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A Study on Existence and Controllability of Conformable Impulsive Equations

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Abstract: We study the existence/uniqueness of conformable fractional type impulsive nonlinear systems as well as the controllability of linear/semilinear conformable fractional type impulsive controlled systems. Using the conformable fractional derivative approach, we introduce the conformable controllability operator and the conformable controllability Gramian matrix in order to obtain the necessary and sufficient conditions for the complete controllability of linear impulsive conformable systems. We present a set of sufficient conditions for the controllability of the conformable semilinear impulsive systems.

Keywords: existence; controllability; conformable derivative; impulsive equation

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1. Introduction

The Riemann–Liouville definition of fractional derivatives is based on repeated integration, while the Caputo definition is based on initial value problems. Both definitions have their own advantages and disadvantages, and the choice of definition depends on the specific application and problem at hand. For example, the Riemann–Liouville definition is well-suited for problems involving initial conditions, while the Caputo definition is better suited for problems involving boundary conditions. Other definitions of fractional derivatives include the Grunwald–Letnikov definition, the Weyl definition, and the Riesz definition, among others (see [1]). Each of these definitions has its own unique features and is used in specific applications and fields. Overall, the study of fractional derivatives has wide-ranging applications in various fields, including physics, engineering, economics, and biology, among others.

The concept of the conformal derivative was introduced in [2–4] and used to extend Newtonian mechanics [5], logistic models [6], and the model webs [7]. The definition of the conformal derivative depends on the basic limit, which is defined for a classical order derivative. The conformal derivative has the product, quotient, and chain rules properties. Hence, this new concept appears to be a natural extension of the conventional order derivative to arbitrary order without memory affect.

A qualitative analysis of linear/semi-linear/non-linear deterministic/stochastic differential equations and delay differential equations with a conformable/classical derivative was studied in [8–25], and the Caputo derivative equations were studied in [26–28]. The concept of conformable derivative is used in the study of nonlinear control systems, where the goal is to find a suitable control input that will steer the system from one state to another in a desired manner. The conformable derivative helps in characterizing the behavior of nonlinear systems, and can be used in developing control strategies for such systems.

A semilinear impulsive differential equation is a mathematical model that describes the evolution of a system with both continuous and impulsive (discontinuous) changes in the state variables. Biological phenomena involving thresholds, optimal control models in economics, and frequently modulated systems, do exhibit impulse effects. Thus, impulsive equations provide a natural description of the observed evolution processes of several real-world problems.

Controllability refers to the ability to manipulate the state of a system to achieve a desired outcome by applying control inputs. The concept of controllability is important in control theory and is used to design control systems that can effectively steer the system to the desired state. The study of the controllability concept for impulsive systems has received significant attention in recent years due to its potential applications in a wide range of fields. The works by Benzaid and Sznaier [29], George et al. [30], Guan et al. [31,32], Xie and Wang [33,34], Zhao and Sun [35,36], Han et al. [37], Muni and George [38], among others, have made significant contributions to the theory of impulsive control systems and have provided new insights into the controllability of such systems. These results have been applied to a wide range of systems, including those with fractal behaviors in complex trigonometric function systems, polynomial systems, switched systems, index function systems, rational function systems, and others, providing new avenues for control design and the development of novel control algorithms.

Impulsive differential equations with a conformable derivative have not yet been studied. Motivated by the mentioned works, in this paper, we study the existence/uniqueness and controllability of solutions for the following semilinear impulsive differential equations with a conformable derivative:

$$\begin{cases} D_0^\alpha y(t) = Ay(t) + Bu(t) + f(t, y(t)), \ \ t \in [0,T] \backslash \{t_1, \ldots, t_p\}, \ 0 < \alpha < 1, \\ y(t_k^+) = (I + C_k) y(t_k^-), \ k \in \mathbb{K} := \{1, 2, \ldots, p\}, \ t_0 = 0, \ \ t_{p+1} = T, \\ y(0) = y_0, \end{cases} \tag{1}$$

where $D_0^\alpha y$ is the conformable derivative with lower index 0 of the function y, $A, C_k \in \mathbb{R}^{d \times d}$ are matrices, $B \in \mathbb{R}^{r \times d}$ is a matrix, $y(t_k) = y(t_k^-)$, $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $u : [0,T] \to \mathbb{R}^r$ is a control function that belong to $L^2([0,T], \mathbb{R}^r)$.

This paper is organized as follows: in Section 2, we recall the definitions of conformable fractional derivatives and conformable integrals and some known results. In Section 3, we study the following conformable linear impulsive Cauchy problem:

$$\begin{cases} D_0^\alpha y(t) = Ay(t) + Bu(t) + f(t), \ \ t \in [0,T], \ 0 < \alpha < 1, \ f \in C\Big([0,T], \mathbb{R}^d\Big), \\ y(t_k^+) = (I + C_k) y(t_k^-), \ k \in \mathbb{K} := \{1, 2, \ldots, p\}, \ t_0 = 0, \ \ t_{p+1} = T, \\ y(0) = y_0. \end{cases} \tag{2}$$

We derive the representation of the solution of the impulsive linear problem with a conformable derivative (2). Section 4 studies the existence and uniqueness of solutions to conformable impulsive semilinear/nonlinear differential equations using the iterative method and the Schauder fixed point method. Section 5 is devoted to the controllability of linear/semilinear conformable impulsive equation.

The main contributions of the paper can be stated as follows: we first find a representation of a solution for inhomogeneous system of (2) and then derive its general solution. Next, we study the existence/uniqueness of a solution of semilinear system (1). Further, we introduce the conformable controllability operator and the conformable controllability Gramian matrix in order to obtain the necessary and sufficient conditions for the complete controllability of linear impulsive conformable systems. Finally, we present a set of sufficient conditions for the controllability of the semilinear conformable impulsive system (1).

2. Preliminaries

We start by defining some function spaces, the conformable derivative, conformable integrals, and the analytic form of a solution to the conformable linear equation, which we will need to use in this paper.

- $\left(\mathbb{R}^d, \|\cdot\|\right)-$ d dimensional Euclidean space.
- $\left(C\left([0,T],\mathbb{R}^d\right), \|\cdot\|_\infty\right)-$ Banach space of continuous functions from $[0,T]$ to $\mathbb{R}^d$ with infinity norm.
- $PC\left([0,T],\mathbb{R}^d\right) := \{y : [0,T] \to \mathbb{R}^d : y \in C\left((t_k,t_{k+1}],\mathbb{R}^d\right),\ k = 0,1,\ldots,\ \exists y(t^+),$ $y(t_k^-) = y(t_k)$ endowed with the norm $\|y\|_{PC} := \sup\{\|y(t)\| : 0 \le t \le T\}$ }.
- $e_A\left(\frac{t^\alpha}{\alpha}\right) = \exp\left(A\frac{t^\alpha}{\alpha}\right) = \sum\limits_{n=0}^{\infty} A^n \dfrac{t^{\alpha n}}{n!\alpha^n}.$

Definition 1 ([3]). *The conformable derivative with lower index 0 of the function $y : [0,\infty) \to \mathbb{R}$ is defined as follows:*

$$\begin{cases} D_0^\alpha y(t) = \lim_{\varepsilon\to 0} \dfrac{y\left(t+\varepsilon t^{1-\alpha}\right) - y(t)}{\varepsilon}, & t > 0,\ 0 < \alpha < 1, \\ D_0^\alpha y(0) = \lim_{t\to 0^+} D_0^\alpha y(t). \end{cases}$$

Remark 1. *We note that the conformable derivative $D_0^\alpha y(t)$, $t > 0$, exists if y is differentiable at t and $D_0^\alpha y(t) = t^{1-\alpha} y'(t)$.*

Definition 2 ([3]). *The conformable integral with lower index a of a function $y : [0,\infty) \to \mathbb{R}$ is defined as follows:*

$$I_0^\alpha y(t) = \int_0^t s^{\alpha-1} y(s) ds,\ \ t \ge 0,\ 0 < \alpha < 1.$$

Lemma 1. *A solution $y \in C\left([0,T],\mathbb{R}^d\right)$ of the linear problem*

$$\begin{cases} D_0^\alpha y(t) = Ay(t) + f(t), & t \in [t_0,T],\ 0 < \alpha < 1,\ f \in C\left([0,T],\mathbb{R}^d\right), \\ y(t_0) = y_0, \end{cases}$$

has the following form:

$$y(t) = e_A\left(\frac{t^\alpha - t_0^\alpha}{\alpha}\right)y_0 + \int_{t_0}^t e_A\left(\frac{t^\alpha}{\alpha} - \frac{t_0^\alpha}{\alpha}\right) f(s) s^{\alpha-1} ds.$$

Proof. It is clear that

$$\begin{aligned} D_0^\alpha e_A\left(\frac{t^\alpha - t_0^\alpha}{\alpha}\right) &= t^{1-\alpha} e'_A\left(\frac{t^\alpha - t_0^\alpha}{\alpha}\right) \\ &= t^{1-\alpha} \sum_{n=1}^{\infty} A^n \frac{\left(t^\alpha - t_0^\alpha\right)^{n-1}}{(n-1)!\alpha^{n-1}} t^{\alpha-1} \\ &= Ae_A\left(\frac{t^\alpha - t_0^\alpha}{\alpha}\right). \end{aligned}$$

Thus,

$$\begin{aligned} D_0^\alpha y(t) &= Ae_A\left(\frac{t^\alpha - t_0^\alpha}{\alpha}\right)y_0 + A\int_{t_0}^t e_A\left(\frac{t^\alpha}{\alpha} - \frac{t_0^\alpha}{\alpha}\right) f(s) s^{\alpha-1} ds + f(t) \\ &= Ay(t). \end{aligned}$$

□

3. Linear System

In this section, we seek the closed form representation of solutions to (2).

Theorem 1. *A solution $y \in PC\Big([0,T],\mathbb{R}^d\Big)$ of the Equation (2) has the following form:*

$$y(t) = \begin{cases} e_A\left(\frac{t^\alpha}{\alpha}\right)y_0 + \int_{t_0}^{t} e_A\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)f(s)(s-t_0)^{\alpha-1}ds, \ \ 0 \le t \le t_1; \\ e_A\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right)\prod_{j=k}^{1}(I+C_j)e_A\left(\frac{\left(t_j - t_{j-1}\right)^\alpha}{\alpha}\right)y_0 \\ +e_A\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}(I+C_j)e_A\left(\frac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right) \\ \quad \times (I+C_i)\int_{t_{i-1}}^{t_i} e_A\left(\frac{t_i^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)f(s)s^{\alpha-1}ds \\ + \int_{t_k}^{t} e_A\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)f(s)s^{\alpha-1}ds, \ \ \ t_k < t \le t_{k+1}, \ k = 1,2,\dots,p. \end{cases} \tag{3}$$

Proof. For $0 \le t \le t_1$, using Lemma 1, we have:

$$y(t) = e_A\left(\frac{t^\alpha}{\alpha}\right)y(0) + \int_0^t e_A\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)f(s)s^{\alpha-1}ds.$$

For $t = t_1^+$, we have

$$\begin{aligned} y\left(t_1^+\right) &= y\left(t_1^-\right) + C_1 y(t_1) \\ &= (I+C_1)e_A\left(\frac{t_1^\alpha}{\alpha}\right)y_0 \\ &\quad + (I+C_1)\int_0^{t_1} e_A\left(\frac{t_1^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)f(s)s^{\alpha-1}ds. \end{aligned} \tag{4}$$

Moreover, for $t_1 < t \le t_2$, we use the following calculation to obtain

$$\begin{aligned} y(t) &= e_A\left(\frac{t^\alpha - t_1^\alpha}{\alpha}\right)y\left(t_1^+\right) + \int_{t_1}^{t} e_A\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)f(s)s^{\alpha-1}ds \\ &= e_A\left(\frac{t^\alpha - t_1^\alpha}{\alpha}\right)(I+C_1)e_A\left(\frac{t_1^\alpha}{\alpha}\right)y_0 \\ &\quad + e_A\left(\frac{t^\alpha - t_1^\alpha}{\alpha}\right)(I+C_1)\int_0^{t_1} e_A\left(\frac{t_1^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)f(s)s^{\alpha-1}ds \\ &\quad + \int_{t_1}^{t} e_A\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)f(s)s^{\alpha-1}ds, \end{aligned}$$

where $y\left(t_1^+\right)$ is given by (4). This means that Theorem 1 holds for $k = 1$. Now, suppose that the Formula (3) is true when $k = m$. Reasoning using the mathematical induction for $k = m+1$, we have

$$\begin{aligned} y(t) &= e_A\left(\frac{t^\alpha - t_{m+1}^\alpha}{\alpha}\right)y\left(t_{m+1}^+\right) + \int_{t_{m+1}}^{t} e_A\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)f(s)s^{\alpha-1}ds \\ &= e_A\left(\frac{t^\alpha - t_{m+1}^\alpha}{\alpha}\right)(I+C_{m+1})y\left(t_{m+1}^-\right) + \int_{t_{m+1}}^{t} e_A\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)f(s)s^{\alpha-1}ds \\ &= e_A\left(\frac{t^\alpha - t_{m+1}^\alpha}{\alpha}\right)(I+C_{m+1})e_A\left(\frac{t^\alpha - t_m^\alpha}{\alpha}\right)\prod_{j=m}^{1}(I+C_j)e_A\left(\frac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right)y_0 \end{aligned}$$

$$
\begin{aligned}
&+e_A\left(\frac{t^\alpha-t_{m+1}^\alpha}{\alpha}\right)(I+C_{m+1})e_A\left(\frac{t^\alpha-t_m^\alpha}{\alpha}\right)\sum_{i=1}^{m}\prod_{j=m}^{i+1}(I+C_j)e_A\left(\frac{t_j^\alpha-t_{j-1}^\alpha}{\alpha}\right)\\
&\quad\times(I+C_i)\int_{t_{i-1}}^{t_i}e_A\left(\frac{t_i^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)f(s)s^{\alpha-1}ds\\
&+e_A\left(\frac{t^\alpha-t_{m+1}^\alpha}{\alpha}\right)(I+C_{m+1})\int_{t_m}^{t_{m+1}}e_A\left(\frac{t_{m+1}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)f(s)s^{\alpha-1}ds\\
&+\int_{t_{m+1}}^{t}e_A\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)f(s)s^{\alpha-1}ds.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
y(t)=&\,e_A\left(\frac{t^\alpha-t_{m+1}^\alpha}{\alpha}\right)\prod_{j=m+1}^{1}(I+C_j)e_A\left(\frac{t_j^\alpha-t_{j-1}^\alpha}{\alpha}\right)y_0\\
&+e_A\left(\frac{t^\alpha-t_{m+1}^\alpha}{\alpha}\right)\sum_{i=1}^{m+1}\prod_{j=m+1}^{i+1}(I+C_j)e_A\left(\frac{t_j^\alpha-t_{j-1}^\alpha}{\alpha}\right)\\
&\quad\times(I+C_i)\int_{t_{i-1}}^{t_i}e_A\left(\frac{t_i^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)f(s)s^{\alpha-1}ds\\
&+\int_{t_{m+1}}^{t}e_A\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)f(s)s^{\alpha-1}ds,\quad t_{m+1}<t\le t_{m+2}.
\end{aligned}
$$

Thus, we can conclude that Theorem 1 is true for any $k=1,2,\ldots$ This completes the proof. □

Theorem 2 ([39]). *Assume that X is a Banach space, $B\subset PC([0,T],X)$. Suppose that*

(i) B is a uniformly bounded subset of $PC([0,T],X)$;
(ii) B is equicontinuous in (t_k,t_{k+1}), $k=0,1,\ldots,p$;
(iii) $B(t):=\{x(t):x\in B,\ t\in[0,T]\backslash\{t_1,\ldots,t_p\}\}$, $B(t_k^+):=\{x(t_k^+):x\in B\}$ and $B(t_k^-):=\{x(t_k^-):x\in B\}$ are relatively compact subset of X.
Then, B is a relatively compact subset of $PC([0,T],X)$.

4. Existence of Solutions

The iterative method and the Schauder fixed point method are two common methods used to study the existence and uniqueness of solutions to conformable impulsive semilinear/nonlinear differential equations. The iterative method can be used to show both existence and uniqueness, while the Schauder fixed point method is typically used to show existence only. These methods are based on different mathematical concepts and techniques, and they provide different types of information about the solutions to these types of equations.

The Picard iterative method is a method used to prove the existence and uniqueness of a solution to an initial value problem for ordinary differential equations. The method is based on the idea of constructing a sequence of functions that converges to the solution of the equation.

The key steps in the Picard iterative method are as follows:

- Start with an initial value for the unknown solution, usually denoted by y_0.
- Use the initial value to define a sequence of approximations, $y_1,y_2,\ldots$, where each approximation is defined in terms of the previous one and the right-hand side of the differential equation.
- Show that the sequence converges to a solution of the differential equation, and that this solution is unique.

If these steps can be successfully carried out, then the Picard approximation method provides a proof of existence and uniqueness for the solution of the differential equation.

Therefore, to prove the first main results in this section, namely the existence and uniqueness theorem, we use the Picard iterative method.

Consider the following assumptions that will be used in this section:

Hypothesis 1 ($\mathbf{H_1}$). $f(\cdot,\cdot) \in C\Big([0,T] \times \mathbb{R}^d, \mathbb{R}^d\Big)$.

Hypothesis 2 ($\mathbf{H_2}$). *$\exists L_f > 0$ such that for any $t \in [0,T]$ and $x, y \in \mathbb{R}^d$ we have*

$$\|f(t,x) - f(t,y)\| \leq L_f \|x - y\|.$$

Define

$$y_0(t) = \begin{cases} e_A\left(\dfrac{t^\alpha}{\alpha}\right) y_0, \;\; 0 \leq t \leq t_1; \\ e_A\left(\dfrac{t^\alpha - t_k^\alpha}{\alpha}\right) \displaystyle\prod_{j=k}^{1} (I + C_j) e_A\left(\dfrac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right) y_0, \quad t_k < t \leq t_{k+1},\; k = 1, 2, \ldots, p. \end{cases} \tag{5}$$

Set

$$C := \prod_{j=p}^{1} (I + \|C_j\|),$$

$$B_r := \{y \in PC([0,T], R^n) : \|y - y_0\|_\infty \leq r\},$$

where

$$r := \left[pCe^2_{\|A\|}\left(\frac{T^\alpha}{\alpha}\right) + 1\right] \frac{1}{\|A\|} M_f \left[e_{\|A\|}\left(\frac{T^\alpha}{\alpha}\right) - 1\right].$$

$$r := \left[pCe_{\|A\|}\left(\frac{2T^\alpha}{\alpha}\right) + 1\right] \frac{1}{\|A\|} M_f \left[e_{\|A\|}\left(\frac{T^\alpha}{\alpha}\right) - 1\right],$$

$$K(T) := C^2 e_{\|A\|}\left(\frac{3T^\alpha}{\alpha}\right) + e_{\|A\|}\left(\frac{T^\alpha}{\alpha}\right).$$

It is clear that

$$\|f(t,y(t))\| \leq \|f(t,0) - f(t,y(t))\| + \|f(t,0)\| \leq L_f \|y(t)\| + \|f(t,0)\|,$$

consequently,

$$M_f := \sup\{\|f(t,y(t))\| : t \in [0,T],\; y \in B_r\}$$

exists.

Theorem 3. *Assume that (H_1) and (H_2) hold. Then, the semilinear Equation (1) has a unique solution in the space of piecewise continuous functions $PC\Big([0,T], \mathbb{R}^d\Big)$.*

Proof. As the zeroth approximation, we choose

$$y_0(t) = \begin{cases} e_A\left(\dfrac{t^\alpha}{\alpha}\right) y_0, \;\; 0 \leq t \leq t_1; \\ e_A\left(\dfrac{t^\alpha - t_k^\alpha}{\alpha}\right) \displaystyle\prod_{j=k}^{1} (I + C_j) e_A\left(\dfrac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right) y_0, \quad t_k < t \leq t_{k+1},\; k = 1, 2, \ldots, p. \end{cases}$$

The nth approximation can be chosen as follows:

$$y_n(t)=\begin{cases} e_A\left(\frac{t^\alpha}{\alpha}\right)y_0+\int_{t_0}^{t}e_A\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)f(s,y_{n-1}(s))s^{\alpha-1}ds,\\ \quad 0\le t\le t_1;\\ e_A\left(\frac{t^\alpha-t_k^\alpha}{\alpha}\right)\prod_{j=k}^{1}(I+C_j)e_A\left(\frac{t_j^\alpha-t_{j-1}^\alpha}{\alpha}\right)y_0\\ +e_A\left(\frac{t^\alpha-t_k^\alpha}{\alpha}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}(I+C_j)e_A\left(\frac{t_j^\alpha-t_{j-1}^\alpha}{\alpha}\right)\\ \qquad\times(I+C_i)\int_{t_{i-1}}^{t_i}e_A\left(\frac{t_i^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)f(s,y_{n-1}(s))s^{\alpha-1}ds\\ +\int_{t_k}^{t}e_A\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)f(s,y_{n-1}(s))s^{\alpha-1}ds,\\ \quad t_k<t\le t_{k+1},\ k=1,2,\dots,p. \end{cases} \tag{6}$$

According to (H_1), (6) is well defined.
Step 1. For any $n\in\mathbb{N}$, we prove that $y_n\in B_r$.
(i) For $n=1$ and $t\in[0,t_1]$, we have

$$\begin{aligned}\|y_1(t)-y_0(t)\|&=\left\|\int_0^t e_A\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)f(s,y_0(s))s^{\alpha-1}ds\right\|\\ &\le\int_0^t e_{\|A\|}\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\|f(s,y_0(s))\|s^{\alpha-1}ds\\ &\le M_f\int_0^t e_{\|A\|}\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)s^{\alpha-1}ds\\ &=\frac{1}{\|A\|}M_f\left[e_{\|A\|}\left(\frac{T^\alpha}{\alpha}\right)-1\right]\le r.\end{aligned}\tag{7}$$

For $n=1$ and $t\in(t_k,t_{k+1}]$, we have

$$\begin{aligned}&\|y_1(t)-y_0(t)\|\\ &\le\left\|e_A\left(\frac{t^\alpha-t_k^\alpha}{\alpha}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}(I+C_j)e_A\left(\frac{t_j^\alpha-t_{j-1}^\alpha}{\alpha}\right)\right.\\ &\qquad\left.\times\ (I+C_i)\int_{t_{i-1}}^{t_i}e_A\left(\frac{t_i^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)f(s,y_0(s))s^{\alpha-1}ds\right\|\\ &+\left\|\int_{t_k}^{t}e_A\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)f(s,y_0(s))s^{\alpha-1}ds\right\|\\ &\le e_{\|A\|}\left(\frac{t^\alpha-t_k^\alpha}{\alpha}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}(1+\|C_j\|)e_{\|A\|}\left(\frac{t_j^\alpha-t_{j-1}^\alpha}{\alpha}\right)\\ &\qquad\times\ (1+\|C_i\|)\int_{t_{i-1}}^{t_i}e_{\|A\|}\left(\frac{t_i^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\|f(s,y_0(s))\|s^{\alpha-1}ds\\ &+\int_{t_k}^{t}e_{\|A\|}\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\|f(s,y_0(s))\|s^{\alpha-1}ds\end{aligned}$$

$$\begin{aligned}&=\sum_{i=1}^{k}\prod_{j=k}^{i}(1+\|C_j\|)e_{\|A\|}\left(\frac{t^\alpha-t_k^\alpha}{\alpha}\right)e_{\|A\|}\left(\frac{t_j^\alpha-t_{j-1}^\alpha}{\alpha}\right)\\ &\times\int_{t_{i-1}}^{t_i}e_{\|A\|}\left(\frac{t_i^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\|f(s,y_0(s))\|s^{\alpha-1}ds\\ &+\int_{t_k}^{t}e_{\|A\|}\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\|f(s,y_0(s))\|s^{\alpha-1}ds\end{aligned}$$

$$\leq e_{\|A\|}\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right)\sum_{i=1}^{k}\prod_{j=k}^{1}(I+\|C_j\|)e_{\|A\|}\left(\frac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right)\frac{1}{\|A\|}M_f\left[e_{\|A\|}\left(\frac{t_i^\alpha - t_{i-1}^\alpha}{\alpha}\right)-1\right]$$
$$+\frac{1}{\|A\|}M_f\left[e_{\|A\|}\left(\frac{T^\alpha - t_k^\alpha}{\alpha}\right)-1\right]$$

$$\begin{aligned}&\leq\left[pCe^2_{\|A\|}\left(\frac{T^\alpha}{\alpha}\right)+1\right]\frac{1}{\|A\|}M_f\left[e_{\|A\|}\left(\frac{T^\alpha}{\alpha}\right)-1\right]\\&\leq r.\end{aligned} \tag{8}$$

From (7) and (8), it follows that for any $t \in [0,T]$

$$\|y_1(t)-y_0(t)\| \leq r.$$

For $t \in [0,T]$ and $n = m$, assume that $\|y_m - y_0\|_\infty \leq r$. We have

$$\begin{aligned}&\|y_{m+1}(t)-y_0(t)\|\\&\leq\left\|e_A\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}(I+C_j)e_A\left(\frac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right)\right.\\&\qquad\left.\times\ (I+C_i)\int_{t_{i-1}}^{t_i}e_A\left(\frac{t_i^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)f(s,y_m(s))s^{\alpha-1}ds\right\|\\&+\left\|\int_{t_k}^{t}e_A\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)f(s,y_m(s))s^{\alpha-1}ds\right\|.\end{aligned}$$

Similar to (8), we have

$$\begin{aligned}\|y_{m+1}(t)-y_0(t)\| &\leq\left[pCe^2_{\|A\|}\left(\frac{T^\alpha}{\alpha}\right)+1\right]\frac{1}{\|A\|}M_f\left[e_{\|A\|}\left(\frac{T^\alpha}{\alpha}\right)-1\right]\\&\leq r.\end{aligned}$$

It follows that for any $n \geq 1$

$$\|y_n - y_0\|_\infty \leq r.$$

Step 2: We claim that the approximating sequence $\{y_n\}$ converges uniformly on $[0,T]$. Consider the following series

$$S(t) = y_0(t) + \sum_{m=1}^{\infty}(y_m(t)-y_{m-1}(t)),\ \ t\in[0,T], \tag{9}$$

and the sequence

$$y_n(t) = y_0(t) + \sum_{m=1}^{n}(y_m(t)-y_{m-1}(t)),\ \ t\in[0,T].$$

We show that (9) is uniformly convergent on $[0,T]$. We have

$$\begin{aligned}&t\in[0,t_1]:\quad \|y_1(t)-y_0(t)\| = \left\|\int_0^t e_A\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)f(s,y_0(s))s^{\alpha-1}ds\right\|\\&\leq\int_0^t e_{\|A\|}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\|f(s,y_0(s))\|s^{\alpha-1}ds\\&\leq M_f\int_0^t e_{\|A\|}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)s^{\alpha-1}ds\\&\leq\frac{1}{\alpha}M_f\,t^\alpha\,e_{\|A\|}\left(\frac{t^\alpha}{\alpha}\right).\end{aligned}$$

$$
\begin{aligned}
&\|y_1(t) - y_0(t)\| \\
&\leq e_{\|A\|}\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}(1 + \|C_j\|)e_{\|A\|}\left(\frac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right) \\
&\quad \times (1 + \|C_i\|)\int_{t_{i-1}}^{t_i} e_{\|A\|}\left(\frac{t_i^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\|f(s, y_0(s))\|s^{\alpha-1}ds \\
&+ \int_{t_k}^{t} e_{\|A\|}\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\|f(s, y_0(s))\|s^{\alpha-1}ds \\
&\leq e_{\|A\|}\left(\frac{3t^\alpha}{\alpha}\right)C^2 M_f \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} s^{\alpha-1}ds \\
&+ e_{\|A\|}\left(\frac{t^\alpha}{\alpha}\right)M_f\int_{t_k}^{t} s^{\alpha-1}ds \\
&\leq \max\left(e_{\|A\|}\left(\frac{3t^\alpha}{\alpha}\right)C^2, e_{\|A\|}\left(\frac{t^\alpha}{\alpha}\right)\right)M_f\frac{t^\alpha}{\alpha} \leq K(T)M_f\frac{t^\alpha}{\alpha}
\end{aligned} \tag{10}
$$

Next, using the Lipschitz condition (H_2), one has:

$$
\begin{aligned}
&\|y_2(t) - y_1(t)\| \\
&\leq \Bigg\| e_A\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}(I + C_j)e_A\left(\frac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right) \\
&\qquad \times (I + C_i)\int_{t_{i-1}}^{t_i} e_A\left(\frac{t_i^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)[f(s, y_1(s)) - f(s, y_0(s))]s^{\alpha-1}ds\Bigg\| \\
&+ \left\|\int_{t_k}^{t} e_A\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)[f(s, y_1(s)) - f(s, y_0(s))]s^{\alpha-1}ds\right\| \\
&\leq e_{\|A\|}\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}(I + \|C_j\|)e_{\|A\|}\left(\frac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right) \\
&\times (I + \|C_i\|)\int_{t_{i-1}}^{t_i} e_{\|A\|}\left(\frac{t_i^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\|f(s, y_1(s)) - f(s, y_0(s))\|s^{\alpha-1}ds \\
&+ \int_{t_k}^{t} e_{\|A\|}\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\|f(s, y_1(s)) - f(s, y_0(s))\|s^{\alpha-1}ds \\
&\leq C^2 e_{\|A\|}\left(\frac{3T^\alpha}{\alpha}\right)L_f\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\|y_1(s) - y_0(s)\|s^{\alpha-1}ds \\
&+ e_{\|A\|}\left(\frac{T^\alpha}{\alpha}\right)L_f\int_{t_k}^{t}\|y_1(s) - y_0(s)\|s^{\alpha-1}ds \\
&\leq L_f K(T)\int_0^t \|y_1(s) - y_0(s)\|s^{\alpha-1}ds \\
&\leq L_f K^2(T)M_f\int_0^t \frac{s^\alpha}{\alpha}s^{\alpha-1}ds = L_f K^2(T)M_f^2\frac{t^{2\alpha}}{2!\,\alpha}, \quad t \in (t_k, t_{k+1}],\ k = 1, 2, \ldots
\end{aligned}
$$

For $0 \leq t \leq t_1$, we have the similar estimate. Thus, for any $t \in [0, T]$

$$
\|y_2(t) - y_1(t)\| \leq \frac{1}{2!\,\alpha^2}L_f K^2(T)M_f t^{2\alpha}. \tag{11}
$$

By mathematical induction, assume that

$$
\|y_n(t) - y_{n-1}(t)\| \leq \frac{1}{n!\,\alpha^n}K^n(T)L_f^{n-1}M_f t^{n\alpha}
$$

holds for a natural number n and $t \in [0, T]$. Then, for $t \in [0, T]$, according to (H_2), we have:

$$
\begin{aligned}
&\|y_{n+1}(t) - y_n(t)\| \\
&\le CL_f \int_0^t e_{\|A\|}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\|y_n(s) - y_{n-1}(s)\|s^{\alpha-1}ds \\
&\le CL_f \frac{1}{n!\,\alpha^n} C^n L_f^{n-1} M_f \int_0^t e_{\|A\|}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) e_{\|A\|}\left(\frac{s^\alpha}{\alpha}\right) s^{n\alpha} s^{\alpha-1} ds \\
&\le \frac{1}{(n+1)!\,\alpha^{n+1}} C^{n+1} L_f^n M_f e_{\|A\|}\left(\frac{t^\alpha}{\alpha}\right) t^{(n+1)\alpha}.
\end{aligned} \tag{12}
$$

Note that

$$
\begin{aligned}
\|S(t)\| \le \|y_0(t)\| + \sum_{m=1}^{\infty} \|y_m(t) - y_{m-1}(t)\| \\
\le Ce_{\|A\|}\left(\frac{t^\alpha}{\alpha}\right)\|y_0\| + M_f \sum_{m=1}^{\infty} \frac{K^m(T) L_f^{m-1} C^{m+1} L_f^m}{m!\,\alpha^m} t^{m\alpha}.
\end{aligned}
$$

Therefore, the sequence of approximating functions $\{y_n(t)\}$ is uniformly convergent on $[0,T]$. So $\exists\, y \in PC\Big([0,T], \mathbb{R}^d\Big)$, such that $y_n(t)$ uniformly converges to $y(t)$ on $[0,T]$.

Step 3: We claim that the limit y is a solution of the semilinear Equation (1).

The sequence $y_n(t) \overset{\text{uniformly}}{\Longrightarrow} y(t)$ on $[0,T]$, so the sequence of functions $f(t, y_n(t))$ converges uniformly to the continuous function $f(t, y(t))$ on $[0,T]$. For all $t \in [0,T]$, we have:

$$
\begin{aligned}
&\lim_{n\to\infty} y_n(t) \\
&= \begin{cases}
e_A\left(\frac{t^\alpha}{\alpha}\right)y_0 + \lim_{n\to\infty} \int_0^t e_A\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, y_{n-1}(s)) s^{\alpha-1} ds, \quad 0 \le t \le t_1; \\
e_A\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right) \prod_{j=k}^{1} (I + C_j) e_A\left(\frac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right) y_0 \\
+ e_A\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right) \sum_{i=1}^{k} \prod_{j=k}^{i+1} (I + C_j) e_A\left(\frac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right) \\
\quad \times (I + C_i) \lim_{n\to\infty} \int_{t_{i-1}}^{t_i} e_A\left(\frac{t_i^\alpha - s^\alpha}{\alpha}\right) f(s, y_{n-1}(s)) s^{\alpha-1} ds \\
+ \lim_{n\to\infty} \int_{t_k}^{t} e_A\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, y_{n-1}(s)) s^{\alpha-1} ds, \quad t_k < t \le t_{k+1},\ k = 1, 2, \ldots, p.
\end{cases} \\
&= \begin{cases}
e_A\left(\frac{t^\alpha}{\alpha}\right)y_0 + \int_0^t e_A\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, y(s)) s^{\alpha-1} ds, \quad 0 \le t \le t_1; \\
e_A\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right) \prod_{j=k}^{1} (I + C_j) e_A\left(\frac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right) y_0 \\
+ e_A\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right) \sum_{i=1}^{k} \prod_{j=k}^{i+1} (I + C_j) e_A\left(\frac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right) \\
\quad \times (I + C_i) \int_{t_{i-1}}^{t_i} e_A\left(\frac{t_i^\alpha - s^\alpha}{\alpha}\right) f(s, y(s)) s^{\alpha-1} ds \\
+ \int_{t_k}^{t} e_A\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, y(s)) s^{\alpha-1} ds, \quad t_k < t \le t_{k+1},\ k = 1, 2, \ldots, p.
\end{cases} \\
&= y(t).
\end{aligned}
$$

Step 4. The solution is unique.

Suppose that z is another solution of (1). Using the condition (H_2) similar to (12) we have

$$\|y(t) - z(t)\| \leq K(T) L_f \int_0^t e_{\|A\|}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) \|y(s) - z(s)\| s^{\alpha-1} ds.$$

Applying Gronwall's inequality (conformable version), we get:

$$\|y(t) - z(t)\| \leq 0 \implies y(t) = z(t), \ \ t \in [0, T].$$

The proof is complete. □

Schauder's fixed point theorem is a result in mathematical analysis that states that, if a continuous and compact operator maps a complete metric space into itself, then it has a fixed point. This theorem can be used to prove the existence of a solution to a variety of problems in mathematics, including differential equations and integral equations. In order to apply Schauder's fixed point theorem, the following assumptions must be met:

- The operator must be continuous and compact.
- The metric space in which the operator maps must be complete.
- The image of the operator must be contained within the metric space.

If these conditions are satisfied, then Schauder's fixed point theorem guarantees the existence of a fixed point of the operator. T

Therefore, we use the Schauder FPT to prove the second main result, namely an existence theorem.

Assume the following conditions:

Hypothesis 3 (H_3). *$f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is measurable in the first variable and continuous in the second variable.*

Hypothesis 4 (H_4). *There exists a positive constant $M_f > 0$ such that, for any $t \in [0, T]$ and $y \in \mathbb{R}^d$, we have*

$$\|f(t, y)\| \leq M_f.$$

Theorem 4. *Assume that (H_3) and (H_4) hold. Then, (1) has at least one solution in $PC\left([0, T], \mathbb{R}^d\right)$.*

Proof. Set

$$B_r := \left\{ y \in PC\left([0, T], \mathbb{R}^d\right) : \|y\|_\infty \leq C e_{\|A\|}\left(\frac{T^\alpha}{\alpha}\right) \|y_0\| + \frac{1}{\alpha} C M_f \left(e_{\|A\|}\left(\frac{T^\alpha}{\alpha}\right) - 1\right) \right\}.$$

Consider the nonlinear operator H defined on B_r as follows:

$$(Hy)(t) := \begin{cases} e_A\left(\frac{t^\alpha}{\alpha}\right) y_0 + \displaystyle\int_0^t e_A\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, y(s)) s^{\alpha-1} ds, \ \ 0 \leq t \leq t_1; \\ e_A\left(\dfrac{t^\alpha - t_k^\alpha}{\alpha}\right) \displaystyle\prod_{j=k}^{1} (I + C_j) e_A\left(\dfrac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right) y_0 \\ + e_A\left(\dfrac{t^\alpha - t_k^\alpha}{\alpha}\right) \displaystyle\sum_{i=1}^{k} \prod_{j=k}^{i+1} (I + C_j) e_A\left(\dfrac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right) \\ \quad \times (I + C_i) \displaystyle\int_{t_{i-1}}^{t_i} e_A\left(\dfrac{t_i^\alpha - s^\alpha}{\alpha}\right) f(s, y(s)) s^{\alpha-1} ds \\ + \displaystyle\int_{t_k}^t e_A\left(\dfrac{t^\alpha - s^\alpha}{\alpha}\right) f(s, y(s)) s^{\alpha-1} ds, \ \ \ t_k < t \leq t_{k+1}, \ k = 1, 2, \ldots, p. \end{cases}$$

Step 1. We prove that $H(B_r) \subset B_r$.

For $y \in B_r$ and any $t \in [0, T]$, we have:

$$\begin{aligned}\|(Hy)(t)\| &\leq \prod_{j=k}^{1}(I+\|C_j\|)e_{\|A\|}\left(\frac{t^\alpha}{\alpha}\right)\|y_0\| \\ &+\prod_{j=p}^{1}(I+\|C_j\|)\int_0^t e_{\|A\|}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\|f(s,y(s))\|s^{\alpha-1}ds \\ &\leq Ce_{\|A\|}\left(\frac{t^\alpha}{\alpha}\right)\|y_0\| \\ &+CM_f\int_0^t e_{\|A\|}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)s^{\alpha-1}ds \\ &= Ce_{\|A\|}\left(\frac{T^\alpha}{\alpha}\right)\|y_0\| + \frac{1}{\alpha}CM_f\left(e_{\|A\|}\left(\frac{T^\alpha}{\alpha}\right)-1\right).\end{aligned}$$

Step 2. We prove the continuity of the nonlinear operator H.
Let y_n be a sequence with $y_n \to y$ in B_r as $n \to \infty$. For any $t \in [0, T]$, we have:

$$\begin{aligned}&\|(Hy_n)(t) - (Hy)(t)\| \\ &\leq \prod_{j=p}^{1}(I+\|C_j\|)\int_0^t e_{\|A\|}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\|f(s,y_n(s)) - f(s,y(s))\|s^{\alpha-1}ds.\end{aligned}$$

From the assumptions (H_3) and (H_4), it follows that

$$\max_{0\leq s\leq T}\|f(s,y_n(s)) - f(s,y(s))\| \to 0 \text{ as } n \to \infty,$$
$$e_{\|A\|}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\|f(s,y_n(s)) - f(s,y(s))\|s^{\alpha-1} \leq 2M_f e_{\|A\|}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)s^{\alpha-1},$$
$$2M_f e_{\|A\|}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)s^{\alpha-1} \text{ is integrable with respect to } s \in [0,T].$$

It remains to apply the Lebesgue dominated theorem to get continuity of H.

Step 3. We prove that the set $H(B_r)$ is equicontinuous.
Let $t', t'' \in (t_k, t_{k+1}]$, $t' < t''$, and $|t' - t''| < \delta$. For any $y \in B_r$, we have

$$\begin{aligned}&\|(Hy)(t'') - (Hy)(t')\| \\ &\leq C\left[e_{\|A\|}\left(\frac{(t'')^\alpha}{\alpha}\right) - e_{\|A\|}\left(\frac{(t')^\alpha}{\alpha}\right)\right]\|y_0\| \\ &+C\left[e_{\|A\|}\left(\frac{(t'')^\alpha}{\alpha}\right) - e_{\|A\|}\left(\frac{(t')^\alpha}{\alpha}\right)\right]\int_0^{t'} e_{\|A\|}\left(\frac{-s^\alpha}{\alpha}\right)\|f(s,y(s))\|s^{\alpha-1}ds \\ &+Ce_{\|A\|}\left(\frac{(t'')^\alpha}{\alpha}\right)\int_0^{t''} e_{\|A\|}\left(\frac{-s^\alpha}{\alpha}\right)\|f(s,y(s))\|s^{\alpha-1}ds.\end{aligned}$$

Uniform continuity of $e_{\|A\|}\left(\frac{t^\alpha}{\alpha}\right)$ on $[0, T]$ implies that $\|(Hy)(t'') - (Hy)(t')\| \to 0$ as $\delta \to 0$. So, $H(B_r)$ is equicontinuous.

Steps 1–3 with Theorem 2 when $X = \mathbb{R}^d$ say that the nonlinear operator $H : B_r \to B_r$ is compact. Therefore, the Schauder FPT implies that H has a fixed point in $PC\left([0,T], \mathbb{R}^d\right)$. The proof is complete. □

5. Complete Controllability

5.1. Linear Systems

Consider

$$\begin{cases} D_0^\alpha y(t) = Ay(t) + Bu(t), \;\; t \in [0,T],\; 0<\alpha<1, \\ y\big(t_k^+\big) = (I+C_k)y\big(t_k^-\big),\; k \in \mathbb{K} := \{1,2,\dots,p\},\; t_0 = 0, \;\; t_{p+1} = T, \\ y(0) = y_0. \end{cases} \tag{13}$$

Definition 3. *The system (13) is said to be completely controllable on $[0,T]$ if, given an arbitrary initial vector function y_0 and a terminal state vector y_T at time T, there exists a control input $u \in L^2([0,T],\mathbb{R}^r)$, such that the state of the system $y \in PC\Big([0,T],\mathbb{R}^d\Big)$ satisfies $y(T) = y_T$.*

In other words, the system can be driven from any initial state to any desired terminal state by means of a suitable control input. Complete controllability is an important property in control theory because it ensures that the system can be effectively controlled and manipulated to achieve a desired behavior.

To define the impulsive controllability operator, we introduce the continuous linear bounded operator $M : L^2([0,T],\mathbb{R}^r) \to \mathbb{R}^d$ as follows:

$$\begin{aligned} Mu &= e_A\left(\frac{T^\alpha - t_k^\alpha}{\alpha}\right)\sum_{i=1}^{p}\prod_{j=p}^{i+1}(I+C_j)e_A\left(\frac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right)(I+C_i)\int_{t_{i-1}}^{t_i} e_A\left(\frac{t_i^\alpha - s^\alpha}{\alpha}\right)Bu(s)ds \\ &+ \int_{t_p}^{T} e_A\left(\frac{T^\alpha - s^\alpha}{\alpha}\right)Bu(s)ds. \end{aligned}$$

Before stating the controllability result, we introduce the adjoint operator M^*.

Lemma 2. *The adjoint operator $M^* : \mathbb{R}^d \to L^2([0,T],\mathbb{R}^r)$ has the following form:*

$$M^*\psi(t) = \begin{cases} B^\intercal e_A^\intercal\left(\dfrac{T^\alpha - s^\alpha}{\alpha}\right)\varphi, & t_p < t \le T, \\ B^\intercal e_A^\intercal\left(\dfrac{t_k^\alpha - t^\alpha}{\alpha}\right)(I+C_k^\intercal) & \\ \times\prod_{i=k+1}^{p} e_A^\intercal\left(\dfrac{t_i^\alpha - t_{i-1}^\alpha}{\alpha}\right)(I+C_i^\intercal)e_A^\intercal\left(\dfrac{T^\alpha - t_p^\alpha}{\alpha}\right)\varphi, & t_{k-1} < t \le t_k. \end{cases}$$

Proof. Letting $y(0) = 0$ in (13) yields $\langle y(T),\varphi\rangle = \langle w, M^*\varphi\rangle = \displaystyle\int_0^T \langle u(s), B^*\psi(s)\rangle ds$,which implies

$$\begin{aligned} &\langle y(T),\varphi\rangle \\ &= \left\langle e_A\left(\frac{T^\alpha - t_k^\alpha}{\alpha}\right)\sum_{i=1}^{p}\prod_{j=p}^{i+1}(I+C_j)e_A\left(\frac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right)(I+C_i)\int_{t_{i-1}}^{t_i} e_A\left(\frac{t_i^\alpha - s^\alpha}{\alpha}\right)Bu(s)ds, \varphi\right\rangle \\ &+ \left\langle \int_{t_p}^{T} e_A\left(\frac{T^\alpha - s^\alpha}{\alpha}\right)Bu(s)ds, \varphi\right\rangle \\ &= \int_{t_p}^{b}\left\langle u(s), B^\intercal e_A^\intercal\left(\frac{T^\alpha - s^\alpha}{\alpha}\right)\varphi\right\rangle ds \\ &+ \sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}\left\langle u(s)ds, B^\intercal e_A^\intercal\left(\frac{t_k^\alpha - t^\alpha}{\alpha}\right)(I+C_k^\intercal)\prod_{i=k+1}^{p} e_A^\intercal\left(\frac{t_i^\alpha - t_{i-1}^\alpha}{\alpha}\right)(I+C_i^\intercal)e_A^\intercal\left(\frac{T^\alpha - t_p^\alpha}{\alpha}\right)\varphi\right\rangle. \end{aligned}$$

□

Lemma 3. *The operator MM^* has the following form:*

$$MM^* = \Theta_0^{t_p} + \Gamma_{t_p}^{b},$$

where $\Gamma_{t_p}^{T}, \Theta_0^{t_p} : \mathbb{R}^d \to \mathbb{R}^d$ are non-negative matrices and defined as follows:

$$\begin{aligned}
\Gamma_{t_p}^{T} &:= \int_{t_p}^{T} e_A\left(\frac{T^\alpha - s^\alpha}{\alpha}\right) BB^\intercal e_A^\intercal\left(\frac{T^\alpha - s^\alpha}{\alpha}\right) ds,\\
\Theta_0^{t_p} &:= e_A\left(\frac{T^\alpha - t_k^\alpha}{\alpha}\right) \sum_{i=1}^{p} \prod_{j=p}^{i+1} (I + C_j) e_A\left(\frac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right)\\
&\quad \times (I + C_i) \int_{t_{i-1}}^{t_i} e_A\left(\frac{t_i^\alpha - s^\alpha}{\alpha}\right) BB^\intercal e_A^\intercal\left(\frac{t_k^\alpha - s^\alpha}{\alpha}\right) ds\\
&\quad \times (I + C_i^\intercal) \prod_{k=i+1}^{p} e_A^\intercal\left(\frac{t_k^\alpha - t_{k-1}^\alpha}{\alpha}\right) (I + C_k^\intercal) e_A^\intercal\left(\frac{T^\alpha - t_p^\alpha}{\alpha}\right).
\end{aligned}$$

Proof. Indeed,

$$\begin{aligned}
MM^*\varphi = e_A\left(\frac{T^\alpha - t_k^\alpha}{\alpha}\right) &\sum_{i=1}^{p} \prod_{j=p}^{i+1} (I + C_j) e_A\left(\frac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right) (I + C_i)\\
&\times \int_{t_{i-1}}^{t_i} e_A\left(\frac{t_i^\alpha - s^\alpha}{\alpha}\right) BB^\intercal e_A^\intercal\left(\frac{t_k^\alpha - s^\alpha}{\alpha}\right) ds\\
&\times (I + C_i^\intercal) \prod_{k=i+1}^{p} e_A^\intercal\left(\frac{t_k^\alpha - t_{k-1}^\alpha}{\alpha}\right) (I + C_k^\intercal) e_A^\intercal\left(\frac{T^\alpha - t_p^\alpha}{\alpha}\right) \varphi\\
&+ \int_{t_p}^{T} e_A\left(\frac{T^\alpha - s^\alpha}{\alpha}\right) BB^\intercal e_A^\intercal\left(\frac{T^\alpha - s^\alpha}{\alpha}\right) ds\varphi.
\end{aligned}$$

Obviously, $\Gamma_{t_p}^{b}, \Theta_0^{t_p} : \mathbb{R}^d \to \mathbb{R}^d$ are non-negative. □

Therefore, we can introduce the controllability Gram matrix as follows:

$$MM^* = \Theta_0^{t_p} + \Gamma_{t_p}^{T}.$$

Theorem 5. *The linear conformable impulsive Equation (13) is completely controllable on $[0, T]$, if and only if the $d \times d$ matrix*

$$MM^* = \Theta_0^{t_p} + \Gamma_{t_p}^{T}$$

is invertible.

Proof. Since the operator $M : L^2([0, T], \mathbb{R}^r) \to R^d$ is linear and bounded. By Proposition 2.2(iii) [40], the complete controllability of (13) is equivalent to the invertibility of the matrix MM^*. □

The matrix MM^* is called the conformable controllability Gramian and it is positive semidefinite, that is,

$$y^\intercal \left(\Theta_0^{t_p} + \Gamma_{t_p}^{T}\right) y \geq 0, \text{ for all } y \in \mathbb{R}^d.$$

Corollary 1. *The conformable impulsive linear Equation (13) is completely controllable on $[0, T]$, if and only if the $d \times d$ conformable controllability Gramian matrix is positive definite.*

Proof. By Theorem 5, the complete controllability of (13) is equivalent to invertibility of the matrix MM^*, which in turn is equivalent to the positivity of MM^*. □

Corollary 2. *The conformable impulsive linear Equation (13) is completely controllable on* $[0,T]$*, if* $\Theta_0^{t_p}$ *or* $\Gamma_{t_p}^T$ *is positive definite.*

Proof. By Theorem 5, the linear conformable impulsive Equation (13) is completely controllable on $[0,T]$, if and only if the $d\times d$ matrix is positive definite:

$$y^{\mathsf{T}}\left(\Theta_0^{t_p}+\Gamma_{t_p}^T\right)y>0,\ \text{for all } 0\neq y\in\mathbb{R}^d.$$

Since $\Theta_0^{t_p}+\Gamma_{t_p}^T$ is positive semidefinite, the positivity of $\Theta_0^{t_p}+\Gamma_{t_p}^T$ is equivalent to the positivity of $\Theta_0^{t_p}$ or $\Gamma_{t_p}^T$. □

Corollary 3. *The conformable impulsive linear Equation (13) is controllable on* $[0,T]$*, if*

$$\operatorname{rank}\left\{B\ \ AB\ \ A^2B\ \ A^{d-1}B\right\}=d.$$

Proof. It is known that the positivity of $\Gamma_{t_p}^T$ is equivalent to the Kalman rank condition:

$$\operatorname{rank}\left\{B\ \ AB\ \ A^2B\ \ A^{d-1}B\right\}=d.$$

Thus, by the Corollary 2, the conformable impulsive linear Equation (13) is controllable on $[0,T]$ □

5.2. Semilinear Systems

We introduce the following assumptions:

Assumption 1 (A$_1$). *Conformable controllability Gramian matrix* $\Theta_0^{t_p}+\Gamma_{t_p}^T$ *is invertible.*

Assumption 2 (A$_2$). *There exists a positive constant* $M_f>0$ *such that, for any* $t\in[0,T]$ *and* $y\in\mathbb{R}^d$*, we have*

$$\|f(t,y)\|\leq M_f.$$

In view of (A$_1$), for any $y\in C\left([0,T]\times\mathbb{R}^d\right)$, consider a control function $u(t;x)$ defined by

$$\begin{aligned}u(t;y):={}&M^*\left(\Theta_0^{t_p}+\Gamma_{t_p}^T\right)^{-1}\left(y_T-e_A\left(\frac{t^\alpha-t_k^\alpha}{\alpha}\right)\prod_{j=k}^{1}(I+C_j)e_A\left(\frac{t_j^\alpha-t_{j-1}^\alpha}{\alpha}\right)y_0\right.\\&-e_A\left(\frac{T^\alpha-t_k^\alpha}{\alpha}\right)\sum_{i=1}^{p}\prod_{j=p}^{i+1}(I+C_j)e_A\left(\frac{t_j^\alpha-t_{j-1}^\alpha}{\alpha}\right)(I+C_i)\int_{t_{i-1}}^{t_i}e_A\left(\frac{t_i^\alpha-s^\alpha}{\alpha}\right)f(s,y(s))ds\\&\left.-\int_{t_p}^{T}e_A\left(\frac{T^\alpha-s^\alpha}{\alpha}\right)f(s,y(s))ds\right).\end{aligned}$$

Next, we prove our main result via FPT. We firstly show that, using control $u(t;y)$, the operator $P:PC\left([0,T],\mathbb{R}^d\right)\to PC\left([0,T],\mathbb{R}^d\right)$ defined by

$$(Py)(t) := \begin{cases} e_A\left(\frac{t^\alpha}{\alpha}\right)y_0 + \int_0^t e_A\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)[f(s,y(s)) + Bu(s;y)]s^{\alpha-1}ds, \ \ 0 \le t \le t_1; \\ e_A\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right)\prod_{j=k}^{1}(I + C_j)e_A\left(\frac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right)y_0 \\ +e_A\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}(I + C_j)e_A\left(\frac{t_j^\alpha - t_{j-1}^\alpha}{\alpha}\right) \\ \quad \times(I + C_i)\int_{t_{i-1}}^{t_i} e_A\left(\frac{t_i^\alpha - s^\alpha}{\alpha}\right)[f(s,y(s)) + Bu(s;y)]s^{\alpha-1}ds \\ +\int_{t_k}^{t} e_A\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)[f(s,y(s)) + Bu(s;y)]s^{\alpha-1}ds, \ \ \ t_k < t \le t_{k+1}, \ k = 1,2,\dots,p \end{cases}$$

has a fixed point y^*. It can be easily checked that $(Py^*)(T) = y_T$ and $(Py^*)(0) = y_0$. In other words, $u(t;y)$ steers system (1) from y_0 to y_T in finite time T. Thus, system (1) is controllable on $[0,T]$.

Theorem 6. *Assumptions (A_1) and (A_2) are satisfied. Then, system (1) is completely controllable on* $[0,T]$.

Proof. Step 1. We prove the continuity of the control $u(t;\cdot)$.

Let y_n be a sequence with $y_n \to y$ in B_r as $n \to \infty$. For any $t \in [0,T]$, we have:

$$\begin{aligned} &\|u(t;y_n) - u(t;y)\| \\ &\le \|M^*\|\left\|\left(\Theta_0^{t_p} + \Gamma_{t_p}^T\right)^{-1}\right\|\prod_{j=p}^{1}(I + \|C_j\|)\int_0^T e_{\|A\|}\left(\frac{T^\alpha - s^\alpha}{\alpha}\right)\|f(s,y_n(s)) - f(s,y(s))\|s^{\alpha-1}ds. \end{aligned}$$

From the assumptions (A_1) and (A_2), it follows that

$$\max_{0\le s\le T}\|f(s,y_n(s)) - f(s,y(s))\| \to 0 \text{ as } n \to \infty,$$

$$e_{\|A\|}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\|f(s,y_n(s)) - f(s,y(s))\|s^{\alpha-1} \le 2M_f e_{\|A\|}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)s^{\alpha-1},$$

$$2M_f e_{\|A\|}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)s^{\alpha-1} \text{ is integrable with respect to } s \in [0,T].$$

It remains to apply the Lebesgue dominated theorem to get the continuity of $u(t;\cdot)$.

Step 1. We prove that the control $u(t;y)$ is bounded.

The boundedness of $u(t;y)$ follows from the same property (A_2) of f.

Now, we can mimic the proof of Theorem 4 to show that P has a fixed point y^* in $PC\left([0,T],\mathbb{R}^d\right)$, in other words, the system (1) is completely controllable on $[0,T]$. □

6. Examples

Example 1. *Consider the following three-dimensional system:*

$$\begin{cases} D_0^\alpha y(t) = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix} y(t) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} u(t), \ \ t \in [0,4]\backslash\{1,2,3\}, \\ \Delta y(t_i) = \frac{1}{4}y(t_i^-), \ \ t_i = i, \ \ ,i = 1,2,3, \\ y(0) = 0. \end{cases} \tag{14}$$

Now, we try to use our criteria to investigate the controllability on $[0,4]$ *of system (14). Denote by*

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C_i = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

One can obtain

$$\begin{aligned} &\operatorname{rank}\Big(B \;\; AB \;\; A^2B\Big) \\ &= \operatorname{rank}\begin{pmatrix} 1 & 0 & 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 4 & 2 \end{pmatrix} = 3. \end{aligned}$$

By Corollary 3, the system (14) is controllable on $[0,4]$*.*

Example 2. *Consider the following three-dimensional system:*

$$\begin{cases} D_0^\alpha y(t) = \begin{pmatrix} -1 & -4 & -2 \\ 0 & 6 & 1 \\ 1 & 7 & -1 \end{pmatrix} y(t) + \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{pmatrix} u(t), \;\; t \in [0,5] \backslash \{1,2,3,4\}, \\ \Delta y(t_i) = \frac{1}{5} y(t_i^-), \;\; t_i = i, \;\; i = 1,2,3,4, \\ y(0) = 0. \end{cases} \tag{15}$$

One can obtain

$$A = \begin{pmatrix} -1 & -4 & -2 \\ 0 & 6 & 1 \\ 1 & 7 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}, C_i = \frac{1}{5}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{aligned} &\operatorname{rank}\Big(B \;\; AB \;\; A^2B\Big) \\ &= \operatorname{rank}\begin{pmatrix} 1 & 0 & -3 & * & * & * \\ 2 & 1 & 19 & * & * & * \\ 0 & 1 & 1 & * & * & * \end{pmatrix} = 3. \end{aligned}$$

By Corollary 3, the system (15) is controllable on $[0,5]$*.*

Example 3. *Consider the following three-dimensional semilinear system:*

$$\begin{cases} D_0^\alpha y(t) = \begin{pmatrix} -1 & -4 & -2 \\ 0 & 6 & 1 \\ 1 & 7 & -1 \end{pmatrix} y(t) + \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{pmatrix} u(t) + \frac{1}{5} t \sin y(t), \;\; t \in [0,5] \backslash \{1,2,3,4\}, \\ \Delta y(t_i) = \frac{1}{5} y(t_i^-), \;\; t_i = i, \;\; , i = 1,2,3,4, \\ y(0) = 0. \end{cases} \tag{16}$$

By Example 2, the linear part is controllable and the nonlinear part is bounded. Using Theorem 6, we say the semilinear system (16) is completely controllable.

7. Conclusions

Fractional impulsive differential equations are mathematical models that describe systems with both fractional derivatives (derivatives of non-integer order) and impulsive (discontinuous) changes in the state variables. The study of the controllability of fractional impulsive differential equations is an active area of research, as these equations can be used to model a wide range of complex physical, biological, and engineering systems. The controllability results for fractional impulsive differential equations depend on various factors such as the fractional order, the type of impulsive changes, and the form of the control inputs. Further research is needed to fully understand the controllability of such systems.

We study the representation of a solution of conformable fractional type impulsive linear systems and investigate the existence/uniqueness of conformable fractional-type impulsive nonlinear systems. To show existence and uniqueness, we use the Picard iterative methods, while for existence, we use the Schauder fixed point theorem. Moreover, we study the complete controllability of a linear/semilinear conformable fractional-type impulsive controlled system. By using the conformable fractional derivative approach, we have introduced the conformable controllability Gramian matrix, which has the potential to provide new insights into the controllability behavior of these systems, and studied the controllability of conformable linear/semilinear impulsive systems. These results are innovative and application-based, and are likely to be highly useful for future research in this field.

For future work, we can present the approximate/null controllability of instantaneous/noninstantaneous impulsive conformable stochastic evolution equations/inclusions with different stochastic perturbations, see [20–22].

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