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On the Geometry of the Riemannian Curvature Tensor of Nearly Trans-Sasakian Manifolds

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Abstract: This paper presents the results of fundamental research into the geometry of the Riemannian curvature tensor of nearly trans-Sasakian manifolds. The components of the Riemannian curvature tensor on the space of the associated G-structure are counted, and the components of the Ricci tensor are calculated. Some identities are obtained that are satisfied by the Riemannian curvature tensors and the Ricci tensor. A number of properties are proved that characterize nearly trans-Sasakian manifolds with a closed contact form. The structure of nearly trans-Sasakian manifolds with a closed contact form is obtained. Several classes are singled out in terms of second-order differential-geometric invariants, and their local structure is obtained. The k-nullity distribution of a nearly trans-Sasakian manifold is studied.

Keywords: nearly trans-Sasakian manifold; Riemannian curvature tensor; k-nullity distribution; closely cosymplectic manifold

1. Introduction

It is known [1,2] that if M is an almost contact metric manifold, then an almost Hermitian structure (called a linear extension of the original almost contact metric structure [3]) is canonically induced on the manifold $M \times \mathbf{R}$. The question of the connection between these structures has been repeatedly studied. The classical result in this direction is the well-known result of Nakayama stating that an almost contact metric structure is normal if and only if its linear extension is a Hermitian structure [4]. The study of this connection makes it possible to single out interesting new classes of almost contact metric structures. Obigney [5] singled out classes of trans-Sasakian and almost trans-Sasakian structures whose linear extensions belong to the classes W_4 and $W_2 \oplus W_4$ of almost Hermitian structures in the Gray–Hervella classification [6], respectively. V.F. Kirichenko and E.V. Rodina in [3] obtained a number of profound results concerning the geometry of trans-Sasakian and almost trans-Sasakian manifolds. The class of almost contact metric structures, which is a linear extension of almost Hermitian structures of the class $W_1 \oplus W_4$ in the Gray–Hervella classification, is also known as nearly trans-Sasakian structures [7].

There are about 15 articles on this topic, and we present them here.

In [8], the author investigates the harmonicity and D-pluriharmonicity of a (φ, J)-holomorphic mapping [9] of a nearly trans-Sasakian manifold into an almost Hermitian manifold. It is proved that any (φ, J)-holomorphic mapping of a nearly trans-Sasakian manifold into a quasi-Kählerian manifold is harmonic, and into a Kählerian manifold it is D-pluriharmonic. Further works have been devoted to the study of submanifolds of a nearly trans-Sasakian manifold. Thus, in [10], some sufficient conditions were obtained under which a submanifold of codimension two of a nearly trans-Sasakian manifold with a trivial normal bundle admits a nearly trans-Sasakian structure. Similar results are formulated for some particular types of nearly trans-Sasakian: nearly Sasakian, nearly Kenmotsu, nearly cosymplectic, trans-Sasakian, Kenmotsu, and cosymplectic.

The paper [11] studies CR-submanifolds of nearly trans-Sasakian manifolds, generalizing the results of trans-Sasakian manifolds. The work [12] studies the skew product of

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CR-submanifolds of nearly trans-Sasakian manifolds. The authors study the relationship between the square of the norm of the second fundamental form (external invariant) and the deformation function (internal invariant) for submanifolds of the skew product. They obtained an inequality, namely,

$$\|h\|^2 \geq 2s\left[\|\nabla lnf\|^2 + \alpha^2 - \beta^2\right], \tag{1}$$

for contact CR-skew products of nearly trans-Sasakian manifolds. The study of properties of CR-submanifolds of nearly trans-Sasakian manifolds with a semisymmetric nonmetric connection is the subject of [13].

In [14–17], semi-invariant submanifolds of a nearly trans-Sasakian manifold are studied, and the Nijenhuis tensor of a nearly trans-Sasakian manifold is calculated. Conditions for the integrability of some distributions on a semi-invariant submanifold of a nearly trans-Sasakian manifold are studied. Completely umbilical, totally contact umbilical, totally geodesic, and totally contact geodesic submanifolds are also studied. A classification of fully umbilical semi-invariant submanifolds of an almost trans-Sasakian manifold is obtained.

The next series of papers [18–22] is devoted to the study of a submanifold of a skew product of nearly trans-Sasakian manifolds. Conditions for the integrability of distributions on these submanifolds are obtained. Some interesting results regarding such manifolds have also been obtained. The articles obtained necessary and sufficient conditions for a totally umbilical proper oblique submanifold and show when it is totally geodesic. They have studied the geometry of oblique submanifolds of a nearly trans-Sasakian manifold when the tensor field Q is parallel. It is proved that Q is not parallel on a submanifold if it is not anti-invariant.

In [23], non-invariant hypersurfaces of a nearly trans-Sasakian manifold endowed with an (f, g, u, V, λ)-structure are studied. Some properties of this structure are obtained, and the second fundamental forms of non-invariant hypersurfaces of nearly trans-Sasakian manifolds and weakly cosymplectic manifolds with (f, g, u, V, λ)-structure are calculated under the condition that f is parallel. In addition, the eigenvalues f are found, and it is proved that a non-invariant hypersurface with the (f, g, u, V, λ)-structure of a weakly cosymplectic manifold with a contact structure becomes completely geodesic. The article ends with a study of the necessary condition for a completely geodesic or completely umbilical non-invariant hypersurface with the (f, g, u, V, λ)-structure of a nearly trans-Sasakian manifold.

Let us mention one more article [24], in which a systematic study of nearly trans-Sasakian manifolds was started. In what follows, we will frequently refer to this work.

It follows from the above review of sources that the authors of these publications did not consider the geometry of the Riemannian curvature tensor.

In this paper, we propose to fill this gap; we present the results of studies of the geometry of the Riemannian curvature tensor of nearly trans-Sasakian manifolds. The article has the following structure.

Section 2 provides the basic information necessary for further research. The method of research is given, the object of research is determined. In Section 3, we provide a definition of a nearly trans-Sasakian manifold and obtain a complete group of structural equations, as well as some identities characterizing a nearly trans-Sasakian structure. The main results of this section are Theorems 3 and 5.

In Section 4, we calculate the components of the Riemannian curvature tensor and the Ricci tensor of a nearly trans-Sasakian manifold on the space of the associated G-structure and calculate the scalar curvature. Some identities that are satisfied by the Riemannian curvature tensor and the Ricci tensor are proved.

In Section 5, we study nearly trans-Sasakian manifolds of constant curvature. It is proved that a nearly trans-Sasakian manifold of constant curvature is either a trans-Sasakian manifold of constant negative curvature or is locally conformal to a closely cosymplectic manifold of constant curvature. In particular, a nearly trans-Sasakian manifold is zero constant curvature if and only if it is a cosymplectic manifold of constant curvature.

In Section 6, we study the contact analogs of the Gray classes of nearly trans-Sasakian manifolds, i.e., classes CR_1, CR_2, and CR_3. It is proved that nearly trans-Sasakian manifolds of the class CR_3 coincide with almost contact metric manifolds that are locally conformal to closely cosymplectic manifolds. Classes CR_4 and CR_5 are introduced. It is proved that nearly trans-Sasakian varieties of classes CR_4 and CR_5 coincide. Additionally, the local structure of this class of manifolds is obtained.

In Section 7, we introduce the concept of the k-nullity distribution. It is proved that a nearly trans-Sasakian manifold on which there exists a k-nullity distribution containing the characteristic vector field ξ is a nearly trans-Sasakian manifold of class CR_4.

2. Preliminary Information

Recall that an almost contact metric structure on a manifold M is a collection $(\xi, \eta, \Phi, g = \langle\cdot,\cdot\rangle)$ of tensor fields on M, where ξ is a vector field, called characteristic; η is a differential 1-form, called a contact form; Φ is an endomorphism of the module $\mathcal{X}(M)$ of smooth vector fields of the manifold M, called a structural endomorphism; and $g = \langle\cdot,\cdot\rangle$ is a Riemannian metric. In this case:

$$\begin{gathered}(1)\ \eta(\xi) = 1;\\ (2)\ \Phi(\xi) = 0;\\ (3)\ \eta \circ \Phi = 0;\\ (4)\ \Phi^2 = -id + \xi \otimes \eta;\\ (5)\ \langle \Phi X, \Phi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y);\ \ X, Y \in \mathcal{X}(M).\end{gathered} \tag{2}$$

Such structures naturally arise on hypersurfaces of almost Hermitian manifolds [25], on spaces of principal T^1 bundles over symplectic manifolds with integral fundamental form (Boothby–Wan bundles [26]), and, more generally, over almost Hermitian manifolds [27], and are natural generalizations of the so-called contact metric manifolds that arise on odd-dimensional manifolds with a fixed 1-form of maximum rank (contact structure).

It is well known that a manifold admitting an almost contact metric structure is odd-dimensional and orientable. In the $C^\infty(M)$-module $\mathcal{X}(M)$ of smooth vector fields on such a manifold, two mutually complementary projectors,

$$\begin{gathered}(1)\ \mathbf{l} = id - \mathbf{m} = -\Phi^2;\\ (2)\ \mathbf{m} = \xi \otimes \eta,\end{gathered} \tag{3}$$

are internally defined. These will be projections onto the distributions

$$\begin{gathered}(1)\ \mathcal{L} = Im\Phi = ker\eta;\\ (2)\ \mathcal{M} = ker\Phi,\end{gathered} \tag{4}$$

respectively, which we call the first and second fundamental distributions of the almost contact metric structure. Thus, for a module $\mathcal{X}(M)$ of smooth vector fields, we have

$$\mathcal{X}(M) = \mathcal{L} \oplus \mathcal{M}, \tag{5}$$

where

$$\begin{gathered}(1)\ dim\mathcal{L} = 2n\\ (2)\ dim\mathcal{M} = 1.\end{gathered} \tag{6}$$

Moreover, if we introduce into consideration

$$\mathcal{X}(M)^C = C \oplus \mathcal{X}(M), \tag{7}$$

the complexification of the module $\mathcal{X}(M)$, then

$$\mathcal{X}(M)^C = D_\Phi^{\sqrt{-1}} \oplus D_\Phi^{-\sqrt{-1}} \oplus D_\Phi^0, \tag{8}$$

where $D_\Phi^{\sqrt{-1}}$, $D_\Phi^{-\sqrt{-1}}$, and D_Φ^0 are the eigendistributions of the structural endomorphism Φ corresponding to the eigenvalues $\sqrt{-1}$, $-\sqrt{-1}$, and 0. Moreover, the projections onto the terms of this direct sum are, respectively, the endomorphisms [28,29]

$$\begin{gathered}(1)\ \pi = -\tfrac{1}{2}\left(\Phi^2 + \sqrt{(-1)}\Phi\right);\\ (2)\ \overline{\pi} = \tfrac{1}{2}(-\Phi^2 + \sqrt{-1}\Phi);\\ (3)\ \mathbf{m} = \eta \otimes \xi.\end{gathered} \tag{9}$$

Defining an almost contact metric structure on the manifold M^{2n+1} is equivalent to defining a G-structure $\mathcal{G}$ on M with the structure group

$$G = U(n) \times 1. \tag{10}$$

The elements of the total space of this G-structure are the complex frames of the manifold M of the form

$$p = (p, \xi_p, \epsilon_1, \ldots, \epsilon_n, \epsilon_{\hat{1}}, \ldots, \epsilon_{\hat{n}}). \tag{11}$$

These frames are characterized by the fact that the matrices of the tensors Φ and g in them have, respectively, the form

$$\begin{gathered}(1)\ (\Phi_i^j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & 0 \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix};\\ (2)\ (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix},\end{gathered} \tag{12}$$

where I_n is the identity matrix of order n. We will assume that the indexes $i, j, k, \ldots$ run from 0 to $2n$, and that the indexes $a, b, c, d, \ldots$ run from 1 to n. Let

$$\hat{a} = a + n. \tag{13}$$

It is well known [28,29] that the first group of structural equations of the G-structure $\mathcal{G}$ has the form:

$$\begin{gathered}(1)\ d\theta^a = -\theta_b^a \wedge \theta^b + C^{ab}{}_c\theta^c \wedge \theta_b + C^{abc}\theta_b \wedge \theta_c + C^a{}_b\theta^b \wedge \theta + C^{ab}\theta_b \wedge \theta;\\ (2)\ d\theta_a = \theta_a^b \wedge \theta_b + C_{ab}{}^c\theta_c \wedge \theta^b + C_{abc}\theta^b \wedge \theta^c + C_a{}^b\theta_b \wedge \theta + C_{ab}\theta^b \wedge \theta;\\ (3)\ d\theta = D_{ab}\theta^a \wedge \theta^b + D^{ab}\theta_a \wedge \theta_b + D_a^b\theta^a \wedge \theta_b + D_a\theta \wedge \theta^a + D^a\theta \wedge \theta_a,\end{gathered} \tag{14}$$

where $\left\{\theta_j^i\right\}$ are the components of the Riemannian connection ∇ form of metrics g; $\{\theta^i\}$ are the components of the solder form,

$$\theta = \theta^0 = \pi^*\eta; \tag{15}$$

π is a natural projection of the G-structure total space to the manifold M, and

$$
\begin{gathered}
(1)\ \Phi^{a}_{b,k} = 0; \\
(2)\ \Phi^{\hat{a}}_{\hat{b},k} = 0; \\
(3)\ \Phi^{0}_{0,k} = 0; \\
(4)\ C^{abc} = \frac{\sqrt{-1}}{2}\Phi^{a}_{[\hat{b},\hat{c}]}; \\
(5)\ C_{abc} = -\frac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{[b,c]}; \\
(6)\ C^{ab}{}_{c} = -\frac{\sqrt{-1}}{2}\Phi^{a}_{\hat{b},c}; \\
(7)\ C_{ab}{}^{c} = \frac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{b,\hat{c}}; \\
(8)\ C^{ab} = \sqrt{-1}(\tfrac{1}{2}\Phi^{a}_{\hat{b},0} - \Phi^{a}_{0,\hat{b}}); \\
(9)\ C_{ab} = -\sqrt{-1}(\tfrac{1}{2}\Phi^{\hat{a}}_{b,0} - \Phi^{\hat{a}}_{0,b}); \\
(10)\ C^{a}{}_{b} = -\sqrt{-1}\Phi_{0,b}; \\
(11)\ C_{a}{}^{b} = \sqrt{-1}\Phi^{\hat{a}}_{0,\hat{b}}; \\
(12)\ D^{ab} = \sqrt{-1}\Phi^{0}_{[\hat{a},\hat{b}]}; \\
(13)\ D_{ab} = -\sqrt{-1}\Phi^{0}_{[a,b]}; \\
(14)\ D^{b}_{a} = -\sqrt{-1}(\Phi^{0}_{a,\hat{b}} + \Phi^{0}_{\hat{b},a}); \\
(15)\ D^{a} = -\sqrt{-1}\Phi^{0}_{\hat{a},0}; \\
(16)\ D_{a} = \sqrt{-1}\Phi^{0}_{a,0}.
\end{gathered} \tag{16}
$$

In this case,

$$
\begin{gathered}
(1)\ C^{abc} = -C^{acb}; \\
(2)\ C_{abc} = -C_{acb}; \\
(3)\ C^{ab}{}_{c} = -C^{ba}{}_{c}; \\
(4)\ C_{ab}{}^{c} = C_{ba}{}^{c}; \\
(5)\ D^{ab} = -D^{ba}; \\
(6)\ D_{ab} = -D_{ba}; \\
(7)\ D^{b}_{a} = C_{a}{}^{b} - C^{b}{}_{a}.
\end{gathered} \tag{17}
$$

We also recall that an almost Hermitian structure on a manifold M is a pair $(J, g = \langle \cdot, \cdot \rangle)$ of tensor fields on M, where J is an almost complex structure,

$$J^2 = -id, \tag{18}$$

and g is a Riemannian metric. Moreover,

$$\langle JX, JY \rangle = \langle X, Y \rangle; \;\; X, Y \in \mathcal{X}(M). \tag{19}$$

Defining an almost Hermitian structure on M^{2n} is equivalent to defining a G-structure on M with structure group $U(n)$. The elements of the total space of this G-structure are complex frames of the manifold M, characterized by the fact that the matrices of the tensors J and g in them have, respectively, the form

$$(1)\ (J_i^j) = \begin{pmatrix} \sqrt{-1}I_n & 0 \\ 0 & -\sqrt{-1}I_n \end{pmatrix},$$
$$(2)\ (g_{ij}) = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}. \quad (20)$$

It is well known that the first group of structural equations of this G-structure has the form [29,30]

$$(1)\ d\omega^a = -\omega_b^a \wedge \omega^b + B^{ab}{}_c\omega^c \wedge \omega_b + B^{abc}\omega_b \wedge \omega_c;$$
$$(2)\ d\omega_a = \omega_a^b \wedge \omega_b + B_{ab}{}^c\omega_c \wedge \omega^b + B_{abc}\omega^b \wedge \omega^c. \quad (21)$$

Here, $\left\{\omega_j^i\right\}$ are the components of the form of the Riemannian connection of the metric g; $\{\omega^i\}$ are the components of the solder form; and

$$(1)\ B^{abc} = \overline{B_{abc}};$$
$$(2)\ B^{ab}{}_c = \overline{B_{ab}{}^c} \quad (22)$$

are the components the so-called structural and virtual tensors on the space of the associated G-structure. In this case,

$$(1)\ B^{abc} = -B^{acb};$$
$$(2)\ B_{abc} = -B_{acb};$$
$$(3)\ B^{ab}{}_c = -B^{ba}{}_c;$$
$$(4)\ B_{ab}{}^c = -B_{ba}{}^c; \quad (23)$$

(for details, see, for example, [29,30]).

We recall [6] that almost contact metric structures of the class $W_1 \oplus W_4$ in the Gray–Hervella classification (Vaisman–Gray structures) on the manifold M^{2n} are defined by the identity

$$\nabla_X(\Psi)(X,Y) = -\frac{1}{2(n-1)}\{\langle X,X\rangle\delta\Psi(Y) - \langle X,Y\rangle\delta\Psi(X) - \langle JX,Y\rangle\delta\Psi(JX)\}, \quad (24)$$

where

$$\Psi(X,Y) = \langle X, JY\rangle \quad (25)$$

is the fundamental form of almost Hermitian structures and δ is the codifferentiation operator. It is verified by direct calculation that this identity is equivalent to the following relations on the space of the associated G-structure:

$$(1)\ B^{[abc]} = B^{abc};$$
$$(2)\ B_{[abc]} = B_{abc};$$
$$(3)\ B^{ab}{}_c = \beta^{[a}\delta_c^{b]};$$
$$(4)\ B_{ab}{}^c = \beta_{[a}\delta_{b]}^c; \quad (26)$$

where $\{\beta_i\}$ are functions on the space of the associated G-structure that are components of the so-called Lee form.

Let us briefly recall the construction of a linear extension of an almost contact metric manifold M (or, similarly, a linear extension of its almost contact metric structure). Note that on the manifold $M \times \mathbf{R}$, a two-dimensional distribution Δ is internally defined such that

$$\Delta_{(p,t)} = M_p \oplus \mathbf{R}. \quad (27)$$

Obviously, this distribution is equipped with a canonical almost Hermitian structure (J_0, g_0), where J_0 is the rotation operator through the angle $\frac{\pi}{2}$ in the positive direction. It is also obvious that, in the pair $(J, \tilde{g})$, where

$$J_{(p,t)} = \Phi|_{L_p} \oplus J_0 \tag{28}$$

and $\tilde{g}$ is the Cartesian product metric, is an almost Hermitian structure on the manifold $M \times \mathbf{R}$. We note that the distribution $\delta^{\perp}$ is invariant under the endomorphism J. The triple $(M \times \mathbf{R}, J, \tilde{g})$ is called a linear extension of the initial almost contact metric manifold [3]. On the manifold $(M \times \mathbf{R})$, the vector field ν generated by the unit vector of the real axis $\mathbf{R}$ and its dual closed 1-form ζ, which defines the completely integrable Pfaff equation

$$\zeta = 0, \tag{29}$$

whose maximal integral manifolds are naturally identified with the manifold M, are defined internally, as well as vector field ξ and covector field η, respectively, by the characteristic vector and the contact form of the manifold M. With their help, frames of type

$$p = \left(p, \xi_p, \epsilon_1, ..., \epsilon_n, \epsilon_{\hat{1}}, ..., \epsilon_{\hat{n}}\right) \tag{30}$$

of the manifold M are naturally complemented to frames of type

$$\tilde{p} = \left(p, \xi_p, \epsilon_1, ..., \epsilon_n, \nu_p, \epsilon_{\hat{1}}, ..., \epsilon_{\hat{n}}\right) \tag{31}$$

of the manifold $(M \times \mathbf{R})$. This variety is naturally associated with the G-structure $\mathbf{G}$ with the structural group

$$G = U(n) \times U(1), \tag{32}$$

the first group of structural equations of which has the form:

$$\begin{aligned} (1)\ d\omega^{\alpha} &= -\omega^{\alpha}_{\beta} \wedge \omega^{\beta} + B^{\alpha\beta}{}_{\gamma}\omega^{\gamma} \wedge \omega_{\beta} + B^{\alpha\beta\gamma}\omega_{\beta} \wedge \omega_{\gamma}; \\ (2)\ d\omega_{\alpha} &= \omega^{\beta}_{\alpha} \wedge \omega_{\beta} + B_{\alpha\beta}{}^{\gamma}\omega_{\gamma} \wedge \omega^{\beta} + B_{\alpha\beta\gamma}\omega^{\beta} \wedge \omega^{\gamma}, \end{aligned} \tag{33}$$

(indexes α, β, γ,... run values from 0 to n). The elements of the total space of this G-structure are complex frames of the form

$$r = \left(p, \epsilon_0, \epsilon_1, ..., \epsilon_n, \epsilon_{\hat{0}}, ..., \epsilon_{\hat{n}}\right), \tag{34}$$

where

$$\begin{aligned} (1)\ \epsilon_0 &= \tfrac{1}{\sqrt{2}}\left(\xi_p - \sqrt{-1}\nu_p\right); \\ (2)\ \epsilon_{\hat{p}} &= \tfrac{1}{\sqrt{2}}\left(\xi_p + \sqrt{-1}\nu_p\right). \end{aligned} \tag{35}$$

Complementing system (14) with the equation

$$d\theta_0 = 0, \tag{36}$$

where

$$\theta_0 = \pi^* \zeta, \tag{37}$$

and using the transition matrix from the frame $\tilde{p}$ to the frame r, it is easy to establish a fundamental connection between the structural objects of the G-structure $\mathcal{G}$ and $\mathbf{G}$:

$$\begin{gathered}
(1)\ B^{ab}{}_{c} = C^{ab}{}_{c};\\
(2)\ B^{ab}{}_{0} = \tfrac{1}{\sqrt{2}}\Big(D^{ab} - C^{[ab]}\Big);\\
(3)\ B^{a0}_{b} = \tfrac{1}{\sqrt{2}}C^{a}{}_{b},\\
(4)\ B^{a0}{}_{0} = \tfrac{1}{2}D^{a};\\
(5)\ B^{abc} = C^{abc};\\
(6)\ B^{ab0} = \tfrac{1}{2\sqrt{2}}C^{ab},\\
(7)\ B^{0ab} = \tfrac{1}{\sqrt{2}}D^{ab};\\
(8)\ B^{00a} = -\tfrac{1}{4}D^{a},
\end{gathered} \tag{38}$$

and complex conjugate formulas.

Let M be a $(2n+1)$-dimensional almost contact metric manifold provided with an almost contact metric structure $(\Phi, \xi, \eta, g = \langle \cdot, \cdot \rangle)$. We will denote by

$$\Omega(X, Y) = \langle X, \Phi Y \rangle \tag{39}$$

the fundamental form of the structure,

$$\Omega(X, Y) = -\Omega(Y, X). \tag{40}$$

We recall [28,30] that an almost contact metric structure is called contact metric or almost Sasakian if

$$d\eta = \Omega \tag{41}$$

and normal if

$$2N + d\eta \otimes \xi = 0, \tag{42}$$

where

$$N(X, Y) = \frac{1}{4}\Big(\Phi^2[X, Y] + [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y]\Big) \tag{43}$$

is the Nijenhuis tensor of the structure operator. We note [28,30] that an almost contact metric structure is normal if and only if, on the space of the associated G-structure,

$$\begin{gathered}
(1)\ C^{abc} = C_{abc} = 0;\\
(2)\ C^{ab}{}_{c} = C_{ab}{}^{c} = 0;\\
(3)\ C^{ab} = C_{ab} = 0;\\
(4)\ D^{ab} = D_{ab} = 0;\\
(5)\ D^{a} = D_{a} = 0.
\end{gathered} \tag{44}$$

A normal contact metric structure is called Sasakian.

We note that an almost contact metric structure is called almost cosymplectic if its contact and fundamental forms are closed. A normal almost cosymplectic structure is called cosymplectic. An almost contact metric structure for which

$$\nabla_X(\Phi)Y + \nabla_Y(\Phi)X = 0 \tag{45}$$

is called nearly cosymplectic. A nearly cosymplectic structure with a closed contact form is called closely cosymplectic. It is known [31] that every closely cosymplectic manifold is locally equivalent to the product of a nearly Kähler manifold and a real straight line.

Definition 1. *The Lee form of an almost Hermitian structure* $(J, \tilde{g})$ *on the manifold* M^{2n+2} *is the form*

$$\alpha = \frac{1}{n}\delta\Psi \circ J, \tag{46}$$

where

$$\Psi(X, Y) = \tilde{g}(X, JY) \tag{47}$$

is the fundamental form of the structure and δ *is the codifferentiation operator. The vector* β *dual to the Lee form is called the Lee vector. In this paper, by the Lee form of an almost contact metric structure, we mean the Lee form of its linear extension.*

It is easy to check that, on the space of the associated G-structure $\mathcal{G}$, the components of the Lee vector (or form) are found by the formula

$$\beta^\alpha = \frac{2}{n}B^{\alpha\gamma}{}_\gamma \tag{48}$$

or, considering (38),

$$\begin{gathered}(1)\ \beta^a = \tfrac{2}{n}C^{ah}{}_h + \tfrac{1}{n}D^a;\\ (2)\ \beta_a = \tfrac{2}{n}C_{ah}{}^h + \tfrac{1}{n}D_a;\\ (3)\ \beta^0 = -\tfrac{\sqrt{2}}{n}C^h{}_h;\\ (4)\ \beta_0 = -\tfrac{\sqrt{2}}{n}C_h{}^h.\end{gathered} \tag{49}$$

3. Definition of a Nearly Trans-Sasakian Structure and Its Structural Equations

Definition 2 ([7])**.** *An almost contact metric structure is called a nearly trans-Sasakian (NTS)-structure if its linear extension belongs to the class* $W_1 \oplus W_4$ *of almost Hermitian structures in the Gray–Hervella classification. An almost contact metric manifold endowed with an NTS-structure is called an NTS-manifold.*

The following theorem is valid.

Theorem 1. *An almost contact metric structure is an NTS-structure if and only if, on the space of the associated G-structure,*

$$\begin{gathered}(1)\ C^{ab}{}_c = C^{ab} = D^{ab} = D^a = 0;\\ (2)\ C^a{}_b = -\tfrac{1}{\sqrt{2}}\beta^0\delta^b_a;\\ (3)\ C_{ab}{}^c = C_{ab} = D_{ab} = D_a = 0;\\ (4)\ C_a{}^b = -\tfrac{1}{\sqrt{2}}\beta_0\delta^b_a;\\ (5)\ D^a_b = \tfrac{1}{\sqrt{2}}(\beta_0 - \beta^0)\delta^a_b;\\ (6)\ C^{[abc]} = C^{abc};\\ (7)\ C_{[abc]} = C_{abc}.\end{gathered} \tag{50}$$

Proof. Let M be an NTS-manifold. By definition, this means that the linear extension of its almost contact metric structure belongs to the Gray–Hervella class $W_1 \oplus W_4$. As already noted, this is equivalent to the relations:

$$\begin{gathered}(1)\ B^{\alpha\beta}{}_\gamma = \beta^{[\alpha}\delta^{\beta]}_\gamma;\\ (2)\ B_{\alpha\beta}{}^\gamma = \beta_{[\alpha}\delta^\gamma_{\beta]};\\ (3)\ B^{[\alpha\beta\gamma]} = B^{\alpha\beta\gamma};\\ (4)\ B_{[\alpha\beta\gamma]} = B_{\alpha\beta\gamma}.\end{gathered} \tag{51}$$

Describing (21) and taking into account these relations, we obtain:

$$(1)\ B^{ab}{}_{c} = \beta^{[a}\delta_{c}^{b]}, \tag{52}$$

i.e.,

$$C^{ab}{}_{c} = \beta^{[a}\delta_{c}^{b]}; \tag{53}$$

$$(2)\ B^{ab}{}_{0} = \beta^{[a}\delta_{0}^{b]} = 0, \tag{54}$$

i.e.,

$$C^{[ab]} = D^{ab}; \tag{55}$$

$$(3)\ B^{a0}{}_{b} = \beta^{[a}\delta_{b}^{0]}, \tag{56}$$

i.e.,

$$C^{a}{}_{b} = -\frac{1}{\sqrt{2}}\beta^{0}\delta_{b}^{a}; \tag{57}$$

$$(4)\ B^{a0}{}_{0} = \beta^{[a}\delta_{0}^{0]} = \frac{1}{2}\beta^{a}, \tag{58}$$

i.e.,

$$\beta^{a} = D^{a}; \tag{59}$$

$$(5)\ B^{[abc]} = B^{abc}, \tag{60}$$

i.e.,

$$C^{[abc]} = C^{abc}; \tag{61}$$

$$(6)\ B^{ab0} = -B^{ba0}, \tag{62}$$

i.e.,

$$C^{ab} = -C^{ba}, \tag{63}$$

which means

$$D^{ab} = C^{ab}; \tag{64}$$

$$(7)\ B^{ab0} = B^{0ab}, \tag{65}$$

i.e.,

$$C^{ab} = 2D^{ab}, \tag{66}$$

which means

$$C^{ab} = D^{ab} = 0; \tag{67}$$

$$(8)\ B^{00a} = 0, \tag{68}$$

i.e.,

$$D^{a} = 0, \tag{69}$$

and hence

$$\beta^{a} = 0, \tag{70}$$

$$C^{ab}{}_{c} = 0. \tag{71}$$

Finally,

$$\begin{cases} D_b^a &= C_b{}^a - C^a{}_b \\ D_b^a &= \frac{1}{\sqrt{2}}(\beta^0 - \beta_0)\delta_b^a. \end{cases} \quad (72)$$

The remaining relations are verified in a similar way. □

Corollary 1. *The first group of structural equations of the NTS-structure on the space of the associated G-structure has the form*

$$\begin{aligned} &(1)\ d\theta^a = -\theta_b^a \wedge \theta^b + C^{abc}\theta_b \wedge \theta_c + \frac{\beta^0}{\sqrt{2}}\theta \wedge \theta^a; \\ &(2)\ d\theta_a = \theta_a^b \wedge \theta_b + C_{abc}\theta^b \wedge \theta^c + \frac{\beta_0}{\sqrt{2}}\theta \wedge \theta_a; \\ &(3)\ d\theta = \frac{1}{\sqrt{2}}(\beta^0 - \beta_0)\delta_a^b\theta^a \wedge \theta_b, \end{aligned} \quad (73)$$

where

$$\begin{aligned} &(1)\ C^{[abc]} = C^{abc}; \\ &(2)\ C_{[abc]} = C_{abc}. \end{aligned} \quad (74)$$

Corollary 2. *The components of the vector (or form) Lee have the form*

$$\begin{aligned} &(1)\ \beta^a = \beta_a = 0; \\ &(2)\ \beta^0 = -\frac{\sqrt{2}}{n}C^h{}_h; \\ &(3)\ \beta_0 = -\frac{\sqrt{2}}{n}C_h{}^h. \end{aligned} \quad (75)$$

Corollary 3. *For the components of the covariant differential of the structural endomorphism of an NTS-structure on the space of the associated G-structure, we have*

$$\begin{aligned} &(1)\ \Phi^a_{0,b} = \frac{1}{\sqrt{2}}\sqrt{-1}\beta^0\delta_b^a; \\ &(2)\ \Phi^{\hat{a}}_{0,\hat{b}} = -\frac{1}{\sqrt{2}}\sqrt{-1}\beta_0\delta_a^b; \\ &(3)\ \Phi^0_{\hat{a},b} = -\frac{1}{\sqrt{2}}\sqrt{-1}\beta^0\delta_b^a; \\ &(4)\ \Phi^0_{a,\hat{b}} = \frac{1}{\sqrt{2}}\sqrt{-1}\beta_0\delta_a^b; \\ &(5)\ \Phi^{\hat{c}}_{a,b} = -\Phi^{\hat{c}}_{b,a}; \\ &(6)\ \Phi^c_{\hat{a},\hat{b}} = -\Phi^c_{\hat{b},\hat{a}}, \end{aligned} \quad (76)$$

and the remaining components are zero.

Corollary 4. *An NTS-structure is a trans-Sasakian structure if and only if*

$$C^{abc} = C_{abc} = 0 \quad (77)$$

on the space of the associated G-structure. At the same time, it is Sasakian if and only if

$$\beta^0 = -\beta_0 = \sqrt{-2}; \quad (78)$$

it is cosymplectic if and only if

$$\beta^0 = \beta_0 = 0; \quad (79)$$

and it is Kenmotsu if and only if

$$\beta^0 = \beta_0 = \sqrt{2}. \quad (80)$$

Corollary 5. *An NTS-structure is a closely cosymplectic structure if and only if*

$$\beta^0 = \beta_0 = 0 \tag{81}$$

on the space of the associated G-structure.

Corollary 6. *An NTS-structure is a special generalized Kenmotsu structure of the second kind if and only if*

$$\beta^0 = \beta_0 = \sqrt{2} \tag{82}$$

on the space of the associated G-structure.

Corollaries 4–6 to Theorem 1 give numerous examples of NTS-manifolds.

The standard procedure for the differential continuation of relations (73) allows us to obtain the second group of structural equations of the NTS-structure,

$$d\theta^a_b + \theta^a_c \wedge \theta^c_b = \left(A^{ad}_{bc} - 2C^{adh}C_{hbc} + \frac{1}{2}\beta^0\left(\beta^0 - \beta_0\right)\delta^a_{[b}\delta^d_{c]} \right)\theta^c \wedge \theta_d, \tag{83}$$

where $\left\{ A^{ad}_{bc} \right\}$ is a family of functions on the space of the associated G-structure that serve as components of the so-called curvature tensor of the associated Q-algebra [28] or the structural tensor of the second kind, and

$$\begin{gathered}(1)\ A^{ad}_{[bc]} = 0;\\ (2)\ A^{[bd]}_{ac} = -\tfrac{1}{2}\left\{ (\beta^0)^2 - (\beta_0)^2 \right\}\delta^{[b}_a \delta^{d]}_c;\\ (3)\ \overline{A^{ad}_{bc}} = A^{bc}_{ad} + \tfrac{1}{2}\left\{ (\beta^0)^2 - (\beta_0)^2 \right\}\delta^{[b}_a \delta^{c]}_d.\end{gathered} \tag{84}$$

Moreover,

$$\begin{gathered}(1)\ dC^{abc} + C^{dbc}\theta^a_d + C^{adc}\theta^b_d + C^{abd}\theta^c_d = C^{abcd}\theta_d + \tfrac{1}{\sqrt{2}}\beta^0 C^{abc}\theta;\\ (2)\ dC_{abc} - C_{dbc}\theta^d_a - C_{adc}\theta^d_b - C_{abd}\theta^d_c = C_{abcd}\theta^d + \tfrac{1}{\sqrt{2}}\beta_0 C_{abc}\theta;\\ (3)\ d\beta^0 = \beta^{00}\theta;\ \ (4)\ d\beta_0 = \beta_{00}\theta,\end{gathered} \tag{85}$$

where C^{abcd}, C_{abcd}, $C^{abc}{}_d$, $C_{abc}{}^d$, β^{00}, and β_{00} are appropriate functions on the space of the associated G-structure, and

$$\begin{gathered}(1)\ (\beta^0 - \beta_0)C_{abc} = 0;\\ (2)\ (\beta^0 - \beta_0)C^{abc} = 0;\\ (3)\ (\beta^{00} - \beta_{00}) = \tfrac{1}{\sqrt{2}}\left\{ (\beta^0)^2 - (\beta_0)^2 \right\};\\ (4)\ C^{a[bcd]} = 0;\\ (5)\ C_{a[bcd]} = 0.\end{gathered} \tag{86}$$

Externally differentiating Equation (83), we obtain:

$$dA^{ad}_{bc} + A^{hd}_{bc}\theta^a_h + A^{ah}_{bc}\theta^d_h - A^{ad}_{hc}\theta^h_b - A^{ad}_{bh}\theta^h_c = A^{ad}_{bch}\theta^h + A^{adh}_{bc}\theta_h + A^{ad}_{bc0}\theta, \tag{87}$$

where

$$\begin{gathered}(1)\ A^{ad}_{b[ch]} = A^{a[dh]}_{bc} = 0;\\ (2)\ \left(A^{a[d}_{bc} - 2C^{a[d|h}C_{hbc} \right) C^{c|fg]} = 0;\\ (3)\ \left(A^{ad}_{b[c} - 2C^{adf}C_{fb[c} \right) C_{|d|hg]} = 0.\end{gathered} \tag{88}$$

Differentiating (85:1) externally, we obtain:

$$\begin{aligned} dC^{abcd} + C^{hbcd}\theta_h^a + C^{abcd}\theta_h^d + C^{abd}\theta_d^c + C^{abch}\theta_h^d = \\ = C^{abcdh}\theta_h + \tfrac{1}{\sqrt{2}}(\beta^0 - \beta_0)C^{abcd}\theta, \end{aligned} \tag{89}$$

where

$$\begin{aligned} &(1)\ C^{abc[dh]} = 0; \\ &(2)\ C^{abcg}C_{gdh} = 0. \end{aligned} \tag{90}$$

Theorem 2. *An NTS-manifold is either a trans-Sasakian manifold or has a closed contact form.*

Proof. 1. It follows from equality (86:1) that

$$\beta^0 = \beta_0 \tag{91}$$

or

$$C_{abc} = 0. \tag{92}$$

In the first case, the NTS-manifold has a closed contact form. In the second case, the NTS-manifold, according to Corollary 4 to Theorem 1, is a trans-Sasakian manifold.

2. Consider the identity (84:1), i.e., identity

$$A_{ac}^{[bd]} = -\frac{1}{2}\left\{\left(\beta^0\right)^2 - (\beta_0)^2\right\}\delta_a^{[b}\delta_c^{d]}. \tag{93}$$

Let us collapse this equality first by indexes a and b and then by indexes c and d; then, by virtue of (84:1), we obtain

$$\left(\beta^0\right)^2 - (\beta_0)^2 = 0, \tag{94}$$

i.e.,

$$\left(\beta^0 - \beta_0\right)\left(\beta^0 + \beta_0\right) = 0. \tag{95}$$

Consider two cases:

$$\begin{aligned} &(1)\ \beta^0 - \beta_0 = 0; \\ &(2)\ \beta^0 + \beta_0 = 0. \end{aligned} \tag{96}$$

In the first case, the NTS-manifold has a closed contact form. In the second case, it follows from (86:1) that

$$C_{abc} = 0, \tag{97}$$

which, according to Corollary 4, means that the NTS-manifold is trans-Sasakian. □

Remark 1. *Consider the identity*

$$A_{bc}^{ad} = A_{[bc]}^{[ad]} + A_{[bc]}^{(ad)} + A_{(bc)}^{[ad]} + A_{(bc)}^{(ad)}. \tag{98}$$

Considering the identities (84:1) and (84:2), this equality can be written as:

$$A_{bc}^{ad} = A_{(bc)}^{(ad)} - \frac{1}{2}\left\{\left(\beta^0\right)^2 - (\beta_0)^2\right\}\delta_a^{[b}\delta_c^{d]}. \tag{99}$$

i.e.,

$$\tilde{A}_{bc}^{ad} = A_{(bc)}^{(ad)} = A_{bc}^{ad} + \frac{1}{2}\left\{\left(\beta^0\right)^2 - (\beta_0)^2\right\}\delta_a^{[b}\delta_c^{d]}, \tag{100}$$

where obviously

$$\tilde{A}^{ad}_{[bc]} = \tilde{A}^{[ad]}_{bc} = 0. \tag{101}$$

Let the contact form η of an NTS-manifold be the Killing form, i.e.,

$$\nabla_X(\eta)Y + \nabla_Y(\eta)X = 0, \tag{102}$$

which on the frame bundle space can be written as

$$\eta_{i,j} + \eta_{j,i} = 0. \tag{103}$$

The components of the contact form η on the frame bundle space over M satisfy the equation

$$d\eta_i - \eta_j\theta^j_i = \eta_{i,j}\theta^j, \tag{104}$$

which on the space of the associated G-structure is equivalent to the following:

$$\begin{aligned} &(1)\ \eta_{0,i} = 0; \\ &(2)\ \eta_{a,i} = \sqrt{-1}\Phi^0_{a,i}; \\ &(3)\ \eta_{\hat{a},i} = -\sqrt{-1}\Phi^0_{\hat{a},i}. \end{aligned} \tag{105}$$

According to Corollary 3 to Theorem 1, we obtain

$$\begin{aligned} &(1)\ \eta_{a,\hat{b}} = \frac{1}{\sqrt{2}}\beta_0\delta^b_a; \\ &(2)\ \eta_{\hat{a},b} = \frac{1}{\sqrt{2}}\beta^0\delta^a_b, \end{aligned} \tag{106}$$

and the rest of the components are zero. From (103) and (106), it follows that, for an NTS-manifold whose contact form is the Killing form, we have that

$$\beta^0 = \beta_0. \tag{107}$$

Thus, an NTS manifold with a Killing contact form is a manifold with a contact form. It is easy to see that the inverse is also true, i.e., the closed form of an NTS-manifold is a Killing form. Thus proved.

Theorem 3. *For an NTS-manifold, the following conditions are equivalent:*
(1) The contact form is closed;
(2) The contact form is a Killing form.

Definition 3 ([24])**.** *An NTS-structure with a closed contact form is called a proper NTS-structure.*

The following theorem is valid.

Theorem 4 ([24])**.** *An almost contact metric structure with a closed contact form on a manifold M is a proper NTS-structure if and only if the identity*

$$\nabla_X(\Phi)Y + \nabla_Y(\Phi)X = \chi\{\eta(X)\Phi Y + \eta(Y)\Phi X\},\ \ X, Y \in \mathcal{X}(M). \tag{108}$$

Theorem 5. *The class of NTS-manifolds with a closed contact form coincides with the class of almost contact metric manifolds with a closed contact form that are locally conformal to closely cosymplectic manifolds.*

Proof. Let $\sigma \in C^{\infty}(M)$. Let us perform a conformal transformation with the defining function σ of the proper NTS-structure:

$$\tilde{g} = e^{-2\sigma} g; \;\; \tilde{\eta} = e^{-\sigma}\eta; \;\; \tilde{\xi} = e^{\sigma}\xi. \tag{109}$$

Let $\tilde{\nabla}$ be the Riemannian connection of the transformed structure. Then, as is well known (see, for example, [6]), the affine deformation tensor T from the connection ∇ to the connection $\tilde{\nabla}$ has the form

$$T(X,Y) = \langle X,Y\rangle\zeta - d\sigma(X)Y - d\sigma(Y)X, \;\; X,Y \in \mathcal{X}(M), \tag{110}$$

where

$$\zeta = grad\sigma. \tag{111}$$

Therefore,

$$\tilde{\nabla}_X Y = \nabla_X Y + \langle X,Y\rangle\zeta - d\sigma(X)Y - d\sigma(Y)X, \tag{112}$$

and hence,

$$\begin{aligned}\tilde{\nabla}_X(\Phi)Y = \nabla_X(\Phi)Y - \Phi(\tilde{\nabla}_X Y) = \nabla_X(\Phi Y) + \langle X,\Phi Y\rangle\zeta - \\ -d\sigma(X)\Phi Y - d\sigma(\Phi Y)X - \Phi(\nabla_X Y) - \langle X,Y\rangle\Phi\zeta + d\sigma(X)\Phi Y + d\sigma(Y)\Phi X = \\ = \nabla_X(\Phi)Y + \langle X,\Phi Y\rangle\zeta - d\sigma(\Phi Y)X - \langle X,Y\rangle\Phi\zeta + d\sigma(Y)\Phi X,\end{aligned} \tag{113}$$

i.e.,

$$\tilde{\nabla}_X(\Phi)Y = \nabla_X(\Phi)Y + \langle X,\Phi Y\rangle\zeta - d\sigma(\Phi Y)X - \langle X,Y\rangle\Phi\zeta + d\sigma(Y)\Phi X. \tag{114}$$

In particular, considering (108),

$$\begin{aligned}\tilde{\nabla}_X(\Phi)X = \nabla_X(\Phi)X - d\sigma(\Phi X)X - \|X\|^2\Phi\zeta + d\sigma(X)\Phi X = \\ = \chi\eta(X)\Phi X - d\sigma(\Phi X)X + d\sigma(X)\Phi X - \|X\|^2\Phi\zeta.\end{aligned} \tag{115}$$

In particular, if the function σ can be chosen so that

$$d\sigma = -\chi\eta; \tag{116}$$

then, obviously,

$$\zeta = -\chi\xi, \tag{117}$$

and, considering the axioms of the almost contact metric structure,

$$\tilde{\nabla}_X(\Phi)X = 0. \tag{118}$$

Moreover, in this case, due to the closedness of the contact form η,

$$d\tilde{\eta} = d\left(e^{-\sigma}\eta\right) = -e^{-\sigma}d\sigma \wedge \eta = e^{-\sigma}\chi\eta \wedge \eta = 0, \tag{119}$$

i.e., the transformed structure is closely cosymplectic, and the manifold M is conformally the closely cosymplectic manifold.

Conversely, let M be an almost contact metric manifold with a closed contact form η local conformal to a closely cosymplectic manifold, and let σ be the determining function of the corresponding conformal transformation of its almost contact metric structure $(\xi,\eta,\Phi,g) \to \left(\tilde{\xi},\tilde{\eta},\Phi,\tilde{g}\right)$. Then,

$$\tilde{\eta} = e^{-\sigma}\eta. \tag{120}$$

Differentiating (120) externally, we obtain

$$0 = d\tilde{\eta} = -e^{-\sigma}d\sigma \wedge \eta \tag{121}$$

and, hence,

$$d\sigma \wedge \eta = 0. \tag{122}$$

Therefore, $\exists \chi \in C^{\infty}(M)$, and

$$d\sigma = -\chi\eta. \tag{123}$$

Accordingly, for the vector ζ dual to the form $d\sigma$, we have

$$\langle\langle \zeta, X\rangle\rangle = d\sigma(X) = -\chi\eta(X) = -\chi\langle \xi, X\rangle, \tag{124}$$

and, due to the non-degeneracy of the metric, $\zeta = -\chi\xi$. Therefore, in view of (115),

$$\begin{gathered} 0 = \tilde{\nabla}_X(\Phi)X = \\ = \nabla_X(\Phi)X - d\sigma(\Phi X)X - \|X\|^2\Phi\zeta + d\sigma(X)\Phi X = \nabla_X(\Phi)X - \chi\eta(X)\Phi X. \end{gathered} \tag{125}$$

Polarizing this identity, we obtain identity (108). By virtue of Theorem 4, the original structure is its own NTS-structure. □

Taking into account the Main Theorem [3], Theorems 2 and 5 can be combined into the following theorem, which provides a complete classification of NTS-manifolds.

Theorem 6. *The class of NTS-manifolds with non-closed contact form coincides with the class of almost contact metric manifolds homothetic to Sasaki varieties. The class of NTS-manifolds with a closed contact form coincides with the class of almost contact metric manifolds with a closed contact form that are locally conformal to closely cosymplectic manifolds.*

4. Riemannian Curvature Tensor of a Nearly Trans-Sasakian Manifold

In this section, we calculate the components of the Riemann–Christoffel tensor, the Ricci tensor, and the scalar curvature of an NTS-manifold on the space of the associated G-structure.

We recall that the tensor components of the form of the Riemannian connection on the space of the associated G-structure have the form [8,9]:

$$\begin{gathered} (1)\ \theta^a_{\hat{b}} = \frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},i}\theta^i; \\ (2)\ \theta^{\hat{a}}_b = -\frac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{b,i}\theta^i; \\ (3)\ \theta^a_0 = \sqrt{-1}\Phi^a_{0,i}\theta^i; \\ (4)\ \theta^{\hat{a}}_0 = -\sqrt{-1}\Phi^{\hat{a}}_{0,i}\theta^i; \\ (5)\ \theta^0_a = -\sqrt{-1}\Phi^0_{a,i}\theta^i; \\ (6)\ \theta^0_{\hat{a}} = \sqrt{-1}\Phi^0_{\hat{a},i}\theta^i; \\ (7)\ \theta^0_0 = 0; \\ (8)\ \theta^i_j + \theta^{\hat{j}}_{\hat{i}} = 0. \end{gathered} \tag{126}$$

Considering (14) and Corollary 3 to Theorem 1, relations (126) on the space of the associated G-structure are rewritten in the form:

$$\begin{gathered}(1)\ \theta^a_{\hat{b}} = C^{abc}\theta_c;\\ (2)\ \theta^{\hat{a}}_b = C_{abc}\theta^c;\\ (3)\ \theta^a_0 = \tfrac{1}{\sqrt{2}}\beta^0\delta^a_b\theta^b;\\ (4)\ \theta^{\hat{a}}_0 = \tfrac{1}{\sqrt{2}}\beta_0\delta^b_a\theta_b;\\ (5)\ \theta^0_a = -\tfrac{1}{\sqrt{2}}\beta_0\delta^b_a\theta_b;\\ (6)\ \theta^0_{\hat{a}} = -\tfrac{1}{\sqrt{2}}\beta^0\delta^a_b\theta^b;\\ (7)\ \theta^0_0 = 0;\ \ 8)\ \theta^i_j + \theta^{\hat{j}}_{\hat{i}} = 0.\end{gathered} \tag{127}$$

Differentiating externally (127), we obtain:

$$\begin{gathered}(1)\ d\theta^a_{\hat{b}} = -C^{dbc}\theta^a_d\wedge\theta_c - C^{adc}\theta^b_d\wedge\theta_c + C^{abh}C_{hcd}\theta^c\wedge\theta^d - C^{ab[cd]}\theta_c\wedge\theta_d;\\ (2)\ d\theta^{\hat{a}}_b = C_{dbc}\theta^d_a\wedge\theta^c + C_{adc}\theta^d_b\wedge\theta^c - C_{ab[cd]}\theta^c\wedge\theta^d + C_{abh}C^{hcd}\theta_c\wedge\theta_d;\\ (3)\ d\theta^a_0 = -\tfrac{1}{\sqrt{2}}\beta^0\theta^a_b\wedge\theta^b + \tfrac{1}{\sqrt{2}}\beta^0C^{abc}\theta_b\wedge\theta_c + \tfrac{1}{\sqrt{2}}\Big\{\beta^{00} - \tfrac{1}{\sqrt{2}}(\beta^0)^2\Big\}\theta\wedge\theta^a;\\ (4)\ d\theta^{\hat{a}}_0 = \tfrac{1}{\sqrt{2}}\beta_0\theta^b_a\wedge\theta_b + \tfrac{1}{\sqrt{2}}\beta_0C_{abc}\theta^b\wedge\theta^c + \tfrac{1}{\sqrt{2}}\Big\{\beta_{00} - \tfrac{1}{\sqrt{2}}(\beta_0)^2\Big\}\theta\wedge\theta_a;\\ (5)\ d\theta^0_a = -\tfrac{1}{\sqrt{2}}\beta_0\theta^b_a\wedge\theta_b - \tfrac{1}{\sqrt{2}}\beta_0C_{abc}\theta^b\wedge\theta^c - \tfrac{1}{\sqrt{2}}\Big\{\beta_{00} - \tfrac{1}{\sqrt{2}}(\beta_0)^2\Big\}\theta\wedge\theta_a;\\ (6)\ d\theta^0_{\hat{a}} = \tfrac{1}{\sqrt{2}}\beta^0\theta^a_b\wedge\theta^b - \tfrac{1}{\sqrt{2}}\beta^0C^{abc}\theta_b\wedge\theta_c - \tfrac{1}{\sqrt{2}}\Big\{\beta^{00} - \tfrac{1}{\sqrt{2}}(\beta^0)^2\Big\}\theta\wedge\theta^a.\end{gathered} \tag{128}$$

Recall that the second group of structural equations of the Riemannian connection has the form [28,29]

$$d\theta^i_j = -\theta^i_k\wedge\theta^k_j + \frac{1}{2}R^i_{jkl}\theta^k\wedge\theta^l, \tag{129}$$

where $\left\{R^i_{jkl}\right\} \subset C^\infty(BM)$ are the components of the Riemann–Christoffel tensor. Describing (129) on the space of the associated G-structure, we obtain:

$$\begin{gathered}(1)\ d\theta^a_{\hat{b}} = -C^{dbc}\theta^a_d\wedge\theta_c - C^{adc}\theta^b_d\wedge\theta_c + \tfrac{1}{2}\Big(R^a_{\hat{b}cd} + (\beta^0)^2\delta^a_{[c}\delta^b_{d]}\Big)\theta^c\wedge\theta^d +\\ +R^a_{\hat{b}c\hat{d}}\theta^c\wedge\theta_d + \tfrac{1}{2}R^a_{\hat{b}\hat{c}\hat{d}}\theta_c\wedge\theta_d + R^a_{\hat{b}0c}\theta\wedge\theta^c + R^a_{\hat{b}0\hat{c}}\theta\wedge\theta_c;\\ (2)\ d\theta^{\hat{a}}_b = C_{dbc}\theta^d_a\wedge\theta^c + C_{adc}\theta^d_b\wedge\theta^c + \tfrac{1}{2}R^{\hat{a}}_{bcd}\theta^c\wedge\theta^d + R^{\hat{a}}_{bc\hat{d}}\theta^c\wedge\theta_d +\\ +\tfrac{1}{2}\Big\{R^{\hat{a}}_{b\hat{c}\hat{d}} + (\beta_0)^2\delta^{[c}_a\delta^{d]}_b\Big\}\theta_c\wedge\theta_d + R^{\hat{a}}_{b0c}\theta\wedge\theta^c + R^{\hat{a}}_{b0\hat{c}}\theta\wedge\theta_c;\\ (3)\ d\theta^a_0 = -\tfrac{1}{\sqrt{2}}\beta^0\theta^a_b\wedge\theta^b + \tfrac{1}{2}R^a_{0bc}\theta^b\wedge\theta^c + R^a_{0b\hat{c}}\theta^b\wedge\theta_c +\\ +\Big(\tfrac{1}{2}R^a_{0\hat{b}\hat{c}} + \tfrac{1}{\sqrt{2}}\beta_0C^{abc}\Big)\theta_b\wedge\theta_c + R^a_{00b}\theta\wedge\theta^b + R^a_{00\hat{b}}\theta\wedge\theta_b;\\ (4)\ d\theta^{\hat{a}}_0 = \tfrac{1}{\sqrt{2}}\beta_0\theta^b_a\wedge\theta_b + \Big(\tfrac{1}{2}R^{\hat{a}}_{0bc} + \tfrac{1}{\sqrt{2}}\beta^0C_{abc}\Big)\theta^b\wedge\theta^c + R^{\hat{a}}_{0b\hat{c}}\theta^b\wedge\theta_c +\\ +\tfrac{1}{2}R^{\hat{a}}_{0\hat{b}\hat{c}}\theta_b\wedge\theta_c + R^{\hat{a}}_{00b}\theta\wedge\theta^b + R^{\hat{a}}_{00\hat{b}}\theta\wedge\theta_b;\\ (5)\ d\theta^0_a = -\tfrac{1}{\sqrt{2}}\beta_0\theta^b_a\wedge\theta_b + \Big(\tfrac{1}{2}R^0_{abc} - \tfrac{1}{\sqrt{2}}\beta^0C_{abc}\Big)\theta^b\wedge\theta^c + R^0_{ab\hat{c}}\theta^b\wedge\theta_c +\\ +\tfrac{1}{2}R^0_{a\hat{b}\hat{c}}\theta_b\wedge\theta_c + R^0_{a0b}\theta\wedge\theta^b + R^0_{a0\hat{b}}\theta\wedge\theta_b;\end{gathered}$$

$$(6)\ d\theta^0_{\hat{a}} = \frac{1}{\sqrt{2}}\beta^0\theta^a_b \wedge \theta^b + \frac{1}{2}R^0_{\hat{a}bc}\theta^b \wedge \theta^c + R^0_{\hat{a}b\hat{c}}\theta^b \wedge \theta_c +$$
$$+\left(\frac{1}{2}R^0_{\hat{0}\hat{b}\hat{c}} - \frac{1}{\sqrt{2}}\beta_0 C^{abc}\right)\theta_b \wedge \theta_c + R^0_{\hat{a}0b}\theta \wedge \theta^b + R^0_{\hat{a}0\hat{b}}\theta \wedge \theta_b;$$
$$(7) d\theta^a_b + \theta^a_c \wedge \theta^c_b = \frac{1}{2}R^a_{bcd}\theta^c \wedge \theta^d + \left(R^a_{bc\hat{d}} - C^{adh}C_{hbc} + \frac{1}{2}\beta^0\beta_0\delta^a_c\delta^d_b\right)\theta^c \wedge \theta_d +$$
$$+\frac{1}{2}R^a_{b\hat{c}\hat{d}}\theta_b \wedge \theta_c + R^a_{b0c}\theta \wedge \theta^c + R^a_{b0\hat{c}}\theta \wedge \theta_c. \tag{130}$$

Comparing (130) with (83) and (128), we obtain that the essential nonzero components of the Riemann–Christoffel tensor of an NTS-manifold on the space of the associated G-structure have the form

$$(1)\ R^a_{\hat{b}cd} = 2C^{abh}C_{hcd} - (\beta^0)^2\delta^a_{[c}\delta^b_{d]};$$
$$(2)\ R^{\hat{a}}_{b\hat{c}\hat{d}} = 2C_{abh}C^{hcd} - (\beta_0)^2\delta^{[c}_a\delta^{d]}_b;$$
$$(3)\ R^{\hat{a}}_{bcd} = -2C_{ab[cd]};$$
$$(4)\ R^a_{\hat{b}\hat{c}\hat{d}} = -2C^{ab[cd]};$$
$$(5)\ R^a_{00b} = \frac{1}{\sqrt{2}}\left\{\beta^{00} - \frac{1}{\sqrt{2}}(\beta^0)^2\right\}\delta^a_b;$$
$$(6)\ R^{\hat{a}}_{00\hat{b}} = \frac{1}{\sqrt{2}}\left\{\beta_{00} - \frac{1}{\sqrt{2}}(\beta_0)^2\right\}\delta^b_a;$$
$$(7)\ R^0_{a0\hat{b}} = -\frac{1}{\sqrt{2}}\left\{\beta_{00} - \frac{1}{\sqrt{2}}(\beta_0)^2\right\}\delta^b_a;$$
$$(8)\ R^0_{\hat{a}0b} = -\frac{1}{\sqrt{2}}\left\{\beta^{00} - \frac{1}{\sqrt{2}}(\beta^0)^2\right\}\delta^a_b;$$
$$(9)\ R^a_{bc\hat{d}} = A^{ad}_{bc} - C^{adh}C_{hbc} + \frac{1}{2}\beta^0(\beta^0 - \beta_0)\delta^a_{[b}\delta^d_{c]} - \frac{1}{2}\beta^0\beta_0\delta^a_c\delta^d_b, \tag{131}$$

in addition to the ratios derived from them, considering the properties of symmetry. The other components are zero.

Theorem 7. *For any NTS-manifold, the following identities hold:*

$$(1)\ R(\Phi^2X, \Phi^2Y)\xi - R(\Phi X, \Phi Y)\xi = 0;$$
$$(2)\ R(\Phi^2X, \Phi^2Y)\xi + R(\Phi X, \Phi Y)\xi = 0;$$
$$(3)\ R(\Phi X, \Phi Y)\xi = 0;$$
$$(4)\ R(\xi, \Phi^2X)\Phi^2Y - R(\xi, \Phi X)\Phi Y = 0; \forall X, Y, Z \in \mathcal{X}(M). \tag{132}$$

Proof. Let us apply the identity recovery procedure [28,29] to the equalities:

$$R^0_{0ab} = R^c_{0ab} = R^{\hat{c}}_{0ab} = 0, \tag{133}$$

i.e., to the equality

$$R^i_{0ab} = 0, \tag{134}$$

which we write as

$$\{R(\epsilon_a, \epsilon_b)\xi\}^i = 0, \tag{135}$$

i.e.,

$$R(\epsilon_a, \epsilon_b)\xi = 0. \tag{136}$$

Because the projections of $\mathcal{X}(M)$ onto the subspaces $D^{\sqrt{-1}}_\Phi$ and D^0_Φ are the endomorphisms

$$\pi = \sigma \circ \mathbf{l} = -\frac{1}{2}\left(\Phi^2 + \sqrt{-1}\Phi\right) \tag{137}$$

and

$$\mathbf{m} = \eta \otimes \xi, \tag{138}$$

then

$$R\left(\Phi^2 X + \sqrt{-1}\Phi X, \Phi^2 Y + \sqrt{-1}\Phi Y\right)\xi = 0, \ \forall X, Y, Z \in \mathcal{X}(M). \tag{139}$$

Separating the real and imaginary parts of the last equality, we obtain identities equivalent to the identity

$$R(\Phi^2 X, \Phi^2 Y)\xi - R(\Phi X, \Phi Y)\xi = 0; \ \forall X, Y, Z \in \mathcal{X}(M). \tag{140}$$

Similarly, applying the identity recovery procedure to the equalities

$$R^0_{0a\hat{b}} = R^c_{0a\hat{b}} = R^{\hat{c}}_{0a\hat{b}} = 0, \tag{141}$$

we obtain the identity

$$R(\Phi^2 X, \Phi^2 Y)\xi + R(\Phi X, \Phi Y)\xi = 0 \tag{142}$$

for all $X, Y, Z \in \mathcal{X}(M)$. From the last two identities, it follows that

$$R(\Phi X, \Phi Y)\xi = 0; \ \forall X, Y, Z \in \mathcal{X}(M). \tag{143}$$

Applying the identity recovery procedure to the equalities

$$R^0_{a0b} = R^c_{a0b} = R^{\hat{c}}_{a0b} = 0, \tag{144}$$

we obtain the identity

$$R(\xi, \Phi^2 X)\Phi^2 Y - R(\xi, \Phi X)\Phi Y = 0 \tag{145}$$

for all $X, Y, Z \in \mathcal{X}(M)$. $\square$

The components of the Ricci tensor are calculated by the formula

$$S_{ij} = -R^k_{ijk}. \tag{146}$$

Let us calculate the components of the Ricci tensor of an NTS-manifold on the space of the associated G-structure:

$$\begin{aligned}
&(1)\ S_{00} = -\sqrt{2}n\left\{\beta^{00} - \tfrac{1}{\sqrt{2}}(\beta^0)^2\right\} = -\sqrt{2}n\left\{\beta_{00} - \tfrac{1}{\sqrt{2}}(\beta_0)^2\right\};\\
&(2)\ S_{a\hat{b}} = \left\{A^{cb}_{ac} - C^{cbd}C_{dac} + \tfrac{1}{2}\beta^0(\beta^0 - \beta_0)\delta^c_{[a}\delta^b_{c]} - \tfrac{1}{2}n\beta^0\beta_0\delta^b_a\right\} -\\
&\quad -\left\{2C_{cad}C^{dbc} - (\beta_0)^2\delta^{[b}_c\delta^{c]}_a\right\} - \sqrt{2}\left\{\beta^{00} - \tfrac{1}{\sqrt{2}}(\beta^0)^2\right\}\delta^b_a;\\
&(3)\ S_{\hat{b}a} = \left\{A^{ac}_{cb} - C^{acd}C_{dcb} + \tfrac{1}{2}\beta^0(\beta^0 - \beta_0)\delta^a_{[c}\delta^c_{b]} - \tfrac{1}{2}n\beta^0\beta_0\delta^a_b\right\} -\\
&\quad -\left\{2C^{cad}C_{dbc} - (\beta^0)^2\delta^c_{[b}\delta^a_{c]}\right\} - \sqrt{2}\left\{\beta^{00} - \sqrt{1}\sqrt{2}(\beta^0)^2\right\}\delta^a_b.
\end{aligned} \tag{147}$$

The other components are zero.

Remark 2. *Using (84:1), (84:2), and (86:3), it is easy to show that*

$$S_{a\hat{b}} = S_{\hat{b}a}. \tag{148}$$

Remark 3. *Because for an NTS-manifold*

$$\begin{aligned}
&(1)\ S_{0a} = S_{0\hat{a}} = S_{a0} = S_{\hat{a}0} = 0,\\
&(2)\ S_{ab} = S_{\hat{a}\hat{b}} = 0,
\end{aligned} \tag{149}$$

it follows from Theorem 6 in [32] that an NTS-manifold has the Φ-invariant Ricci tensor.

Theorem 8. *The Ricci tensor of an NTS-manifold satisfies the identities*

$$\begin{gathered}(1)\ S(\xi, \Phi^2 X) = 0;\\ (2)\ S(\Phi^2 X, \Phi^2 Y) = S(\Phi X, \Phi Y)\end{gathered} \tag{150}$$

for all $X, Y \in \mathcal{X}(M)$.

Proof. Applying the identity recovery procedure to the equalities

$$\begin{gathered}(1)\ S_{0a} = 0,\\ (2)\ S_{ab} = 0,\end{gathered} \tag{151}$$

we obtain the required identities. □

5. NTS-Manifolds of Constant Curvature

Le M be an NTS-manifold. As it is known [29], a pseudo-Riemannian manifold is a manifold of pointwise constant curvature k if and only if its Riemannian curvature tensor has the structure

$$R(X, Y)Z = k(\langle Y, Z\rangle X - \langle X, Z\rangle Y) \tag{152}$$

for all $X, Y, Z \in \mathcal{X}(M)$. This relation on the frame bundle space can be written in the form

$$R_{ijkl} = k\Big(g_{ik}g_{jl} - g_{il}g_{jk}\Big), \tag{153}$$

which, considering (131) and (12) on the space of the associated G-structure, is equivalent to the following relations on the components of the Riemannian curvature tensor:

$$\begin{gathered}(1)\ R^{\hat{a}}_{bcd} = -2C_{ab[cd]} = 0;\\ (2)\ R^{a}_{\hat{b}cd} = k\delta^{ab}_{cd} = 2C^{abh}C_{hcd} - (\beta^0)^2\delta^a_{[c}\delta^b_{d]};\\ (3)\ R^{a}_{bc\hat{d}} = k\delta^a_c\delta^d_b = A^{ad}_{bc} - C^{adh}C_{hbc} + \tfrac{1}{2}\beta^0(\beta^0 - \beta_0)\delta^a_{[b}\delta^d_{c]} - \tfrac{1}{2}\beta^0\beta_0\delta^a_c\delta^d_b;\\ (4)\ R^{a}_{0b0} = k\delta^a_b = -\tfrac{1}{\sqrt{2}}\Big\{\beta^{00} - \tfrac{1}{\sqrt{2}}(\beta^0)^2\Big\}\delta^a_b.\end{gathered} \tag{154}$$

From (86:5) and (154:1), it follows that

$$C_{abcd} = 0. \tag{155}$$

Let us convolve the equality (154:4) by indexes a and b; then, we obtain

$$k = -\frac{1}{\sqrt{2}}\Big\{\beta^{00} - \frac{1}{\sqrt{2}}(\beta^0)^2\Big\}. \tag{156}$$

Let us convolve the equality (154:2) first by indexes a and c, and then by indexes b and d; then, we obtain

$$k = \frac{2}{n(n-1)}C^{abh}C_{hab} - \frac{1}{2}(\beta^0)^2. \tag{157}$$

Now, we convolve the equality (154:3) first by indexes a and c, and then by indexes b and d; we obtain

$$kn^2 = A^{ab}_{ab} - C^{abc}C_{abc} + \frac{1}{4}\beta^0(\beta^0 - \beta_0)n(n-1) - \frac{1}{2}\beta^0\beta_0 n^2. \tag{158}$$

By Theorem 2, an NTS-manifold is either trans-Sasakian or has a closed contact form and, hence, by Theorem 5, it is locally most exactly conformal to a cosymplectic manifold. In the first case, we have that $C^{abc} = 0$. From (157), we have

$$k = -\frac{1}{2}(\beta^0)^2 \leq 0. \tag{159}$$

Thus, we have proved the following theorem.

Theorem 9. *An NTS-manifold of constant curvature is either a trans-Sasakian manifold of constant negative curvature, or it has a closed contact form, and hence it is locally conformal to a closely cosymplectic manifold of constant curvature. In particular, an NTS-manifold is of zero constant curvature if and only if it is a cosymplectic manifold of constant curvature.*

6. Curvature Identities

It is known [33] that A. Gray singled out three classes of almost Hermitian structures according to the symmetry properties of their Riemannian curvature tensor, i.e., by differential-geometric invariants of the second order.

In almost contact geometry, the following analogs are introduced [34] for all $X, Y, Z, W \in \mathcal{X}(M)$:

$$\begin{aligned} &(1)\ CR_1: \ \langle R(\Phi X, \Phi Y)\Phi Z, \Phi W\rangle = \left\langle R(\Phi^2 X, \Phi^2 Y)\Phi Z, \Phi W\right\rangle; \\ &(2)\ CR_2: \ \langle R(\Phi X, \Phi Y)\Phi Z, \Phi W\rangle = \left\langle R(\Phi^2 X, \Phi^2 Y)\Phi Z, \Phi W\right\rangle + \\ &\qquad + \left\langle R(\Phi^2 X, \Phi Y)\Phi^2 Z, \Phi W\right\rangle + \left\langle R(\Phi^2 X, \Phi Y)\Phi Z, \Phi^2 W\right\rangle; \\ &(3)\ CR_3: \ \langle R(\Phi X, \Phi Y)\Phi Z, \Phi W\rangle = \left\langle R(\Phi^2 X, \Phi^2 Y)\Phi^2 Z, \Phi^2 W\right\rangle. \end{aligned} \tag{160}$$

Almost contact metric structures whose tensor R satisfies the identity CR_i are called structures of the class CR_i. The meaning of these curvature identities is most transparently manifested in terms of the components of the Riemannian curvature tensor.

Theorem 10 ([34])**.** *On the space of the associated G-structure, the identities $CR_1 - CR_3$ are equivalent to the following equalities:*

$$\begin{aligned} &(1)\ CR_1 \Leftrightarrow R_{\hat{a}bcd} = R_{abcd} = R_{\hat{a}\hat{b}cd} = 0; \\ &(2)\ CR_2 \Leftrightarrow R_{\hat{a}bcd} = R_{abcd} = 0; \\ &(3)\ CR_3 \Leftrightarrow R_{\hat{a}bcd} = 0. \end{aligned} \tag{161}$$

Remark 4. *According to formula (161), the inclusions $CR_1 \subset CR_2 \subset CR_3$ are obvious.*

Theorem 11. *An NTS-manifold is a manifold of class CR_3.*

Proof. Since
$$R_{\hat{a}bcd} = 0 \tag{162}$$
for an NTS-manifold, the required assertion follows by Theorem 10. □

Theorem 12. *An NTS-manifold is a manifold of class CR_2 if and only if*
$$C_{abcd} = 0 \tag{163}$$
on the space of the associated G-structure.

Proof. Let an NTS-manifold be a manifold of class CR_2, then
$$R_{abcd} = -C_{ab[cd]} = 0, \tag{164}$$
i.e.,
$$C_{abcd} = C_{abdc}. \tag{165}$$
This equality, together with the equality
$$C_{a[bcd]} = 0, \tag{166}$$

gives

$$C_{abcd} = 0. \tag{167}$$

Conversely, let the equality

$$C_{abcd} = 0 \tag{168}$$

hold for an NTS-manifold; then,

$$R_{abcd} = -C_{ab[cd]} = -\frac{1}{2}(C_{abcd} - C_{abdc}) = 0, \tag{169}$$

i.e., an NTS-manifold is a manifold of class CR_2. □

Theorem 13. *NTS-manifolds of class CR_3 coincide with almost contact metric manifolds locally conformal to closely cosymplectic manifolds.*

Proof. Let an NTS-manifold be a manifold of class CR_3; then

$$R^{a}_{\hat{b}cd} = 2C^{abh}C_{hcd} - (\beta^0)^2\delta^a_{[c}\delta^b_{d]} = 0. \tag{170}$$

Let us convolve the obtained equality first by indexes a and c, and then by indexes b and d; then,

$$2C^{abc}C_{abc} - \frac{1}{2}n(n-1)(\beta^0)^2 = 0 \Rightarrow \beta^0 = \sqrt{\frac{4C^{abc}C_{abc}}{n(n-1)}}. \tag{171}$$

Similarly, from the condition

$$R^{\hat{a}}_{b\hat{c}\hat{d}} = 2C_{abh}C^{hcd} - (\beta_0)^2\delta^{[c}_a\delta^{d]}_b = 0, \tag{172}$$

we obtain

$$\beta_0 = \sqrt{\frac{4C^{abc}C_{abc}}{n(n-1)}}, \tag{173}$$

i.e.,

$$\beta^0 = \sqrt{\frac{4C^{abc}C_{abc}}{n(n-1)}} = \beta_0, \tag{174}$$

and the NTS-manifold has a closed contact form. Hence, according to Theorem 5, NTS-manifolds of class CR_3 coincide with almost contact metric manifolds with closed contact form locally conformal to closely cosymplectic manifolds. □

Definition 4. *An NTS-manifold whose curvature tensor satisfies the condition*

$$R(\xi, \Phi^2 X)\xi = 0, \ \forall X \in \mathcal{X}(M), \tag{175}$$

is called an NTS-manifold of the class CR_4.

Identity (175) is equivalent to the identity

$$R(\xi, X)\xi = 0, \ \forall X \in \mathcal{X}(M). \tag{176}$$

Definition 5. *An NTS-manifold whose curvature tensor satisfies the condition*

$$R(\xi, \Phi^2 X)\Phi^2 Y + R(\xi, \Phi X)\Phi Y = 0, \ \forall X, Y \in \mathcal{X}(M) \tag{177}$$

is called an NTS-manifold of the class CR_5.

Identity (177), considering identity (145), is equivalent to the identities

$$R(\xi, \Phi^2 X)\Phi^2 Y = R(\xi, \Phi X)\Phi Y = 0, \ \ \forall X, Y \in \mathcal{X}(M). \tag{178}$$

Theorem 14. *The class of NTS-manifolds of the class CR_4 coincides with the class of NTS-manifolds of the class CR_5.*

Proof. Let M be an NTS-manifold of class CR_4; then, identity (176) holds, which, considering (131) on the space of the associated G-structure, is equivalent to the following equalities:

$$\begin{aligned} &(1)\ R^a_{00b} = R^{\hat{a}}_{00\hat{b}} = 0; \\ &(2)\ R^a_{0b0} = R^{\hat{a}}_{0\hat{b}0} = 0. \end{aligned} \tag{179}$$

Due to the symmetry properties of the Riemannian curvature tensor from (179), we obtain:

$$\begin{aligned} &(1)\ R^0_{\hat{a}0b} = R^0_{a0\hat{b}} = 0; \\ &(2)\ R^c_{\hat{a}0b} = R^{\hat{c}}_{a0\hat{b}} = 0; \\ &(3)\ R^{\hat{c}}_{\hat{a}0b} = R^c_{a0\hat{b}} = 0. \end{aligned} \tag{180}$$

Applying the identity recovery procedure to the equality

$$R^i_{\hat{a}0b} = 0, \tag{181}$$

we obtain the identity (177). Thus, an NTS-manifold of class CR_4 is a manifold of class CR_5.

Similarly, one can prove that an NTS-manifold of class CR_5 is an NTS-manifold of class CR_4. □

Remark 5. *On the space of the associated G-structure, identity (175) is equivalent to the equalities*

$$R^i_{00b} = 0, \tag{182}$$

and it is equivalent to the equality

$$R^a_{00b} = R^{\hat{a}}_{00b} = 0. \tag{183}$$

Identity (177) on the space of the associated G-structure is equivalent to the equalities

$$R^i_{\hat{a}0b} = R^i_{a0\hat{b}} = 0, \tag{184}$$

which is equivalent to the equalities

$$R^0_{\hat{a}0b} = R^0_{a0\hat{b}} = 0. \tag{185}$$

Let M be an NTS-manifold of class CR_5; then,

$$R^0_{a0\hat{b}} = -\frac{1}{\sqrt{2}}\left\{\beta_{00} - \frac{1}{\sqrt{2}}(\beta_0)^2\right\}\delta^b_a = 0, \tag{186}$$

i.e.,

$$\beta_{00} = \frac{1}{\sqrt{2}}(\beta_0)^2. \tag{187}$$

Differentiate externally (85:4); then, considering (187) and (73:3), we have

$$d\beta_{00} \wedge \theta + \beta_{00} d\theta = d\left\{\frac{1}{\sqrt{2}}(\beta_0)^2\right\} \wedge \theta + \frac{1}{\sqrt{2}}\beta_{00}(\beta^0 - \beta_0)\delta^b_a \theta^a \wedge \theta_b = 0, \tag{188}$$

i.e.,

$$\sqrt{2}\beta_0 d\beta_0 \wedge \theta + \frac{1}{\sqrt{2}}\beta_{00}(\beta^0 - \beta_0)\delta_a^b \theta^a \wedge \theta_b = 0, \tag{189}$$

i.e.,

$$\beta_{00}(\beta^0 - \beta_0)\delta_a^b \theta^a \wedge \theta_b = 0. \tag{190}$$

Due to the linear independence of the basic forms, we have

$$\beta_{00}(\beta^0 - \beta_0)\delta_a^b = 0, \tag{191}$$

i.e.,

$$\beta_{00}(\beta^0 - \beta_0) = 0. \tag{192}$$

Hence,

(1) either

$$\beta_{00} = 0, \tag{193}$$

i.e.,

$$\beta_0 = 0, \tag{194}$$

i.e., the manifold is cosymplectic;

(2) or

$$\beta^0 - \beta_0 = 0, \tag{195}$$

i.e., the manifold has a closed contact form. This means that it is locally conformal to a closely cosymplectic manifold.

Note that a manifold locally conformal to a closely cosymplectic manifold is a special generalized Kenmotsu manifold of the second kind [35]. A special generalized Kenmotsu manifold of the second kind is an NTS-manifold with a closed contact form [24]. Thus, we have proved the following theorem.

Theorem 15. *Let M be an NTS-manifold; then, it is a manifold of class CR_5 if and only if it is a special generalized Kenmotsu manifold of the second kind, i.e., it is locally conformal to a closely cosymplectic manifold.*

7. k-Nullity Distribution

Definition 6 ([36])**.** *The k-nullity-distribution on a Riemannian manifold (M,g) for a real number k is a distribution*

$$N(k) : p \to N_p(k). \tag{196}$$

Therefore,

$$N(k) = \{Z \in T_pM | R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y), X, Y, Z \in T_pM\}, \tag{197}$$

where R is the curvature operator of the metric g.

Let M^{2n+1} with (Φ, η, ξ, g) be an NTS-manifold, or the characteristic vector field ξ, which belongs to the distribution $N(k)$; then,

$$R(X,Y)\xi = k(g(Y,\xi)X - g(X,\xi)Y) = k(\eta(Y)X - \eta(X)Y). \tag{198}$$

For example, Sasakian manifolds are characterized by the identity

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{199}$$

Therefore, a Sasakian manifold can be defined as a contact metric manifold whose vector field ξ belongs to the distribution $N(1)$. A Kenmotsu manifold is characterized by the identity

$$R(X,Y)\xi = -(\eta(Y)X - \eta(X)Y), \tag{200}$$

which means that a Kenmotsu manifold can be defined as a Riemannian manifold whose vector field ξ belongs to the distribution $N(-1)$.

Theorem 16. *An NTS-manifold on which there exists an N(k)-distribution containing a characteristic vector field ξ is a space of constant curvature k.*

The proof of the theorem follows directly from (152), (154), and (198).

Theorem 17. *An NTS-manifold on which there exists an N(k)-distribution containing the characteristic vector field ξ is an NTS-manifold of class CR_4.*

The proof of the theorem follows directly from (178) and Theorem 14.

8. Conclusions

In this paper, it is proved that the class of NTS-manifolds with non-closed contact form coincides with the class of almost contact metric manifolds homothetic to Sasaki varieties. Additionally, the class of NTS-manifolds with a closed contact form coincides with the class of almost contact metric manifolds with a closed contact form locally conformal to closely cosymplectic manifolds. This article presents some interesting identities that are satisfied by the Riemannian curvature tensor and the Ricci tensor. It is proved that an NTS-manifold of constant curvature is either a trans-Sasakian manifold of constant negative curvature, or it has a closed contact form, and hence it is locally conformal to a closely cosymplectic manifold of constant curvature. In particular, an NTS-manifold is of zero constant curvature if and only if it is a cosymplectic manifold of constant curvature. Several classes of NTS-manifolds are distinguished. It is proved that NTS-manifolds of class CR_3 coincide with almost contact metric manifolds locally conformal to closely cosymplectic manifolds. An NTS-manifold is a manifold of class CR_5 if and only if it is a special generalized Kenmotsu manifold of the second kind, i.e., it is locally conformal to a closely cosymplectic manifold. It is proved that an NTS-manifold on which there exists an N(k)-distribution containing a characteristic vector field ξ is a space of constant curvature k. Further, an NTS-manifold on which there exists an N(k)-distribution containing the characteristic vector field ξ is an NTS-manifold of class CR_4.

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