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Further Accurate Numerical Radius Inequalities

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Abstract: The goal of this study is to refine some numerical radius inequalities in a novel way. The new improvements and refinements purify some famous inequalities pertaining to Hilbert space operators numerical radii. The inequalities that have been demonstrated in this work are not only an improvement over old inequalities but also stronger than them. Several examples supporting the validity of our results are provided as well.

Keywords: numerical radius; norm; inequalities; Hermite–Hadamard inequality

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1. Introduction

Let $\mathscr{A}(\mathscr{J})$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(\mathscr{J};\langle\cdot,\cdot\rangle)$ with the identity operator $1_{\mathscr{J}}$ in $\mathscr{A}(\mathscr{J})$. Then, for a bounded linear operator $\mathfrak{F}$ on a Hilbert space $\mathscr{J}$, the numerical range $W(\mathfrak{F})$ of a bounded operator $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$ is defined by $W(\mathfrak{F}) = \{\langle\mathfrak{F}\mu,\mu\rangle : \mu \in \mathscr{J}, \|\mu\| = 1\}$. Additionally, the numerical radius is defined to be

$$\omega(\mathfrak{F}) = \sup_{\beta\in W(\mathfrak{F})} |\beta| = \sup_{\|\mu\|=1} |\langle\mathfrak{F}\mu,\mu\rangle|.$$

We recall that the usual operator norm of an operator $\mathfrak{F}$ is defined to be

$$\|\mathfrak{F}\| = \sup\{\|\mathfrak{F}\mu\| : \mu \in \mathscr{J}, \|\mu\| = 1\}.$$

It is well known that the numerical radius $\omega(\cdot)$ defines an operator norm on $\mathscr{A}(\mathscr{J})$, which is equivalent to the operator norm $\|\cdot\|$. Moreover, we have

$$\frac{1}{2}\|\mathfrak{F}\| \le \omega(\mathfrak{F}) \le \|\mathfrak{F}\| \tag{1}$$

for any $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$.

In 2003, Kittaneh [1] refined the right-hand side of (1) by obtaining that

$$\omega(\mathfrak{F}) \le \frac{1}{2}\||\mathfrak{F}| + |\mathfrak{F}^*|\| \le \frac{1}{2}\left(\|\mathfrak{F}\| + \|\mathfrak{F}^2\|^{1/2}\right) \tag{2}$$

for any $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$. Two years later, Kittaneh [2] proved his celebrated two-sided inequality

$$\frac{1}{4}\|\mathfrak{F}^*\mathfrak{F} + \mathfrak{F}\mathfrak{F}^*\| \le \omega^2(\mathfrak{F}) \le \frac{1}{2}\|\mathfrak{F}^*\mathfrak{F} + \mathfrak{F}\mathfrak{F}^*\| \tag{3}$$

for any $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$. These inequalities are sharp.

In [3], Dragomir established an upper bound for the numerical radius of the product of two Hilbert space operators, as follows:

$$\omega^r(\mathfrak{G}^*\mathfrak{H}) \leq \frac{1}{2}\left\| |\mathfrak{H}|^{2r} + |\mathfrak{G}|^{2r} \right\| \qquad (r \geq 1). \tag{4}$$

In his recent work [4], Alomari refined the right-hand side of (3) and the recent results of Kittaneh and Moradi [5], as follows:

$$\begin{aligned}\omega^{2p}(\mathfrak{F}) &\leq \frac{1}{4}\delta\left\| |\mathfrak{F}|^{2p\delta} + |\mathfrak{F}^*|^{2p(1-\delta)} \right\|^2 + \frac{1}{2}(1-\delta)\omega^p(\mathfrak{F})\left\| |\mathfrak{F}|^{2p\delta} + |\mathfrak{F}^*|^{2p(1-\delta)} \right\| \\ &\leq \frac{1}{2}\delta\left\| |\mathfrak{F}|^{4p\delta} + |\mathfrak{F}^*|^{4p(1-\delta)} \right\| + \frac{1}{2}(1-\delta)\omega^p(\mathfrak{F})\left\| |\mathfrak{F}|^{2p\delta} + |\mathfrak{F}^*|^{2p(1-\delta)} \right\| \\ &\leq \frac{1}{2}\left\| |\mathfrak{F}|^{4p\delta} + |\mathfrak{F}^*|^{4p(1-\delta)} \right\|\end{aligned} \tag{5}$$

for any operator $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$, $p \geq 1$, and $\delta \in [0,1]$. In particular, it was shown that

$$\begin{aligned}\omega^2(\mathfrak{F}) &\leq \frac{1}{12}\| |\mathfrak{F}| + |\mathfrak{F}^*| \|^2 + \frac{1}{3}\omega(\mathfrak{F})\| |\mathfrak{F}| + |\mathfrak{F}^*| \| \\ &\leq \frac{1}{6}\left\| |\mathfrak{F}|^2 + |\mathfrak{F}^*|^2 \right\| + \frac{1}{3}\omega(\mathfrak{F})\| |\mathfrak{F}| + |\mathfrak{F}^*| \| \\ &\leq \frac{1}{4}\| |\mathfrak{F}| + |\mathfrak{F}^*| \|^2.\end{aligned} \tag{6}$$

The first inequality in (6) was proved by Alomari in [4] and the second inequality by Kittaneh and Moradi in [5].

In the same work [4], a refinement of (4) was proved, as follows:

$$\begin{aligned}\omega^{2r}(\mathfrak{G}^*\mathfrak{H}) &\leq \frac{1}{4}\delta\left\| |\mathfrak{H}|^{2r} + |\mathfrak{G}|^{2r} \right\|^2 + \frac{1}{2}(1-\delta)\omega^r(\mathfrak{H})\left\| |\mathfrak{H}|^{2r} + |\mathfrak{G}|^{2r} \right\| \\ &\leq \frac{1}{2}\delta\left\| |\mathfrak{F}|^{4r} + |\mathfrak{G}|^{4r} \right\| + \frac{1}{2}(1-\delta)\omega^r(\mathfrak{H})\left\| |\mathfrak{H}|^{2r} + |\mathfrak{G}|^{2r} \right\| \\ &\leq \frac{1}{2}\left\| |\mathfrak{H}|^{4r} + |\mathfrak{G}|^{4r} \right\|.\end{aligned} \tag{7}$$

In particular, it was shown that

$$\begin{aligned}\omega^2(\mathfrak{G}^*\mathfrak{H}) &\leq \frac{1}{12}\left\| |\mathfrak{H}|^2 + |\mathfrak{G}|^2 \right\|^2 + \frac{1}{3}\omega(\mathfrak{G}^*\mathfrak{H})\left\| |\mathfrak{H}|^2 + |\mathfrak{G}|^2 \right\| \\ &\leq \frac{1}{6}\left\| |\mathfrak{H}|^4 + |\mathfrak{G}|^4 \right\| + \frac{1}{3}\omega(\mathfrak{G}^*\mathfrak{H})\left\| |\mathfrak{H}|^2 + |\mathfrak{G}|^2 \right\| \\ &\leq \frac{1}{2}\left\| |\mathfrak{H}|^4 + |\mathfrak{G}|^4 \right\|.\end{aligned} \tag{8}$$

In [6], Sababheh and Moradi presented some new numerical radius inequalities. Among others, the well-known Hermite–Hadamard inequality was used to perform the following result.

$$\varphi(\omega(\mathfrak{F})) \leq \left\| \int_0^1 \varphi((1-s)|\mathfrak{F}| + s|\mathfrak{F}^*|)ds \right\| \leq \frac{1}{2}\| \varphi(|\mathfrak{F}|) + \varphi(|\mathfrak{F}^*|) \| \tag{9}$$

for every $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$ and increasing operator convex function $\varphi : [0,\infty) \to [0,\infty)$.

On the other hand, Moradi and Sababheh, in [7], proved the following refinement of (9):

$$\varphi(\omega(\mathfrak{F})) \leq \frac{1}{2}\left\| \varphi\left(\frac{3|\mathfrak{F}| + |\mathfrak{F}^*|}{4}\right) + \varphi\left(\frac{|\mathfrak{F}| + 3|\mathfrak{F}^*|}{4}\right) \right\| \tag{10}$$

for all increasing convex functions $\varphi : [0, \infty) \to [0, \infty)$. In particular, they proved

$$\omega^2(\mathfrak{F}) \leq \frac{1}{32}\left\| (3|\mathfrak{F}| + |\mathfrak{F}^*|)^2 + (|\mathfrak{F}| + 3|\mathfrak{F}^*|)^2 \right\|. \tag{11}$$

The constant $\frac{1}{32}$ is the best possible.

For more generalizations, counterparts, and recent related results, the reader may refer to [8–22]. Further results can be found in [23–36].

In this work, further refinements of the previously mentioned inequalities are presented. Some new improvements and refinements purify the inequalities (4)–(11). The inequalities demonstrated in this work are not only an improvement over previous inequalities but also stronger than them. We presented examples that proved these novel inequalities and demonstrated that the refinements are more precise than existing ones.

2. Refinements of the Numerical Radius Inequalities

Lemma 1 (Theorem 1.4, [37])**.** *If $\mathfrak{P} \in \mathscr{A}(\mathscr{J})^+$, then*

$$\langle \mathfrak{P}c, c\rangle^p \leq \langle \mathfrak{P}^p c, c\rangle, \qquad p \geq 1 \tag{12}$$

for any vector $c \in \mathscr{J}$. The inequality (12) is reversed if $0 \leq p \leq 1$.

Lemma 2 ([38])**.** *If $\mathfrak{G} \in \mathscr{A}(\mathscr{J})$, then*

$$|\langle \mathfrak{G}\lambda, \mu\rangle|^2 \leq \left\langle |\mathfrak{G}|^{2\eta}\lambda, \lambda \right\rangle \left\langle |\mathfrak{G}^*|^{2(1-\eta)}\mu, \mu \right\rangle, \qquad 0 \leq \eta \leq 1, \tag{13}$$

for any vectors $\lambda, \mu \in \mathscr{J}$, where $|\mathfrak{G}| = (\mathfrak{G}^*\mathfrak{G})^{1/2}$.

The following lemma is an operator version of the classical Jensen inequality.

Lemma 3 (Theorem 1.2, [37])**.** *Let $\mathfrak{G}$ be a selfadjoint operator whose spectrum $\mathfrak{G} \subset [m, M]$ for some scalars $m \leq M$. If $f(t)$ is a convex function on $[m, M]$, then*

$$\varphi(\langle \mathfrak{G}\mu, \mu\rangle) \leq \langle \varphi(\mathfrak{G})\mu, \mu\rangle \tag{14}$$

for any unit vector $\mu \in \mathscr{J}$.

We are in a position to state our first main result.

Theorem 1. *If $\varphi : [0, \infty) \to [0, \infty)$ is an increasing and convex function, then*

$$\varphi(\omega(\mathfrak{F})) \leq \frac{1}{2}\left\| \varphi\left(\frac{2|\mathfrak{F}| + |\mathfrak{F}^*|}{3}\right) + \varphi\left(\frac{|\mathfrak{F}| + 2|\mathfrak{F}^*|}{3}\right) \right\| \tag{15}$$

for any Hilbert space operator $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$.

Proof. Since φ is increasing and operator convex, then by Jensen's inequality (13), we have

$$\begin{aligned}
&\varphi(|\langle \mathfrak{F}\mu, \mu\rangle|) \\
&\leq \varphi\left(\sqrt{\langle |\mathfrak{F}|\mu, \mu\rangle \langle |\mathfrak{F}^*|\mu, \mu\rangle}\right) \\
&\leq \varphi\left(\left\langle \frac{|\mathfrak{F}| + |\mathfrak{F}^*|}{2}\mu, \mu \right\rangle\right) \\
&= \varphi\left(\frac{1}{2} \cdot \left[\left\langle \left(\frac{2|\mathfrak{F}| + |\mathfrak{F}^*|}{3}\right)\mu, \mu \right\rangle + \left\langle \left(\frac{|\mathfrak{F}| + 2|\mathfrak{F}^*|}{3}\right)\mu, \mu \right\rangle\right]\right)
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}\left[\varphi\left(\left\langle\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right)\mu,\mu\right\rangle\right)+\varphi\left(\left\langle\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\mu,\mu\right\rangle\right)\right]\\
&\le \frac{1}{2}\left[\left\langle\varphi\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right)\mu,\mu\right\rangle+\left\langle\varphi\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\mu,\mu\right\rangle\right]\\
&= \frac{1}{2}\left\langle\left[\varphi\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right)+\varphi\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\right]\mu,\mu\right\rangle.
\end{aligned}
$$

Taking the supremum over all unit vectors $\mu \in \mathscr{J}$ in all previous inequalities, we obtain the required result. □

Corollary 1. *If $\varphi : [0,\infty) \to [0,\infty)$ is increasing and convex, then*

$$\omega^p(\mathfrak{F}) \le \frac{1}{2}\left\|\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right)^p+\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)^p\right\|$$

for any Hilbert space operator $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$. In a particular case,

$$\omega^2(\mathfrak{F}) \le \frac{1}{18}\left\|(2|\mathfrak{F}|+|\mathfrak{F}^*|)^2+(|\mathfrak{F}|+2|\mathfrak{F}^*|)^2\right\|. \tag{16}$$

The constant $\frac{1}{18}$ is the best possible.

Proof. Considering $f(s) = s^p$, $s \ge 0$ $(p \ge 1)$ in (15), we obtain the desired result. The particular case in (16) follows directly by setting $p = 2$. To prove the sharpness of (16), assume that (16) holds with another constant $c > 0$, i.e.,

$$\omega^2(\mathfrak{F}) \le c\left\|(2|\mathfrak{F}|+|\mathfrak{F}^*|)^2+(|\mathfrak{F}|+2|\mathfrak{F}^*|)^2\right\|. \tag{17}$$

Assuming that $\mathfrak{F}$ is a normal operator and employing the fact that for normal operators we have $\omega(\mathfrak{F}) = \|\mathfrak{F}\|$, then by (17), we deduce that $\frac{1}{18} \le c$, and this shows that the constant $\frac{1}{18}$ is the best possible, and thus, the inequality is sharp. □

A non-trivial refinement of (15) is considered in the following result.

Theorem 2. *If $\varphi : [0,\infty) \to [0,\infty)$ is increasing and operator convex, then*

$$
\begin{aligned}
\varphi(\omega(\mathfrak{F})) &\le \left\|\int_0^1 \varphi\left((1-s)\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right)+s\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\right)ds\right\|\\
&\le \frac{1}{2}\left\|\varphi\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right)+\varphi\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\right\|\\
&\le \left\|\frac{\varphi(|\mathfrak{F}|)+\varphi(|\mathfrak{F}^*|)}{2}\right\|
\end{aligned}
\tag{18}
$$

for any Hilbert space operator $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$.

Proof. Since φ is increasing and operator convex, then by Jensen's inequality, we have

$$
\begin{aligned}
&\varphi(|\langle\mathfrak{F}\mu,\mu\rangle|)\\
&\le \varphi\left(\sqrt{\langle|\mathfrak{F}|\mu,\mu\rangle\langle|\mathfrak{F}^*|\mu,\mu\rangle}\right)\\
&\le \varphi\left(\left\langle\frac{|\mathfrak{F}|+|\mathfrak{F}^*|}{2}\mu,\mu\right\rangle\right)\\
&= \varphi\left(\frac{1}{2}\cdot\left[\left\langle\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right)\mu,\mu\right\rangle+\left\langle\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\mu,\mu\right\rangle\right]\right)
\end{aligned}
$$

$$\leq \int_0^1 \varphi\left((1-s)\left\langle\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right)\mu,\mu\right\rangle + s\left\langle\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\mu,\mu\right\rangle\right)ds.$$

On the other hand,

$$\begin{aligned}
&\int_0^1 \varphi\left((1-s)\left\langle\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right)\mu,\mu\right\rangle + s\left\langle\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\mu,\mu\right\rangle\right)ds \\
&= \int_0^1 \varphi\left(\left\langle(1-s)\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right)\mu,\mu\right\rangle + \left\langle s\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\mu,\mu\right\rangle\right)ds \\
&\leq \int_0^1 \varphi\left(\left\langle(1-s)\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right) + s\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\mu,\mu\right\rangle\right)ds \\
&\leq \left\langle\left(\int_0^1 \left((1-s)f\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right) + sf\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\right)ds\right)\mu,\mu\right\rangle \\
&\leq \frac{1}{2}\left\langle\left[\varphi\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right) + \varphi\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\right]\mu,\mu\right\rangle \\
&\leq \left\langle\frac{\varphi(|\mathfrak{F}|)+\varphi(|\mathfrak{F}^*|)}{2}\mu,\mu\right\rangle.
\end{aligned}$$

Taking the supremum over all unit vectors $\mu \in \mathscr{J}$ in all previous inequalities, we obtain the required result. □

Corollary 2. *Let $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$. Then,*

$$\begin{aligned}
\omega^p(\mathfrak{F}) &\leq \left\|\int_0^1 \left((1-s)\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right) + s\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\right)^p ds\right\| \\
&\leq \frac{1}{2}\left\|\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right)^p + \left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)^p\right\| \\
&\leq \left\|\frac{|\mathfrak{F}|^p+|\mathfrak{F}^*|^p}{2}\right\|
\end{aligned} \tag{19}$$

for all $1 \leq p \leq 2$.

Proof. The result follows by applying the increasing operator function $\varphi(t) = t^p, 1 \leq p \leq 2$, to inequality (18). □

Corollary 3. *Let $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$. Then,*

$$\begin{aligned}
\omega^2(\mathfrak{F}) &\leq \left\|\int_0^1 \left((1-s)\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right) + s\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\right)^2 ds\right\| \\
&\leq \frac{1}{18}\left\|(2|\mathfrak{F}|+|\mathfrak{F}^*|)^2 + (|\mathfrak{F}|+2|\mathfrak{F}^*|)^2\right\| \\
&\leq \frac{1}{2}\left\||\mathfrak{F}|^2+|\mathfrak{F}^*|^2\right\|.
\end{aligned} \tag{20}$$

Proof. Set $p = 2$ in (19). □

Example 1. *Consider $\mathfrak{F} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$. It is easy to observe that $\omega(\mathfrak{F}) = 1.5$. Applying the inequalities in (20), we obtain*

$$2.25 = \omega^2(\mathfrak{F}) \leq \left\|\int_0^1 \left((1-s)\left(\frac{2|\mathfrak{F}|+|\mathfrak{F}^*|}{3}\right) + s\left(\frac{|\mathfrak{F}|+2|\mathfrak{F}^*|}{3}\right)\right)^2 ds\right\| = 2.25$$

$$\leq \frac{1}{18}\left\|(2|\mathfrak{F}|+|\mathfrak{F}^*|)^2+(|\mathfrak{F}|+2|\mathfrak{F}^*|)^2\right\| = 2.27$$
$$\leq \frac{1}{2}\left\||\mathfrak{F}|^2+|\mathfrak{F}^*|^2\right\| = 2.5.$$

As we can see, the first inequality turns into an equality in this example and gives the exact value of the numerical radius. Moreover, the second inequality improves the Sabaheh–Mordai inequality (11). Indeed, applying (11), we obtain

$$2.25 = \omega^2(\mathfrak{F}) \leq \frac{1}{32}\left\|(3|\mathfrak{F}|+|\mathfrak{F}^*|)^2+(|\mathfrak{F}|+3|\mathfrak{F}^*|)^2\right\| = 2.3125,$$

and this shows that our first two inequalities are much better than (11). Practically and more preciously, the first two inequalities in (20) are stronger than the upper bound in (3), and the inequalities in (9), (10), and (11).

Theorem 3. *If $\varphi : [0,\infty) \to [0,\infty)$ is an increasing and operator convex, then*

$$\begin{aligned}\varphi(\omega(\mathfrak{G}^*\mathfrak{H})) &\leq \left\|\int_0^1 \varphi\left((1-s)\left(\frac{2|\mathfrak{H}|^2+|\mathfrak{G}|^2}{3}\right)+s\left(\frac{|\mathfrak{H}|^2+2|\mathfrak{G}|^2}{3}\right)\right)ds\right\| \\ &\leq \frac{1}{2}\left\|\varphi\left(\frac{2|\mathfrak{H}|^2+|\mathfrak{G}|^2}{3}\right)+\varphi\left(\frac{|\mathfrak{H}|^2+2|\mathfrak{G}|^2}{3}\right)\right\| \\ &\leq \left\|\frac{\varphi\left(|\mathfrak{H}|^2\right)+\varphi\left(|\mathfrak{G}|^2\right)}{2}\right\|\end{aligned} \tag{21}$$

for any two operators $\mathfrak{H},\mathfrak{G} \in \mathscr{A}(\mathscr{J})$.

Proof. Let $\mu \in \mathscr{J}$ be a unit vector. Then, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\varphi(|\langle\mathfrak{G}^*\mathfrak{H}\mu,\mu\rangle|) &= \varphi(|\langle\mathfrak{H}\mu,\mathfrak{G}\mu\rangle|) \\ &\leq \varphi(\|\mathfrak{H}\mu\|\|\mathfrak{G}\mu\|) \\ &= \varphi\left(\left\langle|\mathfrak{H}|^2\mu,\mu\right\rangle^{\frac{1}{2}}\left\langle|\mathfrak{G}|^2\mu,\mu\right\rangle^{\frac{1}{2}}\right) \\ &\leq \varphi\left(\frac{\left\langle|\mathfrak{H}|^2\mu,\mu\right\rangle+\left\langle|\mathfrak{G}|^2\mu,\mu\right\rangle}{2}\right) \qquad \text{(by AM-GM inequality).}\end{aligned}$$

The rest of the proof is typically similar to that given in the proof of Theorem 1; by replacing $|\mathfrak{F}|$ and $|\mathfrak{F}^*|$ by $|\mathfrak{H}|^2$ and $|\mathfrak{G}|^2$, respectively, we obtain the required result. □

We finish this work by introducing some refined improvements to numerical radius inequalities. Among others, Sababheh and Moradi in [6,7], presented some new general forms of numerical radius inequalities for Hilbert space operators. In fact, Sababheh and Moradi used the classical Hermite–Hadamard inequality and its operator version to prove their results. We refine and extend these inequalities in tlight of the Alomari refinement extension of the Hermite–Hadamard inequality [39].

Theorem 4. *Let $\Psi : \mathscr{A}(\mathscr{J}) \to \mathscr{A}(\mathscr{R})$ be a positive unital linear map and $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$. If $\varphi : [0,\infty) \to [0,\infty)$ is an increasing and convex function, then*

$$\begin{aligned}&\varphi\left(\omega^2(\Psi(\mathfrak{F}))\right) \\ &\leq \varphi\left(\frac{1}{2}\left\|\Psi\left(|\mathfrak{F}|^2+|\mathfrak{F}^*|^2\right)\right\|\right)\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}\left[\varphi\left(\left\|\Psi\left(\frac{3|\mathfrak{F}|^2+|\mathfrak{F}^*|^2}{4}\right)\right\|\right)+\varphi\left(\left\|\Psi\left(\frac{|\mathfrak{F}|^2+3|\mathfrak{F}^*|^2}{4}\right)\right\|\right)\right]\\
&\le \sup_{\substack{\mu\in\mathscr{J}\\ \|\mu\|=1}}\int_0^1\varphi\left(\left\|\Psi^{1/2}\left((1-t)|\mathfrak{F}|^2+t|\mathfrak{F}^*|^2\right)\mu\right\|^2\right)dt \qquad (22)\\
&\le \frac{1}{2}\left[\varphi\left(\left\|\Psi\left(\frac{|\mathfrak{F}|^2+|\mathfrak{F}^*|^2}{2}\right)\right\|\right)+\frac{1}{2}\left\|\Psi\left(\varphi\left(|\mathfrak{F}|^2\right)+\varphi\left(|\mathfrak{F}^*|^2\right)\right)\right\|\right]\\
&\le \frac{1}{2}\left\|\Psi\left(\varphi\left(|\mathfrak{F}|^2\right)+\varphi\left(|\mathfrak{F}^*|^2\right)\right)\right\|
\end{aligned}
$$

for any unit vector $\mu\in\mathscr{J}$.

Proof. In [40], Alomari proved the following refinement of the classical Hermite–Hadamard inequality:

$$
\begin{aligned}
\frac{(b-a)}{2}\left[g\left(\frac{3a+b}{4}\right)+g\left(\frac{a+3b}{4}\right)\right] &\le \int_a^b g(t)dt \qquad (23)\\
&\le \frac{(b-a)}{2}\left[g\left(\frac{a+b}{2}\right)+\frac{g(a)+g(b)}{2}\right]
\end{aligned}
$$

for every convex function $g:[a,b]\to\mathbb{R}$. Moreover, since g is convex, then we may rewrite (23), as follows

$$
\begin{aligned}
g\left(\frac{a+b}{2}\right)=g\left(\frac{1}{2}\left[\frac{3a+b}{4}+\frac{a+3b}{4}\right]\right) &\le \frac{1}{2}\left[g\left(\frac{3a+b}{4}\right)+g\left(\frac{a+3b}{4}\right)\right]\\
&\le \int_0^1 g((1-t)a+tb)dt \qquad (24)\\
&\le \frac{1}{2}\left[g\left(\frac{a+b}{2}\right)+\frac{g(a)+\varphi(b)}{2}\right]\\
&\le \frac{g(a)+g(b)}{2}.
\end{aligned}
$$

Let $\mathfrak{F}=K+iL$ be the Cartesian decomposition of $\mathfrak{F}\in\mathscr{A}(\mathscr{J})$. Then, we have

$$
|\mathfrak{F}|^2+|\mathfrak{F}^*|^2=\mathfrak{F}^*\mathfrak{F}+\mathfrak{F}\mathfrak{F}^*=2\left(K^2+L^2\right), \qquad (25)
$$

and

$$
|\langle\mathfrak{F}\mu,\mu\rangle|^2=\langle K\mu,\mu\rangle+\langle L\mu,\mu\rangle, \qquad \forall\mu\in\mathscr{J}. \qquad (26)
$$

Replacing a and b with $\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle$ and $\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle$ in (24), for $\mu\in\mathscr{J}$ such that $\|\mu\|=1$, we obtain

$$
\begin{aligned}
&\varphi\left(\frac{\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle+\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle}{2}\right)\\
&\le \frac{1}{2}\left[\varphi\left(\frac{3\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle+\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle}{4}\right)+\varphi\left(\frac{\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle+3\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle}{4}\right)\right]\\
&\le \int_0^1\varphi\left(\left\langle\Psi\left((1-t)|\mathfrak{F}|^2+t|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle\right)dt
\end{aligned}
$$

$$\leq \frac{1}{2}\left[\varphi\left(\frac{\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle+\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle}{2}\right)+\frac{\varphi\left(\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle\right)+\varphi\left(\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle\right)}{2}\right]$$
$$\leq \frac{\varphi\left(\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle\right)+\varphi\left(\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle\right)}{2}.$$

However, since φ is convex and Ψ is a positive unital linear map, the last two inequalities can be refined, respectively, as follows:

$$\frac{1}{2}\left[\varphi\left(\frac{\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle+\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle}{2}\right)+\frac{\varphi\left(\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle\right)+\varphi\left(\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle\right)}{2}\right]$$
$$\leq \frac{1}{2}\left[\varphi\left(\frac{\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle+\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle}{2}\right)+\frac{\left\langle\Psi\left(\varphi\left(|\mathfrak{F}|^2\right)\right)\mu,\mu\right\rangle+\left\langle\Psi\left(\varphi\left(|\mathfrak{F}^*|^2\right)\right)\mu,\mu\right\rangle}{2}\right],$$

and

$$\begin{aligned}
&\frac{\varphi\left(\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle\right)+\varphi\left(\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle\right)}{2}\\
&\leq \frac{\left\langle\Psi\left(\varphi\left(|\mathfrak{F}|^2\right)\right)\mu,\mu\right\rangle+\left\langle\Psi\left(\varphi\left(|\mathfrak{F}^*|^2\right)\right)\mu,\mu\right\rangle}{2}\\
&= \frac{1}{2}\left\langle\Psi\left(\varphi\left(|\mathfrak{F}|^2\right)+\varphi\left(|\mathfrak{F}^*|^2\right)\right)\mu,\mu\right\rangle.
\end{aligned}$$

Combining the above two inequalities together, we obtain

$$\begin{aligned}
&\sup_{\substack{\mu\in\mathscr{J}\\ \|\mu\|=1}} \int_0^1 \varphi\left(\left\|\Psi^{1/2}\left((1-t)|\mathfrak{F}|^2+t|\mathfrak{F}^*|^2\right)\mu\right\|^2\right)dt\\
&\leq \frac{1}{2}\left[\varphi\left(\left\|\Psi\left(\frac{|\mathfrak{F}|^2+|\mathfrak{F}^*|^2}{2}\right)\right\|\right)+\frac{1}{2}\left\|\Psi\left(\varphi\left(|\mathfrak{F}|^2\right)+\varphi\left(|\mathfrak{F}^*|^2\right)\right)\right\|\right]\\
&\leq \frac{1}{2}\left\|\Psi\left(\varphi\left(|\mathfrak{F}|^2\right)+\varphi\left(|\mathfrak{F}^*|^2\right)\right)\right\|.
\end{aligned}$$

Now, since φ is increasing, we have

$$\begin{aligned}
\varphi\left(|\langle\Psi(\mathfrak{F})\mu,\mu\rangle|^2\right) &= \varphi\left(\langle\Psi(K)\mu,\mu\rangle^2+\langle\Psi(L)\mu,\mu\rangle^2\right)\\
&\leq \varphi\left(\left\langle\Psi^2(K)\mu,\mu\right\rangle+\left\langle\Psi^2(L)\mu,\mu\right\rangle\right)\\
&= \varphi\left(\left\langle\Psi\left(K^2+L^2\right)\mu,\mu\right\rangle\right)\\
&= \varphi\left(\frac{\left\langle\Psi\left(|\mathfrak{F}|^2+|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle}{2}\right)\\
&= \varphi\left(\frac{\left\langle\Psi\left(|\mathfrak{F}|^2\right)\mu,\mu\right\rangle+\left\langle\Psi\left(|\mathfrak{F}^*|^2\right)\mu,\mu\right\rangle}{2}\right).
\end{aligned}$$

Taking the supremum over all unit vectors $\mu\in\mathscr{J}$ in all previous inequalities, we obtain the required result. □

The following example ensures that the inequalities in (22) refine the Sababheh–Moradi inequality (Theorem 2.2, [6]).

Example 2. *Consider* $\mathfrak{F} = \begin{bmatrix} 4 & 2 \\ 0 & 1 \end{bmatrix}$, $\varphi(t) = t^2$ $(t \geq 0)$, *and the unital positive linear map* $\Psi : \mathfrak{M}_2(\mathbb{C}) \to \mathfrak{M}_2(\mathbb{C})$ *given by* $\Psi(\mathfrak{F}) = \frac{1}{2}(\mathrm{tr}(\mathfrak{F}))I$ *for all matrices* $\mathfrak{F} \in \mathfrak{M}_2(\mathbb{C})$. *Employing* (22), *we obtain*

$$\begin{aligned}
&\omega^4(\Psi(\mathfrak{F})) = 39.0625 \\
&\leq \frac{1}{4}\left\|\Psi\left(|\mathfrak{F}|^2 + |\mathfrak{F}^*|^2\right)\right\|^2 = 56.25 \\
&\leq \frac{1}{2}\left[\left\|\Psi\left(\frac{3|\mathfrak{F}|^2 + |\mathfrak{F}^*|^2}{4}\right)\right\|^2 + \left\|\Psi\left(\frac{|\mathfrak{F}|^2 + 3|\mathfrak{F}^*|^2}{4}\right)\right\|^2\right] = 175.78 \\
&\leq \sup_{\substack{\mu \in \mathscr{J} \\ \|\mu\|=1}} \int_0^1 \left\|\Psi^{1/2}\left((1-t)|\mathfrak{F}|^2 + t|\mathfrak{F}^*|^2\right)\mu\right\|^4 dt \\
&\leq \frac{1}{8}\left\|\Psi\left(|\mathfrak{F}|^2 + |\mathfrak{F}^*|^2\right)\right\|^2 + \frac{1}{4}\left\|\Psi\left(|\mathfrak{F}|^4 + |\mathfrak{F}^*|^4\right)\right\| = 251.125 \\
&\leq \frac{1}{2}\left\|\Psi\left(|\mathfrak{F}|^4 + |\mathfrak{F}^*|^4\right)\right\| = 446.
\end{aligned}$$

The following result gives an extensive alternative proof of (Theorem 2.2, [6]). The approach presented in the proof is completely different and motivated by the concept of the Cartesian decomposition of an arbitrary Hilbert space operator. At the same time, a chain of inequalities improves the result in [6] and refines the lower bound of the celebrated Kittaneh inequality (3).

Theorem 5. *Let* $K + iL$ *be the Cartesian decomposition of an operator* $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$. *If* $\varphi : [0, \infty) \to [0, \infty)$ *is a non-negative increasing operator convex function, then we have*

$$\begin{aligned}
\varphi\left(\omega^2(S)\right) &\geq \frac{\varphi\left(\|K\|^2\right) + \varphi\left(\|L\|^2\right)}{2} \\
&\geq \frac{\varphi(\|K^2\|) + \varphi(\|L^2\|)}{2} \\
&\geq \int_0^1 \left\|\delta\varphi\left(K^2\right) + (1-\delta)\varphi\left(L^2\right)\right\| d\delta \\
&\geq \int_0^1 \left\|\varphi\left(\delta K^2 + (1-\delta)L^2\right)\right\| d\delta \\
&\geq \left\|\int_0^1 \varphi\left(\delta K^2 + (1-\delta)L^2\right) d\delta\right\| \\
&\geq \left\|\varphi\left(\frac{S^*S + SS^*}{4}\right)\right\|.
\end{aligned} \tag{27}$$

Proof. Since $\mathfrak{F} = K + iL$, then we have

$$|\langle \mathfrak{F}\mu, \mu\rangle|^2 = \langle K\mu, \mu\rangle^2 + \langle L\mu, \mu\rangle^2, \qquad \mu \in \mathscr{J}.$$

The monotonicity of φ and the above identity imply that

$$\delta\varphi\left(|\langle \mathfrak{F}\mu, \mu\rangle|^2\right) \geq \delta\varphi\left(\langle K\mu, \mu\rangle^2\right),$$

and

$$(1-\delta)\varphi\left(|\langle \mathfrak{F}\mu, \mu\rangle|^2\right) \geq (1-\delta)\varphi\left(\langle L\mu, \mu\rangle^2\right)$$

for all $\delta \in [0,1]$. Therefore,

$$\begin{aligned}\varphi\left(|\langle \mathfrak{F}\mu,\mu\rangle|^2\right) &= \delta\varphi\left(|\langle \mathfrak{F}\mu,\mu\rangle|^2\right) + (1-\delta)\varphi\left(|\langle \mathfrak{F}\mu,\mu\rangle|^2\right)\\ &\geq \delta\varphi\left(\langle K\mu,\mu\rangle^2\right) + (1-\delta)\varphi\left(\langle L\mu,\mu\rangle^2\right).\end{aligned}$$

Taking the supremum over all unit vectors $\mu \in \mathscr{J}$, since φ is increasing, we obtain

$$\begin{aligned}\varphi\left(\omega^2(\mathfrak{F})\right) &\geq \delta\varphi\left(\|K\|^2\right) + (1-\delta)\varphi\left(\|L\|^2\right)\\ &\geq \delta\varphi\left(\left\|K^2\right\|\right) + (1-\delta)\varphi\left(\left\|L^2\right\|\right) \qquad (\text{since } \|\mathfrak{F}^2\| \leq \|\mathfrak{F}\|^2, \text{ for all } \mathfrak{F} \in \mathscr{A}(\mathscr{J}))\\ &= \delta\left\|\varphi\left(K^2\right)\right\| + (1-\delta)\left\|\varphi\left(L^2\right)\right\| \qquad (\text{since } \varphi(\|\mathfrak{F}\|) = \|\varphi(|\mathfrak{F}|)\|)\\ &\geq \left\|\delta\varphi\left(K^2\right) + (1-\delta)\varphi\left(L^2\right)\right\| \qquad (\text{by triangle inequality})\\ &\geq \left\|\varphi\left(\delta K^2 + (1-\delta)L^2\right)\right\| \qquad (\varphi \text{ is operator convex}).\end{aligned}$$

Integrating with respect to δ over $[0,1]$, we have

$$\begin{aligned}\varphi\left(\omega^2(\mathfrak{F})\right) &\geq \frac{\varphi\left(\|K\|^2\right) + \varphi\left(\|L\|^2\right)}{2}\\ &\geq \frac{\varphi(\|K^2\|) + \varphi(\|L^2\|)}{2}\\ &= \frac{\|\varphi(K^2)\| + \|\varphi(L^2)\|}{2}\\ &\geq \int_0^1 \left\|\delta\varphi\left(K^2\right) + (1-\delta)\varphi\left(L^2\right)\right\| d\delta\\ &\geq \int_0^1 \left\|\varphi\left(\delta K^2 + (1-\delta)L^2\right)\right\| d\delta\\ &\geq \left\|\int_0^1 \varphi\left(\delta K^2 + (1-\delta)L^2\right) d\delta\right\| \qquad (\text{by triangle inequality})\\ &\geq \left\|\varphi\left(\frac{K^2+L^2}{2}\right)\right\| \qquad (\varphi \text{ is operator convex})\\ &= \left\|\varphi\left(\frac{\mathfrak{F}^*\mathfrak{F} + \mathfrak{F}\mathfrak{F}^*}{4}\right)\right\|,\end{aligned}$$

and this proves the required result. □

The following result refines (27) and gives a better estimate of the numerical radius.

Theorem 6. *Let $K + iL$ be the Cartesian decomposition of an operator $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$. If $\varphi : [0,\infty) \to [0,\infty)$ is a non-negative increasing operator convex function, then*

$$\varphi\left(\omega^2(\mathfrak{F})\right) \geq \frac{r}{r+s}\varphi\left(\|K\|^2\right) + \frac{s}{r+s}\varphi\left(\|L\|^2\right) \geq \left\|\varphi\left(\frac{rK^2 + sL^2}{r+s}\right)\right\|, \tag{28}$$

for all real numbers $r, s > 0$.

Proof. Our proof is similar to that presented in the proof of Theorem 5. Let $r, s > 0$, and $\mathfrak{F} = K + iL$. Then,

$$|\langle \mathfrak{F}\mu,\mu\rangle|^2 = \langle K\mu,\mu\rangle^2 + \langle L\mu,\mu\rangle^2, \qquad \mu \in \mathscr{J}.$$

The monotonicity of φ and the above identity imply that

$$\frac{r}{r+s}\varphi\Big(|\langle\mathfrak{F}\mu,\mu\rangle|^2\Big) \geq \frac{r}{r+s}\varphi\Big(\langle K\mu,\mu\rangle^2\Big),$$

and

$$\frac{s}{r+s}\varphi\Big(|\langle\mathfrak{F}\mu,\mu\rangle|^2\Big) \geq \frac{s}{r+s}\varphi\Big(\langle L\mu,\mu\rangle^2\Big)$$

for all positive real numbers $r, s > 0$. Therefore,

$$\begin{aligned}\varphi\Big(|\langle\mathfrak{F}\mu,\mu\rangle|^2\Big) &= \frac{r}{r+s}\varphi\Big(|\langle\mathfrak{F}\mu,\mu\rangle|^2\Big) + \frac{s}{r+s}\varphi\Big(|\langle\mathfrak{F}\mu,\mu\rangle|^2\Big) \\ &\geq \frac{r}{r+s}\varphi\Big(\langle K\mu,\mu\rangle^2\Big) + \frac{s}{r+s}\varphi\Big(\langle L\mu,\mu\rangle^2\Big).\end{aligned}$$

Taking the supremum over all unit vector $\mu \in \mathscr{J}$, since φ is increasing, we obtain

$$\begin{aligned}\varphi\Big(\omega^2(\mathfrak{F})\Big) &\geq \frac{r}{r+s}\varphi\Big(\|K\|^2\Big) + \frac{s}{r+s}\varphi\Big(\|L\|^2\Big) \\ &\geq \frac{r}{r+s}\varphi\Big(\Big\|K^2\Big\|\Big) + \frac{s}{r+s}\varphi\Big(\Big\|L^2\Big\|\Big) && (\text{since } \|\mathfrak{F}^2\| \leq \|\mathfrak{F}\|^2, \text{ for all } \mathfrak{F} \in \mathscr{A}(\mathscr{J})) \\ &= \frac{r}{r+s}\Big\|\varphi\Big(K^2\Big)\Big\| + \frac{s}{r+s}\Big\|\varphi\Big(L^2\Big)\Big\| && (\text{since } \varphi(\|\mathfrak{F}\|) = \|\varphi(|\mathfrak{F}|)\|) \\ &\geq \left\|\frac{r}{r+s}\varphi\Big(K^2\Big) + \frac{s}{r+s}\varphi\Big(L^2\Big)\right\| && (\text{by triangle inequality}) \\ &\geq \left\|\varphi\left(\frac{rK^2+sL^2}{r+s}\right)\right\| && (\varphi \text{ is operator convex}),\end{aligned}$$

which yields the desired result. □

Example 3. *Consider* $\mathfrak{F} = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$. *It is easy to observe that* $\omega(\mathfrak{F}) = 6$. *Then, define the function* $\varphi(t) = t^2$ $(t \geq 0)$. *By applying the first inequality in* (28) *(which is the same result given in* (Theorem 2.2, [6]), *we have that* $\omega(\mathfrak{F}) \geq 5.04635$ *(the case when* $r = s = 1$*)*

$$\omega(\mathfrak{F}) = 6 \geq \begin{cases} 5.04635, & \text{if } \ r=1,\ s=1; \\ 5.42213, & \text{if } \ r=2,\ s=1; \\ 5.84414, & \text{if } \ r=9,\ s=1; \\ 5.97039, & \text{if } \ r=50,\ s=1; \\ 5.99850, & \text{if } \ r=1000,\ s=1. \end{cases}$$

It is possible to improve estimations by changing the values of r and s. In this example, since r is greater than s, we obtain a better estimate (lower bound), which improves the Mordai–Sabaheh inequality (the case of $r = s = 1$*). Generally, once the values of* $\|K\|$ *and r are large (small) enough, and once the values of* $\|L\|$ *and s are small (large), we obtain better estimates. In light of this, it is convenient to note that* (30) *is always an accurate lower bound. This suggests that the Mordai–Sabaheh inequality can be improved by finding the appropriate values of r and s. In practice, this can be performed via numerical optimization.*

In [7], Moradi and Sabaheh used the interesting inequality

$$\left(\frac{\mathfrak{H}+\mathfrak{G}}{2}\right)^2 \leq \left(\frac{\mathfrak{H}+\mathfrak{G}}{2}\right)^2 + \left(\frac{|\mathfrak{H}-\mathfrak{G}|}{2}\right)^2 = \frac{\mathfrak{H}^2+\mathfrak{G}^2}{2} \tag{29}$$

for every self-adjoint operator $\mathfrak{H}, \mathfrak{G} \in \mathscr{A}(\mathscr{J})$ to prove the following refinement of the left-hand side of (3), as follows:

$$\frac{1}{4}\|\mathfrak{F}^*\mathfrak{F} + \mathfrak{F}\mathfrak{F}^*\| \leq \frac{1}{4}\left\|(\mathfrak{F}^*\mathfrak{F} + \mathfrak{F}\mathfrak{F}^*)^2 + \left|\mathfrak{F}^2 + (\mathfrak{F}^*)^2\right|^2\right\|^{\frac{1}{2}} \leq \omega^2(\mathfrak{F}). \tag{30}$$

By recalling the original result in [7], an interesting improvement to (30) holds. Namely, we have

$$\begin{aligned}\frac{1}{4}\|\mathfrak{F}^*\mathfrak{F} + \mathfrak{F}\mathfrak{F}^*\| &\leq \frac{1}{4}\left\|(\mathfrak{F}^*\mathfrak{F} + \mathfrak{F}\mathfrak{F}^*)^2 + \left(\mathfrak{F}^2 + (\mathfrak{F}^*)^2\right)^2\right\|^{\frac{1}{2}} \\ &\leq \frac{1}{4\sqrt{2}}\left(\|\mathfrak{F} + \mathfrak{F}^*\|^4 + \|\mathfrak{F} - \mathfrak{F}^*\|^4\right)^{\frac{1}{2}} \\ &\leq \omega^2(\mathfrak{F}).\end{aligned} \tag{31}$$

The next result extends and refines inequality (31) as follows:

Theorem 7. *Let $K + iL$ be the Cartesian decomposition of $\mathfrak{F} \in \mathscr{A}(\mathscr{J})$. Then,*

$$\begin{aligned}&\frac{1}{4}\left\|\left(\frac{r-s}{r+s}\right) \cdot \left(\mathfrak{F}^2 + (\mathfrak{F}^*)^2\right) + (\mathfrak{F}\mathfrak{F}^* + \mathfrak{F}^*\mathfrak{F})\right\| \\ &\leq \frac{1}{4}\left\|\left[\left(\frac{r-s}{r+s}\right) \cdot \left(\mathfrak{F}^2 + (\mathfrak{F}^*)^2\right) + (\mathfrak{F}\mathfrak{F}^* + \mathfrak{F}^*\mathfrak{F})\right]^2\right. \\ &\qquad\left. + \left[\left(\mathfrak{F}^2 + (\mathfrak{F}^*)^2\right) + \left(\frac{r-s}{r+s}\right) \cdot (\mathfrak{F}\mathfrak{F}^* + \mathfrak{F}^*\mathfrak{F})\right]^2\right\|^{\frac{1}{2}} \\ &\leq \frac{1}{2\sqrt{2}} \cdot \left(\frac{r^2\|\mathfrak{F} + \mathfrak{F}^*\|^4 + s^2\|\mathfrak{F} - \mathfrak{F}^*\|^4}{(r+s)^2}\right)^{\frac{1}{2}} \\ &\leq \omega^2(\mathfrak{F}),\end{aligned} \tag{32}$$

for all positive real numbers r, s.

Proof. Since $K + iL$ is the Cartesian decomposition of $\mathfrak{F}$, then for all real numbers $r, s > 0$, we obtain

$$\frac{rK^2 + sL^2}{r+s} = \left(\frac{r-s}{r+s}\right) \cdot \frac{\mathfrak{F}^2 + (\mathfrak{F}^*)^2}{4} + \frac{\mathfrak{F}\mathfrak{F}^* + \mathfrak{F}^*\mathfrak{F}}{4},$$

and

$$\frac{rK^2 - sL^2}{r+s} = \frac{\mathfrak{F}^2 + (\mathfrak{F}^*)^2}{4} + \left(\frac{r-s}{r+s}\right) \cdot \frac{\mathfrak{F}\mathfrak{F}^* + \mathfrak{F}^*\mathfrak{F}}{4}.$$

Replacing $\mathfrak{H}$ and $\mathfrak{G}$ by $\frac{2r}{r+s}K^2$ and $\frac{2s}{r+s}L^2$ $(\forall r, s > 0)$, respectively, in (29), we obtain

$$\begin{aligned}\left(\frac{rK^2 + sL^2}{r+s}\right)^2 &\leq \left(\frac{rK^2 + sL^2}{r+s}\right)^2 + \left(\frac{\left|rK^2 - sL^2\right|}{r+s}\right)^2 \\ &= \frac{2r^2K^4 + 2s^2L^4}{(r+s)^2}.\end{aligned}$$

Consequently,

$$\begin{aligned}
\left\| \left(\frac{r-s}{r+s}\right) \cdot \frac{\mathfrak{F}^2 + (\mathfrak{F}^*)^2}{4} + \frac{\mathfrak{F}\mathfrak{F}^* + \mathfrak{F}^*\mathfrak{F}}{4} \right\|^2 &= \left\| \frac{rK^2 + sL^2}{r+s} \right\|^2 \\
&= \left\| \left(\frac{rK^2 + sL^2}{r+s} \right)^2 \right\| \\
&\leq \left\| \left(\frac{rK^2 + sL^2}{r+s} \right)^2 + \left(\frac{|rK^2 - sL^2|}{r+s} \right)^2 \right\| \\
&= \left\| \frac{2r^2K^4 + 2s^2L^4}{(r+s)^2} \right\| \\
&\leq \frac{2r^2\|K\|^4 + 2s^2\|L\|^4}{(r+s)^2} \\
&\leq \omega^4(\mathfrak{F}),
\end{aligned}$$

which gives the desired result in (32). □

Remark 1. *In particular, choosing $r = s$ in* (32)*, we can refer to* (31)*.*

Remark 2. *However,* (32) *still offers a better estimate than* (31)*. By choosing specific values for r and s, we would then obtain a better lower bound. To verify this assertion, consider Example 3. We leave the investigation of this note to the interested reader. Nevertheless, once the values of $\|K\|$ and r are large (small) enough and the values of $\|L\|$ and s are small (large) enough, we obtain a better estimation than* (31)*.*

3. Conclusions

This study refines several well-known and sharp numerical radius inequalities obtained in the literature using more accurate numerical radius inequalities. Namely, as shown, inequality (12) refines the Sababheh–Moradi inequality (9). In fact, (16) is sharper than both (14) and (11). An extensive alternative proof of (Theorem 2.2, [6]) is also provided. Among other inequalities, two interesting and novel results are established. Namely, it is shown that

$$\varphi\left(\omega^2(\mathfrak{F})\right) \geq \frac{r}{r+s}\varphi\left(\|K\|^2\right) + \frac{s}{r+s}\varphi\left(\|L\|^2\right) \geq \left\| \varphi\left(\frac{rK^2 + sL^2}{r+s} \right) \right\|$$

for every increasing operator convex function φ and all real numbers $r, s > 0$. Additionally,

$$\begin{aligned}
&\frac{1}{4}\left\| \left(\frac{r-s}{r+s}\right) \cdot \left(\mathfrak{F}^2 + (\mathfrak{F}^*)^2\right) + (\mathfrak{F}\mathfrak{F}^* + \mathfrak{F}^*\mathfrak{F}) \right\| \\
&\leq \frac{1}{4}\left\| \left[\left(\frac{r-s}{r+s}\right) \cdot \left(\mathfrak{F}^2 + (\mathfrak{F}^*)^2\right) + (\mathfrak{F}\mathfrak{F}^* + \mathfrak{F}^*\mathfrak{F}) \right]^2 \right. \\
&\qquad\qquad \left. + \left[\left(\mathfrak{F}^2 + (\mathfrak{F}^*)^2\right) + \left(\frac{r-s}{r+s}\right) \cdot (\mathfrak{F}\mathfrak{F}^* + \mathfrak{F}^*\mathfrak{F}) \right]^2 \right\|^{\frac{1}{2}} \\
&\leq \frac{1}{2\sqrt{2}} \cdot \left(\frac{r^2\|\mathfrak{F} + \mathfrak{F}^*\|^4 + s^2\|\mathfrak{F} - \mathfrak{F}^*\|^4}{(r+s)^2} \right)^{\frac{1}{2}} \\
&\leq \omega^2(\mathfrak{F}),
\end{aligned}$$

is valid for all $r, s > 0$.

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References

1. Kittaneh, F. Numerical radius inequalities for Hilbert space operators. *Studia Math.* **2005**, *168*, 73–80. [CrossRef]
2. Kittaneh, F. A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix. *Studia Math.* **2003**, *158*, 11–17. [CrossRef]
3. Dragomir, S.S. Power inequalities for the numerical radius of a product of two operators in Hilbert spaces. *Sarajevo J. Math.* **2009**, *5*, 269–278.
4. Alomari, M.W. On Cauchy-Schwarz type inequalities and applications to numerical radius inequalities. *Ricerche Mat.* **2022**. [CrossRef]
5. Kittaneh, F.; Moradi, H.R. Cauchy-Schwarz type inequalities and applications to numerical radius inequalities. *Math. Ineq. Appl.* **2020**, *23*, 1117–1125. [CrossRef]
6. Sababheh, M.; Moradi, H.R. More accurate numerical radius inequalities (I). *Linear Multilinear Algebra* **2021**, *69*, 1964–1973. [CrossRef]
7. Moradi, H.R.; Sababheh, M. More accurate numerical radius inequalities (II). *Linear Multilinear Algebra* **2021**, *69*, 921–933. [CrossRef]
8. Burqan, A.; Alkhalely, S.; Conde, C. Some generalizations of numerical radius inequalities including the off-diagonal parts of block matrices. *Filomat* **2023**, *37*, 6355–6363.
9. Aici, S.; Frakis, A.; Kittaneh, F. Refinements of some numerical radius inequalities for operators. *Rend. Circ. Mat. Palermo II Ser.* **2023**. [CrossRef]
10. Feki, K. Some $\mathbb{A}$-numerical radius inequalities for $d \times d$ operator matrices. *Rend. Circ. Mat. Palermo II Ser.* **2022**, *71*, 85–103. [CrossRef]
11. Altwaijry, N.; Dragomir, S.S.; Feki, K. On the joint A-numerical radius of operators and related inequalities. *Mathematics* **2023**, *11*, 2293. [CrossRef]
12. Conde, C.; Feki, K. Some numerical radius inequality for several semi-Hilbert space operators. *Linear Multilinear Algebra* **2023**, *71*, 1054–1071. [CrossRef]
13. Altwaijry, N.; Feki, K.; Minculete, N. On some generalizations of Cauchy-Schwarz inequalities and their applications. *Symmetry* **2023**, *15*, 304. [CrossRef]
14. Kittaneh, F.; Moradi, H.R.; Sababheh, M. Sharper bounds for the numerical radius. *Linear Multilinear Algebra* **2023**, 1–11. [CrossRef]
15. Bhunia, P.; Paul, K. Refinement of numerical radius inequalities of complex Hilbert space operators. *Acta Sci. Math.* **2023**, 1–10. [CrossRef]
16. Jana, S.; Bhunia, P.; Paul, K. Euclidean operator radius inequalities of a pair of bounded linear operators and their applications. *Bull. Braz. Math. Soc. New Ser.* **2023**, *54*, 1. [CrossRef]
17. Hatamleh, R.; Zolotarev, V.A. Triangular Models of Commutative Systems of Linear Operators Close to Unitary Ones. *Ukr. Math. J.* **2016**, *68*, 791–811. [CrossRef]
18. Bhunia, P.; Paul, K. Refinements of norm and numerical radius inequalities. *Rocky Mountain J. Math.* **2021**, *51*, 1953–1965. [CrossRef]
19. Bhunia, P.; Paul, K. New upper bounds for the numerical radius of Hilbert space operators. *Rocky Mountain J. Math.* **2021**, *167*, 102959. [CrossRef]
20. Kian, M. Operator Jensen inequality for superquadratic functions. *Linear Algebra Appl.* **2014**, *456*, 82–87. [CrossRef]
21. Kian, M.; Dragomir, S.S. Inequalities involving superquadratic functions and operators. *Mediterr. J. Math.* **2014**, *11*, 1205–1214. [CrossRef]
22. Kian, M.; Alomari, M.W. Improvements of trace inequalities for convex functions. *Ann. Funct. Anal.* **2022**, *13*, 64. [CrossRef]
23. Tafazoli, S.; Moradi, H.R.; Furuichi, S.; Harikrishnan, P. Further inequalities for the numerical radius of Hilbert space operators. *J. Math. Ineq.* **2019**, *13*, 955–967. [CrossRef]
24. Furuichi, S.; Minculete, N. *Refined inequalities on the weighted logarithmic mean. J. Math. Ineq.* **2020**, *13*, 1347–1357. [CrossRef]
25. Burqan, A.; Abu-Snainah, A.; Saadeh, R. Improvements of logarithmic and identric mean inequalities for scalars and operators. *J. Appl. Math.* **2023**, *2023*, 5195233. [CrossRef]
26. Altwaijry, N.; Feki, K.; Minculete, N. Further inequalities for the weighted numerical radius of operators. *Mathematics* **2022**, *10*, 3576. [CrossRef]
27. Bhunia, P.; Bhanja, A.; Bag, S.; Paul, K. Bounds for the Davis–Wielandt radius of bounded linear operators. *Ann. Funct. Anal.* **2021**, *12*, 18. [CrossRef]

28. Bhunia, P.; Paul, K. Some improvements of numerical radius inequalities of operators and operator matrices. *Linear Multilinear Algebra* **2020**, *70*, 1995–2013. [CrossRef]
29. Feki, K.; Mahmoud, S.A.O.A. Davis-Wielandt shells of semi-Hilbertian space operators and its applications. *Banach J. Math. Anal.* **2020**, *14*, 1281–1304. [CrossRef]
30. Hajmohamadi, M.; Lashkaripour, R.; Bakherad, M. Some generalizations of numerical radius on off-diagonal part of 2×2 operator matrices. *J. Math. Inequalities* **2018**, *12*, 447–457. [CrossRef]
31. Hajmohamadi, M.; Lashkaripour, R.; Bakherad, M. Further refinements of generalized numerical radius inequalities for Hilbert space operators. *Georgian Math. J.* **2021**, *28*, 83–92. [CrossRef]
32. Moghaddam, S.F.; Mirmostafaee, A.K.; Janfada, M. Some Sharp Estimations for Davis-Wielandt Radius in $B(H)$. *Mediterr. J. Math.* **2022**, *19*, 283. [CrossRef]
33. Alomari, M.W. On the Davis-Wielandt radius inequalities of Hilbert space operators. *Linear Multilinear Algebra* **2022**, 1–25. [CrossRef]
34. Alomari, M.W. Numerical radius inequalities for Hilbert space operators. *Complex Anal. Oper. Theory* **2021**, *15*, 111. [CrossRef]
35. Hatamleh, R. On the form of correlation function for a class of nonstationary field with a zero spectrum. *Rocky Mt. J. Math.* **2003**, *33*, 159–173. [CrossRef]
36. Burqan, A.; Dbabesh, H.; Qazza, A.; Khandaqi, M. New bounds for the eigenvalues of Matrix polynomials. *Eur. J. Pure Appl. Math.* **2023**, *16*, 808–818. [CrossRef]
37. Furuta, T.; Mićić, J.; Pečarić, J.; Seo, Y. *Mond–Pečarić Method in Operator Inequalities*; Ele-Math Element Publishinhg House: Zagreb, Croatia, 2005.
38. Kato, T. Notes on some inequalities for linear operators. *Math. Ann.* **1952**, *125*, 208–212. [CrossRef]
39. Alomari, M.W. A generalization of Hermite–Hadamard's inequality. *Kragujevac J. Math.* **2017**, *41*, 313–328. [CrossRef]
40. Alomari, M.W.; Shebrawi, K.; Chesneau, C. Some generalized Euclidean operator radius inequalities. *Axioms* **2022**, *11*, 285. [CrossRef]

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