}{5}, g = \frac{1}{5}, r = \frac{18}{95}$, the phase portrait of cusp of codimension 2 of is described with the help of Matlab; see Figure 1.

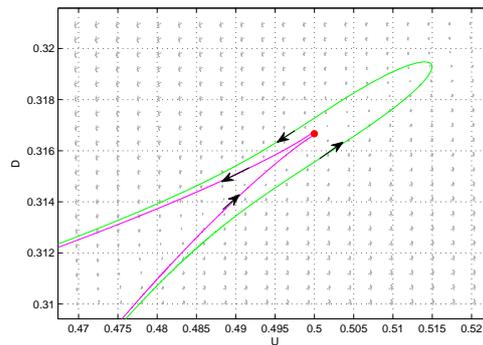


Figure 1. When $\alpha = \frac{3}{25}, \beta = \frac{13}{60}, f = \frac{3}{5}, g = \frac{1}{5}, r = \frac{18}{95}$, equilibrium E_* is a cusp of codimension 2 in System (2).

Remark 1. From [11], if doubt destroys comprehension, then $f > 0$ and $g > 0$; if doubt drives comprehension, then $f < 0$ and $g < 0$. Therefore, according to sufficient condition for existence of cusp of codimension 2, doubt destroys comprehension.

3. Analysis of Bogdanov–Takens Bifurcation

From Section 2, we know that E_* is a cusp of codimension 2. In this section, we investigate Bogdanov–Takens bifurcation around E_* .

Theorem 4. Under the condition for the existence of equilibrium E_* , which is a cusp of codimension 2, we choose α and r as bifurcation parameters. As the parameters (α, r) disturb around (α_1, r_1) , system (2) undergoes Bogdanov–Takens bifurcation in a small neighborhood of E_* .

Proof. We choose α and r as two bifurcation parameters and replace α and r with $\alpha + \varepsilon_1$ and $r + \varepsilon_2$, respectively, in System (2). Then, we acquire the unfolding system of System (2):

$$\begin{aligned} \dot{U} &= \{U(U - 1)(\alpha + \varepsilon_1 - U) - fUD\}\{1 - (U + D)\}, \\ \dot{D} &= (r + \varepsilon_2)\{D(\beta - D) + gUD\}\{1 - (U + D)\}, \end{aligned} \tag{9}$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2)$ is a parameter vector in a small neighborhood of origin. Obviously, when $\varepsilon_1 = \varepsilon_2 = 0$, System (9) has a unique positive equilibrium E_* , which is a cusp of codimension 2. For simplicity, the universal unfolding will be obtained with the help of a series of transformation in the following.

Making a translation transformation $u_1 = U - U_*, v_1 = D - \beta - gU_*$, equilibrium E_* will be translated into $(0, 0)$, then System (9) becomes

$$\begin{aligned} \frac{du_1}{dt} &= \hat{a}_{00}(\varepsilon) + \hat{a}_{10}(\varepsilon)u_1 + \hat{a}_{01}(\varepsilon)v_1 + \hat{a}_{20}(\varepsilon)u_1^2 \\ &\quad + \hat{a}_{11}(\varepsilon)u_1v_1 + \hat{a}_{02}(\varepsilon)v_1^2 + O(|(u_1, v_1)|^3), \\ \frac{dv_1}{dt} &= \hat{b}_{10}(\varepsilon)u_1 + \hat{b}_{01}(\varepsilon)v_1 + \hat{b}_{20}(\varepsilon)u_1^2 \\ &\quad + \hat{b}_{11}(\varepsilon)u_1v_1 + \hat{b}_{02}(\varepsilon)v_1^2 + O(|(u_1, v_1)|^3), \end{aligned} \tag{10}$$

where

$$\begin{aligned} \hat{a}_{00}(\varepsilon) &= -\frac{U_* \varepsilon_1 (U_* - 1)^2 (fg + f + U_* - 1)}{f}, \\ \hat{a}_{10}(\varepsilon) &= -\frac{(U_* - 1)(2U_* fg - fg + 3fU_* + 2U_*^2 - f - 3U_* + 1)\varepsilon_1}{f} \\ &\quad - \frac{(U_* - 1)(f^2g^2U_* + f^2gU_* + fgU_*^2 - U_* fg)}{f}, \\ \hat{a}_{01}(\varepsilon) &= -U_* (U_* - 1)\varepsilon_1 + U_* (U_* - 1)(fg + f + U_* - 1), \end{aligned}$$

$$\begin{aligned}
 \hat{a}_{20}(\varepsilon) &= -\frac{(U_*fg - fg + 3fU_* + U_*^2 - 2f - 2U_* + 1)\varepsilon_1}{f} \\
 &\quad \frac{U_*^3 - (2-f)U_*^2 - (f^2g^2 + 2f^2g - fg + f - 1) + f^2g^2 + f^2g - fg}{f}, \\
 \hat{a}_{11}(\varepsilon) &= (1 - 2U_*)\varepsilon_1 - fg + 2fU_* + U_*^2 - f - 2U_* + 1, \\
 \hat{a}_{02}(\varepsilon) &= fU_*, \\
 \hat{b}_{10}(\varepsilon) &= -\frac{(U_* - 1)(fg + f + U_* - 1)g(U_*fg - fg + U_*^2 - 2U_* + 1)\varepsilon_2}{f^2} \\
 &\quad - (U_* - 1)(fg + f + U_* - 1)g^2U_*, \\
 \hat{b}_{01}(\varepsilon) &= \frac{(U_* - 1)(fg + f + U_* - 1)(U_*fg - fg + U_*^2 - 2U_* + 1)\varepsilon_2}{f^2} \\
 &\quad + (U_* - 1)(fg + f + U_* - 1)gU_*, \\
 \hat{b}_{20}(\varepsilon) &= -\frac{g(U_*fg - fg + U_*^2 - 2U_* + 1)\varepsilon_2}{f} - fg^2U_*, \\
 \hat{b}_{11}(\varepsilon) &= -\frac{(2fg^2 + 2gU_* - 2g - U_* + 1)(U_*fg - fg + U_*^2 - 2U_* + 1)\varepsilon_2}{(fg + U_* - 1)f} \\
 &\quad - \frac{(2fg^2 + 2gU_* - 2g - U_* + 1)gU_*f}{fg + U_* - 1}, \\
 \hat{b}_{02}(\varepsilon) &= \frac{(2fg + f + 2U_* - 2)(U_*fg - fg + U_*^2 - 2U_* + 1)\varepsilon_2}{(fg + U_* - 1)f} \\
 &\quad + \frac{(2fg + f + 2U_* - 2)gU_*f}{fg + U_* - 1}.
 \end{aligned}$$

To save space, the current expression consists of a previous expression when System (10) undergoes each approximate identity transformation in the following.

Letting $u_2 = u_1, v_2 = \frac{du_1}{dt}$, System (10) can be written as

$$\begin{aligned}
 \frac{du_2}{dt} &= v_2, \\
 \frac{dv_2}{dt} &= \hat{c}_{00}(\varepsilon) + \hat{c}_{10}(\varepsilon)u_2 + \hat{c}_{01}(\varepsilon)v_2 + \hat{c}_{20}(\varepsilon)u_2^2 \\
 &\quad + \hat{c}_{11}(\varepsilon)u_2v_2 + \hat{c}_{02}(\varepsilon)v_2^2 + O(|(u_2, v_2)|^3),
 \end{aligned} \tag{11}$$

where

$$\begin{aligned}
 \hat{c}_{00}(\varepsilon) &= -\frac{1}{\hat{a}_{01}^3(\varepsilon)}(\hat{a}_{00}^3(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{a}_{11}(\varepsilon) - \hat{a}_{00}^2(\varepsilon)\hat{a}_{01}^2(\varepsilon)\hat{b}_{02}(\varepsilon) \\
 &\quad - \hat{a}_{00}^2(\varepsilon)\hat{a}_{01}(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{b}_{01}(\varepsilon) + 2\hat{a}_{00}^2(\varepsilon)\hat{a}_{02}^2(\varepsilon)\hat{b}_{00}(\varepsilon) \\
 &\quad + 2\hat{a}_{00}(\varepsilon)\hat{a}_{01}^2(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{b}_{00}(\varepsilon) - \hat{a}_{01}^4(\varepsilon)\hat{b}_{00}(\varepsilon) + \hat{a}_{00}(\varepsilon)\hat{a}_{01}^3(\varepsilon)\hat{b}_{01}(\varepsilon)), \\
 \hat{c}_{10}(\varepsilon) &= \frac{1}{\hat{a}_{01}^3(\varepsilon)}(2\hat{a}_{00}^2(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{a}_{11}(\varepsilon) - \hat{a}_{00}(\varepsilon)\hat{a}_{01}^2(\varepsilon)\hat{a}_{11}(\varepsilon) \\
 &\quad - 2\hat{a}_{00}(\varepsilon)\hat{a}_{01}^2(\varepsilon)\hat{b}_{02}(\varepsilon) - 2\hat{a}_{00}(\varepsilon)\hat{a}_{01}(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{b}_{01}(\varepsilon) + 4\hat{a}_{00}(\varepsilon)\hat{a}_{02}^2(\varepsilon)\hat{b}_{00}(\varepsilon) \\
 &\quad + \hat{a}_{01}^3(\varepsilon)\hat{a}_{10}(\varepsilon) + \hat{a}_{01}^3(\varepsilon)\hat{b}_{01}(\varepsilon) + 2\hat{a}_{01}^2(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{b}_{00}(\varepsilon)), \\
 \hat{c}_{01}(\varepsilon) &= -\frac{1}{\hat{a}_{01}^3(\varepsilon)}(\hat{a}_{00}(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{a}_{11}(\varepsilon) - \hat{a}_{01}^2(\varepsilon)\hat{a}_{11}(\varepsilon) - \hat{a}_{01}^2(\varepsilon)\hat{b}_{02}(\varepsilon) \\
 &\quad - \hat{a}_{01}(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{b}_{01}(\varepsilon) + 2\hat{a}_{02}^2(\varepsilon)\hat{b}_{00}(\varepsilon)), \\
 \hat{c}_{20}(\varepsilon) &= -\frac{1}{\hat{a}_{01}^3(\varepsilon)}(\hat{a}_{00}^2(\varepsilon)\hat{a}_{01}(\varepsilon)\hat{a}_{11}^2(\varepsilon) - 2\hat{a}_{00}^2(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{a}_{10}(\varepsilon)\hat{a}_{11}(\varepsilon) - \hat{a}_{00}(\varepsilon)\hat{a}_{01}^3(\varepsilon)\hat{b}_{11}(\varepsilon) \\
 &\quad - 2\hat{a}_{00}(\varepsilon)\hat{a}_{01}^2(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{b}_{10}(\varepsilon) + 2\hat{a}_{00}(\varepsilon)\hat{a}_{01}^2(\varepsilon)\hat{a}_{10}(\varepsilon)\hat{b}_{02}(\varepsilon) \\
 &\quad + 2\hat{a}_{00}(\varepsilon)\hat{a}_{01}(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{a}_{10}(\varepsilon)\hat{b}_{01}(\varepsilon) + 2\hat{a}_{00}(\varepsilon)\hat{a}_{01}(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{a}_{11}(\varepsilon)\hat{b}_{00}(\varepsilon) \\
 &\quad - 4\hat{a}_{00}(\varepsilon)\hat{a}_{02}^2(\varepsilon)\hat{a}_{10}(\varepsilon)\hat{b}_{00}(\varepsilon) + \hat{a}_{01}^4(\varepsilon)\hat{b}_{10}(\varepsilon) - \hat{a}_{01}^3(\varepsilon)\hat{a}_{10}(\varepsilon)\hat{b}_{01}(\varepsilon) \\
 &\quad + \hat{a}_{01}^3(\varepsilon)\hat{a}_{11}(\varepsilon)\hat{b}_{00}(\varepsilon) - 2\hat{a}_{01}^2(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{a}_{10}(\varepsilon)\hat{b}_{00}(\varepsilon)), \\
 \hat{c}_{11}(\varepsilon) &= -\frac{1}{\hat{a}_{01}^3(\varepsilon)}(\hat{a}_{00}(\varepsilon)\hat{a}_{01}(\varepsilon)\hat{a}_{11}^2(\varepsilon) - 2\hat{a}_{00}(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{a}_{10}(\varepsilon)\hat{a}_{11}(\varepsilon) \\
 &\quad - 2\hat{a}_{01}^3(\varepsilon)\hat{a}_{20}(\varepsilon) - \hat{a}_{01}^3(\varepsilon)\hat{b}_{11}(\varepsilon) - 2\hat{a}_{01}^2(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{b}_{10}(\varepsilon) \\
 &\quad + \hat{a}_{01}^2(\varepsilon)\hat{a}_{10}(\varepsilon)\hat{a}_{11}(\varepsilon) + 2\hat{a}_{01}^2(\varepsilon)\hat{a}_{10}(\varepsilon)\hat{b}_{02}(\varepsilon) - 4\hat{a}_{02}^2(\varepsilon)\hat{a}_{10}(\varepsilon)\hat{b}_{00}(\varepsilon) \\
 &\quad + 2\hat{a}_{01}(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{a}_{10}(\varepsilon)\hat{b}_{01}(\varepsilon) + 2\hat{a}_{01}(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{a}_{11}(\varepsilon)\hat{b}_{00}(\varepsilon)),
 \end{aligned}$$

$$\begin{aligned} \hat{c}_{02}(\varepsilon) = & -\frac{1}{\hat{a}_{01}^3(\varepsilon)} (\hat{a}_{00}(\varepsilon)\hat{a}_{01}^2(\varepsilon)\hat{a}_{11}(\varepsilon)\hat{a}_{20}(\varepsilon) - \hat{a}_{00}(\varepsilon)\hat{a}_{01}(\varepsilon)\hat{a}_{10}(\varepsilon)\hat{a}_{11}^2(\varepsilon)) \\ & + \hat{a}_{00}(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{a}_{10}^2(\varepsilon)\hat{a}_{11}(\varepsilon) - \hat{a}_{01}^4(\varepsilon)\hat{b}_{20}(\varepsilon) + \hat{a}_{01}^3(\varepsilon)\hat{a}_{10}(\varepsilon)\hat{b}_{11}(\varepsilon) \\ & - \hat{a}_{01}^3(\varepsilon)\hat{a}_{11}(\varepsilon)\hat{b}_{10}(\varepsilon) + \hat{a}_{01}^3(\varepsilon)\hat{a}_{20}(\varepsilon)\hat{b}_{01}(\varepsilon) \\ & + 2\hat{a}_{01}^2(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{a}_{10}(\varepsilon)\hat{b}_{10}(\varepsilon) + 2\hat{a}_{01}^2(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{a}_{20}(\varepsilon)\hat{b}_{00}(\varepsilon) \\ & - \hat{a}_{01}^2(\varepsilon)\hat{a}_{10}^2(\varepsilon)\hat{b}_{02}(\varepsilon) - \hat{a}_{01}(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{a}_{10}^2(\varepsilon)\hat{b}_{01}(\varepsilon) \\ & - 2\hat{a}_{01}(\varepsilon)\hat{a}_{02}(\varepsilon)\hat{a}_{10}(\varepsilon)\hat{a}_{11}(\varepsilon)\hat{b}_{00}(\varepsilon) + 2\hat{a}_{02}^2(\varepsilon)\hat{a}_{10}^2(\varepsilon)\hat{b}_{00}(\varepsilon). \end{aligned}$$

To remove v_2^2 from System (11), introduce a new time variable τ with $dt = (1 - \hat{c}_{02}(\varepsilon)u_2)d\tau$ and make $u_3 = u_2, v_3 = v_2(1 - \hat{c}_{02}(\varepsilon)u_2)$, then System (11) becomes (rewriting τ as t):

$$\begin{aligned} \frac{du_3}{dt} &= v_3, \\ \frac{dv_3}{dt} &= \hat{d}_{00}(\varepsilon) + \hat{d}_{10}(\varepsilon)u_3 + \hat{d}_{01}(\varepsilon)v_3 + \hat{d}_{20}(\varepsilon)u_3^2 \\ &\quad + \hat{d}_{11}(\varepsilon)u_3v_3 + O(|(u_3, v_3)|^3), \end{aligned} \tag{12}$$

where

$$\begin{aligned} \hat{d}_{00}(\varepsilon) &= \hat{c}_{00}(\varepsilon), \quad \hat{d}_{10}(\varepsilon) = \hat{c}_{10}(\varepsilon) - 2\hat{c}_{00}(\varepsilon)\hat{c}_{02}(\varepsilon), \\ \hat{d}_{01}(\varepsilon) &= \hat{c}_{01}(\varepsilon), \quad \hat{d}_{20}(\varepsilon) = \hat{c}_{20}(\varepsilon) - 2\hat{c}_{02}(\varepsilon)\hat{c}_{10}(\varepsilon) + \hat{c}_{00}(\varepsilon)\hat{c}_{02}^2(\varepsilon), \\ \hat{d}_{11}(\varepsilon) &= -\hat{c}_{01}(\varepsilon)\hat{c}_{02}(\varepsilon) + \hat{c}_{11}(\varepsilon). \end{aligned}$$

From $\hat{d}_{20}(0) = -\frac{(U_*-1)^2(fg+f+U_*-1)^2gU_*^2}{f}$, when ε_i ($i = 1, 2$) are small, we have $\hat{d}_{20}(\varepsilon) < 0$. Therefore, $\hat{d}_{20}(\varepsilon)$ can be reduced into -1 with the following coordinates change:

$$u_4 = u_3, \quad v_4 = \frac{v_3}{\sqrt{-\hat{d}_{20}(\varepsilon)}}, \quad \tau = \sqrt{-\hat{d}_{20}(\varepsilon)}t,$$

then System (12) can be transformed into (rewriting τ as t)

$$\begin{aligned} \frac{du_4}{dt} &= v_4, \\ \frac{dv_4}{dt} &= \hat{e}_{00}(\varepsilon) + \hat{e}_{10}(\varepsilon)u_4 + \hat{e}_{01}(\varepsilon)v_4 - u_4^2 + \hat{e}_{11}(\varepsilon)u_4v_4 + O(|(u_4, v_4)|^3), \end{aligned} \tag{13}$$

where

$$\hat{e}_{00}(\varepsilon) = -\frac{\hat{d}_{00}(\varepsilon)}{\hat{d}_{20}(\varepsilon)}, \quad \hat{e}_{10}(\varepsilon) = -\frac{\hat{d}_{10}(\varepsilon)}{\hat{d}_{20}(\varepsilon)}, \quad \hat{e}_{01}(\varepsilon) = \frac{\hat{d}_{01}(\varepsilon)}{\sqrt{-\hat{d}_{20}(\varepsilon)}}, \quad \hat{e}_{11}(\varepsilon) = \frac{\hat{d}_{11}(\varepsilon)}{\sqrt{-\hat{d}_{20}(\varepsilon)}}.$$

Making the transformation

$$u_5 = u_4 - \frac{\hat{e}_{10}(\varepsilon)}{2}, \quad v_5 = v_4,$$

u_4 in System (13) can be removed, then System (13) becomes

$$\begin{aligned} \frac{du_5}{dt} &= v_5, \\ \frac{dv_5}{dt} &= \hat{f}_{00}(\varepsilon) + \hat{f}_{01}(\varepsilon)v_5 - u_5^2 + \hat{f}_{11}(\varepsilon)u_5v_5 + O(|(u_5, v_5)|^3), \end{aligned} \tag{14}$$

where

$$\hat{f}_{00}(\varepsilon) = \hat{e}_{00}(\varepsilon) + \frac{1}{4}\hat{e}_{10}^2(\varepsilon), \quad \hat{f}_{01}(\varepsilon) = \hat{e}_{01}(\varepsilon) + \frac{1}{2}\hat{e}_{10}(\varepsilon)\hat{e}_{11}(\varepsilon), \quad \hat{f}_{11}(\varepsilon) = \hat{e}_{11}(\varepsilon).$$

From $\hat{f}_{11}(0) = \frac{(fg+f+U_*-1)(fg(fg+U_*^2-1)+2U_*(U_*-1)^2)}{(fg+U_*-1)f} \neq 0$, when ε_i ($i = 1, 2$) are small, we have $\hat{f}_{11}(\varepsilon) \neq 0$. Employing the change of variables again,

$$u_6 = \hat{f}_{11}^2(\varepsilon)u_5, \quad v_6 = -\hat{f}_{11}^3(\varepsilon)v_5, \quad \tau = -\frac{1}{\hat{f}_{11}(\varepsilon)}t,$$

we obtain the universal unfolding of System (9),

$$\begin{aligned} \frac{du_6}{d\tau} &= v_6, \\ \frac{dv_6}{d\tau} &= \hat{g}_{00}(\varepsilon) + \hat{g}_{01}(\varepsilon)v_6 - u_6^2 + u_6v_6 + O(|(u_6, v_6)|^3), \end{aligned} \tag{15}$$

where

$$\hat{g}_{00}(\varepsilon) = -\hat{f}_{00}(\varepsilon)\hat{f}_{11}^4(\varepsilon), \quad \hat{g}_{01}(\varepsilon) = -\hat{f}_{01}(\varepsilon)\hat{f}_{11}(\varepsilon).$$

The transversality condition

$$\left| \frac{\partial(\hat{g}_{00}, \hat{g}_{01})}{\partial(\varepsilon_1, \varepsilon_2)} \right|_{\varepsilon_1=\varepsilon_2=0} = \frac{(fg(fg+U_*^2-1)+2U_*(U_*-1)^2)^5}{(fg+U_*-1)^4(U_*-1)^3f^4U_*^6g^3} \neq 0$$

holds. Therefore, from the Bogdanov–Takens bifurcation theorem [19,20], System (2) undergoes a Bogdanov–Takens bifurcation of codimension 2, which includes a sequence of bifurcations of codimension 1: saddle-node bifurcation, Hopf bifurcation and homoclinic bifurcation, when the parameters ε_1 and ε_2 vary in a small neighborhood of $(0, 0)$. □

Turning to Matlab, we provide some data groups to simulate the dynamic process of Bogdanov–Takens bifurcation to support the theoretical results. Let $U_* = \frac{1}{2}, f = \frac{3}{5}$ and $g = \frac{1}{5}$, then $\alpha_1 = \frac{3}{25}, \beta_1 = \frac{13}{60}, r_1 = \frac{18}{95}$ and System (9) becomes

$$\begin{aligned} \dot{U} &= \{U(U-1)(\frac{3}{25} + \varepsilon_1 - U) - \frac{3}{5}UD\}\{1 - (U + D)\}, \\ \dot{D} &= (\frac{18}{95} + \varepsilon_2)\{D(\frac{13}{60} - D) + \frac{1}{5}UD\}\{1 - (U + D)\}. \end{aligned} \tag{16}$$

System (16) will present different phase portraits according to following six examples for disturbance of ε_1 and ε_2 .

Example 1: System (16) admits a cusp of codimension 2 if $(\varepsilon_1, \varepsilon_2) = (0, 0)$; see Figure 1.

Example 2: there exists no equilibrium in System (16) if $(\varepsilon_1, \varepsilon_2) = (0.001, -0.001)$; see Figure 2a.

Example 3: set $(\varepsilon_1, \varepsilon_2) = (-0.001, -0.074)$, System (16) admits an unstable focus and a saddle; see Figure 2b.

Example 4: take $(\varepsilon_1, \varepsilon_2) = (-0.001, -0.06)$, System (16) has a stable focus surrounded by an unstable limit cycle; see Figure 2c.

Example 5: System (16) has an unstable homoclinic loop consisted of a saddle and an unstable homoclinic orbit if $(\varepsilon_1, \varepsilon_2) = (-0.001, -0.05)$; see Figure 2d.

Example 6: if $(\varepsilon_1, \varepsilon_2) = (-0.001, -0.04)$, there is a stable focus and a saddle in System (16); see Figure 2e.

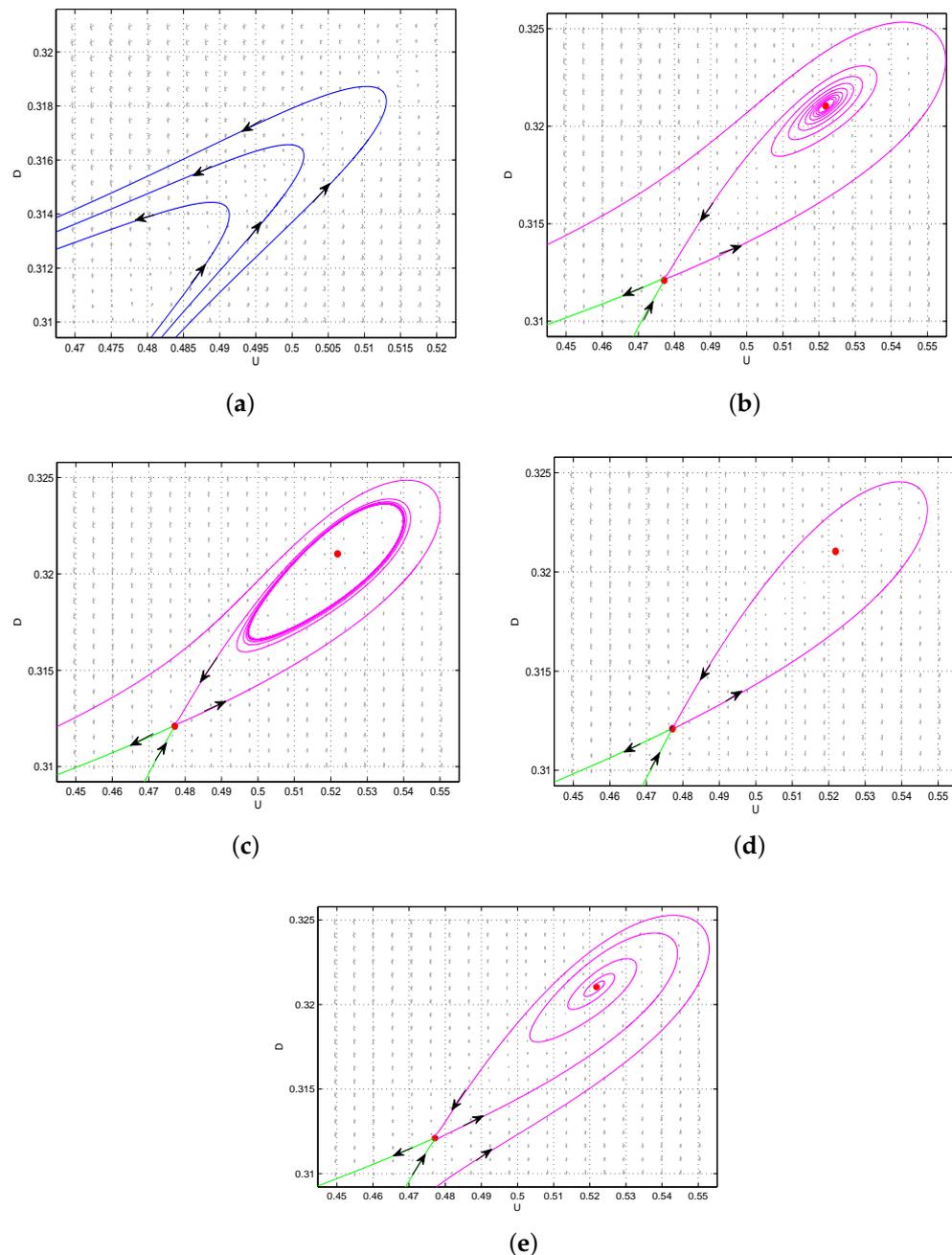


Figure 2. Phase portrait of (16) under disturbance of ε_1 and ε_2 . (a) There exists no equilibrium in System (16) if $(\varepsilon_1, \varepsilon_2) = (0.001, -0.001)$; (b) System (16) admits an unstable focus and a saddle if $(\varepsilon_1, \varepsilon_2) = (-0.001, -0.074)$; (c) System (16) has an unstable limit cycle if $(\varepsilon_1, \varepsilon_2) = (-0.001, -0.06)$; (d) System (16) has an unstable homoclinic loop if $(\varepsilon_1, \varepsilon_2) = (-0.001, -0.05)$; (e) there is a stable focus and a saddle in System (16) if $(\varepsilon_1, \varepsilon_2) = (-0.001, -0.04)$.

4. Conclusions

The dynamical behavior of System (1) around the unique positive equilibrium E_* was investigated in this manuscript. The case that the Jacobian matrix J_{E_*} has two zero eigenvalues is focused. We proved that equilibrium E_* is a cusp of codimension 2 through linear transformation and normal form theory [20].

From Theorems 1 and 3, in order to ensure that System (1) admits a cusp of codimension 2, parameters f and g must satisfy $f > 0$ and $g > 0$. Therefore, doubt should be regarded as restraining force and destroys comprehension in the learning process if

System (1) has a cusp of codimension 2 and undergoes Bogdanov–Takens bifurcation of codimension 2.

By Theorem 4, System (1) undergoes Bogdanov–Takens bifurcation of codimension 2, which leads to potentially dramatic changes in the system, hence bifurcation of codimension 2 should be not ignored. Due to a saddle-node bifurcation, the number of the interior equilibrium is zero, one, or two as parameters α and r are disturbed. Thus, there are some values of parameters such as that understanding and doubt co-exist in the form of interior equilibrium for different initial values. Through a Hopf bifurcation, there is a limit cycle appearing in System (1). As seen in Figure 2c, an unstable limit cycle is produced by the Hopf bifurcation. When all initial values lying inside the limit cycle, the orbit will tend to the interior equilibrium at last. That is, understanding and doubt will reach a steady state. In other words, knowledge can not be absorbed fully; once the learner decreases the time to understand, doubts regarding the knowledge will increase. From homoclinic bifurcation, a homoclinic loop appears in System (1). As seen in Figure 2d, an unstable homoclinic loop is consisted of a saddle and an unstable homoclinic orbit, which means that understanding and doubt co-exist in the form of an interior equilibrium for all initial values lying inside the homoclinic loop.

It is noteworthy that when all initial values lying outside the unstable limit cycle and unstable homoclinic loop, combining with Theorem 2, the orbit will tend to the equilibrium E_1 . That is, if enough time was taken, the learner will feel confident with the knowledge at last.

Above all, the occurrences of Bogdanov–Takens bifurcation yields to more complex dynamical behavior, including bifurcations of codimension one. It is very sensitive to the choice of coefficient of difficulty in studying material and the speed of the learning process, which indicates that the selection of learning and teaching methods should be put forward higher requirements for students and teachers in education.

However, the existence condition of Bogdanov–Takens of codimension 2 is sufficient and the Bogdanov–Takens bifurcation of higher codimensions in the distribution functions [21] is worth studying in the future.

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