

Article

Constraint Qualifications for Vector Optimization Problems in Real Topological Spaces

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Abstract: In this paper, we introduce a series of definitions of generalized affine functions for vector-valued functions by use of “linear set”. We prove that our generalized affine functions have some similar properties to generalized convex functions. We present examples to show that our generalized affinities are different from one another, and also provide an example to show that our definition of presubaffinelikeness is non-trivial; presubaffinelikeness is the weakest generalized affinity introduced in this article. We work with optimization problems that are defined and taking values in linear topological spaces. We devote to the study of constraint qualifications, and derive some optimality conditions as well as a strong duality theorem. Our optimization problems have inequality constraints, equality constraints, and abstract constraints; our inequality constraints are generalized convex functions and equality constraints are generalized affine functions.

Keywords: real linear topological spaces; affine functions; generalized affine functions; convex functions; generalized convex functions; constraint qualifications

MSC: 90C26; 90C29; 90C48



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1. Introduction and Preliminary

The theory of vector optimization is at the crossroads of many subjects. The terms “minimum,” “maximum,” and “optimum” are in line with a mathematical tradition, while words such as “efficient” or “non-dominated” find larger use in business-related topics. Historically, linear programs were the focus in the optimization community, and initially, it was thought that the major divide was between linear and nonlinear optimization problems; later, people discovered that some nonlinear problems were much harder than others, and the “right” divide was between convex and nonconvex problems. The author has determined that affinity and generalized affinities are also very useful for the subject “optimization”.

Suppose X, Y are real linear topological spaces [1].

A subset $B \subseteq X$ is called a linear set if B is a nonempty vector subspace of X .

A subset $B \subseteq X$ is called an affine set if the line passing through any two points of B is entirely contained in B (i.e., $\alpha x_1 + (1 - \alpha)x_2 \in B$ whenever $x_1, x_2 \in B$ and $\alpha \in \mathbb{R}$);

A subset $B \subseteq X$ is called a convex set if any segment with endpoints in B is contained in B (i.e., $\alpha x_1 + (1 - \alpha)x_2 \in B$ whenever $x_1, x_2 \in B$ and $\alpha \in [0, 1]$).

Each linear set is affine, and each affine set is convex. Moreover, any translation of an affine (convex, respectively) set is affine (convex, resp.). It is known that a set B is linear if and only if B is affine and contains the zero point 0_X of X ; a set B is affine if and only if B is a translation of a linear set.

A subset Y_+ of Y is said to be a cone if $\lambda y \in Y_+$ for all $y \in Y_+$ and $\lambda \geq 0$. We denote by 0_Y the zero element in the topological vector space Y and simply by 0 if there is no confusion. A convex cone is one for which $\lambda_1 y_1 + \lambda_2 y_2 \in Y_+$ for all $y_1, y_2 \in Y_+$ and $\lambda_1, \lambda_2 \geq 0$. A pointed cone is one for which $Y_+ \cap (-Y_+) = \{0\}$. Let Y be a real topological

vector space with pointed convex cone Y_+ . We denote the partial order induced by Y_+ as follows:

$$y_1 \succ y_2 \text{ iff } y_1 - y_2 \in Y_+, \text{ or, } y_1 \prec y_2 \text{ iff } y_1 - y_2 \in -Y_+$$

$$y_1 \succ\!\succ y_2 \text{ iff } y_1 - y_2 \in \text{int}Y_+, \text{ or } y_1 \prec\!\prec y_2 \text{ iff } y_1 - y_2 \in -\text{int}Y_+$$

where $\text{int}Y_+$ denotes the topological interior of a set Y_+ .

A function $f: X \rightarrow Y$ is said to be linear if

$$f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2)$$

whenever $x_1, x_2 \in X$ and $\alpha, \beta \in R$; f is said to be affine if

$$f(\alpha x_1 + (1 - \alpha)x_2) = \alpha f(x_1) + (1 - \alpha)f(x_2)$$

whenever $x_1, x_2 \in D, \alpha \in R$; and f is said to be convex if

$$\alpha f(x_1) + (1 - \alpha)f(x_2) \prec f(\alpha x_1 + (1 - \alpha)x_2)$$

whenever $x_1, x_2 \in D, \alpha \in [0, 1]$.

In the next section, we generalize the definition of affine function, prove that our generalized affine functions have some similar properties with generalized convex functions, and present some examples which show that our generalized affinities are not equivalent to one another.

In Section 3, we recall some existing definitions of generalized convexities, which are very comparable with the definitions of generalized affinities introduced in this article.

Section 4 works with optimization problems that are defined and taking values in linear topological spaces, devotes to the study of constraint qualifications, and derives some optimality conditions as well as a strong duality theorem.

2. Generalized Affinities

A function $f: D \subseteq X \rightarrow Y$ is said to be affine on D if $\forall x_1, x_2 \in D, \forall \alpha \in R$, there holds

$$\alpha f(x_1) + (1 - \alpha)f(x_2) = f(\alpha x_1 + (1 - \alpha)x_2)$$

We introduce here the following definitions of generalized affine functions.

Definition 1. A function $f: D \subseteq X \rightarrow Y$ is said to be *affinelinear* on D if $\forall x_1, x_2 \in D, \forall \alpha \in R, \exists x_3 \in D$ such that

$$\alpha f(x_1) + (1 - \alpha)f(x_2) = f(x_3)$$

Definition 2. A function $f: D \subseteq X \rightarrow Y$ is said to be *preaffinelinear* on D if $\forall x_1, x_2 \in D, \forall \alpha \in R, \exists x_3 \in D, \exists \tau \in R \setminus \{0\}$ such that

$$\alpha f(x_1) + (1 - \alpha)f(x_2) = \tau f(x_3)$$

In the following Definitions 3 and 4, we assume that $B \subseteq Y$ is any given linear set.

Definition 3. A function $f: D \subseteq X \rightarrow Y$ is said to be *B-subaffinelinear* on D if $\forall x_1, x_2 \in D, \forall \alpha \in R, \exists u \in B, \exists x_3 \in D$ such that

$$u + \alpha f(x_1) + (1 - \alpha)f(x_2) = f(x_3)$$

Definition 4. A function $f: D \subseteq X \rightarrow Y$ is said to be B -presubaffinelike on D if $\forall x_1, x_2 \in D, \forall \alpha \in R, \exists u \in B, \exists x_3 \in D, \exists \tau \in R \setminus \{0\}$ such that

$$u + \alpha f(x_1) + (1 - \alpha)f(x_2) = \tau f(x_3)$$

For any linear set B , since $0 \in B$, we may take $u = 0$. So, affineness implies subaffineness, and preaffineness implies presubaffineness.

It is obvious that affineness implies preaffineness, and the following Example 1 shows that the converse is not true.

Example 1. An example of an affinelike function which is not an affine function.

It is known that a function is an affine function if and only if it is in the form of $f(x) = ax + b$; therefore

$$f(x) = x^3, x \in R$$

is not an affine function.

However, f is affinelike. $\forall x_1, x_2 \in R, \forall \alpha \in R$, taking

$$x_3 = [\alpha f(x_1) + (1 - \alpha)f(x_2)]^{1/3}$$

then

$$\alpha f(x_1) + (1 - \alpha)f(x_2) = f(x_3)$$

Similarly, affineness implies preaffineness ($\tau = 1$), and presubaffineness implies subaffineness. The following Example 2 shows that a preaffinelike function is not necessary to be an affinelike function.

Example 2. An example of a preaffinelike function which is not an affinelike function.

Consider the function $f(x) = x^2, x \in R$.

Take $x_1 = 0, x_2 = 1, \alpha = 2$, then $\alpha f(x_1) + (1 - \alpha)f(x_2) = -1$; but

$$\forall x_3 \in R, f(x_3) = x_3^2 \geq 0$$

therefore

$$\alpha f(x_1) + (1 - \alpha)f(x_2) \neq f(x_3), \forall x_3 \in R$$

So f is not affinelike.

But f is a preaffinelike function. For $\forall x_1, x_2 \in R, \forall \alpha \in R$, taking $\tau = 1$ if $\alpha f(x_1) + (1 - \alpha)f(x_2) \geq 0, \tau = -1$ if $\alpha f(x_1) + (1 - \alpha)f(x_2) < 0$, then

$$\alpha f(x_1) + (1 - \alpha)f(x_2) = \tau f(x_3)$$

where $x_3 = |\alpha f(x_1) + (1 - \alpha)f(x_2)|^{1/2}$.

Example 3. An example of a subaffinelike function which is not an affinelike function.

Consider the function $f(x) = x^3 + 8, x \in D = [0, 1]$, and the linear set $B = R$.

$\forall x_1, x_2 \in D = [0, 1], \forall \alpha \in R$, taking $x_3 = 1 \in D, u = 8 - [\alpha f(x_1) + (1 - \alpha)f(x_2)] \in B$, then

$$u + \alpha f(x_1) + (1 - \alpha)f(x_2) = f(x_3)$$

therefore $f(x) = x^3 + 8, x \in [0, 1]$ is B -subaffinelike on $D = [0, 1]$.

$f(x) = x^3 + 8, x \in [0, 1]$ is not affineline on $D = [0, 1]$. Actually, for $\alpha = -8 \in R$, $x_1 = 1 \in D, x_2 = 0 \in D = [0, 1]$, one has $\alpha f(x_1) + (1 - \alpha)f(x_2) = 0$, but

$$f(x_3) = x_3^3 + 8 \neq 0, \forall x \in [0, 1]$$

hence

$$\alpha f(x_1) + (1 - \alpha)f(x_2) \neq f(x_3), \forall x_3 \in D = [0, 1]$$

Example 4. An example of a presubaffineline function which is not a preaffineline function.

Actually, the function in Example 3 is subaffineline, therefore it is presubaffineline on D .

However, for $\alpha = 9 \in R, x_1 = 0 \in D, x_2 = 1 \in D$, one has

$$\alpha f(x_1) + (1 - \alpha)f(x_2) = 0$$

but

$$f(x_3) = x_3^3 - 8 \neq 0, \forall x \in [0, 1]$$

Hence

$$\alpha f(x_1) + (1 - \alpha)f(x_2) \neq \tau f(x_3), \forall x_3 \in D = [0, 1], \forall \tau \neq 0$$

This shows that the function f is not preaffineline on D .

Example 5. An example of a presubaffineline function which is not a subaffineline function.

Consider the function $f(x, y) = (x^2, y^2), x, y \in R$.

Take the 2-dimensional linear set $B = \{(x, y) : y = -x, x \in R\}$.

Take $\alpha = 3, (x_1, y_1) = (0, 0), (x_2, y_2) = (1, 1)$, then

$$\alpha f(x_1, y_1) + (1 - \alpha)f(x_2, y_2) = (-2, -2)$$

Either $x - 2$ or $-x - 2$ must be negative; but $x_3^2 \geq 0, y_3^2 \geq 0, \forall u = (x, -x) \in B$; therefore

$$u + \alpha f(x_1, y_1) + (1 - \alpha)f(x_2, y_2) = (x - 2, -x - 2) \neq f(x_3, y_3) = (x_3^2, y_3^2)$$

And so, $f(x, y) = (x^2, y^2)$ is not B -subaffineline.

However, $f(x, y) = (x^2, y^2)$ is B -presubaffineline.

$$\forall x_1, x_2 \in [0, 1], \forall \alpha \in R$$

$$\alpha f(x_1, y_1) + (1 - \alpha)f(x_2, y_2) = (\alpha x_1^2 + (1 - \alpha)x_2^2, \alpha y_1^2 + (1 - \alpha)y_2^2)$$

Case 1. If both of $\alpha x_1^2 + (1 - \alpha)x_2^2, \alpha y_1^2 + (1 - \alpha)y_2^2$ are positive, we take $u = (0, 0)$, $\tau = 1, x_3 = |\alpha x_1^2 + (1 - \alpha)x_2^2|^{1/2}, y_3 = |\alpha y_1^2 + (1 - \alpha)y_2^2|^{1/2}$, then

$$u + \alpha f(x_1) + (1 - \alpha)f(x_2) = \tau f(x_3)$$

Case 2. If both of $\alpha x_1^2 + (1 - \alpha)x_2^2, \alpha y_1^2 + (1 - \alpha)y_2^2$ are negative, we take $u = (0, 0)$, $\tau = -1, x_3 = |\alpha x_1^2 + (1 - \alpha)x_2^2|^{1/2}, y_3 = |\alpha y_1^2 + (1 - \alpha)y_2^2|^{1/2}$, then

$$u + \alpha f(x_1) + (1 - \alpha)f(x_2) = \tau f(x_3)$$

Case 3. If one of $\alpha x_1^2 + (1 - \alpha)x_2^2$, $\alpha y_1^2 + (1 - \alpha)y_2^2$ is negative, and the other is non-negative, we take

$$x = [(\alpha y_1^2 + (1 - \alpha)y_2^2) - (\alpha x_1^2 + (1 - \alpha)x_2^2)]/2, \text{ and } u = (x, -x) \in B$$

Then

$$\begin{aligned} & x + \alpha x_1^2 + (1 - \alpha)x_2^2 \\ &= -x + \alpha y_1^2 + (1 - \alpha)y_2^2 \\ &= [\alpha x_1^2 + (1 - \alpha)x_2^2 + \alpha y_1^2 + (1 - \alpha)y_2^2]/2 \end{aligned}$$

And so $x + \alpha x_1^2 + (1 - \alpha)x_2^2$, $-x + \alpha y_1^2 + (1 - \alpha)y_2^2$ are both non-negative or both negative; taking $\tau = 1$ or $\tau = -1$, respectively, one has

$$u + \alpha f(x_1) + (1 - \alpha)f(x_2) = \tau f(x_3)$$

where

$$x_3 = |x + \alpha x_1^2 + (1 - \alpha)x_2^2|^{1/2}, y_3 = | -x + \alpha y_1^2 + (1 - \alpha)y_2^2 |^{1/2}$$

Therefore, $f(x, y) = (x^2, y^2)$ is B -presubaffineline.

Example 6. An example of a subaffineline function which is not a preaffineline function.

Consider the function $f(x, y) = (x^2, y^2)$, $x, y \in R$.

Take the 2-dimensional linear set $B = \{(x, y) : y = x, x \in R\}$.

Take $x_1 = 0, x_2 = 1, \alpha = 2$, then

$$\begin{aligned} & \alpha f(x_1, y_1) + (1 - \alpha)f(x_2, y_2) \\ &= (\alpha x_1^2 + (1 - \alpha)x_2^2, \alpha y_1^2 + (1 - \alpha)y_2^2) \\ &= (-2, 3) \\ &\neq \tau f(x_3, y_3) = (\tau x_3^2, \tau y_3^2). \end{aligned}$$

In the above inequality, we note that either $\tau x_3^2 \geq 0, \tau y_3^2 \geq 0$ or $\tau x_3^2 \leq 0, \tau y_3^2 \leq 0$, $\forall \tau \neq 0$.

Therefore, $f(x, y) = (x^2, y^2)$ is not preaffineline.

However, $f(x, y) = (x^2, y^2)$, $x, y \in R$ is B -subaffineline.

In fact, $\forall x_1, x_2 \in R, \forall \alpha \in R$, we may choose $u = (x, x) \in B$ with x large enough such that

$$u + \alpha f(x_1, y_1) + (1 - \alpha)f(x_2, y_2) = (x + \alpha x_1^2 + (1 - \alpha)x_2^2, x + \alpha y_1^2 + (1 - \alpha)y_2^2) \succ 0$$

Then,

$$u + \alpha f(x_1, y_1) + (1 - \alpha)f(x_2, y_2) = f(x_3, y_3)$$

where

$$x_3 = (x + \alpha x_1^2 + (1 - \alpha)x_2^2)^{1/2} \text{ and } y_3 = (x + \alpha y_1^2 + (1 - \alpha)y_2^2)^{1/2}$$

Example 7. An example of a preaffineline function which is not a subaffineline function.

Consider the function $f(x, y) = (x^2, -x^2)$, $x, y \in R$.

Take the 2-dimensional linear set $B = \{(x, y) : y = x, x \in R\}$.

Take $x_1 = 0, x_2 = 1, \alpha = 2$, then

$$\alpha f(x_1, y_1) + (1 - \alpha)f(x_2, y_2) = (\alpha x_1^2 + (1 - \alpha)x_2^2, -(\alpha x_1^2 + (1 - \alpha)x_2^2)) = (-1, 1)$$

So, $\forall u = (x, x) \in B$,

$$u + \alpha f(x_1, y_1) + (1 - \alpha)f(x_2, y_2) = (x + 1, x - 1)$$

However, for $f(x_3, y_3) = (x_3^2, -x_3^2), \forall x_3 \in R$,

$$(x_3^2, -x_3^2) \neq (x-1, x+1), \forall x, x_3 \in R \quad (1)$$

Actually, if $x = 0$, it is obvious that $(x_3^2, -x_3^2) \neq (-1, 1)$; if $x \neq 0$, the right side of (1) implies that $x_3^2 + (-x_3^2) = 0$, and the left side of (1) is $(x-1) + (x+1) = 2x \neq 0$. This proves that the inequality (1) must be true. Consequently,

$$u + \alpha f(x_1, y_1) + (1 - \alpha)f(x_2, y_2) \neq f(x_3, y_3), \forall \alpha \in R, \forall x_1, x_2, x_3, y_1, y_2, y_3 \in R$$

So $f(x, y) = (x^2, -x^2), x, y \in R$ is not B -subaffinelike.

On the other hand, $\forall x_1, x_2 \in R, \forall \alpha \in R$, we may take $\tau = 1$ if $\alpha x_1^2 + (1 - \alpha)x_2^2 \geq 0$ or $\tau = -1$ if $\alpha x_1^2 + (1 - \alpha)x_2^2 \leq 0$, then

$$\begin{aligned} & \alpha f(x_1, y_1) + (1 - \alpha)f(x_2, y_2) \\ &= (\alpha x_1^2 + (1 - \alpha)x_2^2, -(\alpha x_1^2 + (1 - \alpha)x_2^2)) \\ &= \tau(x_3^2, -x_3^2) \\ &= \tau f(x_3, y_3) \end{aligned}$$

where $x_3 = |\alpha x_1^2 + (1 - \alpha)x_2^2|^{1/2}$.

Therefore, $f(x, y) = (x^2, -x^2), x, y \in R$ is preaffinelike.

So far, we have showed the following relationships (where subaffinelikeness and presubaffinelikeness are related to “a given linear set B ”):

$$\begin{array}{ccc} \text{affineness} & \xrightleftharpoons[\text{not true}]{\text{true}} & \text{affinelikeness} & \xrightleftharpoons[\text{not true}]{\text{true}} & \text{preaffinelikeness} \\ & & \text{nottrue} \uparrow \downarrow \text{true} & \text{nottrue} \swarrow \nearrow \text{nottrue} & \text{nottrue} \uparrow \downarrow \text{true} \\ & & \text{subaffinelikeness} & \xrightleftharpoons[\text{not true}]{\text{true}} & \text{presubaffinelikeness} \end{array}$$

The following Proposition 1 is very similar to the corresponding results for generalized convexities (see Proposition 2).

Proposition 1. Suppose $f: D \subseteq X \rightarrow Y$ is a function, $B \subseteq Y$ a given linear set, and t is any real scalar.

- f is affinelike on D if and only if $f(D)$ is an affine set;
- f is preaffinelike on D if and only if $\cup_{t \in R \setminus \{0\}} t f(D)$ is an affine set;
- f is B -subaffinelike on D if and only if $f(D) + B$ is an affine set;
- f is B -presubaffinelike on D if and only if $\cup_{t \in R \setminus \{0\}} t f(D) + B$ is an affine set.

Proof. (a) If f is affinelike on D , $\forall f(x_1), f(x_2) \in f(D), \forall \alpha \in R, \exists x_3 \in D$ such that

$$\alpha f(x_1) + (1 - \alpha)f(x_2) = f(x_3) \in f(D)$$

Therefore, $f(D)$ is an affine set.

On the other hand, assume that $f(D)$ is an affine set. $\forall x_1, x_2 \in D, \forall \alpha \in R$, we have

$$\alpha f(x_1) + (1 - \alpha)f(x_2) \in f(D)$$

Therefore, $\exists x_3 \in D$ such that

$$\alpha f(x_1) + (1 - \alpha)f(x_2) = f(x_3)$$

And hence f is affinelike on D .

(b) Assume f is a preaffinelike function.

$\forall y_1, y_2 \in \cup_{t \in R \setminus \{0\}} tf(D)$, $\forall \alpha \in R$, $\exists x_1, x_2 \in D$, $\exists t_1, t_2 \in R \setminus \{0\}$ for $\exists x_3 \in D$, $\exists t \in R \setminus \{0\}$ such that

$$\begin{aligned} & \alpha y_1 + (1 - \alpha)y_2 \\ &= \alpha t_1 f(x_1) + (1 - \alpha)t_2 f(x_2) \\ &= (\alpha t_1 + (1 - \alpha)t_2) \left[\frac{\alpha t_1}{\alpha t_1 + (1 - \alpha)t_2} f(x_1) + \frac{(1 - \alpha)t_2}{\alpha t_1 + (1 - \alpha)t_2} f(x_2) \right]. \end{aligned}$$

Since f is preaffinelike, $\exists x_3 \in D$, $\exists t \in R \setminus \{0\}$ such that

$$\frac{\alpha t_1}{\alpha t_1 + (1 - \alpha)t_2} f(x_1) + \frac{(1 - \alpha)t_2}{\alpha t_1 + (1 - \alpha)t_2} f(x_2) = tf(x_3)$$

Therefore

$$\begin{aligned} & \alpha y_1 + (1 - \alpha)y_2 \\ &= \alpha t_1 f(x_1) + (1 - \alpha)t_2 f(x_2) \\ &= (\alpha t_1 + (1 - \alpha)t_2) \left[\frac{\alpha t_1}{\alpha t_1 + (1 - \alpha)t_2} f(x_1) + \frac{(1 - \alpha)t_2}{\alpha t_1 + (1 - \alpha)t_2} f(x_2) \right] \\ &= (\alpha t_1 + (1 - \alpha)t_2) tf(x_3) \\ &= \tau f(x_3) \in \cup_{t \in R \setminus \{0\}} tf(D) \end{aligned}$$

where $\tau = (\alpha t_1 + (1 - \alpha)t_2)t$. Consequently, $\cup_{t \in R \setminus \{0\}} tf(D)$ is an affine set.

On the other hand, suppose that $\cup_{t \in R \setminus \{0\}} tf(D)$ is an affine set. Then, $\forall x_1, x_2 \in D$, $\forall \alpha \in R$, since $f(x_1), f(x_2) \in \cup_{t \in R \setminus \{0\}} tf(D)$,

$$\alpha f(x_1) + (1 - \alpha)f(x_2) \in \cup_{t \in R \setminus \{0\}} tf(D)$$

Therefore, $\exists x_3 \in D$, $\exists \tau \neq 0$ such that

$$\alpha f(x_1) + (1 - \alpha)f(x_2) = \tau f(x_3)$$

Then, f is an affinelike function.

(c) Assume that f is B -subaffinelike.

$\forall y_1, y_2 \in f(D) + B$, $\exists x_1, x_2 \in D$, $\exists b_1, b_2 \in B$, such that $y_1 = f(x_1) + b_1$ and $y_2 = f(x_2) + b_2$. The subaffinelikeness of f implies that $\forall \alpha \in R$, $\exists x_3 \in D$, and $\exists v \in B$ such that

$$v + \alpha f(x_1) + (1 - \alpha)f(x_2) = f(x_3)$$

i.e.,

$$\alpha f(x_1) + (1 - \alpha)f(x_2) = f(x_3) - v$$

Therefore

$$\begin{aligned} & \alpha y_1 + (1 - \alpha)y_2 \\ &= \alpha(f(x_1) + b_1) + (1 - \alpha)(f(x_2) + b_2) \\ &= f(x_3) - v + \alpha b_1 + (1 - \alpha)b_2 \\ &= f(x_3) + u \in f(D) + B \end{aligned}$$

where $u = -v + \alpha b_1 + (1 - \alpha)b_2 \in B$

Then, $f(D) + B$ is an affine set.

On the other hand, assume that $f(D) + B$ is an affine set.

$\forall x_1, x_2 \in D$, $\forall \alpha \in R$, $\exists b_1, b_2, b_3 \in B$, $\exists x_3 \in D$, such that

$$\alpha(f(x_1) + b_1) + (1 - \alpha)(f(x_2) + b_2) = f(x_3) + b_3$$

i.e.,

$$u + \alpha f(x_1) + (1 - \alpha)f(x_2) = f(x_3)$$

where $\alpha b_1 + (1 - \alpha)b_2 - b_3 \in B$. And hence f is B -subaffinelike.

(d) Suppose f is a B -presubaffinelike function.

$\forall y_1, y_2 \in \cup_{t \in R \setminus \{0\}} tf(D) + B$, similar to the proof of (b), $\forall \alpha \in R, \exists x_1, x_2, x_3 \in D, \exists b_1, b_2, b_3, u \in B, \exists t_1, t_2, t_3 \in R \setminus \{0\}$, for which $y_1 = t_1 f(x_1) + b_1, y_2 = t_2 f(x_2) + b_2$, and

$$\begin{aligned} & \alpha y_1 + (1 - \alpha)y_2 \\ &= \alpha t_1 f(x_1) + (1 - \alpha)t_2 f(x_2) + \alpha b_1 + (1 - \alpha)b_2 \\ &= (\alpha t_1 + (1 - \alpha)t_2)[t_3 f(x_3) + b_3 - u] + \alpha b_1 + (1 - \alpha)b_2 \\ &= (\alpha t_1 + (1 - \alpha)t_2)t_3 f(x_3) + \alpha b_1 + (1 - \alpha)b_2 + (\alpha t_1 + (1 - \alpha)t_2)(b_3 - u) \\ &\in tf(D) + B \subseteq \cup_{t \in R \setminus \{0\}} tf(D) + B \end{aligned}$$

where $t = (\alpha t_1 + (1 - \alpha)t_2)t_3$. This proves that $\cup_{t \in R \setminus \{0\}} tf(D) + B$ is an affine set.

On the other hand, assume that $\cup_{t \in R \setminus \{0\}} tf(D) + B$ is an affine set.

$\forall x_1, x_2 \in D, \forall b_1, b_2 \in B, \forall \alpha \in R$, since $f(x_1) + b_1, f(x_2) + b_2 \in \cup_{t \in R \setminus \{0\}} tf(D) + B, \exists x_3 \in D, \exists b_3 \in B, \exists t \in R \setminus \{0\}$ such that

$$\alpha(f(x_1) + b_1) + (1 - \alpha)(f(x_2) + b_2) = tf(x_3) + b_3$$

Therefore,

$$\alpha b_1 + (1 - \alpha)b_2 - b_3 + \alpha f(x_1) + (1 - \alpha)f(x_2) = tf(x_3)$$

i.e.,

$$u + \alpha f(x_1) + (1 - \alpha)f(x_2) = tf(x_3)$$

where $u = \alpha b_1 + (1 - \alpha)b_2 - b_3 \in B$. And so f is B -presubaffinelike. $\square$

The presubaffineness is the weakest one in the series of the generalized affinences introduced here. The following example shows that our definition of presubaffinelikeness is not trivial.

Example 8. An example of non-presubaffinelike function.

Consider the function $f(x, y, z) = (x^2, y^2, z^2), x, y, z \in R$.

Take the linear set $B = \{(x, -x, 0) : x \in R\}$.

Take $\alpha = 5, (x_1, y_1, z_1) = (0, 0, 1), (x_2, y_2, z_2) = (1, 1, 0)$, then

$$\alpha f(x_1, y_1, z_1) + (1 - \alpha)f(x_2, y_2, z_2) = (-4, -4, 5)$$

Either $x - 4$ or $-x - 4$ must be negative, but $x_3^2 \geq 0, y_3^2 \geq 0$ hold for $\forall u = (x, -x, 0) \in B$; therefore, for any scalar $\tau \neq 0$

$$u + \alpha f(x_1, y_1, z_1) + (1 - \alpha)f(x_2, y_2, z_2) = (x - 4, -x - 4, 5) \neq \tau f(x_3, y_3, z_3) = \tau(x_3^2, y_3^2, z_3^2)$$

(Actually, $\forall \tau < 0$, one has $\tau z_3^2 \leq 0 < 5$; and $\forall \tau > 0$, either $\tau(x - 4) < 0$ or $\tau(-x - 4) < 0$, then, either $\tau(x - 4) < 0 \leq \tau x_3^2$ or $\tau(-x - 4) < 0 \leq \tau y_3^2$).

And so, $f(x, y) = (x^2, y^2)$ is not B -presubaffinelike.

3. Generalized Convexities

In this section, we recall some existing definitions of generalized convexities, which are very comparable with the definitions of generalized affinences introduced in this article.

Let Y be a topological vector space, $D \subseteq X$ be a nonempty set, and Y_+ be a convex cone in Y and $\text{int}Y_+ \neq \emptyset$.

It is known that a function $f: D \rightarrow Y$ is said to be Y_+ -convex on D if, for all $x_1, x_2 \in D, \alpha \in [0, 1]$, there holds

$$\alpha f(x_1) + (1 - \alpha)f(x_2) \prec f(\alpha x_1 + (1 - \alpha)x_2)$$

The following Definition 5 was introduced in Fan [2].

Definition 5. A function $f: D \rightarrow Y$ is said to be Y_+ -convexlike on D if $\forall, \forall \alpha \in [0, 1], \exists x_3 \in D$ such that

$$\alpha f(x_1) + (1 - \alpha)f(x_2) \prec f(x_3)$$

We may define Y_+ -preconvexlike functions as follows.

Definition 6. A function $f: D \rightarrow Y$ is said to be Y_+ -preconvexlike on D if $\forall x_1, x_2 \in D, \forall \alpha \in [0, 1], \exists x_3 \in D, \exists \tau > 0$ such that

$$\alpha f(x_1) + (1 - \alpha)f(x_2) \prec \tau f(x_3).$$

Definition 7 was introduced by Jeyakumar [3].

Definition 7. A function $f: D \rightarrow Y$ is said to be Y_+ -subconvexlike on D if $\forall u \in \text{int}Y_+, \forall x_1, x_2 \in D, \forall \alpha \in [0, 1], \exists x_3 \in D$ such that

$$u + \alpha f(x_1) + (1 - \alpha)f(x_2) \prec f(x_3)$$

In fact, in Jeyakumar [3], the definition of subconvexlike was introduced as the following form Definition 8.

Definition 8. A function $f: D \rightarrow Y$ is said to be Y_+ -subconvexlike on D if $\exists u \in \text{int}Y_+, \forall \varepsilon > 0, \forall x_1, x_2 \in D, \forall \alpha \in [0, 1], \exists x_3 \in D$ such that

$$\varepsilon u + \alpha f(x_1) + (1 - \alpha)f(x_2) \prec f(x_3)$$

Li and Wang ([4]) proved that: A function $f: D \rightarrow Y$ is Y_+ -subconvexlike on D by Definition 8 if and only if $\forall u \in \text{int}Y_+, \forall x_1, x_2 \in D, \forall \alpha \in [0, 1], \exists x_3 \in D$ such that

$$u + \alpha f(x_1) + (1 - \alpha)f(x_2) \prec f(x_3)$$

From the definitions above, one may introduce the following definition of presubconvexlike functions.

Definition 9. A function $f: D \rightarrow Y$ is said to be Y_+ -presubconvexlike on D if $\forall u \in \text{int}Y_+, \forall x_1, x_2 \in D, \forall \alpha \in [0, 1], \exists x_3 \in D, \exists \tau > 0$ such that

$$u + \alpha f(x_1) + (1 - \alpha)f(x_2) \prec \tau f(x_3)$$

And, similar to ([4]), one can prove that a function $f: D \rightarrow Y$ is Y_+ -presubconvexlike on D if and only if $\exists u \in \text{int}Y_+, \forall \varepsilon > 0, \forall x_1, x_2 \in D, \forall \alpha \in [0, 1], \exists x_3 \in D, \exists \tau > 0$ such that

$$\varepsilon u + \alpha f(x_1) + (1 - \alpha)f(x_2) \prec \tau f(x_3)$$

Our Definitions 7 and 9 are more comparable with our definitions of generalized affineness. Similar to the proof of the above Proposition 1, we present the following Proposition 2. Some examples of generalized convexities were given in [5,6].

Proposition 2. Let $f: X \rightarrow Y$ be function, and $t > 0$ be any positive scalar, then

- f is Y_+ -convexlike on D if and only if $f(D) + Y_+$ is convex;
- f is Y_+ -subconvexlike on D if and only if $f(D) + \text{int}Y_+$ is convex;
- f is Y_+ -preconvexlike on D if and only if $\cup_{t>0} tf(D) + Y_+$ is convex;
- f is Y_+ -presubconvexlike on D if and only if $\cup_{t>0} tf(D) + \text{int}Y_+$ is convex.

4. Constraint Qualifications

Consider the following vector optimization problem:

$$(VP) \quad \begin{aligned} & Y_+ - \min f(x) \\ & g_i(x) \prec 0, i = 1, 2, \dots, m; \\ & h_j(x) = 0, j = 1, 2, \dots, n; \\ & x \in D \end{aligned}$$

where $f: X \rightarrow Y$, $g_i: X \rightarrow Z_i$, $h_j: X \rightarrow W_j$, Y_+ , Z_{i+} are closed convex cones in Y and Z_i , respectively, and D is a nonempty subset of X .

Throughout this paper, the following assumptions will be used (τ_i, t_j are real scalars).

$$(A1) \forall x_1, x_2 \in D, \forall \alpha \in [0, 1], \exists u_0 \in \text{int} Y_+, \exists u_i \in \text{int} Z_{i+} (i = 1, 2, \dots, n), \exists x_3 \in D$$

$$\exists \tau_i > 0 (i = 0, 1, 2, \dots, m), \exists t_j \neq 0 (j = 1, 2, \dots, n) \text{ such that}$$

$$\begin{aligned} u_0 + \alpha f(x_1) + (1 - \alpha)f(x_2) &\prec \tau_0 f(x_3) \\ u_i + \alpha g_i(x_1) + (1 - \alpha)g_i(x_2) &\prec \tau_i g_i(x_3) \\ \alpha h_j(x_1) + (1 - \alpha)h_j(x_2) &= t_j h_j(x_3) \end{aligned}$$

$$(A2) \text{int} h_j(D) \neq \emptyset, (j = 1, 2, \dots, n)$$

$$(A3) W_j (j = 1, 2, \dots, n) \text{ are finite dimensional spaces.}$$

Remark 1. We note that the condition (A1) says that f and $g_i (i = 1, 2, \dots, m)$ are presubconvexlike, and $h_j (j = 1, 2, \dots, n)$ are preaffinelike.

Let F be the feasible set of (VP), i.e.,

$$F := \{x \in D : g_i(x) \prec 0, i = 1, 2, \dots, m; h_j(x) = 0, j = 1, 2, \dots, n\}$$

The following is the well-known definition of a weakly efficient solution.

Definition 10. A point $\bar{x} \in F$ is said to be a weakly efficient solution of (VP) with a weakly efficient value $\bar{y} \in f(\bar{x})$ if for every $x \in F$ there exists no $y \in f(x)$ satisfying $\bar{y} \succ y$.

We first introduce the following constraint qualification which is similar to the constraint qualification in the differentiate form from nonlinear programming.

Definition 11. Let $\bar{x} \in F$. We say that (VP) satisfies the No Nonzero Abnormal Multiplier Constraint Qualification (NNAMCQ) at $\bar{x}$ if there is no nonzero vector $(\eta, \varsigma) \in \Pi_{i=1}^m Z_i^* \times \Pi_{j=1}^n W_j^*$ satisfying the system

$$\begin{aligned} \min_{x \in D \cap U(\bar{x})} [\sum_{i=1}^m \eta_i g_i(x) + \sum_{j=1}^n \varsigma_j h_j(x)] &= 0 \\ \sum_{i=1}^m \eta_i g_i(\bar{x}) &= 0 \end{aligned}$$

where $U(\bar{x})$ is some neighborhood of $\bar{x}$.

It is obvious that NNAMCQ holds at $\bar{x} \in F$ with $U(\bar{x})$ being the whole space X if and only if for all $(\eta, \varsigma) \in (\Pi_{i=1}^m Z_i^* \times \Pi_{j=1}^n W_j^* \setminus \{0\})$ satisfying $\min \sum_{i=1}^m \eta_i g_i(\bar{x}) = 0$, there exists $x \in D$ such that

$$(\sum_{i=1}^m \eta_i g_i(x) + \sum_{j=1}^n \varsigma_j h_j(x)) \prec \prec 0$$

Hence, NNAMCQ is weaker than ([7], (CQ1)) (in [7], CQ1 was for set-valued optimization problems) in the constraint $\min \sum_{i=1}^m \eta_i g_i(\bar{x}) = 0$, which means that only the

binding constraints are considered. Under the NNAMCQ, the following KuhnTucker type necessary optimality condition holds.

Theorem 1. Assume that the generalized convexity assumption (A1) is satisfied and either (A2) or (A3) holds. If $\bar{x} \in F$ is a weakly efficient solution of (VP) with $\bar{y} \in f(\bar{x})$, then exists a vector $(\xi, \eta, \varsigma) \in Y^* \times \Pi_{i=1}^m Z_i^* \times \Pi_{j=1}^n W_j^*$ with $\xi \neq 0$ such that

$$\begin{aligned} \xi(\bar{y}) &= \min_{x \in D \cap U(\bar{x})} [\xi(f(x)) + \sum_{i=1}^m \eta_i(g_i(x)) + \sum_{j=1}^n \varsigma_j(h_j(x))] \\ \sum_{i=1}^m \eta_i(g_i(\bar{x})) &= 0 \end{aligned} \quad (2)$$

for a neighborhood $U(\bar{x})$ of $\bar{x}$.

Proof. Since $\bar{x}$ is a weakly efficient solution of (VP) with $\bar{y} \in f(\bar{x})$ there exists a nonzero vector $(\xi, \eta, \varsigma) \in Y^* \times \Pi_{i=1}^m Z_i^* \times \Pi_{j=1}^n W_j^*$ such that (2) holds. Since NNAMCQ holds at $\bar{x}$, ξ must be nonzero. Otherwise if $\xi = 0$ then (η, ς) must be a nonzero solution of

$$\begin{aligned} 0 &= \min_{x \in D \cap U(\bar{x})} [\sum_{i=1}^m \eta_i(g_i(x)) + \sum_{j=1}^n \varsigma_j(h_j(x))] \\ \sum_{i=1}^m \eta_i(g_i(\bar{x})) &= 0 \end{aligned}$$

But this is impossible, since the NNAMCQ holds at $\bar{x}$. $\square$

Similar to ([7], (CQ2)) which is slightly stronger than ([7], (CQ1)), we define the following constraint qualification which is stronger than the NNAMCQ.

Definition 12. (SNNAMCQ) Let $\bar{x} \in F$. We say that (VP) satisfies the No Nonzero Abnormal Multiplier Constraint Qualification (NNAMCQ) at $\bar{x}$ provided that

$$(i) \quad \forall \eta \in \Pi_{i=1}^m Z_i^* \setminus \{0\} \text{ satisfying } \sum_{i=1}^m \eta_i(g_i(\bar{x})) = 0,$$

$$\exists x \in D, \text{ s.t. } h_j(x) = 0, \eta_i(g_i(x)) \prec \prec 0$$

$$(ii) \quad \forall \varsigma \in \Pi_{j=1}^n W_j^* \setminus \{0\}, \exists x \in D, \text{ s.t. } \varsigma_j(h_j(x)) \prec \prec 0 \text{ for all } j = 1, 2, \dots, n.$$

We now quote the Slater condition introduced in ([7], (CQ3)).

Definition 13 (Slater Condition CQ). Let $\bar{x} \in F$. We say that (VP) satisfies the Slater condition at $\bar{x}$ if the following conditions hold:

$$(i) \quad \exists x \in D, \text{ s.t. } h_j(x) = 0, g_i(x) \prec \prec 0;$$

$$(ii) \quad 0 \in \text{int} h_j(D) \text{ for all } j.$$

Similar to ([7], Proposition 2) (again, in [7], discussions are made for set-valued optimization problems), we have the following relationship between the constraint qualifications.

Proposition 3. The following statements are true:

(i) Slater CQ $\Rightarrow$ SNNAMCQ $\Rightarrow$ NNAMCQ with $U(\bar{x})$ being the whole space X ;

(ii) Assume that (A1) and (A2) (or (A1) and (A3)) hold and the NNAMCQ with $U(\bar{x})$ being the whole space X without the restriction of $\sum_{i=1}^m \eta_i(g_i(\bar{x})) = 0$ at $\bar{x}$. Then, the Slater condition (CQ) holds.

Proof. The proof of (i) is similar to ([7], Proposition 2). Now we prove (ii). By the assumption (A1), the following sets C_1 and C_2 are convex:

$$\begin{aligned} C_1 &= \left\{ (z, w) \in \Pi_{i=1}^m Z_i^* \times \Pi_{j=1}^n W_j^* : \exists x \in D, \tau_i, t_j > 0, \text{ s.t. } z_i \in \tau_i g_i(x) + \text{int} Z_{i+}, w_j \in t_j h_j(x) \right\} \\ C_2 &= \cup_{t > 0} t h(D) \end{aligned}$$

Suppose to the contrary that the Slater condition does not hold. Then, $0 \notin C_1$ or $0 \notin C_2$. If the former $0 \notin C_1$ holds, then by the separation theorem [1], there exists a nonzero vector $(\eta, \varsigma) \in \Pi_{i=1}^m Z_i^* \times \Pi_{j=1}^n W_j^*$ such that

$$\sum_{i=1}^m \eta_i(\tau_i z_i + z_i^0) + \sum_{j=1}^n \varsigma_j(t_j w_j) \geq 0$$

for all $x \in D$, $\tau_i, t_j > 0$, $z_i = g_i(x)$, $z_i^0 \in \text{int}Z_{i+}$, $w_j = h_j(x)$. Since $\text{int}Z_{i+}$ are convex cones, consequently we have

$$\sum_{i=1}^m \eta_i(\tau_i z_i + s_i z_i^0) + \sum_{j=1}^n \varsigma_j(t_j w_j) \geq 0 \quad (3)$$

for all $x \in D$, $\tau_i, t_j, s_i > 0$, $z_i \in g_i(x)$, $z_i^0 \in \text{int}Z_{i+}$, $w_j \in h_j(x)$ and take $s_i \rightarrow 0$ in (3), we have

$$\sum_{i=1}^m \eta_i(z_i) + \sum_{j=1}^n \varsigma_j(w_j) \geq 0, \quad x \in D, z_i \in g_i(x), w_j \in h_j(x)$$

which contradicts the NNAMCQ. Similarly if the latter $0 \notin \text{int}h_j(D)$ holds then there exists $\varsigma \in \Pi_{j=1}^n W_j^* \setminus \{0\}$ such that $\varsigma_j(h_j(x)) \geq 0, \forall x \in D$, which contradicts NNAMCQ. $\square$

Definition 14 (Calmness Condition). Let $\bar{x} \in F$. Let $Z := \sum_{i=1}^m Z_i$ and $W := \sum_{j=1}^n W_j$. We say that (VP) satisfies the calmness condition at $\bar{x}$ provided that there exist $U(\bar{x}, 0_Z, 0_W)$, a neighborhood of $(\bar{x}, 0_Z, 0_W)$, and a map $\psi(p, q) : Z \times W \rightarrow Y_+$ with $\psi(0_Z, 0_W) = 0_Y$ such that for each

$$(x, p, q) \in U(\bar{x}, 0_Z, 0_W) \setminus \{(\bar{x}, 0_Z, 0_W)\}$$

Satisfying

$$(g_i(x) + p_i) \prec 0, q_j = h_j(x), x \in D$$

there is no $y \in f(x)$, such that

$$\bar{y} \in y + \psi(p, q) + \text{int}Y_+$$

Theorem 2. Assume that (A1) is satisfied and either (A2) or (A3) holds. If $\bar{x} \in F$ is a weakly efficient solution of (VP) with $\bar{y} = f(\bar{x})$, and the calmness condition holds at $\bar{x}$, then there exists $U(\bar{x})$, a neighborhood of $\bar{x}$, and a vector $(\xi, \eta, \varsigma) \in Y_+^* \times Z_+^* \times W^*$ with $\xi \neq 0$ such that

$$\begin{aligned} \xi(\bar{y}) &= \min_{x \in D \cap U(\bar{x})} [\xi(f(x)) + \sum_{i=1}^m \eta_i(g_i(x)) + \sum_{j=1}^n \varsigma_j(h_j(x))] \\ \sum_{i=1}^m \eta_i(g_i(\bar{x})) &= 0 \end{aligned} \quad (4)$$

Proof. It is easy to see that under the calmness condition, $\bar{x}$ being a weakly efficient solution of (VP) implies that $(\bar{x}, 0_Z, 0_W)$ is a weakly efficient solution of the perturbed problem: $VP(p, q)$

$$\begin{aligned} VP(p, q) \quad & Y_+ - \min f(x) + \psi(p, q) \\ & s.t. (g_i(x) + p_i) \prec 0, \\ & q_j = h_j(x), x \in D, \\ & (x, p, q) \in U(\bar{x}, 0_Z, 0_W) \end{aligned}$$

By assumption, the above optimization problem satisfies the generalized convexity assumption (A1). Now we prove that the NNAMCQ holds naturally at $(\bar{x}, 0_Z, 0_W)$. Suppose that $(\eta, \varsigma) \in Z_+^* \times W^*$ satisfies the system:

$$\begin{aligned} \min_{x \in D(x, p, q) \in U(\bar{x}, 0_Z, 0_W)} & [\sum_{i=1}^m \eta_i(g_i(x) + p_i) + \sum_{j=1}^n \varsigma_j(-q_j + h_j(x))] \\ \sum_{i=1}^m \eta_i(g_i(\bar{x})) &= 0 \end{aligned} \quad (5)$$

If $\varsigma \neq 0$, then there exists $q_j \in W_j$ small enough such that $\sum_{j=1}^n \varsigma_j(-q_j) < 0$. Since $\bar{x} \in F$, $0 \in h_j(\bar{x})$, and there exists $z_i^x \in g_i(x) \cap (-Z_{i+})$, which implies that $\eta(z_i^x) \leq 0$, hence

$$\sum_{i=1}^m \eta_i(z_i^x) + \sum_{j=1}^n \varsigma_j(-q_j) < 0$$

which contradicts (5). Hence, $\varsigma = 0$ and (5) becomes

$$\min_{x \in D, (x,p,q) \in U(\bar{x}, 0_Z, 0_W)} \sum_{i=1}^m \eta_i(g_i(x) + p_i) - \sum_{i=1}^m \eta_i(g_i(\bar{x})) = 0$$

If $\eta \neq 0$, then there exists p small enough such that $\sum_{i=1}^m \eta_i(p_i) < 0$. Let $z_i^x = g_i(x)$, then

$$\sum_{i=1}^m \eta_i(z_i^x) \leq 0$$

and hence

$$\sum_{i=1}^m \eta_i(z_i^x + p_i) = \sum_{i=1}^m \eta_i(z_i^x) + \sum_{i=1}^m \eta_i(p_i) < 0$$

which is impossible. Consequently, $\eta = 0$ as well. Hence, there exists $(\xi, \eta, \varsigma) \in Y^* \times Z_+^* \times W_+^*$ with $\xi \neq 0$ such that

$$\min_{x \in D, (x,p,q) \in U(\bar{x}, 0_Z, 0_W)} [\xi(f(x) + \psi(p, q)) + \sum_{i=1}^m \eta_i(g_i(x) + p_i) + \sum_{j=1}^n \varsigma_j(-q_j + h_j(x))] - \sum_{i=1}^m \eta_i(g_i(\bar{x})) = 0 \quad (6)$$

It is obvious that (6) implies (4) and hence the proof of the theorem is complete. $\square$

Definition 15. Let $Z_i (i = 1, 2, \dots, m)$, $W_j (j = 1, 2, \dots, n)$ be normed spaces. We say that (VP) satisfies the error bound constraint qualification at a feasible point $\bar{x}$ if there exist positive constants λ, δ , and ε such that

$$d(\bar{x}, \Sigma(0_Z, 0_W)) \leq \lambda \| (p, q) \|, \forall (p, q) \in \varepsilon B_X, x \in \Sigma(p, q) \cap U_\delta(\bar{x})$$

where B_X is the unit ball of X , and

$$\Sigma(p, q) := \{x \in D : (g_i(x) + p_i) \cap (-Z_{i+}) \neq \emptyset, q_j \in h_j(x)\}$$

Remark 2. Note that the error bound constraint qualification is satisfied at a feasible point $\bar{x}$ if and only if the function $\Sigma(p, q)$ is pseudo upper-Lipschitz continuous around $(0_Z, 0_W, \bar{x})$ in the terminology of ([8]) (which is referred to as being calm at $\bar{x}$ in [9]). Hence, $\Sigma(p, q)$ being either pseudo-Lipschitz continuous around $(0_Z, 0_W, \bar{x})$ in the terminology of [10] or upper-Lipschitz continuous at $\bar{x}$ in the terminology of [11] implies that the error bound constraint qualification holds at $\bar{x}$. Recall that a function $F(x) : R^n \rightarrow R^m$ is called a polyhedral multifunction if its graph is a union of finitely many polyhedral convex sets. This class of function is closed under (finite) addition, scalar multiplication, and (finite) composition. By ([12], Proposition 1), a polyhedral multifunction is upper-Lipschitz. Hence, the following result provides a sufficient condition for the error bound constraint qualification.

Proposition 4. Let $X = R^n$ and $W = R^m$. Suppose that D is polyhedral and h is a polyhedral multifunction. Then, the error bound constraint qualification always holds at any feasible point $\bar{x} \in F := \{x \in D : 0 = h(x)\}$.

Proof. Since D is polyhedral and h is a polyhedral multifunction, its inverse map $S(q) = \{x \in R^n : q \in h(x)\}$ is a polyhedral multifunction. That is, the graph of S is a union of polyhedral convex sets. Since

$$gph \Sigma(p, q) := \{(q, x) \in R^m \times D : q \in h(x)\} = gph S \cap (R^m \times D)$$

which is also a union of polyhedral convex sets, Σ is also a polyhedral multifunction and hence upper-Lipschitz at any point of $\bar{x} \in R^n$ by ([12], Proposition 1). Therefore, the error bound constraint qualification holds at $\bar{x}$. $\square$

Definition 16. Let X be a normed space, $f(x) : X \rightarrow Y$ be a function, and $\bar{x} \in X$. f is said to be Lipschitz near $\bar{x}$ if there exist $U(\bar{x})$, a neighborhood of $\bar{x}$, and a constant $L_f > 0$ such that for all $x_1, x_2 \in U(\bar{x})$,

$$f(x_1) \subseteq f(x_2) + L_f \|x_1 - x_2\| B_Y$$

where B_Y is the unit ball of Y .

Definition 17. Let X be a normed space, $f(x) : X \rightarrow Y$ be a function and $\bar{x} \in X$. f is said to be strongly Lipschitz on $S \subseteq X$ if there exist a constant $L_f > 0$ such that for all $x_1, x_2 \in S$, $y_1 = f(x_1)$, $y_2 = f(x_2)$ and $e \in B_Y \cap Y_+$,

$$y_1 \prec y_2 + L_f \|x_1 - x_2\| e$$

The following result generalizes the exact penalization [13].

Proposition 5. Let X be a normed space, $f(x) : X \rightarrow Y$ be a function which is strongly Lipschitz of rank L_f on a set $S \subseteq X$. Let $C \subseteq X$ and suppose that $\bar{x}$ is a weakly efficient solution of

$$Y_+ - \min_{x \in S} f(x)$$

with $\bar{y} = f(\bar{x})$. Then, for all $K \geq L_f$, $\bar{x}$ is a weakly efficient solution of the exact penalized optimization problem

$$Y_+ - \min_{x \in S} f(x) + K d_C(x) B_Y \cap Y_+$$

where $d_C(x) := \min\{\|x - c\|, c \in C\}$.

Proof. Let us prove the assertion by supposing the contrary. Then, there is a point $S \subseteq X$, $y = f(x)$, and $e \in B_Y \cap Y_+$ satisfying $y + K d_C(x) e \prec \bar{y}$. Let $\varepsilon > 0$ and $c \in C$ be a point such that $\|x - c\| \leq d_C(x) + \varepsilon$. Then, for any $c^* \in f(c)$,

$$c^* \prec y + K \|x - c\| e \prec y + K(d_C(x) + \varepsilon) e \prec \bar{y} + K \varepsilon e$$

Since $\varepsilon > 0$ is arbitrary, it contradicts the fact that $\bar{x}$ is a weakly efficient solution of

$$Y_+ - \min_{x \in S} f(x)$$

$\square$

Proposition 6. Suppose $X \times Z \times W$ is a normed space and f is strongly Lipschitz on D . If $\bar{x}$ is a weakly efficient solution of (VP) and the error bound constraint qualification is satisfied at $\bar{x}$, then (VP) satisfies the calmness condition at $\bar{x}$.

Proof. By the exact penalization principle in Proposition 5 $\bar{x}$ is a weakly efficient solution of the penalized problem

$$Y_+ - \min_{x \in D} f(x) + K d_{\Sigma(0,0)}(x) B_Y \cap Y_+$$

The results then follow from the definitions of the calmness and the error bound constraint qualification. $\square$

Theorem 3. Assume that the generalized convexity assumption (A1) is satisfied with f replaced by $f + K d_C(x) B_Y \cap Y_+$ and either (A2) or (A3) holds. Suppose $X \times Z \times W$ is a normed space and f is strongly Lipschitz on D . If $\bar{x}$ is a weakly efficient solution of (VP) and the error bound

constraint qualification is satisfied at $\bar{x}$, then there exist $U(\bar{x})$, a neighborhood of $\bar{x}$, and a vector $(\xi, \eta, \varsigma) \in Y_+^* \times Z_+^* \times W^*$ with $\xi \neq 0$ such that (4) holds.

Using Proposition 4, Theorem 3 has the following easy corollary.

Corollary 1. Suppose Y is a normed space, $X = \mathbb{R}^n$, $W = \mathbb{R}^m$ and D is polyhedral, and f is strongly Lipschitz on D . Assume that the generalized convexity assumption (A1) is satisfied with f replaced by $f + Kd_C(x)B_Y \cap Y_+$ and either (A2) or (A3) holds. If $\bar{x}$ is a weakly efficient solution of (VP) without the inequality constraint $g(x) \succ 0$, and h is a polyhedral multifunction, then there exist $U(\bar{x})$, a neighborhood of $\bar{x}$ a vector $(\xi, \varsigma) \in Y_+^* \times W^*$ with $\xi \neq 0$ such that

$$\xi(\bar{y}) = \min_{x \in D \cap U(\bar{x})} [\xi(f(x)) + \varsigma_j(h_j(x))]$$

Our last result Theorem 4 is a strong duality theorem, which generalizes a result in Fang, Li, and Ng [14].

For two topological vector spaces Z and Y , let $B(Z; Y)$ be the set of continuous linear transformations from Z to Y and

$$B^+(Z, Y) := \{S \in B(Z, Y) : S(Z_+) \subseteq Y_+\}$$

The Lagrangian map for (VP) is the function

$$L : X \times \prod_{i=1}^m B^+(Z_i, Y) \times \prod_{j=1}^n B^+(W_j, Y) \rightarrow Y$$

defined by

$$L(x, S, T) := f(x) + \sum_{i=1}^m S_i(g_i(x)) + \sum_{j=1}^n T_j(h_j(x))$$

Given $(S, T) \in \prod_{i=1}^m B^+(Z_i, Y) \times \prod_{j=1}^n B^+(W_j, Y)$, consider the vector minimization problem induced by (VP):

$$(VPST) \quad \begin{array}{l} Y_+ - \min L(x, S, T) \\ s.t. x \in D \end{array}$$

and denote by $\Phi(S, T)$ the set of weakly efficient value of the problem (VPST). The Lagrange dual problem associated with the primal problem (VP) is

$$(VD) \quad \begin{array}{l} Y_+ - \max \Phi(S, T) \\ s.t. (S, T) \in \prod_{i=1}^m B^+(Z_i, Y) + \prod_{j=1}^n B^+(W_j, Y) \end{array}$$

The following strong duality result holds which extends the strong duality theorem in ([7], Theorem 7) (which was for set-valued optimization problems), to allow weaker convexity assumptions. We omit the proof since it is similar to [7].

Theorem 4. Assume that (A1) is satisfied, either (A2) or (A3) is satisfied, and a constraint qualification such as NNAMCQ is satisfied. If $\bar{x}$ is a weakly efficient solution of (VP), then there exists

$$(\bar{S}, \bar{T}) \in \prod_{i=1}^m B^+(Z_i, Y) \times \prod_{j=1}^n B^+(W_j, Y)$$

such that

$$\Phi(\bar{S}, \bar{T}) \cap f(x) \neq \emptyset$$

5. Conclusions

We introduce the following definitions of generalized affine functions: affinelikeness, preaffinelikeness, subaffinelikeness, and presubaffinelikeness. Examples 1 to 7 show that definitions of affine, affinelike, preaffinlike, subaffinlike, and presubaffinlike functions are all different. Example 8 is an example of non-presubaffinlike function; presubaffineness is the weakest one in the series. Proposition 1 demonstrates that our generalized affine functions have some similar properties with generalized convex functions.

And then, we work with vector optimization problems in real linear topological spaces, and obtain necessary conditions, sufficient conditions, or necessary and sufficient conditions for weakly efficient solutions, which generalize the corresponding classical results in [13,15] and some recent results in [7,9,16–18]. We note that the constraint qualifications in [13,17,18] are in the differentiation form. Compared with the results in [19] and ([20], p. 297) in discussions of convex constraints, we only required weakened convexities for constraint qualifications in this article. We note that [17] works with semi-definite programming. In [17], two groups of functions $g_i(x) \geq 0, i \in I$ and $h_j(x) = 0, j \in J$ can be just considered as two topological spaces (I and J do not have to be finite sets). We also note that f is supposed to be “proper convex” in [18]; and in [18], functions are required to be “quasiconvex”.

Generalized affine functions and generalized convex functions can be used for other discussions of optimization problems, e.g., dualities, scalarizations, as well as saddle points, etc.

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