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Approximation Characteristics of Gel'fand Type in Multivariate Sobolev Spaces with Mixed Derivative Equipped with Gaussian Measure

Yuqi Liu , Huan Li * and Xuehua Li

College of Science, North China University of Technology, Beijing 100144, China; lyq304560@163.com (Y.L.); lixh@ncut.edu.cn (X.L.)

* Correspondence: lhan00@ncut.edu.cn

Abstract: In this paper, we study the probabilistic Gel'fand (N, δ) -width of multivariate Sobolev spaces $MW_2^r(\mathbb{T}^d)$ with mixed derivative that are equipped with Gaussian measure μ in $L_q(\mathbb{T}^d)$. The sharp asymptotic estimates are determined by employing the discretization method.

Keywords: probabilistic width; Gel'fand width; Sobolev space with mixed derivative; Gaussian measure

MSC: 41A46; 42A61; 41A63



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1. Introduction and Main Result

First, we recall the definitions of N -widths: Let X be a normed linear space with norm $\|\cdot\|$ and $\mathcal{C}$ be a bounded subset of X . Assume $\mathcal{L}_N$ to be the family of subspaces in X of dimensions at most N .

Definition 1 ([1]). The Kolmogorov N -width of $\mathcal{C}$ in X is defined as follows:

$$d_N(\mathcal{C}, X) := \inf_{\mathcal{L} \in \mathcal{L}_N} \sup_{x \in \mathcal{C}} \inf_{y \in \mathcal{L}} \|x - y\|, \quad (1)$$

where the leftmost infimum is taken from all N -dimensional linear subspaces of X .

Definition 2 ([1]). The linear N -width of $\mathcal{C}$ in X is defined as follows:

$$d'_N(\mathcal{C}, X) := \inf_{P_N} \sup_{x \in \mathcal{C}} \|x - P_N x\|, \quad (2)$$

where P_N runs over all linear operators from X to X of rank at most N .

Definition 3 ([1]). If there are N linear independent continuous linear functions $f_1, \dots, f_N$ on X , then $L^N = \{x \in X : f_i(x) = 0, i = 1, \dots, N\}$ is a complementary subspace of X with co-dimension N .

The Gel'fand N -width of $\mathcal{C}$ in X is defined as follows:

$$d^N(\mathcal{C}, X) := \inf_{L^N} \sup_{x \in \mathcal{C} \cap L^N} \|x\|, \quad (3)$$

where L^N runs over all subspaces of X with co-dimension at most N .

Definition 4 ([2–4]). Let W be a bounded subset of X , equipped with a probability measure μ . For any $\delta \in (0, 1]$, the probabilistic Kolmogorov (N, δ) -width and probabilistic linear (N, δ) -width of W in X are defined as follows:

$$d_{N,\delta}(W, \mu, X) := \inf_{G_\delta} d_N(W \setminus G_\delta, X); \quad (4)$$

$$d'_{N,\delta}(W, \mu, X) := \inf_{G_\delta} d'_N(W \setminus G_\delta, X), \quad (5)$$

where G_δ satisfies the condition that $\mu(G_\delta) \leq \delta$.

In order to define the probabilistic Gel'fand (N, δ) -width, we need to introduce some related concepts. Let Y be a Hilbert space. And it is equipped with a probabilistic measure μ . Let A be a closed subspace in Y . The orthogonal complement of A is denoted by $A^\perp$. Then,

$$x = y + z, \quad y \in A, \quad z \in A^\perp.$$

This form is unique.

If we introduce a projection operator P from Y to A , then the element y will be denoted by Px . For any Borel set G_A in A , let $\mu_A(G_A) := \mu(\{x : x \in A, Px \in G_A\})$. Then, μ_A is a probabilistic measure on A . After the above preparation, we can start to define the probabilistic Gel'fand (N, δ) -width.

Definition 5 ([2–4]). Let H be a Hilbert space and X be the same as Definition 1. Let H be equipped with a probabilistic measure μ . For any $\delta \in (0, 1]$, the probabilistic Gel'fand (N, δ) -width is defined as follows:

$$d_\delta^N(H, \mu, X) := \inf_{G_\delta} \inf_{L^N} \sup_{x \in (H \setminus G_\delta) \cap L^N} \|x\|, \quad (6)$$

where L^N satisfies the condition that $\text{codim} L^N \leq N$, G_δ runs over all subsets of H with measure at most δ , which satisfies the condition that for any closed subspace F in H ,

$$\mu_F(G_\delta \cap F) \leq \delta. \quad (7)$$

Remark 1. In this article, we give the definition of the Gel'fand N -width in the probabilistic setting. By comparing to the Kolmogorov N -width, the linear N -width, and the Gel'fand N -width in the probabilistic setting, we find that the first two widths are the infimum of all G_δ with measure at most δ . However, Tan et al. [5] present that the probabilistic Gel'fand (N, δ) -widths are the infimum for a part of G_δ satisfying the condition.

From [5], we know that they add the condition in order to make sure that $(W \setminus G_\delta) \cap L^N$ has enough elements. In addition, we define the probabilistic Gel'fand (N, δ) -widths as follows:

$$d_\delta^N(W, \mu, X) := \inf_{G_\delta} \inf_{L^N} \sup_{x \in (W \setminus G_\delta) \cap L^N} \|x\|, \quad (8)$$

where G_δ runs over all subsets of W with the measure at most δ .

We obtained the following result in paper [6]:

$$d'_{N,\delta}(H, \mu, X) = d_\delta^N(H, \mu, X), \quad (9)$$

where H is a Hilbert space and can be imbedded continuously into X .

Therefore, in the current situation, it is difficult to study d_δ^N . We have to add this condition (7) in Definition 5.

Widths theory is an important part of the approximation theory of functions. Different N -widths represent different approximation methods. In mathematical terms, for example, Kolmogorov N -width gives the optimal approximation degree of the “worst” elements of the sets that will be approximated by N -dimensional subspaces. The linear N -width reflects

the optimal error of the “worst” elements that will be approximated by linear operators in the approximation. However, in the worst-case setting, widths are defined by the optimal approximation degree of the “worst” elements of the approximated sets. Therefore, in the worst-case setting, errors reflected by widths cannot represent the best approximation of the whole elements of approximated sets. If we define widths in the probabilistic or average setting, the information will be optimized. In the probabilistic setting, errors are defined by widths in the worst-case setting of the sets, which can remove some elements for algorithmic “mistakes”. Therefore, the probabilistic setting, compared to the worst-case setting, allows one to give the better analysis of the approximation for the function classes.

Probabilistic and average widths have attracted much attention in recent years. We refer to the literature [7–9] for a survey. The usual widths may be found in the books [1,10]. V. E. Maiorov studied the probabilistic and average Kolmogorov and linear widths of one-dimensional Sobolev space in the L_q -norm, $1 \leq q \leq \infty$ [2–4]. Subsequently, some scholars studied the probabilistic and average widths of $\mathbb{R}^m$ in the l_q^m . They also studied the probabilistic and average linear widths of Sobolev space. These results were fully analogous to those of Maiorov, Fang, and Ye [2,11,12]. Chen and Fang studied the probabilistic and average widths of multivariate Sobolev space with mixed derivative in the L_q -norm, $1 \leq q < \infty$ [13,14]. After 2010, Wang studied the probabilistic and average widths of Sobolev spaces in weighted Sobolev spaces [15,16]. Tan et al. studied the Gel’fand N -width in the probabilistic setting of one-dimensional Sobolev space in the L_q -norm, $1 \leq q < \infty$ [5]. Dai and Wang studied the probabilistic and average linear n -widths of diagonal matrices [17]. Zhai and Hu estimated the sharp bound of the probabilistic and average linear widths of Sobolev spaces with Jacobi weight [9]. Vasil’eva studied the Kolmogorov widths of intersections and Sobolev weighted classes [18,19].

Next, we define two equivalence relations.

Assume that c, c_1, c_2 are positive constants depending only on the parameters r, q, ρ, d . For two positive functions $a(y)$ and $b(y)$ satisfying $a(y) \leq cb(y)$ for all y from the domain of the functions a and b , we write $a(y) \ll b(y)$. If $a(y)$ and $b(y)$ satisfy $c_1b(y) \leq a(y) \leq c_2b(y)$, then we write $a(y) \asymp b(y)$.

In this article, we will continue the research of [5] and estimate the exact bounds of the probabilistic Gel’fand widths of multivariate Sobolev spaces with mixed derivative in the L_q -norm, $1 < q < \infty$. However, our results cannot be derived directly from the results of [5]. This phenomenon shows that the results of multivariate Sobolev spaces are not an ordinary generalization from the results of univariate Sobolev spaces. There are differences in methods and techniques between the results of multivariate Sobolev spaces and univariate Sobolev spaces. We determine the asymptotic order of the probabilistic Gel’fand widths of multivariate Sobolev space with mixed derivative $MW_2^r(\mathbb{T}^d)$ in the space $L_q(\mathbb{T}^d)$, $1 < q < \infty$, where $\mathbb{T} = [0, 2\pi)$.

Theorem 1. Let $r = (r_1, \dots, r_d)$, $\frac{1}{2} \leq r_1 = \dots = r_v < r_{v+1} \leq \dots \leq r_d$, $1 \leq q < 2$, $\rho > 1$, $\delta \in (0, \frac{1}{2}]$. Then,

$$d_\delta^N(MW_2^r(\mathbb{T}^d), \mu, L_q(\mathbb{T}^d)) \asymp (N^{-1} \ln^{v-1} N)^{r_1 + \frac{\rho-1}{2}} (\ln^{\frac{v-1}{2}} N) \sqrt{1 + \frac{1}{N} \ln\left(\frac{1}{\delta}\right)}. \quad (10)$$

Theorem 2. Let $r = (r_1, \dots, r_d)$, $\frac{1}{2} \leq r_1 = \dots = r_v < r_{v+1} \leq \dots \leq r_d$, $2 \leq q < \infty$, $\rho > 1$, $\delta \in (0, \frac{1}{2}]$. Then,

$$\ln^{(v-1)/q} N \ll \frac{d_\delta^N(MW_2^r(\mathbb{T}^d), \mu, L_q(\mathbb{T}^d))}{(N^{-1} \ln^{v-1} N)^{r_1 + (\rho-1)/2} (1 + N^{-1/q} \sqrt{\ln(1/\delta)})} \ll \ln^{(v-1)/2} N. \quad (11)$$

2. Discretization

Consider Hilbert space $L_2(\mathbb{T}^d)$, where $\mathbb{T}^d = [0, 2\pi)^d$, consisting of all 2π -periodic functions $x(t)$ with the Fourier series

$$x(t) = \sum_{k \in \mathbb{Z}^d} c_k \exp(i(k, t)) = \sum_{k \in \mathbb{Z}^d} c_k e_k(t).$$

Define the inner product by

$$\langle x, y \rangle := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} x(t) \overline{y(t)} dt, x, y \in L_2(\mathbb{T}^d).$$

Denoted by $L_q(\mathbb{T}^d)$, $1 \leq q < \infty$, the classical q -integral Lebesgue space of 2π -periodic functions with the usual norm $\|\cdot\|_{L_q} := \|\cdot\|_{L_q(\mathbb{T}^d)}$. Let $y = (y_1, \dots, y_d)$, $t = (t_1, \dots, t_d) \in \mathbb{R}^d$, $s \in \mathbb{R}$ and $r = (r_1, \dots, r_d) \in \mathbb{R}^d$. We denote that $|y|^s = |y_1|^s \cdots |y_d|^s$, $y + s = (y_1 + s, \dots, y_d + s)$, and we define that $y > s \Leftrightarrow y_j > s (\forall j = 1, 2, \dots, d)$, $(y, t) = \sum_{j=1}^d y_j t_j$, $(y, s) = \sum_{j=1}^d y_j s_j$. In summary, we can define the r th order derivative of x for an arbitrary vector $y = (y_1, \dots, y_d) \in \mathbb{R}^d$ in the sense of Weyl by

$$x^{(r)}(t) := (D^r x)(t) = \sum_{k \in \mathbb{Z}^d} (ik)^r c_k \exp(i(k, t)),$$

where $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$, $(ik)^r = \prod_{j=1}^d |k_j|^{r_j} \exp\left(\frac{\pi i}{2} r_j\right)$. It is well known that if $r > \max\{0, 1/2 - 1/q\}$, then the space $MW_2^r(\mathbb{T}^d)$ can be imbedded continuously into $L_q(\mathbb{T}^d)$, $1 \leq q < \infty$.

Sobolev space $MW_2^r(\mathbb{T}^d)$ is a Hilbert space given by

$$MW_2^r(\mathbb{T}^d) := \left\{ x \in L_2(\mathbb{T}^d) : x^{(r)} \in L_2(\mathbb{T}^d), \int_0^{2\pi} x(t) dt_i = 0, i = 1, \dots, d \right\}.$$

The inner product of $MW_2^r(\mathbb{T}^d)$ is $\langle x, y \rangle_r := \langle x^{(r)}, y^{(r)} \rangle$, and the norm in $MW_2^r(\mathbb{T}^d)$ is

$$\|x\|_{MW_2^r} := \langle x^{(r)}, x^{(r)} \rangle^{\frac{1}{2}}.$$

Now, we equip $MW_2^r(\mathbb{T}^d)$ with Gaussian measure μ , whose mean is zero and whose correlation operator C_μ has eigenfunctions $e_k = \exp(i(k, \cdot))$ and eigenvalues $\lambda_k = |k|^{-\rho}$, $\rho > 1$, i.e., $C_\mu e_k = \lambda_k e_k$, $k \in \mathbb{Z}_0^d$, where $\mathbb{Z}_0^d = \mathbb{Z} \setminus \{0 = (0, \dots, 0) \in \mathbb{Z}^d\}$.

Let $y_1, \dots, y_n$ be any orthogonal system of functions in $L_2(\mathbb{T}^d)$, $\sigma_j = \langle C_\mu y_j, y_j \rangle$, $j = 1, \dots, n$, and B be an arbitrary Borel subset of $\mathbb{R}^n$. Then, the Gaussian measure μ on the cylindrical subsets in the space $MW_2^r(\mathbb{T}^d)$,

$$G = \left\{ x \in MW_2^r(\mathbb{T}^d) : \left(\langle x, y_1^{(-r)} \rangle_r, \dots, \langle x, y_n^{(-r)} \rangle_r \right) \in B \right\},$$

is given by

$$\mu(G) = \prod_{j=1}^n (2\pi\sigma_j)^{-\frac{1}{2}} \int_B \exp\left(-\sum_{j=1}^n \frac{|u_j|^2}{2\sigma_j}\right) du_1 \cdots du_n.$$

A result about the average error bounds in Banach spaces equipped with a Gaussian measure can be found in an article by Yongsheng Sun [20].

Tan et al. [5] studied the probabilistic Gel'fand width of univariate Sobolev space $W_2^r(\mathbb{T})$. They proved three lemmas, as follows:

Lemma 1 ([5]). Let $2N \leq m$ and $\delta \in (0, \frac{1}{2}]$. Then,

$$d_\delta^N(\mathbb{R}^m, v, l_q^m) \asymp \begin{cases} m^{1/q-1/2} \sqrt{m + \ln \frac{1}{\delta}}, & 1 \leq q < 2. \\ m^{1/q} + \sqrt{\ln(1/\delta)}, & 2 \leq q < \infty. \end{cases} \quad (12)$$

Here, the upper bounds hold if $N \leq m$.

Lemma 2 ([5]). Let H be a Hilbert space and X be a linear space with norm $\|\cdot\|$. If H can be imbedded continuously into X , $\delta \in (0, 1]$ and μ is the Gaussian measure in H , then

$$d'_{N,\delta}(H, \mu, X) \leq d_\delta^N(H, \mu, X). \quad (13)$$

Lemma 3 ([5]). Let $r > 1/2$, $\delta \in (0, 1/2]$, $1 \leq q < \infty$, $N \in \mathbb{N}_+$. Then,

$$d_\delta^N(W_2^r(\mathbb{T}), \mu, L_q(\mathbb{T})) \asymp \begin{cases} N^{-(r+\frac{\rho-1}{2})} \sqrt{1 + \frac{1}{N} \ln\left(\frac{1}{\delta}\right)}, & 1 \leq q < 2. \\ N^{-(r+\frac{\rho-1}{2})} \left(1 + N^{-\frac{1}{q}} \sqrt{\ln\left(\frac{1}{\delta}\right)}\right), & 2 \leq q < \infty. \end{cases} \quad (14)$$

Remark 2. In the case of one dimension $d = 1$, Theorems 1 and 2 are the same as Lemma 3.

Chen and Fang [14] studied the probabilistic linear N -width of multivariate Sobolev space with mixed derivative. They proved the following:

Lemma 4 ([14]). Let $r = (r_1, \dots, r_d)$, $\frac{1}{2} \leq r_1 = \dots = r_v < r_{v+1} \leq \dots \leq r_d$, $1 \leq q < \infty$, $\rho > 1$, $\delta \in (0, \frac{1}{2}]$. Then,

$$d'_{N,\delta}(MW_2^r(\mathbb{T}^d), \mu, L_q(\mathbb{T}^d)) \asymp (N^{-1} \ln^{v-1} N)^{r_1 + \frac{\rho-1}{2}} \left(\ln^{\frac{v-1}{2}} N\right) \sqrt{1 + \frac{1}{N} \ln\left(\frac{1}{\delta}\right)}, \quad 1 \leq q < 2; \quad (15)$$

$$\ln^{(v-1)/q} N \ll \frac{d'_{N,\delta}(MW_2^r(\mathbb{T}^d), \mu, L_q(\mathbb{T}^d))}{(N^{-1} \ln^{v-1} N)^{r_1 + (\rho-1)/2} (1 + N^{-1/q} \sqrt{\ln(1/\delta)})} \ll \ln^{(v-1)/2} N, \quad 2 \leq q < \infty. \quad (16)$$

Now, we use the discretization method to prove Theorems 1 and 2. This discretization method aims to transform functional spaces into finite-dimensional space. Therefore, discretization can reduce the calculation of the sharp bounds of the probabilistic (N, δ) -widths.

First, we recall some definitions, and we introduce some notations. We also need some results of the standard Gaussian measure of finite-dimensional spaces. Let the l_p^m be m -dimensional normed space of vectors $x = (a_1, \dots, a_m) \in \mathbb{R}^m$, with norm

$$\|a\|_{l_p^m} = \begin{cases} \left(\sum_{i=1}^m |a_i|^p\right)^{\frac{1}{p}}, & 1 \leq p < \infty. \\ \max_{1 \leq i \leq m} |a_i|, & p = \infty. \end{cases}$$

Consider in $\mathbb{R}^m$ the standard Gaussian measure, which is defined as

$$v(G) = (2\pi)^{-\frac{m}{2}} \int_G \exp\left(-\frac{1}{2}\|x\|_2^2\right) dx,$$

where G is any Borel subset in $\mathbb{R}^m$. Evidently, $v(\mathbb{R}^m) = 1$. In order to establish the discretization theorem, we need to split the Fourier series of functions into the sum of dyadic blocks. Then, we introduce some lemmas and notations. For $s = (s_1, \dots, s_d) \in \mathbb{N}^d$, let $\mathcal{K}_s$ be the “block” of $\mathbb{N}^d$, where

$$\mathcal{K}_s := \left\{ n = (n_1, \dots, n_d) \in \mathbb{Z}_0^d : 2^{s_j-1} \leq |n_j| < 2^{s_j}, j = 1, \dots, d \right\}.$$

Assume $\eta_s(x)$ to be the “block” of the Fourier series for $x(t)$: that is,

$$\eta_s(x(t)) := \sum_{n \in \mathcal{K}_s} c_n \exp(i(n, t)).$$

After introducing these necessary concepts, we have

Lemma 5 ([21]). Let S be a subset of $\mathbb{N}^d$, $r = (r_1, \dots, r_d) \in \mathbb{R}^d$, $1 < q < \infty$, $x = \sum_{s \in S} \eta_s x \in F_S$. Then,

$$|S|^{(1/2-1/q)_-} \left(\sum_{s \in S} \left\| 2^{(s,r)} \eta_s x \right\|_{L_q^d}^q \right)^{1/q} \ll \|x^{(r)}\|_{L_q^d} \ll |S|^{(1/2-1/q)_+} \left(\sum_{s \in S} \left\| 2^{(s,r)} \eta_s x \right\|_{L_q^d}^q \right)^{1/q}, \quad (17)$$

where $a_- = \min\{0, a\}$, $b_+ = \max\{0, b\}$, $|S|$ is the cardinality of the set S , and

$$F_S = \text{span}\{\exp(i(n, t)) : n \in \mathcal{K}_s, s \in S\}.$$

Lemma 6 ([2,3]). For $\forall \delta \in (0, \frac{1}{2}]$, there is a positive c_0 , such that

$$v\left(\left\{x \in \mathbb{R}^m : \|x\|_2 \geq c_0 \left(\sqrt{m} + \sqrt{\ln \frac{1}{\delta}}\right)\right\}\right) \leq \delta. \quad (18)$$

For $2 \leq q < \infty$ and any $\delta \in (0, \frac{1}{2}]$, there exists a positive constant c_q that depends only on q , such that

$$v\left(\left\{x \in \mathbb{R}^m : \|x\|_{l_q^m} \geq c_q \left(m^{\frac{1}{q}} + \sqrt{\ln \frac{1}{\delta}}\right)\right\}\right) \leq \delta. \quad (19)$$

Given $\alpha, \beta \in \mathbb{N}$, we define

$$\begin{aligned} \mathcal{S}_{\alpha, \beta} &= \left\{ s \in \mathbb{N}^d : \alpha - 1 \leq (s, \gamma') < \alpha, (s, 1) = \beta \right\}, \\ \mathcal{F}_{\alpha, \beta} &= \text{span}\left\{ e^{i(n, t)} : n \in \mathcal{K}_s, s \in \mathcal{S}_{\alpha, \beta} \right\}, \end{aligned} \quad (20)$$

where $\gamma = r + \rho/2$, $\gamma' = \gamma/(r_1 + \rho/2)$. Note that $\beta \geq d$, $\mathcal{S}_{\alpha, \beta} \neq \emptyset$ for $\beta \geq \alpha$.

Let $\|\mathcal{S}_{\alpha, \beta}\| = \sum_{s \in \mathcal{S}_{\alpha, \beta}} |\mathcal{K}_s|$. We obtain $\|\mathcal{S}_{\alpha, \beta}\| = 2^\beta |\mathcal{S}_{\alpha, \beta}|$. And we define $\Delta_{\alpha, \beta} x := \sum_{s \in \mathcal{S}_{\alpha, \beta}} \eta_s x$.

The following theorem reflects the upper bounds of Theorems 1 and 2:

Theorem 3. Let $r = (r_1, \dots, r_d)$, $\frac{1}{2} \leq r_1 = \dots = r_v < r_{v+1} \leq \dots \leq r_d$, $1 < q < \infty$, $\rho > 1$, $\delta \in (0, \frac{1}{2}]$, $N = 0, 1, \dots$. Let the sequence of natural numbers $\{N_{\alpha, \beta}\}$ be such that $0 \leq N_{\alpha, \beta} \leq \|\mathcal{S}_{\alpha, \beta}\|$, $\sum_{\alpha, \beta} N_{\alpha, \beta} \leq N$. And let the sequence of positive numbers $\{\delta_{\alpha, \beta}\}$ be such that $\sum_{\alpha, \beta} \delta_{\alpha, \beta} \leq \delta$. Then,

$$d_{\delta}^N \left(MW_2^r(\mathbb{T}^d), \mu, L_q(\mathbb{T}^d) \right) \ll \sum_{\alpha, \beta} |\mathcal{S}_{\alpha, \beta}|^{(1/2-1/q)+2^{-(r_1+\rho/2)\alpha+\beta/2-\beta/q}} d_{\delta_{\alpha, \beta}}^{N_{\alpha, \beta}}, \quad (21)$$

where $d_{\delta_{\alpha, \beta}}^{N_{\alpha, \beta}} := d_{\delta_{\alpha, \beta}}^{N_{\alpha, \beta}} \left(\mathbb{R}^{\|\mathcal{S}_{\alpha, \beta}\|}, v, l_q^{\|\mathcal{S}_{\alpha, \beta}\|} \right)$.

Proof. From Lemma 1, there is a positive constant c_q' , such that

$$d_{\delta_{\alpha, \beta}}^{N_{\alpha, \beta}} \left(\mathbb{R}^{\|\mathcal{S}_{\alpha, \beta}\|}, v, l_q^{\|\mathcal{S}_{\alpha, \beta}\|} \right) = \begin{cases} c_q' \|\mathcal{S}_{\alpha, \beta}\|^{1/q-1/2} \sqrt{\|\mathcal{S}_{\alpha, \beta}\| + \ln \frac{1}{\delta_{\alpha, \beta}}}, & 1 \leq q < 2. \\ c_q' \left(\|\mathcal{S}_{\alpha, \beta}\|^{1/q} + \sqrt{\ln \frac{1}{\delta_{\alpha, \beta}}} \right), & 2 \leq q < \infty. \end{cases} \quad (22)$$

Let $Q_{\alpha, \beta} = \left\{ y \in l_q^{\|\mathcal{S}_{\alpha, \beta}\|} : \|y\|_{l_q^{\|\mathcal{S}_{\alpha, \beta}\|}} > c' c_q'^{-1} d_{\delta_{\alpha, \beta}}^{N_{\alpha, \beta}} \right\}$, where

$$c' = \begin{cases} \sqrt{2} c_0, & 1 \leq q < 2. \\ c_q, & 2 \leq q < \infty. \end{cases}$$

By $\|x\|_{l_q^m}^m \leq m^{1/q-1/2} \|x\|_{\mathbb{R}^m}$ [5], we can establish that

$$y \in Q_{\alpha, \beta} \Rightarrow y \in \left\{ x \in \mathbb{R}^{\|\mathcal{S}_{\alpha, \beta}\|} : \|x\|_{\mathbb{R}^{\|\mathcal{S}_{\alpha, \beta}\|}} > c_0 \left(\|\mathcal{S}_{\alpha, \beta}\|^{1/2} + \sqrt{\ln \frac{1}{\delta_{\alpha, \beta}}} \right) \right\}, \quad 1 \leq q < 2;$$

$$y \in Q_{\alpha, \beta} \Rightarrow y \in \left\{ x \in \mathbb{R}^{\|\mathcal{S}_{\alpha, \beta}\|} : \|x\|_{\mathbb{R}^{\|\mathcal{S}_{\alpha, \beta}\|}} > c_q \left(\|\mathcal{S}_{\alpha, \beta}\|^{1/q} + \sqrt{\ln \frac{1}{\delta_{\alpha, \beta}}} \right) \right\}, \quad 2 \leq q < \infty.$$

From Lemma 6, we have $v(Q_{\alpha, \beta}) \leq \delta_{\alpha, \beta}$.

Let $L^{\alpha, \beta}$ be a subspace of $\mathbb{R}^{\|\mathcal{S}_{\alpha, \beta}\|}$ with co-dimension at most $N_{\alpha, \beta}$. Then,

$$y \in L^{\alpha, \beta} \cap \left(\mathbb{R}^{\|\mathcal{S}_{\alpha, \beta}\|} \setminus Q_{\alpha, \beta} \right) \Rightarrow \|y\|_{l_q^{\|\mathcal{S}_{\alpha, \beta}\|}} \leq c' c_q'^{-1} d_{\delta_{\alpha, \beta}}^{N_{\alpha, \beta}}.$$

Now, we consider the polynomials in the space $F_{\alpha, \beta}$:

$$\varphi_{s, m, j}^{\alpha, \beta}(t) = \sum_{n \in \mathcal{K}_s, \text{sgn} n = \text{sgn} m} e_n(t - t_j), \quad s \in \mathcal{S}_{\alpha, \beta},$$

where $m = (m_1, \dots, m_d) = (\pm 1, \dots, \pm 1) \in \mathbb{R}^d$, $j_i = 1, \dots, 2^{s_i-1}$, $i = 1, \dots, d$.

Obviously, these polynomials are orthogonal. And, for any $x \in F_{\alpha, \beta}$, we have

$$(D^r x)_{s, m}(t_j) = \left\langle D^r x, \varphi_{s, m, j}^{\alpha, \beta} \right\rangle, \quad \forall s, m, j.$$

For $\forall \alpha, \beta \in \mathbb{N}$ and $d \leq \beta \leq \alpha$, we consider a mapping,

$$I_{\alpha, \beta} : F_{\alpha, \beta} \rightarrow l_q^{\|\mathcal{S}_{\alpha, \beta}\|}, \quad x \mapsto \left\{ \left\langle D^r x, \varphi_{s, m, j}^{\alpha, \beta} \right\rangle \right\}_{s, m, j}.$$

From the result of Chen and Fang [13], $I_{\alpha,\beta}$ is a linear isomorphic from the space $F_{\alpha,\beta}$ to l_q^d , and

$$\|x\|_{L_q^d} \ll |\mathcal{S}_{\alpha,\beta}|^{(1/2-1/q)+2^{-(r_1+\rho/2)\alpha+\beta\rho/2-\beta/q}} \left\| \left\{ \langle D^r x, \varphi_{s,m,j}^{\alpha,\beta} \rangle \right\}_{s,m,j} \right\|_{l_q} \|\mathcal{S}_{\alpha,\beta}\|. \quad (23)$$

We have

$$\|\Delta_{\alpha,\beta} x\|_{L_q^d} = \left\| \sum_{s \in \mathcal{S}_{\alpha,\beta}} \eta_s x \right\|_{L_q^d} \leq \|x\|_{L_q^d};$$

therefore,

$$\|\Delta_{\alpha,\beta} x\|_{L_q^d} \ll |\mathcal{S}_{\alpha,\beta}|^{(1/2-1/q)+2^{-(r_1+\rho/2)\alpha+\beta\rho/2-\beta/q}} \left\| \left\{ \langle D^r x, \varphi_{s,m,j}^{\alpha,\beta} \rangle \right\}_{s,m,j} \right\|_{l_q} \|\mathcal{S}_{\alpha,\beta}\|.$$

That is, there is a positive constant $c'' > 0$, such that

$$\|\Delta_{\alpha,\beta} x\|_{L_q^d} \leq c'' |\mathcal{S}_{\alpha,\beta}|^{(1/2-1/q)+2^{-(r_1+\rho/2)\alpha+\beta\rho/2-\beta/q}} \left\| \left\{ \langle D^r x, \varphi_{s,m,j}^{\alpha,\beta} \rangle \right\}_{s,m,j} \right\|_{l_q} \|\mathcal{S}_{\alpha,\beta}\|. \quad (24)$$

Let $\sigma_{s,m,j}^{\alpha,\beta} = \langle C_\mu \varphi_{s,m,j}^{\alpha,\beta}, \varphi_{s,m,j}^{\alpha,\beta} \rangle$. From the results of Chen and Fang [13], there is a positive constant $c_1 > 0$, such that $\sigma = \sigma_{s,m,j}^{\alpha,\beta} = c_1 2^{-k(\rho-1)}$.

Now, we consider the subset

$$E_{\alpha,\beta} := \left\{ x \in MW_2^r(\mathbb{T}^d) : \|\Delta_{\alpha,\beta} x\|_{L_q^d} > c'' c' c_q'^{-1} |\mathcal{S}_{\alpha,\beta}|^{(1/2-1/q)+2^{-(r_1+\rho/2)\alpha+\beta\rho/2-\beta/q}} \sigma^{1/2} d_{\delta_{\alpha,\beta}}^{N_{\alpha,\beta}} \right\}.$$

Then,

$$\begin{aligned} & \mu(E_{\alpha,\beta}) \\ & \leq \mu \left(\left\{ x \in MW_2^r(\mathbb{T}^d) : \left\| \left\{ \langle D^r x, \varphi_{s,m,j}^{\alpha,\beta} \rangle \right\}_{s,m,j} \right\|_{l_q} \|\mathcal{S}_{\alpha,\beta}\| > c' c_q'^{-1} \sigma^{1/2} d_{\delta_{\alpha,\beta}}^{N_{\alpha,\beta}} \right\} \right) \\ & = v \left(\left\{ y \in \mathbb{R}^{\|\mathcal{S}_{\alpha,\beta}\|} : \left\| \sigma^{1/2} y \right\|_{l_q} \|\mathcal{S}_{\alpha,\beta}\| > c' c_q'^{-1} \sigma^{1/2} d_{\delta_{\alpha,\beta}}^{N_{\alpha,\beta}} \right\} \right) \\ & = v \left(\left\{ y \in \mathbb{R}^{\|\mathcal{S}_{\alpha,\beta}\|} : \|y\|_{l_q} \|\mathcal{S}_{\alpha,\beta}\| > c' c_q'^{-1} d_{\delta_{\alpha,\beta}}^{N_{\alpha,\beta}} \right\} \right) = v(Q_{\alpha,\beta}) \leq \delta_{\alpha,\beta}. \end{aligned} \quad (25)$$

Let $E = \bigcup_{\alpha,\beta} E_{\alpha,\beta}$. Then, $\mu(E) \leq \sum_{\alpha,\beta} \mu(E_{\alpha,\beta}) \leq \sum_{\alpha,\beta} \delta_{\alpha,\beta} \leq \delta$. Let $F^{\alpha,\beta} = D^{-r} I_{\alpha,\beta}^{-1} L^{\alpha,\beta}$, and $F = \sum_{\alpha,\beta} F^{\alpha,\beta}$, where the sum is the direct sum. Therefore, F is a subspace of L_q^d with co-dimension as follows:

$$\text{co dim } F = \text{co dim } \sum_{\alpha,\beta} F^{\alpha,\beta} \leq \sum_{\alpha,\beta} \text{co dim } F^{\alpha,\beta} \leq \sum_{\alpha,\beta} \text{co dim } L^{\alpha,\beta} \leq \sum_{\alpha,\beta} N_{\alpha,\beta} \leq N.$$

From the definition of the probabilistic Gel'fand (N, δ) -width, we obtain

$$\begin{aligned} d_{\delta}^N \left(MW_2^r(\mathbb{T}^d), \mu, L_q(\mathbb{T}^d) \right) &\leq \sup_{x \in (MW_2^r(\mathbb{T}^d) \setminus E) \cap F} \|x\|_{L_q^d} \\ &\leq \sup_{x \in (MW_2^r(\mathbb{T}^d) \setminus E) \cap F} \sum_{\alpha, \beta} \|\Delta_{\alpha, \beta} x\|_{L_q^d} \\ &\leq \sum_{\alpha, \beta} \sup_{x \in (MW_2^r(\mathbb{T}^d) \setminus E) \cap F} \|\Delta_{\alpha, \beta} x\|_{L_q^d} \\ &\ll \sum_{\alpha, \beta} |\mathcal{S}_{\alpha, \beta}|^{(1/2-1/q)+} 2^{-(r_1+\rho/2)\alpha+\beta/2-\beta/q} d_{\delta, \beta}^{N_{\alpha, \beta}}, \end{aligned}$$

which completes the proof of Theorem 3. $\square$

Remark 3. In order to prove Theorems 1 and 2, we use the discretization method [3]. The discretization method is the classical method of studying widths theory. To estimate the sharp bounds of probabilistic Kolmogorov (N, δ) -widths and probabilistic linear (N, δ) -widths, scholars like Maiorov, Fang, Ye, and Chen [2–4,11–14] also used the discretization method. Therefore, discretization is an effective method of calculating probabilistic (n, δ) -widths. However, our discretization theorem is different from the aforementioned. The difference between the proof of Theorem 3 and other discretization theorems is the difference between the definitions of Kolmogorov N -widths, linear N -widths, and Gel'fand N -widths in the probabilistic setting. We need to structure a new set, to enlarge the probabilistic Gel'fand (N, δ) -widths of multivariate Sobolev spaces. This set is related to the norm of $\Delta_{\alpha, \beta} x$. Therefore, we need to estimate the upper bound of $\|\Delta_{\alpha, \beta} x\|_{L_q^d}$; that is the innovation of our discretization theorem.

3. Proof of Main Result

To establish the upper bound of Theorems 1 and 2, we also need the following lemma:

Lemma 7 (Romanyuk [22]). For $\forall d \in \mathbb{N}$, $\beta > 0$, $v > 1$, we define

$$N_{\alpha, \beta} = \begin{cases} \|S_{\alpha, \beta}\|, d \leq \beta \leq \alpha, \alpha \leq u. \\ \lfloor |S_{\alpha, \beta}| 2^{u+\theta u-2\theta\alpha+\theta\beta} \rfloor, d \leq \beta \leq \alpha, \alpha > u. \\ 0, \text{otherwise.} \end{cases} \quad (26)$$

where $\alpha, \beta, S_{\alpha, \beta}$, as in (20), and $\lfloor a \rfloor$ means the largest integer not greater than a . Then, there is a positive constant $c > 0$, such that $\sum_{\alpha, \beta} N_{\alpha, \beta} \leq c 2^u u^{v-1}$.

For a given $N \in \mathbb{N}$, we select u according to the condition $N \asymp 2^u u^{v-1}$ and define $N_{\alpha, \beta}$ as that in Lemma 7. Let

$$\delta_{\alpha, \beta} = \begin{cases} \delta N_{\alpha, \beta} / N, d \leq \beta \leq \alpha, \alpha > u. \\ 0, \text{otherwise.} \end{cases} \quad (27)$$

From Lemmas 2 and 4, we can obtain the lower bound of Theorems 1 and 2. Now, we estimate the upper bound of Theorems 1 and 2.

Proof of Theorem 1. From [3] we can establish that $d_{N, \delta}(\mathbb{R}^m, v, l_q^m) = d_{\delta}^N(\mathbb{R}^m, v, l_q^m)$ if $1 \leq q < 2$. Consequently, we can obtain the result that $d_{N, \delta}$ and d_{δ}^N of $MW_2^r(\mathbb{T}^d)$ have the same upper bound when $1 \leq q < 2$.

Therefore, from [13] we can establish that

$$d_{\delta}^N \left(MW_2^r(\mathbb{T}^d), \mu, L_q(\mathbb{T}^d) \right) \ll \left(N^{-1} \ln^{v-1} N \right)^{r_1+(\rho-1)/2} \left(\ln^{(v-1)/2} N \right) \sqrt{1 + \frac{1}{N} \ln \frac{1}{\delta}}.$$

From Lemma 2, when $1 \leq q < 2$, we obtain

$$d_{\delta}^N \left(MW_2^r(\mathbb{T}^d), \mu, L_q(\mathbb{T}^d) \right) \asymp \left(N^{-1} \ln^{v-1} N \right)^{r_1 + (\rho-1)/2} \left(\ln^{(v-1)/2} N \right) \sqrt{1 + \frac{1}{N} \ln \frac{1}{\delta}},$$

which completes the proof of Theorem 1. $\square$

Proof of Theorem 2. From Theorem 3, Lemma 1, for $2 \leq q < \infty$, we have

$$\begin{aligned} d_{\delta}^N \left(MW_2^r(\mathbb{T}^d), \mu, L_q(\mathbb{T}^d) \right) &\ll \sum_{\alpha, \beta} |\mathcal{S}_{\alpha, \beta}|^{(1/2-1/q)+} 2^{-(r_1+\rho/2)\alpha+\beta/2-\beta/q} d_{\delta_{\alpha, \beta}}^{N_{\alpha, \beta}} \left(\mathbb{R} \|\mathcal{S}_{\alpha, \beta}\|, v, l_q^{\|\mathcal{S}_{\alpha, \beta}\|} \right) \\ &= \sum_{\alpha > u} \sum_{d \leq \beta \leq \alpha} |\mathcal{S}_{\alpha, \beta}|^{1/2-1/q} 2^{-(r_1+\rho/2)\alpha+\beta/2-\beta/q} d_{\delta_{\alpha, \beta}}^{N_{\alpha, \beta}} \left(\mathbb{R} \|\mathcal{S}_{\alpha, \beta}\|, v, l_q^{\|\mathcal{S}_{\alpha, \beta}\|} \right) \\ &\ll \sum_{\alpha > u} \sum_{d \leq \beta \leq \alpha} |\mathcal{S}_{\alpha, \beta}|^{1/2-1/q} 2^{-(r_1+\rho/2)\alpha+\beta/2-\beta/q} \left(\|\mathcal{S}_{\alpha, \beta}\|^{1/q} + \sqrt{\ln(1/\delta_{\alpha, \beta})} \right) \\ &= \sum_{\alpha > u} \sum_{d \leq \beta \leq \alpha} |\mathcal{S}_{\alpha, \beta}|^{1/2-1/q} 2^{-(r_1+\rho/2)\alpha+\beta/2-\beta/q} 2^{\beta/q} |\mathcal{S}_{\alpha, \beta}|^{1/q} \\ &\quad + \sum_{\alpha > u} \sum_{d \leq \beta \leq \alpha} |\mathcal{S}_{\alpha, \beta}|^{1/2-1/q} 2^{-(r_1+\rho/2)\alpha+\beta/2-\beta/q} \sqrt{\ln(1/\delta_{\alpha, \beta})} \\ &= \sum_{\alpha > u} \sum_{d \leq \beta \leq \alpha} |\mathcal{S}_{\alpha, \beta}|^{1/2} 2^{-(r_1+\rho/2)l+\beta/2} \\ &\quad + \sum_{\alpha > u} \sum_{d \leq \beta \leq \alpha} |\mathcal{S}_{\alpha, \beta}|^{1/2-1/q} 2^{-(r_1+\rho/2)\alpha+\beta/2-\beta/q} \sqrt{\ln(1/\delta_{\alpha, \beta})} \\ &:= I_1 + I_2. \end{aligned}$$

First, we discuss I_1 :

$$I_1 = \sum_{\alpha > u} \sum_{d \leq \beta \leq \alpha} |\mathcal{S}_{\alpha, \beta}|^{1/2} 2^{-(r_1+\rho/2)l+\beta/2} = \sum_{\alpha > u} 2^{-(r_1+\rho/2)\alpha} \sum_{d \leq \beta \leq l} |\mathcal{S}_{\alpha, \beta}|^{1/2} 2^{\beta/2}.$$

Let

$$\sum_{d \leq \beta \leq \alpha} |\mathcal{S}_{\alpha, \beta}|^{1/2} 2^{\beta/2} = \sum_{d \leq \beta \leq \alpha} ' |\mathcal{S}_{\alpha, \beta}|^{1/2} 2^{\beta/2} + \sum_{d \leq \beta \leq \alpha} '' |\mathcal{S}_{\alpha, \beta}|^{1/2} 2^{\beta/2},$$

where $\sum_{d \leq \beta \leq \alpha} '$ is carried out over β for $|\mathcal{S}_{\alpha, \beta}| \leq \alpha^{v-1}$, and $\sum_{d \leq \beta \leq \alpha} ''$ is carried out over β for $|\mathcal{S}_{\alpha, \beta}| > \alpha^{v-1}$. Then, we have

$$\begin{aligned} \sum_{d \leq \beta \leq \alpha} ' |\mathcal{S}_{\alpha, \beta}|^{1/2} 2^{\beta/2} &\leq \alpha^{(v-1)/2} \sum_{d \leq \beta \leq \alpha} ' 2^{\beta/2} \ll \alpha^{(v-1)/2} 2^{\alpha/2}; \\ \sum_{d \leq \beta \leq \alpha} '' |\mathcal{S}_{\alpha, \beta}|^{1/2} 2^{\beta/2} &= \sum_{d \leq \beta \leq \alpha} '' |\mathcal{S}_{\alpha, \beta}|^{-1/2} |\mathcal{S}_{\alpha, \beta}|^{3/2} 2^{\beta/2} \\ &\leq \alpha^{-(v-1)/2} \sum_{d \leq \beta \leq \alpha} '' |\mathcal{S}_{\alpha, \beta}|^{3/2} 2^{\beta/2} \\ &\ll \alpha^{-(v-1)/2} \sum_{(s, \gamma') \leq \alpha} 2^{(s, 1)/2} \\ &\ll \alpha^{-(v-1)/2} 2^{\alpha/2} \alpha^{v-1} \\ &= \alpha^{(v-1)/2} 2^{\alpha/2}. \end{aligned}$$

Therefore, $\sum_{d \leq \beta \leq \alpha} |\mathcal{S}_{\alpha, \beta}|^{1/2} 2^{\beta/2} \ll \alpha^{(v-1)/2} 2^{\alpha/2}$.

Consequently, we establish that

$$\begin{aligned} I_1 &\ll \sum_{\alpha > u} 2^{-(r_1+\rho/2)\alpha} 2^{\alpha/2} \alpha^{(v-1)/2} = \sum_{\alpha > u} 2^{-(r_1+(\rho-1)/2)\alpha} \alpha^{(v-1)/2} \\ &\ll 2^{-(r_1+(\rho-1)/2)u} u^{(v-1)/2} \ll \left(N^{-1} \ln^{v-1} N\right)^{r_1+(\rho-1)/2} \left(\ln^{(v-1)/2} N\right). \end{aligned}$$

Next, we discuss I_2 :

$$\begin{aligned} I_2 &= \sum_{\alpha > u} \sum_{d \leq \beta \leq \alpha} |\mathcal{S}_{\alpha, \beta}|^{1/2-1/q} 2^{-(r_1+\rho/2)\alpha + \beta/2 - \beta/q} \sqrt{\ln(1/\delta_{\alpha, \beta})} \\ &= \sum_{\alpha > u} 2^{-(r_1+\rho/2)\alpha} \sum_{d \leq \beta \leq \alpha} |\mathcal{S}_{\alpha, \beta}|^{1/2-1/q} 2^{\beta/2 - \beta/q} \sqrt{\ln(N/\delta N_{\alpha, \beta})} \\ &\ll \sum_{\alpha > u} 2^{-(r_1+\rho/2)\alpha} \sum_{d \leq \beta \leq \alpha} |\mathcal{S}_{\alpha, \beta}|^{1/2-1/q} 2^{\beta/2 - \beta/q} \left(\sqrt{\ln(1/\delta)} + (N/N_{\alpha, \beta})^{1/2}\right) \\ &= \sum_{\alpha > u} 2^{-(r_1+\rho/2)\alpha} \sum_{d \leq \beta \leq \alpha} |\mathcal{S}_{\alpha, \beta}|^{1/2-1/q} 2^{\beta/2 - \beta/q} \sqrt{\ln(1/\delta)} \\ &\quad + \sum_{\alpha > u} 2^{-(r_1+\rho/2)\alpha} \sum_{d \leq \beta \leq \alpha} |\mathcal{S}_{\alpha, \beta}|^{1/2-1/q} 2^{\beta/2 - \beta/q} (N/N_{\alpha, \beta})^{1/2} \\ &= I_{2,1} + I_{2,2}. \end{aligned}$$

First, we discuss $I_{2,1}$, as follows:

$$I_{2,1} = \sqrt{\ln(1/\delta)} \sum_{\alpha > u} 2^{-(r_1+\rho/2)\alpha} \sum_{d \leq \beta \leq \alpha} |\mathcal{S}_{\alpha, \beta}|^{1/2-1/q} 2^{\beta/2 - \beta/q}.$$

Let

$$\begin{aligned} &\sum_{d \leq \beta \leq \alpha} |\mathcal{S}_{\alpha, \beta}|^{1/2-1/q} 2^{\beta/2 - \beta/q} \\ &= \sum_{d \leq \alpha \leq \beta} ' |\mathcal{S}_{\alpha, \beta}|^{1/2-1/q} 2^{\beta/2 - \beta/q} + \sum_{d \leq \alpha \leq \beta} '' |\mathcal{S}_{\alpha, \beta}|^{1/2-1/q} 2^{\beta/2 - \beta/q}, \end{aligned}$$

where $\sum_{d \leq \beta \leq \alpha} '$ is carried out β for $|\mathcal{S}_{\alpha, \beta}| \leq \alpha^{v-1}$ and $\sum_{d \leq \beta \leq \alpha} ''$ is carried out β for $|\mathcal{S}_{\alpha, \beta}| > \alpha^{v-1}$.

We can establish that

$$\begin{aligned} \sum_{d \leq \beta \leq \alpha} ' |\mathcal{S}_{\alpha, \beta}|^{1/2-1/q} 2^{\beta/2 - \beta/q} &\leq \sum_{d \leq \beta \leq \alpha} ' \alpha^{(1/2-1/q)(v-1)} 2^{\beta/2 - \beta/q} \\ &\ll \alpha^{(v-1)/2} \alpha^{(v-1)/q} 2^{\alpha/2} 2^{-\alpha/q}, \end{aligned}$$

and

$$\begin{aligned}
\sum_{d \leq \beta \leq \alpha} \|\mathcal{S}_{\alpha, \beta}\|^{1/2-1/q} 2^{\beta/2-\beta/q} &\leq \alpha^{-(v-1)/q} \sum_{d \leq \beta \leq \alpha} \|\mathcal{S}_{\alpha, \beta}\|^{1/2} 2^{\beta/2-\beta/q} \\
&= \alpha^{-(v-1)/q} \sum_{d \leq \beta \leq \alpha} \|\mathcal{S}_{\alpha, \beta}\|^{-1/2} \|\mathcal{S}_{\alpha, \beta}\| 2^{\beta/2-\beta/q} \\
&\leq \alpha^{-(v-1)/q} \alpha^{-(v-1)/2} \sum_{d \leq \beta \leq \alpha} \|\mathcal{S}_{\alpha, \beta}\| 2^{\beta/2-\beta/q} \\
&\ll \alpha^{-(v-1)/q} \alpha^{-(v-1)/2} \sum_{(s, \gamma') \leq \alpha} 2^{(1/2-1/q)(s,1)} \\
&\ll \alpha^{-(v-1)/q} \alpha^{-(v-1)/2} 2^{(1/2-1/q)\alpha} \alpha^{v-1} \\
&= \alpha^{(v-1)/2} \alpha^{-(v-1)/q} 2^{\alpha/2-\alpha/q}.
\end{aligned}$$

Therefore, we obtain

$$\sum_{d \leq \beta \leq \alpha} \|\mathcal{S}_{\alpha, \beta}\|^{1/2-1/q} 2^{\beta/2-\beta/q} \ll \alpha^{(v-1)/2} \alpha^{-(v-1)/q} 2^{\alpha/2-\alpha/q}.$$

Consequently, we obtain

$$\begin{aligned}
I_{2,1} &\ll \sqrt{\ln(1/\delta)} \sum_{\alpha > u} 2^{-(r_1+\rho/2)\alpha} \alpha^{(v-1)/2} \alpha^{-(v-1)/q} 2^{\alpha/2-\alpha/q} \\
&= \sqrt{\ln(1/\delta)} \sum_{\alpha > u} 2^{-(r_1+(\rho-1)/2)\alpha} \alpha^{(v-1)/2} \alpha^{-(v-1)/q} 2^{-\alpha/q} \\
&\ll \sqrt{\ln(1/\delta)} 2^{-(r_1+(\rho-1)/2)u} u^{(v-1)/2} u^{-(v-1)/q} 2^{-u/q} \\
&\ll (N^{-1} \ln^{v-1} N)^{r_1+(\rho-1)/2} (\ln^{(v-1)/2} N) N^{-1/q} \sqrt{\ln(1/\delta)}.
\end{aligned}$$

Next, we discuss $I_{2,2}$, as follows:

$$\begin{aligned}
I_{2,2} &= \sum_{\alpha > u} 2^{-(r_1+\rho/2)\alpha} \sum_{d \leq \beta \leq \alpha} \|\mathcal{S}_{\alpha, \beta}\|^{1/2-1/q} 2^{\beta/2-\beta/q} (N/N_{\alpha, \beta})^{1/2} \\
&= N^{1/2} \sum_{\alpha > u} 2^{-(r_1+\rho/2)\alpha} \sum_{d \leq \beta \leq \alpha} \|\mathcal{S}_{\alpha, \beta}\|^{1/2-1/q} 2^{\beta/2-\beta/q} N_{\alpha, \beta}^{-1/2} \\
&\ll N^{1/2} \sum_{\alpha > u} 2^{-(r_1+\rho/2)\alpha} \sum_{d \leq \beta \leq \alpha} \|\mathcal{S}_{\alpha, \beta}\|^{1/2-1/q} 2^{\beta/2-\beta/q} \left(\|\mathcal{S}_{\alpha, \beta}\| 2^{u+\theta u-2\theta l+\theta \beta} \right)^{-1/2} \\
&\ll N^{1/2} 2^{-(\theta+1)u/2} \sum_{\alpha > u} 2^{-(r_1+\rho/2)\alpha} \sum_{d \leq \beta \leq \alpha} \|\mathcal{S}_{\alpha, \beta}\|^{-1/q} 2^{\beta/2-\beta/q-\theta \beta/2} \\
&\ll N^{1/2} 2^{-(\theta+1)u/2} \sum_{\alpha > u} 2^{-(r_1+\rho/2)\alpha} \sum_{d \leq \beta \leq \alpha} 2^{\beta/2-\beta/q-\theta \beta/2} \\
&\ll N^{1/2} 2^{-(\theta+1)u/2} \sum_{\alpha > u} 2^{-(r_1+(\rho-1)/2)\alpha-\alpha/q-\theta \alpha/2} \\
&\ll N^{1/2} 2^{-(\theta+1)u/2} 2^{-(r_1+(\rho-1)/2)u-u/q-\theta u/2} \\
&\ll 2^{u/2} u^{(v-1)/2} 2^{-(\theta+1)u/2} 2^{-(r_1+(\rho-1)/2)u-u/q-\theta u/2} \\
&= u^{(v-1)/2} 2^{-(r_1+(\rho-1)/2)u} 2^{-u/q} \\
&\ll u^{(v-1)/2} 2^{-(r_1+(\rho-1)/2)u} \\
&\ll (N^{-1} \ln^{v-1} N)^{r_1+(\rho-1)/2} (\ln^{(v-1)/2} N).
\end{aligned}$$

Therefore,

$$I_2 = I_{2,1} + I_{2,2} \ll (N^{-1} \ln^{v-1} N) (\ln^{(v-1)/2} N) \left(1 + N^{-1/q} \sqrt{\ln(1/\delta)}\right).$$

In summary, we can obtain, for $1 \leq q < 2$,

$$d_\delta^N(MW_2^r(\mathbb{T}^d), \mu, L_q(\mathbb{T}^d)) \asymp (N^{-1} \ln^{v-1} N)^{r_1 + \frac{v-1}{2}} (\ln^{\frac{v-1}{2}} N) \sqrt{1 + \frac{1}{N} \ln\left(\frac{1}{\delta}\right)},$$

and, for $2 \leq q < \infty$,

$$(\ln^{(v-1)/q} N) \ll \frac{d_\delta^N(MW_2^r(\mathbb{T}^d), \mu, L_q(\mathbb{T}^d))}{(N^{-1} \ln^{v-1} N)^{r_1 + (v-1)/2} (1 + N^{-1/q} \sqrt{\ln(1/\delta)})} \ll (\ln^{(v-1)/2} N),$$

which completes the upper bound of Theorem 2. $\square$

4. Summary

In this article, we have obtained the sharp bounds of probabilistic Gel'fand (N, δ) -widths of the multivariate Sobolev space $MW_2^r(\mathbb{T}^d)$ with mixed derivative. The results of the manuscript are interesting and important, especially for information algorithms. And the proof is complete and clear. Our manuscript should play an important role in research on approximation theory and width theory.

In paper [23], we obtained the sharp bounds of probabilistic Kolmogorov and linear (N, δ) -widths, and p -average Kolmogorov and linear N -widths of multivariate Sobolev spaces $W_2^A(\mathbb{T}^d)$ with common smoothness in the S_q norm. Due to the difference in norm, the discretization method used in the calculation was different from this article. Therefore, we could study the sharp bounds of probabilistic Gel'fand (N, δ) -width and p -average Gel'fand N -widths of $W_2^A(\mathbb{T}^d)$ in S_q norm and L_q norm. The above results have subsequently been obtained.

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References

1. Pinkus, A. *n*-Widths in Approximation Theory; Springer: Berlin/Heidelberg, Germany, 1985.
2. Maiorov, V.E. Linear widths of function spaces equipped with the Gaussian measure. *J. Approx. Theory* **1994**, *77*, 74–88. [\[CrossRef\]](#)
3. Maiorov, V.E. Kolmogorov's (n, δ) -widths of the spaces of the smooth functions. *Russ. Acad. Sci. Sb. Math.* **1994**, *79*, 265–279.
4. Maiorov, E.; Wasilkowski, G.W. Probabilistic and average linear widths in L_∞ -norm with respect to r -fold Wiener measure. *J. Approx. Theory* **1996**, *84*, 31–40. [\[CrossRef\]](#)
5. Tan, X.; Wang, Y.; Sun, L.; Shao, X.; Chen, G. Gel'fand- N -width in probabilistic setting. *J. Inequalities Appl.* **2020**, *143*, 143. [\[CrossRef\]](#)
6. Liu, Y.; Li, H.; Li, X. Gel'fand widths of Sobolev classes of functions in the average setting. *Ann. Funct. Anal.* **2023**, *142*, 14–31.
7. Chen, G.; Nie, P.; Luo, X. The approximation characteristic of diagonal matrix in probabilistic setting. *J. Complex.* **2010**, *26*, 336–343.
8. Zhou, J.; Li, Y. Estimates of probabilistic widths of the diagonal operator of finite-dimensional sets with the Gaussian measure. *J. Inequalities Appl.* **2013**, *277*, 1–12. [\[CrossRef\]](#)
9. Zhai, X.; Hu, X. Probabilistic linear widths of Sobolev space with Jacobi weights on $[-1, 1]$. *J. Inequalities Appl.* **2017**, *2017*, 262. [\[CrossRef\]](#)

10. Temlyakov, V.N. Approximation of functions with bounded mixed derivate. *Tr. Mat. Inst. Akad. Nauk SSSR* **1986**, *178*, 3–113.
11. Fang, G.; Ye, P. Probabilistic and average linear widths of Sobolev spaces with Gaussian measure. *J. Complex.* **2003**, *19*, 73–84.
12. Fang, G.; Ye, P. Probabilistic and average linear widths of Sobolev spaces with Gaussian measure in L_∞ -norm. *Constr. Approx.* **2004**, *20*, 159–172.
13. Chen, G.; Fang, G. Probabilistic and average widths of multivariate Sobolev spaces with mixed derivative equipped with Gaussian measure. *J. Complex.* **2004**, *6*, 858–875.
14. Chen, G.; Fang, G. Linear widths of multivariate function spaces equipped with Gaussian measure. *J. Approx. Theory* **2005**, *132*, 77–96.
15. Wang, H. Probabilistic and average linear widths of weighted Sobolev spaces on the ball equipped with a Gaussian measure. *J. Approx. Theory* **2019**, *241*, 11–32. [[CrossRef](#)]
16. Wang, H. Probabilistic and average linear widths of Sobolev spaces on compact two-point homogeneous spaces equipped with a Gaussian measure. *Constr. Approx.* **2014**, *39*, 485–516. [[CrossRef](#)]
17. Dai, F.; Wang, H. Linear n -widths of diagonal matrices in the average and probabilistic settings. *J. Funct. Anal.* **2012**, *262*, 4103–4119. [[CrossRef](#)]
18. Vasil'eva, A.A. Kolmogorov widths of intersections of finite-dimensional balls. *J. Complex.* **2022**, *72*, 101649. [[CrossRef](#)]
19. Vasil'eva, A.A. Bounds for the Kolmogorov widths of the Sobolev weighted classes with conditions on the zero and highest derivatives. *Russ. J. Math. Phys.* **2022**, *29*, 249–279. [[CrossRef](#)]
20. Sun, Y. Average error bounds of best approximation in a Banach space with Gaussian measure. *East J. Approx.* **1995**.
21. Galeev, E.M. Kolmogorov widths of classes of periodic functions of many variables $\tilde{W}_p^\alpha$ and $\tilde{H}_p^\alpha$ in the space L_q . *Izv. Akad. Nauk SSSR Ser. Mat.* **1985**, *49*, 916–934.
22. Romanyuk, A.S. On estimate of the Kolmogorov widths of the classes $B_{p,q}^r$ in the space L_q . *Ukrainian Math. J.* **2001**, *53*, 1189–1196. [[CrossRef](#)]
23. Liu, Y.; Li, X.; Li, H. N -widths of multivariate Sobolev spaces with common smoothness in probabilistic and average setting in the S_q norm. *Axioms* **2023**, *12*, 698. [[CrossRef](#)]

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