

Article

Phase Portraits of Families VII and VIII of the Quadratic Systems

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Abstract: The quadratic polynomial differential systems in a plane are the easiest nonlinear differential systems. They have been studied intensively due to their nonlinearity and the large number of applications. These systems can be classified into ten classes. Here, we provide all topologically different phase portraits in the Poincaré disc of two of these classes.

Keywords: quadratic vector fields; quadratic systems; phase portraits

MSC: 34C05; 34A34; 34C14

1. Introduction and Statement of the Main Results

A quadratic polynomial differential system (or simply, a quadratic system) is a differential system of the following form:

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (1)$$

where P and Q are real polynomials in variables x and y and the maximum degree of the polynomials P and Q is two.

At the beginning of the 20th century, the study of quadratic systems began. In [1], Coppel noted how Büchel [2], in 1904, published the first work on quadratic systems. Two short surveys on quadratic systems were published, i.e., by Coppel [1] in 1966 and by Chicone and Tian [3] in 1982.

In recent decades, quadratic systems were intensively studied and many good results were obtained, see references [4–6]. In the second reference, one can find many applications for quadratic systems. Although quadratic systems have been studied in more than one thousand papers, we do not have a complete understanding of these systems.

In [7], the authors prove that any quadratic system is affine-equivalent, scaling the time variable, if necessary, to a quadratic system of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y) = d + ax + by + \ell x^2 + mxy + ny^2,$$

where $\dot{x} = P(x, y)$ is one of the following ten:

- | | |
|----------------------------|---------------------------|
| (I) $\dot{x} = 1 + xy,$ | (VI) $\dot{x} = 1 + x^2,$ |
| (II) $\dot{x} = xy,$ | (VII) $\dot{x} = x^2,$ |
| (III) $\dot{x} = y + x^2,$ | (VIII) $\dot{x} = x,$ |
| (IV) $\dot{x} = y,$ | (IX) $\dot{x} = 1,$ |
| (V) $\dot{x} = -1 + x^2,$ | (X) $\dot{x} = 0.$ |

Roughly speaking, the Poincaré disc is the disc centered at the origin of \mathbb{R}^2 and the radius, where the interior of this disc is identified with the whole plane \mathbb{R}^2 and its boundary



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circle \mathbb{S}^1 is identified with the infinity of the plane, \mathbb{R}^2 . This is due to the fact that in the plane, we can go to infinity in as many directions as points on the circle \mathbb{S}^1 . For more details on the Poincaré compactification, see Section 2.2; for the definition of topologically equivalent phase portraits in the Poincaré disc, see Section 2.3.

We note that quadratic system X has straight lines with constant $x =$ coordinates formed by orbits, and the conic $Q(x, y) = 0$ is filled with equilibrium points, so the phase portraits are trivial. On the other hand, quadratic systems IX does not contain any equilibrium points, thus making this quadratic system a subclass of the so-called chordal quadratic system. The phase portraits of these systems in the Poincaré disc have been completely studied in [7]. Thus, the aim of this paper is to classify the different topological phase portraits in the Poincaré disc of the classes of quadratic systems VII and VIII, i.e., of systems

$$\dot{x} = x^2, \quad \dot{y} = d + ax + by + \ell x^2 + mxy + ny^2, \tag{2}$$

and

$$\dot{x} = x, \quad \dot{y} = d + ax + by + \ell x^2 + mxy + ny^2, \tag{3}$$

respectively.

Our main result is as follows:

Theorem 1. *The following two statements hold:*

- (a) *The family of quadratic systems VII has 27 topologically different phase portraits in the Poincaré disc.*
- (b) *The family of quadratic systems VIII has 25 topologically different phase portraits in the Poincaré disc.*

Statements (a) and (b) of Theorem 1 are proved in Sections 3 and 4, respectively.

The paper is organized as follows. In Section 2, we present the basic results of equilibrium points and the Poincaré compactification. In Sections 3 and 4, we first study the local phase portraits of the finite equilibrium points, and then explore the local phase portraits of the infinite equilibrium points. Finally, we analyze the phase portraits of quadratic systems (2) and (3) in the Poincaré disc, respectively.

2. Preliminary Definitions

The study of the phase portraits of quadratic systems always begins with the study of the finite and infinite equilibria of the local phase portraits, followed by the study of their separatrix connections and limit cycles.

In this section, we introduce the basic notations and definitions that we use for the analysis of the finite and infinite equilibrium points of the local phase portraits.

2.1. Equilibrium Points

A point $q \in \mathbb{R}^2$ is said to be an *equilibrium point* of a polynomial differential system (1) if $P(q) = Q(q) = 0$. If the real parts of these eigenvalues (of the linear part of system (1)) are non-zero, the equilibrium point, q , is considered a *hyperbolic* equilibrium point and its possible phase portraits are well known; for instance, see Theorem 2.15 of [8]. If only one of the eigenvalues of the linear part of system (1) at equilibrium point q is zero, then q is considered a *semi-hyperbolic* equilibrium point, whose possible local phase portraits are also well known; see, among others, Theorem 2.19 of [8]. When both eigenvalues of the linear part of system (1) at equilibrium point q are zero, but the linear part is not identically null, then q is a *nilpotent* equilibrium point, and again, its local phase portraits are known; see, for instance, Theorem 3.5 of [8]. Finally, if the linear part of system (1) at equilibrium point q is entirely zero, then q is *degenerate* or q is *linearly zero*. The local phase portraits of such equilibrium points can be studied using the change of variables called blow-ups; see, for instance, [9].

2.2. Poincaré Compactification

Let $X = (P, Q)$ be the vector field defined by the polynomial differential system (1). Roughly speaking, the Poincaré compactification consists of creating a vector field $p(X)$ in a 2-dimensional sphere, \mathbb{S}^2 , such that its phase portraits (in the open northern and southern hemispheres) is a copy of the phase portrait of the vector field X , and the equator of the sphere plays the role of the infinity of the phase portrait of X ; for details, see [10], or Section 5 of [8]. In this way, we can study the orbits of the vector field X , which go to or come from infinity.

Let $\mathbb{S}^2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ be the *Poincaré sphere*. We denote by $T_x\mathbb{S}^2$ the tangent plane to \mathbb{S}^2 at a point $x \in \mathbb{S}^2$. We consider the vector field X defined on the plane $T_{(0,0,1)}\mathbb{S}^2$. Then the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \rightarrow \mathbb{S}^2$ defines two copies of X in \mathbb{S}^2 , one in the northern hemisphere and the other in the southern hemisphere. Obviously the equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$ represents the *infinity* of \mathbb{R}^2 . The projection of the closed northern hemisphere of \mathbb{S}^2 on $x_3 = 0$ under $(x_1, x_2, x_3) \mapsto (x_1, x_2)$ is called the *Poincaré disc*, and it is denoted by \mathbb{D}^2 . As \mathbb{S}^2 is a differentiable manifold, we define six local charts, $U_i = \{x \in \mathbb{S}^2 : x_i > 0\}$, and $V_i = \{x \in \mathbb{S}^2 : x_i < 0\}$ for $i = 1, 2, 3$; with the corresponding diffeomorphisms, $F_i : U_i \rightarrow \mathbb{R}^2$ and $G_i : V_i \rightarrow \mathbb{R}^2$ for $i = 1, 2, 3$ which are the inverses of the central projections from the tangent planes at points $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$ and $(0, 0, -1)$, respectively.

We denote by (u, v) the value of $F_i(x)$ or $G_i(x)$ for any $i = 1, 2, 3$, so a few simple calculations (for $p(X)$) lead to the following formulae in the corresponding local charts (see Section 5 of [8]):

$$\begin{aligned} v^d \left(Q \left(\frac{1}{v}, \frac{u}{v} \right) - uP \left(\frac{1}{v}, \frac{u}{v} \right), -vP \left(\frac{1}{v}, \frac{u}{v} \right) \right) & \text{ in } U_1, \\ v^d \left(P \left(\frac{u}{v}, \frac{1}{v} \right) - uQ \left(\frac{u}{v}, \frac{1}{v} \right), -vQ \left(\frac{u}{v}, \frac{1}{v} \right) \right) & \text{ in } U_2, \\ (P(u, v), Q(u, v)) & \text{ in } U_3, \end{aligned}$$

where d is the degree of the polynomial differential system (1). The formulae for V_i are similar to the formulae for U_i with a multiplicative factor of $(-1)^{d-1}$. In the coordinates for $i = 1, 2$, points (u, v) of the infinity \mathbb{S}^1 satisfy $v = 0$.

2.3. Phase Portraits on the Poincaré Disc

The *separatrix* of $p(X)$ denotes all the orbits of the circle at infinity, the equilibrium points, the limit cycles, and the orbits that lie in the boundary of hyperbolic sectors, i.e., the two separatrices of a hyperbolic sector.

Neumann, [11], showed that the set of all separatrices $S(p(X))$ of the vector field, $p(X)$, was closed.

When there is an orientation preserving or reversing homeomorphism, which maps the trajectories of $p(X)$ into the trajectories of $p(Y)$, we can say that the two differential systems defined by $p(X)$ and $p(Y)$ in the Poincaré disc are *topologically equivalent*.

The *canonical regions* of $p(X)$ are the openly connected components of $\mathbb{D}^2 \setminus S(p(X))$. The set formed by the union of $S(p(X))$ plus one orbit chosen from each canonical region is called a *separatrix configuration* of $p(X)$. When there is an orientation preserving or reversing homeomorphism, which maps the trajectories of $S(p(X))$ into the trajectories of $S(p(Y))$, we can say that the two separatrix configurations, $S(p(X))$ and $S(p(Y))$, are *topologically equivalent*.

The next result is mainly due to work by Markus [12], Neumann [11], and Peixoto [13].

Theorem 2. *Phase portraits in the Poincaré disc of two compactified polynomial differential systems ($p(\mathcal{X})$ and $p(\mathcal{Y})$) with (finitely) many separatrices are topologically equivalent if and only if their separatrix configurations $S(p(\mathcal{X}))$ and $S(p(\mathcal{Y}))$ are topologically equivalent.*

3. Proof of Statement (a) of Theorem 1

3.1. Finite Equilibrium Points

We will determine the local phase portrait at the finite equilibrium points of the quadratic system (2).

Assume that $n \neq 0$. If $b^2 - 4dn > 0$, then the finite equilibrium points of system (2) are as follows:

$$p_{\pm} = \left(0, \frac{-b \pm \sqrt{b^2 - 4dn}}{2n} \right),$$

The eigenvalues of the Jacobian matrix of system (2) at p_{\pm} are 0 and $\pm\sqrt{b^2 - 4dn}$. Thus, from Theorem 2.19 of [8], we have that p_+ and p_- are semi-hyperbolic saddle-nodes.

If $b^2 - 4dn < 0$, there are no finite equilibrium points.

If $b^2 - 4dn = 0$, then $d = b^2/(4n)$ and $p_+ = p_- = p = (0, -b/(2n))$. The Jacobian matrix of the differential system at p is

$$\begin{pmatrix} 0 & 0 \\ a - \frac{bm}{2n} & 0 \end{pmatrix}.$$

If $a - bm/(2n) \neq 0$, then this equilibrium point is nilpotent, and from Theorem 3.5 of [8], this equilibrium point is a saddle-node.

If $a = bm/(2n)$, the linear part of the differential system at the equilibrium point p is identically zero, and the differential system becomes a homogeneous quadratic differential system. Using the results by Date in [14], who classified the phase portraits of all the homogeneous quadratic systems, we can see that the phase portraits of system (3) when $(m - 1)^2 - 4\ell n > 0$ are given in Figure 1, according to the sign of n . If $(m - 1)^2 - 4\ell n = 0$, then the phase portraits of system (3) are given in Figure 2, determined by the sign of n . Finally, if $(m - 1)^2 - 4\ell n < 0$, the phase portraits of system (3) are given in Figure 3, determined by the sign of n .

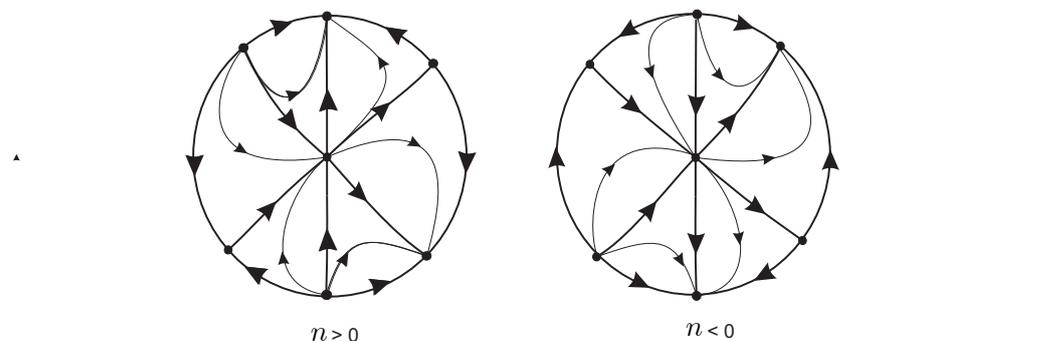


Figure 1. $(m - 1)^2 - 4\ell n > 0$.

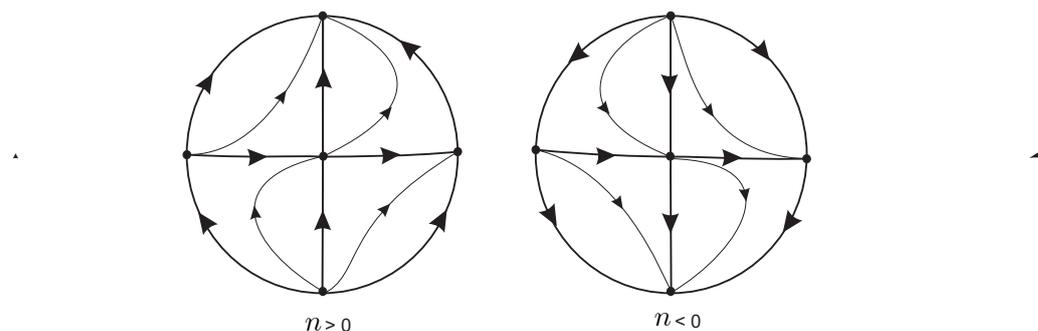


Figure 2. $(m - 1)^2 - 4\ell n = 0$.

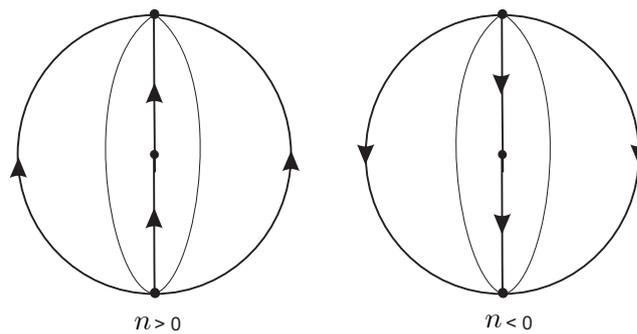


Figure 3. $(m - 1)^2 - 4\ell n < 0$.

We assume that $n = 0$. In this case, if $b \neq 0$, there exists a unique equilibrium point, namely $q = (0, -d/b)$, and the eigenvalues of the Jacobian matrix at q are 0 and b . If $b \neq 0$, then q is a semi-hyperbolic saddle-node (by Theorem 2.19 of [8]). If $b = 0$ and $d \neq 0$, the differential system has no finite equilibria. If $b = d = 0$, then the system has a straight line filled with equilibria; we do not consider this kind of differential system because this case can be reduced to a linear differential system, involving the rescaling of the independent variable.

In summary, we proved the following proposition.

Proposition 1. Assume that $n \neq 0$.

- (a) If $b^2 - 4dn > 0$, the differential system (2) has two finite equilibria p_{\pm} that are semi-hyperbolic saddle-nodes.
- (b) If $b^2 - 4dn < 0$, the differential system (2) has no finite equilibria.
- (c) $b^2 - 4dn = 0$.
 - (c.1) If $a - bm / (2n) \neq 0$, the differential system (2) has one finite equilibrium point p that is a nilpotent saddle-node.
 - (c.2) $a - bm / (2n) = 0$.
 - (c.2.1) If $(m - 1)^2 - 4\ell n > 0$, the phase portrait of the differential system (2) is topologically equivalent to the ones in Figure 1, determined by the sign of n .
 - (c.2.2) If $(m - 1)^2 - 4\ell n = 0$, the phase portrait of the differential system (2) is topologically equivalent to the ones in Figure 2 determined by the sign of n .
 - (c.2.3) If $(m - 1)^2 - 4\ell n < 0$, the phase portrait of the differential system (2) is topologically equivalent to the ones in Figure 3, determined by the sign of n .

Assume that $n = 0$.

- (d) If $b \neq 0$, the differential system (2) has one finite equilibria q , which is a semi-hyperbolic saddle-node.
- (e) If $b = 0$, then the differential system (2) has no finite equilibria if $d \neq 0$, and one straight line is filled with equilibria if $d = 0$.

3.2. The Infinite Equilibrium Points in Chart U_1

System (2) in the local U_1 chart can be expressed as follows:

$$\dot{u} = \ell - u + mu + nu^2 + av + buv + dv^2, \quad \dot{v} = -v. \tag{4}$$

Assume $n \neq 0$ the infinite equilibrium points are

$$P_{\pm} = \left(0, \frac{1 - m \pm \sqrt{(1 - m)^2 - 4\ell n}}{2n} \right).$$

The eigenvalues of the Jacobian matrix at P_{\pm} are $S_{\pm} = (-1, \pm \sqrt{(1 - m)^2 - 4\ell n})$. If they are real, then $(1 - m)^2 - 4\ell n > 0$ and P_+ are hyperbolic saddles and P_- is a hyperbolic

stable node. If $(1 - m)^2 - 4\ell n = 0$, then $P_+ = P_- = P = (0, (1 - m)/(2n))$. In this case, the Jacobian matrix can be expressed as follows:

$$\begin{pmatrix} 0 & a + \frac{b(1 - m)}{2n} \\ 0 & -1 \end{pmatrix},$$

and the eigenvalues are -1 and 0 , which means that the unique equilibrium point in chart U_1 is semi-hyperbolic, and from Theorem 2.19 of [8], is a semi-hyperbolic saddle-node.

Assume that $n = 0$. Then, the unique infinite equilibrium point in the local U_1 chart is $P = (-\ell/(m - 1), 0)$, and the eigenvalues of the Jacobian matrix of system (4) at P are -1 and $m - 1$. If $m \neq 1$, from Theorem 2.15 of [8], P is a hyperbolic saddle if $m > 1$ and a hyperbolic node if $m < 1$. If $m = 1$, there are no equilibrium points in U_1 .

3.3. The Infinite Equilibrium Point at the Origin of Chart U_2

Studying the infinite equilibrium points in the local U_1 chart, we also studied the infinite equilibrium points in the local V_1 chart. Thus, we must see whether the origins of the local U_2 and V_2 charts are infinite equilibrium points or not.

System (2) in the local U_2 chart can be expressed as follows:

$$\begin{aligned} \dot{u} &= -u(n + (m - 1)u + bv + \ell u^2 + auv + dv^2) = P(u, v), \\ \dot{v} &= -v(n + mu + bv + \ell u^2 + auv + dv^2) = Q(u, v), \end{aligned} \tag{5}$$

so the origin of U_2 is an infinite equilibrium point. The eigenvalues of the Jacobian matrix of system (5) at the origin are $-n$ with a multiplicity of two. Therefore, the origin is a hyperbolic stable node if $n > 0$, and an unstable node if $n < 0$.

If $n = 0$, then the Jacobian matrix of the system at the origin of the local U_2 chart is the zero matrix, and we need to make blow-ups in order to study its local phase portrait. Before conducting a vertical blow-up, we need to be sure that $u = 0$ is not a characteristic direction. If $u = 0$ is a characteristic direction, then u would be a factor of the polynomial $\Pi = vP_2(u, v) - uQ_2(u, v)$, where $P_2(u, v)$ and $Q_2(u, v)$ represent the terms of degree two in $P(u, v)$ and $Q(u, v)$. In our case, $\Pi = u^2v$. Thus, $u = 0$ is a characteristic direction and, consequently, before conducting a vertical blow-up, we must perform a twist so $u = 0$ no longer acts as a characteristic direction. This is conducted through the change of variables $(u, v) \rightarrow (u_1, v_1)$, where $u_1 = u + v$, $v_1 = v$. By conducting this change of variables, the differential system (5) can be expressed as follows:

$$\begin{aligned} \dot{u}_1 &= (1 - m)u_1^2 - (b + 2 - m)u_1v_1 + v_1^2 - \ell u_1^3 - (a - 2\ell)u_1^2v_1 + (a - d - \ell)u_1v_1^2 \\ \dot{v}_1 &= -mu_1v_1 + (m - b)v_1^2 - \ell u_1^2v_1 + (2\ell - a)u_1v_1^2 + (a - d - \ell)v_1^3, \end{aligned} \tag{6}$$

Since $u_1 = 0$ is not a characteristic direction, we can conduct a vertical blow-up. This vertical blow-up is given by the change of variables $(u_1, v_1) \rightarrow (u_2, v_2)$, where $u_2 = u_1$, $v_2 = v_1/u_1$. Then, system (6) becomes

$$\begin{aligned} \dot{u}_2 &= u_2^2(m - 1 + \ell u_2 + (b - m + 2)v_2 + (a - 2\ell)u_2v_2 - v_2^2 - (a - d - \ell)u_2v_2^2), \\ \dot{v}_2 &= -u_2v_2(-1 + v_2)^2. \end{aligned}$$

Now, conducting the rescaling of the time with factor u_2 , we obtain the system

$$\begin{aligned} \dot{u}_2 &= u_2(m - 1 + \ell u_2 + (b - m + 2)v_2 + (a - 2\ell)u_2v_2 - v_2^2 - (a - d - \ell)u_2v_2^2), \\ \dot{v}_2 &= -v_2(-1 + v_2)^2. \end{aligned}$$

The equilibrium points of the previous system on $u_2 = 0$ are $(0, 0)$ and $(0, 1)$ (this is double). The eigenvalues of the Jacobian matrix at $(0, 0)$ are -1 and $1 - m$. Thus, the point $(0, 0)$ is a hyperbolic stable node if $m > 1$, a hyperbolic saddle if $m < 1$, and for $m = 1$,

a semi-hyperbolic saddle-node, according to Theorem 2.19 in [8]. The eigenvalues of the Jacobian matrix at $(0, 1)$ are 0 and $-b$. Thus, the local phase portrait of the origin of the local U_2 chart is shown in Figure 4A when $n = 0, b > 0$, and $m > 1$.

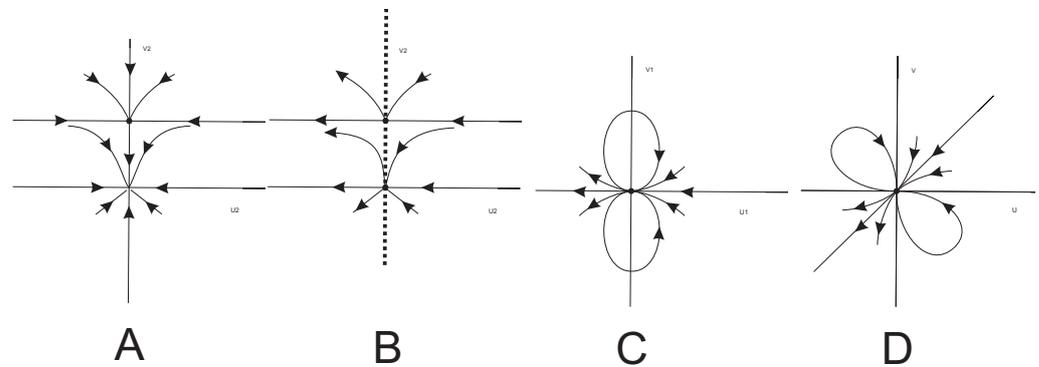


Figure 4. The sequences of blow-ups for obtaining the local phase portrait at the origin of the local U_2 chart when $n = 0, b > 0$, and $m > 1$.

Starting from Figure 4A, we obtain the local phase portrait at the axis $u_2 = 0$ of system (10); see Figure 4B. Going back through the vertical blow, taking into account the value of $\dot{u}_1|_{u_1=0} = v_1^2$, we obtain the local phase portrait at the origin of system (5) in Figure 4C. Finally, undoing the twist, we obtain the local phase portrait at the origin of the local U_2 chart, which is shown in Figures 4D and 5A.

Working in a similar way to the preceding case, conducting the convenient blow-ups and using Theorems 2.15 and 2.19 of [8], we obtain all the local phase portraits at the origin of the local U_2 chart in Figure 5. All the local phase portraits are the following

- $n = 0, b > 0$ and $m > 1$ in Figure 5A;
- $n = 0, b < 0$ and $m > 1$ in Figure 5B;
- $n = 0, b > 0$ and $m < 1$ in Figure 5C;
- $n = 0, b < 0$ and $m < 1$ in Figure 5D;
- $n = 0, b > 0, m = 1$ and $l \neq 0$ in Figure 5E;
- $n = 0, b > 0, m = 1$ and $l = 0$, then $v = 0$ is a straight line of the equilibrium points;
- $n = 0, b < 0, m = 1$ and $l \neq 0$ in Figure 5F;
- $n = 0, b < 0, m = 1$ and $l = 0$, then $v = 0$ is a straight line of the equilibrium points;
- $n = 0, b = 0, d > 0$ and $m > 1$ in Figure 5G;
- $n = 0, b = 0, d < 0$ and $m > 1$ in Figure 5H;
- $n = 0, b = 0, d > 0$ and $m < 1$ in Figure 5I;
- $n = 0, b = 0, d < 0$ and $m < 1$ in Figure 5J;
- $n = 0, b = 0, d > 0$ and $m = 1$ in Figure 5K;
- $n = 0, b = 0, d < 0$ and $m = 1$ in Figure 5L;
- $n = 0, b = 0$ and $d = 0$, then $u = 0$ is a straight line of the equilibrium points.

3.4. The Global Phase Portraits

The preceding results of the finite and infinite equilibrium points allow us to obtain the global phase portraits quite easily, taking into account that the straight line $x = 0$ is invariant.

First, we consider the case satisfying the following conditions: $n > 0, b^2 - 4dn > 0$, and $(1 - m)^2 > 4\ell n$. We can see that if $n > 0$, then there is a stable hyperbolic node at the origin of chart U_2 . Since $b^2 - 4dn > 0$, there exist two real finite equilibrium points, p_+ and p_- , which are semi-hyperbolic saddle-nodes. Finitely, $(1 - m)^2 > 4\ell n$ implies the existence of two infinite equilibrium points in chart U_1 (P_+ is a hyperbolic saddle and P_- a hyperbolic node). The local phase portraits at all these equilibrium points are shown in Figure 6. The tools for studying the phase portraits were employed for all possible configurations that appear in Figure 7.

$n > 0, b^2 - 4dn > 0$ and $(1 - m)^2 > 4ln$ in Figures 7(1–4), but the phase portrait in Figure 7(2) appears by continuity between the phase portraits in Figure 7(1–3);
 $n > 0, b^2 - 4dn > 0$ and $(1 - m)^2 < 4ln$ from Figure 7(5);
 $n > 0, b^2 - 4dn > 0$ and $(1 - m)^2 = 4ln$ in Figure 7(6–8);
 $n > 0, b^2 - 4dn < 0$, and $(1 - m)^2 > 4ln$ in Figure 7(9);
 $n > 0, b^2 - 4dn < 0$, and $(1 - m)^2 < 4ln$ in Figure 7(10);
 $n > 0, b^2 - 4dn < 0$, $(1 - m)^2 = 4ln$ in Figure 7(11);
 $n > 0, b^2 - 4dn = 0$, and $(1 - m)^2 > 4ln$ from Figure 7(12,13);
 $n > 0, b^2 - 4dn = 0$, and $(1 - m)^2 < 4ln$ in Figure 7(14,15);
 $n > 0, b^2 - 4dn = 0$, $(1 - m)^2 = 4ln$ in Figure 7(16–20).

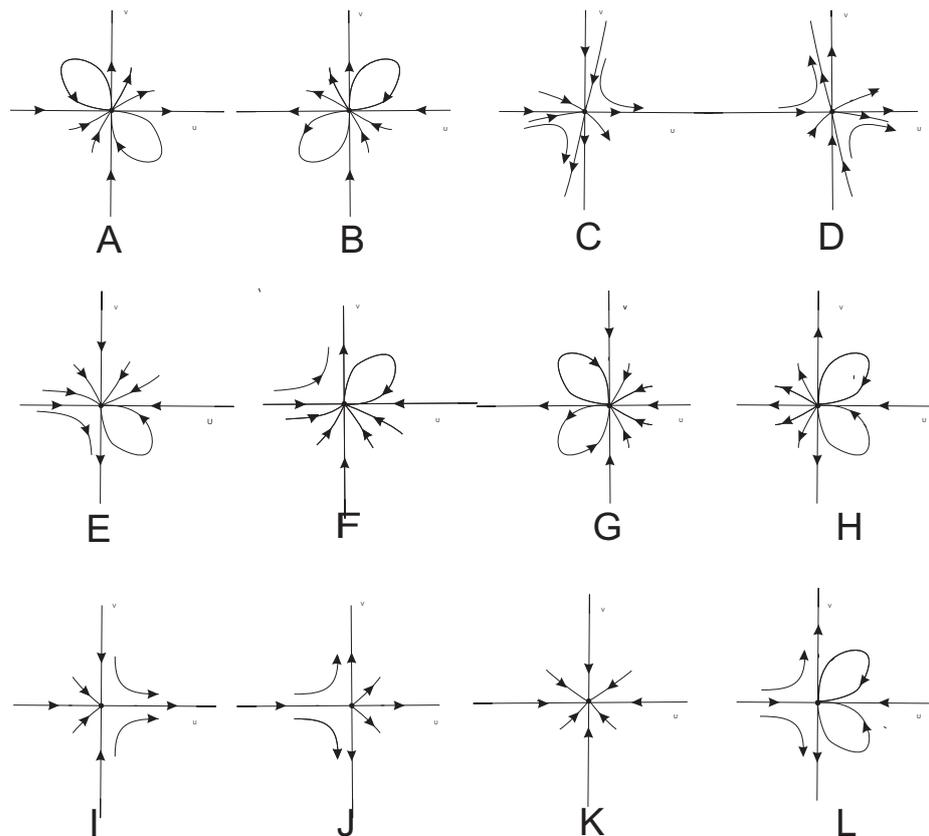


Figure 5. The distinct topological local phase portraits at the origin of the local U_2 chart.

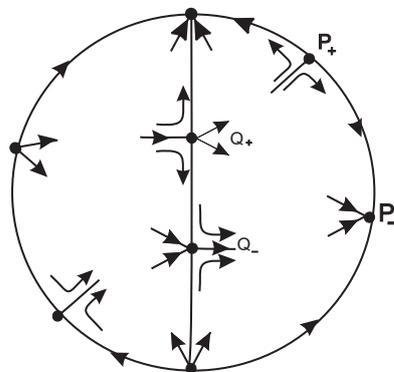


Figure 6. The local phase portraits at the finite and infinite equilibrium points for $n > 0, b^2 - 4dn > 0$ and $(1 - m)^2 > 4ln$.

Phase portraits with $n < 0$ are symmetric with respect to the origins of the coordinates of the preceding eight cases.

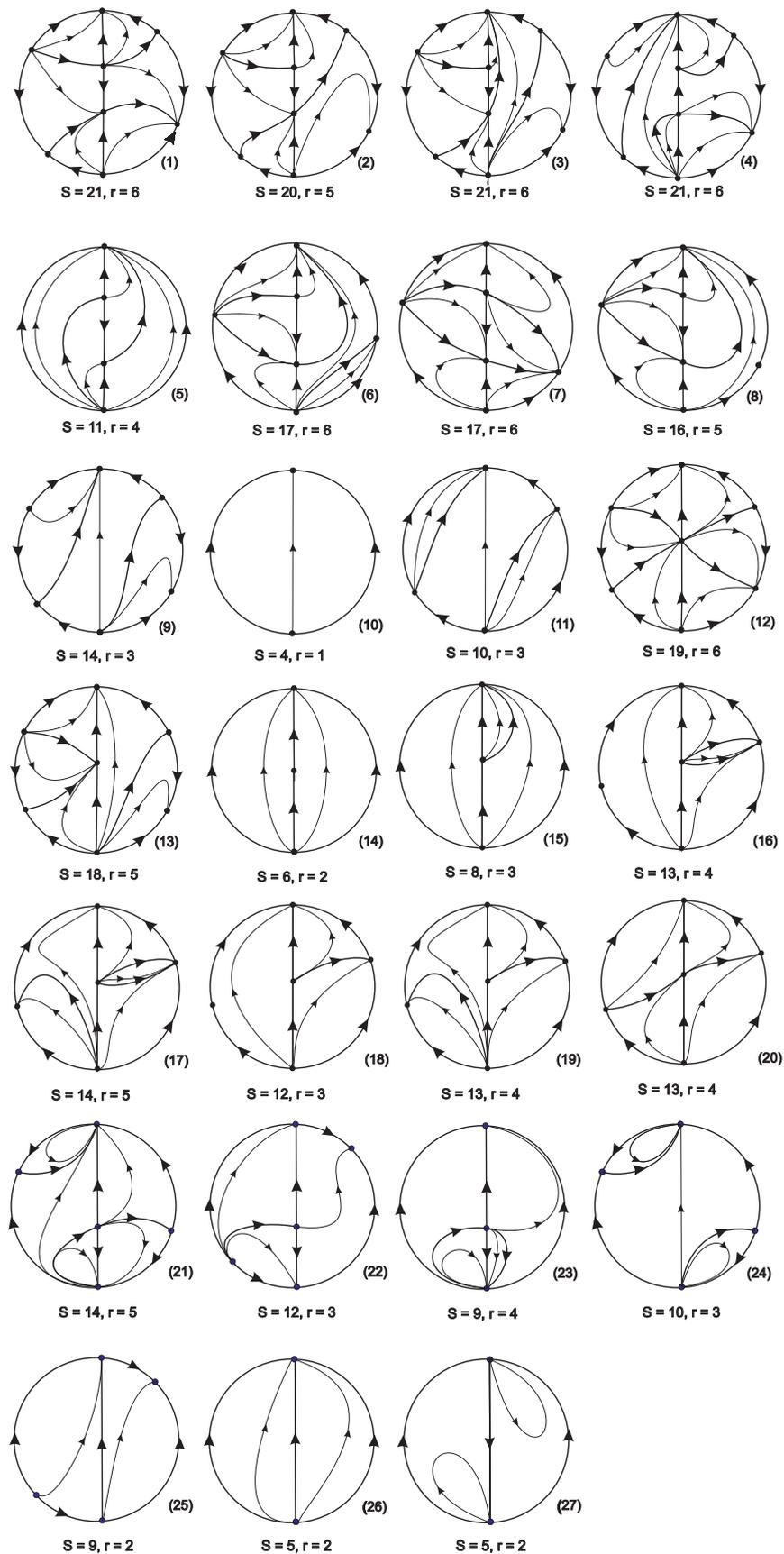


Figure 7. All the distinct topological phase portraits of quadratic system VII. Here, s (respectively, r) denotes the number of separatrices of a phase portrait in the Poincaré disc (respectively, canonical regions).

Now, we study the phase portraits when $n = 0$.

$n = 0, b > 0$ and $m > 1$ in Figure 7(21);

$n = 0, b > 0$ and $m < 1$ in Figure 7(22);

$n = 0, b > 0$ and $m = 1$ in Figure 7(23);

The cases with $b < 0$ are symmetric with respect to the origins of the coordinates of the preceding three cases.

$n = 0, b = 0, d > 0$, and $m > 1$ in Figure 7(24); The cases with $d < 0$ are symmetric with respect to the origins of coordinates of all the preceding cases.

$n = 0, b = 0, d > 0$ and $m < 1$ in Figure 7(25); The cases with $d < 0$ are symmetric with respect to the origins of coordinates of all the preceding cases.

$n = 0, b = 0, d > 0$ and $m = 1$ in Figure 7(26);

$n = 0, b = 0, d < 0$ and $m = 1$ in Figure 7(27).

Of course, from Table 1, the phase portraits with different numbers of separatrices and canonical regions are topologically distinct. Now, we shall see that the phase portraits with the same number of separatrices and canonical regions in Table 1 are also topologically different.

Table 1. Here, p.p. denotes the phase portrait in the Poincaré disc, s denotes the number of separatrices of the phase portrait, and r denotes the number of canonical regions of the phase portrait.

s	4	5	6	8	9	9	10	11	12
r	1	2	2	3	4	2	3	4	3
p.p.	10	26, 27	14	15	23	25	11, 24	5	18, 22
s	13	14	14	16	17	18	19	20	21
r	4	3	5	5	6	5	6	5	6
p.p.	16, 19, 20	9	17, 21	8	6, 7	13	12	2	1, 3, 4

Phase portraits 26 and 27 of Figure 7 are topologically different because phase portrait 27 has two elliptic sectors and phase portrait 26 has no elliptic sectors.

Phase portraits 11 and 24 of Figure 7 are topologically different because phase portrait 24 has two elliptic sectors and phase portrait 11 has no elliptic sectors.

Phase portraits 18 and 22 of Figure 7 are topologically different because phase portrait 18 has orbits starting at the origin of the local U_2 chart and ending at the origin of the local U_1 chart, and these kinds of orbits do not exist in phase portrait 22.

Phase portraits 16, 19, and 20 of Figure 7 are topologically different. First, phase portrait 16 has orbits starting at the origin of the local U_2 chart and ending at the origin of the local U_1 chart, and these kinds of orbits do not exist in phase portraits 19 and 20. Phase portrait 19 has a separatrix starting at the origin of the local U_2 chart and ending at an infinite equilibrium point in the local V_1 chart; this kind of separatrix does not exist in phase portrait 20.

Phase portraits 17 and 21 of Figure 7 are topologically different because phase portrait 21 has two elliptic sectors and phase portrait 17 has no elliptic sectors.

Phase portraits 1, 3, and 4 of Figure 7 are topologically different because the unstable separatrix of the lower equilibrium point on the straight line $x = 0$ contained in $x > 0$ has different ending infinite equilibrium points in the three phase portraits.

4. Proof of Statement (b) Theorem 1

4.1. Finite Equilibrium Points

We are going to analyze the equilibrium points of the quadratic system (3).

Assume that $n \neq 0$. The finite equilibrium points of system (3) are

$$p_{\pm} = \left(0, \frac{-b \pm \sqrt{b^2 - 4dn}}{2n} \right).$$

If $b^2 - 4dn > 0$, the eigenvalues of the Jacobian matrix of system (3) at p_{\pm} are 1 and $\pm\sqrt{b^2 - 4dn}$. Thus, from Theorem 2.15 of [8], p_+ is a hyperbolic unstable node and p_- is a hyperbolic saddle. If $b^2 - 4dn = 0$, then $p_+ = p_- = p = (0, -b/(2n))$. The eigenvalues of the Jacobian matrix of system (3) at p are 1, 0; therefore, by Theorem 2.19 of [8], p is a semi-hyperbolic saddle-node. Of course, if $b^2 - 4dn < 0$, there are no finite equilibrium points.

We assume that $n = 0$. In this case, if $b \neq 0$, there exists a unique equilibrium point, namely $p = (0, -d/b)$, and the eigenvalues of the Jacobian matrix at p are 1 and b . If $b > 0$, then p is a hyperbolic unstable node. If $b < 0$, then p is a hyperbolic saddle. If $b = 0$, there are no finite equilibrium points.

4.2. The Infinite Equilibrium Points in Chart U_1

System (3) in the local U_1 chart can be expressed as follows:

$$\dot{u} = \ell + mu + av + nu^2 + (b - 1)uv + dv^2, \quad \dot{v} = -v^2, \tag{7}$$

Assuming $n \neq 0$, the infinite equilibrium points are

$$P_{\pm} = \left(0, \frac{-m \pm \sqrt{m^2 - 4\ell n}}{2n} \right),$$

if $m^2 - 4\ell n > 0$. If $m^2 - 4\ell n = 0$, then $P_+ = P_- = P = (0, -m/(2n))$. The eigenvalues of the Jacobian matrix at P_{\pm} are 0 and $\pm\sqrt{m^2 - 4\ell n}$. By Theorem 2.19 of [8], we obtain that P_{\pm} are semi-hyperbolic saddle-nodes. The Jacobian matrix at P is

$$\begin{pmatrix} 0 & \frac{2an - bm + m}{2n} \\ 0 & 0 \end{pmatrix}.$$

If $2an + (1 - b)m \neq 0$, then P is a nilpotent equilibrium point, and by Theorem 3.5 of [8], is a saddle-node. If $2an + (1 - b)m = 0$, then P is degenerate. If we translate the equilibrium point P to the origin, it becomes a homogeneous quadratic system; the phase portraits have been classified by Date in [14]. It follows that if $b^2 - 4dn \geq 0$, we obtain that the local phase portrait at P on the Poincaré sphere is formed by two hyperbolic sectors separated by two parabolic ones, and infinity separates the two hyperbolic sectors, which have one separatrix at infinity. If $b^2 - 4dn < 0$, then the local phase portrait at P is a node, unstable if $n < 0$, and stable if $n > 0$.

Assume that $n = 0$. Then the unique infinite equilibrium point in the local U_1 chart is $P = (-l/m, 0)$, and the eigenvalues of the Jacobian matrix of system (7) at P are 0 and m . If $m \neq 0$, from Theorem 2.19 of [8], P is a semi-hyperbolic saddle-node. If $m = 0$, there are no infinite equilibrium points in the local U_1 chart.

4.3. The Infinite Equilibrium Point at the Origin of Chart U_2

Studying the infinite equilibrium points in the local U_1 chart, we have also studied the infinite equilibrium points in the local V_1 chart. Thus, we must see whether the origins of the local U_2 and V_2 charts are infinite equilibrium points or not.

System (3) in the local U_2 chart can be expressed as follows:

$$\begin{aligned} \dot{u} &= -u(n + mu + (b - 1)v + \ell u^2 + auv + dv^2) = P(u, v), \\ \dot{v} &= -v(n + mu + bv + \ell u^2 + auv + dv^2) = Q(u, v), \end{aligned} \tag{8}$$

so the origin of U_2 is an infinite equilibrium point. The eigenvalues of the Jacobian matrix of the system at the origin are $-n$ with a multiplicity of two. Therefore, the origin is a hyperbolic node, stable if $n > 0$, and unstable if $n < 0$.

If $n = 0$, then the Jacobian matrix of the system at the origin is the zero matrix, and we need to make blow-ups in order to study the local phase portrait at the origin

of U_2 . Before conducting a vertical blow-up, we need to be sure that $u = 0$ is not a characteristic direction. If $u = 0$ is a characteristic direction then u is a factor of the polynomial $\Pi = vP_2(u, v) - uQ_2(u, v)$, where $P_2(u, v)$ and $Q_2(u, v)$ are the terms of the lowest degrees of $P(u, v)$ and $Q(u, v)$; in our case, $\Pi = uv^2$. Thus, $u = 0$ is characteristic direction; consequently, before conducting a vertical blow-up, we must conduct a twist so that $u = 0$ no longer acts as a characteristic direction. We accomplish this through the change of variables $(u, v) \rightarrow (u_1, v_1)$, where $u_1 = u + v, v_1 = v$. By making this change of variables, the differential system (8) can be expressed as follows:

$$\begin{aligned} \dot{u}_1 &= -mu_1^2 - v_1^2 + (1 - b + m)u_1v_1 + (a - d - \ell)u_1v_1^2 - \ell u_1^3 + (2\ell - a)u_1^2v_1 \\ \dot{v}_1 &= -v_1(mu_1 + (b - m)v_1 + \ell u_1^2 + (a - 2\ell)u_1v_1 + (d - a + \ell)v_1^2), \end{aligned} \tag{9}$$

The characteristic directions of this system are given by the polynomial $\Pi = (u_1 - v_1)v_1^2$, so $u_1 = 0$ is not a characteristic direction, and we can conduct a vertical blow-up. This vertical blow-up is given by the change of variables $(u_1, v_1) \rightarrow (u_2, v_2)$, where $u_2 = u_1, v_2 = v_1/u_1$. Then, system (9) becomes

$$\begin{aligned} \dot{u}_2 &= u_2^2(-m - \ell u_2 + (1 - b + m)v_2 + (2\ell - a)u_2v_2 - v_2^2 + (a - d - \ell)u_2v_2^2), \\ \dot{v}_2 &= u_2(-1 + v_2)v_2^2. \end{aligned} \tag{10}$$

Now, conducting the rescaling of the time with factor u_2 , we obtain the system

$$\begin{aligned} \dot{u}_2 &= u_2(-m - \ell u_2 + (1 - b + m)v_2 + (2\ell - a)u_2v_2 - v_2^2 + (a - d - \ell)u_2v_2^2), \\ \dot{v}_2 &= (-1 + v_2)v_2^2. \end{aligned} \tag{11}$$

The equilibrium points of system (11) on $u_2 = 0$ are $(0, 0)$, which is double, and $(0, 1)$. The eigenvalues of the Jacobian matrix at $(0, 0)$ are 0 and $-m$. Thus, $(0, 0)$ is a semi-hyperbolic equilibrium point; by applying Theorem 2.19 of [8] to it, it is a saddle-node. The eigenvalues of the Jacobian matrix at $(0, 1)$ are 1 and $-b$. Thus, this equilibrium point is hyperbolic, a saddle if $b > 0$, and an unstable node if $b < 0$; see Figure 8A, when $n = 0, m < 0$, and $b < 0$.

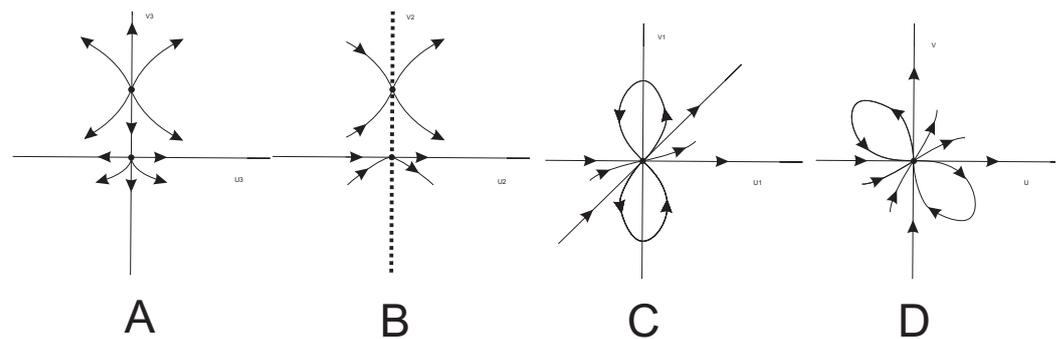


Figure 8. The sequences of blow-ups for obtaining the local phase portrait at the origin of the local U_2 chart when $n = 0, b < 0$, and $m < 0$.

From Figure 8A, we can see that the local phase portrait at the axis $u_2 = 0$ of system (10) is given in Figure 8B. Now, going back through the vertical blow-up and taking into account the value of $\dot{u}_1|_{u_1=0} = -v_1^2$, we obtain the local phase portrait at the origin of system (8) in Figure 8C. Finally, ending the twist, we obtain the local phase portrait at the origin of the local U_2 chart, which is shown in Figures 8D and 9A.

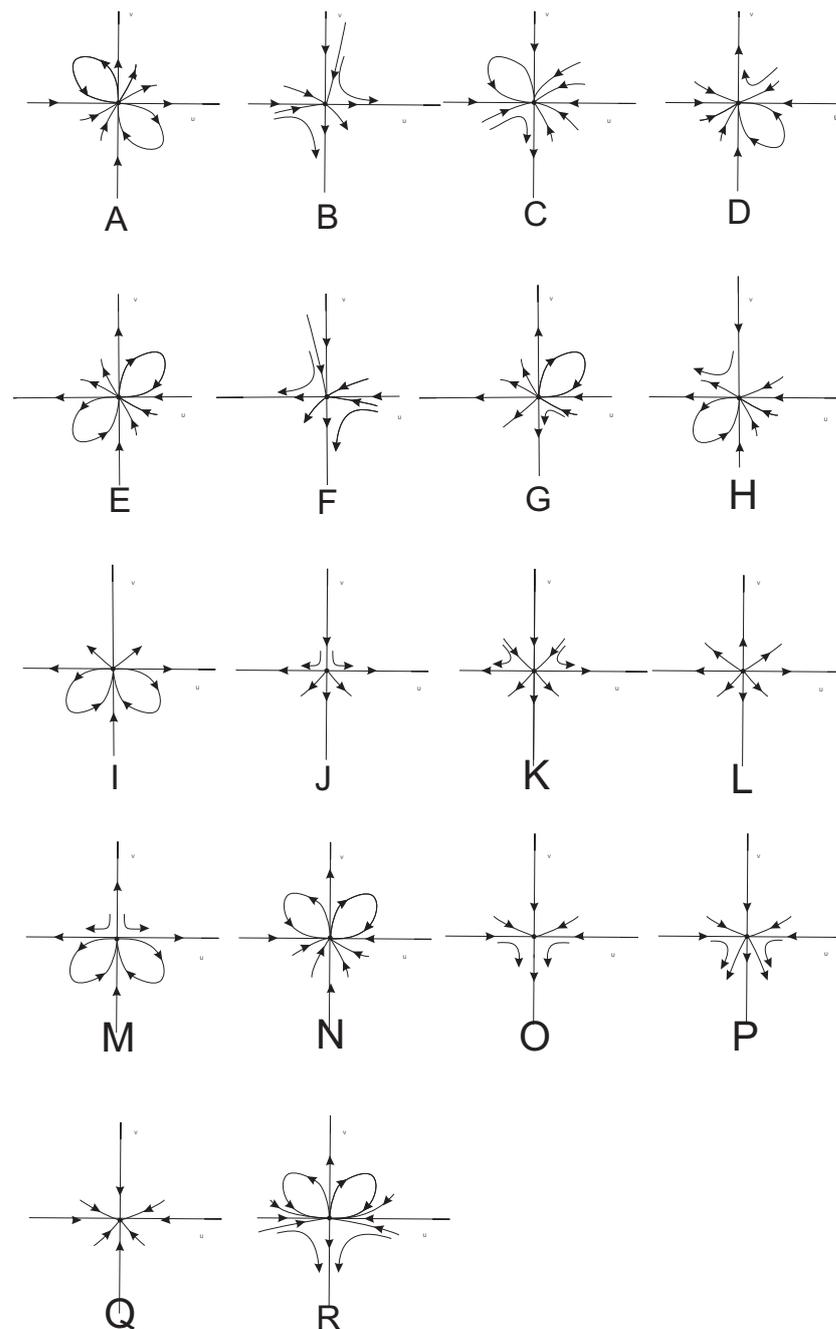


Figure 9. The distinct topological local phase portraits at the origin of the local U_2 chart.

Working in a similar fashion to $n = 0, b < 0$, and $m < 0$, i.e., performing the convenient blow-ups and using Theorems 2.15 and 2.19 of [8], we obtain all the local phase portraits at the origin of the local U_2 chart in Figure 9 for the following cases:

- $n = 0, m < 0$ and $b > 0$ in Figure 9B;
- $n = 0, m < 0, b = 0$, and $d < 0$ in Figure 9C;
- $n = 0, m < 0, b = 0$ and $d > 0$, in Figure 9D;
- $n = 0, m > 0$ and $b < 0$ in Figure 9E;
- $n = 0, m > 0$ and $b > 0$ in Figure 9F;
- $n = 0, m > 0, b = 0$ and $d < 0$ in Figure 9G;
- $n = 0, m > 0, b = 0$ and $d > 0$ in Figure 9H;
- $n = 0, m = 0, \ell < 0$ and $b < 0$ in Figure 9I;
- $n = 0, m = 0, \ell < 0$ and $0 < b \leq \ell + 2$ in Figure 9J;
- $n = 0, m = 0, \ell < 0$ and $b > \ell + 2$ in Figure 9K;

- $n = 0, m = 0, \ell < 0, b = 0$ and $d < 0$ in Figure 9L;
- $n = 0, m = 0, \ell < 0, b = 0$ and $d > 0$ in Figure 9M;
- $n = 0, m = 0, \ell > 0, b < 0$ and in Figure 9N;
- $n = 0, m = 0, \ell > 0$ and $0 < b \leq \ell + 2$ in Figure 9O;
- $n = 0, m = 0, \ell > 0$ and $b > \ell + 2$ in Figure 9P;
- $n = 0, m = 0, \ell > 0, b = 0$ and $d < 0$ in Figure 9Q;
- $n = 0, m = 0, \ell > 0, b = 0$ and $d > 0$ in Figure 9R.

4.4. The Global Phase Portraits

The preceding results for the finite and infinite equilibrium points allowed us to obtain the global phase portraits quite easily, taking into account that the straight line $x = 0$ is invariant.

First, we consider the case satisfying the following conditions: $n > 0, b^2 - 4dn > 0$, and $m^2 > 4\ell n$. We have seen that $n > 0$ denotes a stable hyperbolic node at the origin of chart $U_2, b^2 - 4dn > 0$ indicates the existence of two real finite equilibrium points (p_+ , which is a hyperbolic unstable node, and p_- , which is a hyperbolic saddle), and $m^2 > 4\ell n$ implies two infinite equilibrium points in chart U_1 (P_+ and P_- , which are nilpotent saddle-nodes). The local phase portraits at all these equilibrium points are shown in Figure 10.

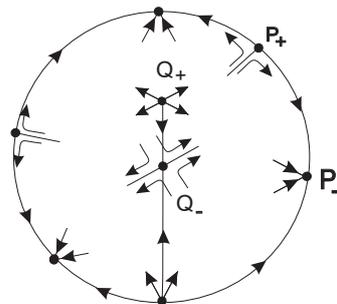


Figure 10. The local phase portraits at the finite and infinite equilibrium points for $n > 0, b^2 - 4dn > 0$, and $m^2 > 4\ell n$.

With the help of Mathematica, we proved that in order for the conditions $n > 0, b^2 - 4dn > 0$, and $m^2 > 4\ell n$ to hold, the parameters of the differential system (3) must satisfy one of the following conditions:

- (i) $b < 0, d \leq 0, \ell < 0$ and $n > 0$;
- (ii) $b < 0, d \leq 0, \ell \geq 0, n > 0$ and $m < -2\sqrt{\ell n}$;
- (iii) $b < 0, d \leq 0, \ell \geq 0, n > 0$ and $m > -2\sqrt{\ell n}$;
- (iv) $b < 0, d > 0, \ell < 0$ and $0 < n < b^2/(4d)$;
- (v) $b < 0, d > 0, \ell \geq 0, 0 < n < b^2/(4d)$ and $m < -2\sqrt{\ell n}$;
- (vi) $b < 0, d > 0, \ell \geq 0, 0 < n < b^2/(4d)$ and $m > -2\sqrt{\ell n}$;
- (vii) $b = 0, d < 0, \ell < 0$ and $n > 0$;
- (viii) $b = 0, d < 0, \ell \geq 0, n > 0$ and $m < -2\sqrt{\ell n}$;
- (ix) $b = 0, d < 0, \ell \geq 0, n > 0$ and $m > -2\sqrt{\ell n}$;
- (x) $b > 0, d \leq 0, \ell < 0$ and $n > 0$;
- (xi) $b > 0, d \leq 0, \ell \geq 0, n > 0$ and $m < -2\sqrt{\ell n}$;
- (xii) $b > 0, d \leq 0, \ell \geq 0, n > 0$ and $m > -2\sqrt{\ell n}$;
- (xiii) $b > 0, d > 0, \ell < 0$ and $0 < n < b^2/(4d)$;
- (xiv) $b > 0, d > 0, \ell \geq 0, 0 < n < b^2/(4d)$ and $m < -2\sqrt{\ell n}$;
- (xv) $b > 0, d > 0, \ell \geq 0, 0 < n < b^2/(4d)$ and $m > -2\sqrt{\ell n}$.

We proved that in cases (i), (ii), (iv), and (vii), and from (ix) to (xv), we could obtain phase portrait (1) of Figure 11; in cases (iii), (vi), and (viii) we obtain phase portrait (2) of Figure 11; finally, in case (v), we obtain the phase portrait that is symmetric to phase portrait

(2), with respect to the straight line $x = 0$. For instance, phase portrait (1) of Figure 11 is obtained when the parameters of system (3) are $d = a = 0, b = -1, \ell = -1, m = -3,$ and $n = 1$; phase portrait (2) of Figure 11 is obtained when the parameters are $d = a = 0, b = -1, \ell = 1, m = 3,$ and $n = 1$. Phase portrait (3) of Figure 11 exists by continuity, from phase portrait (1) to phase portrait (2).

We recall that the *separatrices* of a polynomial’s differential system in the Poincaré disc are all orbits at infinity, the finite equilibria, and the two orbits at the boundary of a hyperbolic sector. Also, the limit cycles are separatrices but quadratic system VIII has no limit cycles. In a phase portrait of the Poincaré disc, if we remove all separatrices, the open components that remain are called the *canonical regions* of the phase portrait. For more details on the separatrices and canonical regions, see [11,12].

The tools used for studying the phase portraits of system (3) for $n > 0, b^2 - 4dn > 0$ and $m^2 > 4\ell n$ are used in the following cases, leading to the following results:

- $n > 0, b^2 - 4dn > 0,$ and $m^2 < 4\ell n$ in Figure 11(4);
- $n > 0, b^2 - 4dn > 0, m^2 = 4\ell n,$ and $2an + (1 - b)m > 0$ in Figure 11(5);
- $n > 0, b^2 - 4dn > 0, m^2 = 4\ell n,$ and $2an + (1 - b)m < 0,$ in this case, the phase portrait is symmetric with respect to the straight line $x = 0$ of the phase portrait of the previous case;
- $n > 0, b^2 - 4dn > 0, m^2 = 4\ell n,$ and $2an + (1 - b)m = 0$ in Figure 11(6);
- $n > 0, b^2 - 4dn < 0,$ and $m^2 > 4\ell n$ in Figure 11(7);
- $n > 0, b^2 - 4dn < 0,$ and $m^2 < 4\ell n$ in Figure 11(8);
- $n > 0, b^2 - 4dn < 0, m^2 = 4\ell n,$ and $2an + (1 - b)m > 0$ in Figure 11(9);
- $n > 0, b^2 - 4dn < 0, m^2 = 4\ell n,$ and $2an + (1 - b)m < 0;$ this case is a symmetric phase portrait with respect to the straight line $x = 0$ of the previous phase portrait;
- $n > 0, b^2 - 4dn < 0, m^2 = 4\ell n,$ and $2an + (1 - b)m = 0$ in Figure 11(10,11);
- $n > 0, b^2 - 4dn = 0,$ and $m^2 > 4\ell n$ from Figure 11(12–14);
- $n > 0, b^2 - 4dn = 0,$ and $m^2 < 4\ell n$ in Figure 11(15);
- $n > 0, b^2 - 4dn = 0, m^2 = 4\ell n,$ and $2an + (1 - b)m > 0$ in Figure 11(16,17);
- $n > 0, b^2 - 4dn = 0, m^2 = 4\ell n,$ and $2an + (1 - b)m = 0$ in Figure 11(18); The cases with $n < 0$ are symmetric with respect to the straight line $y = 0$ in all preceding cases;
- $n = 0, m > 0, b > 0$ in Figure 11(19);
- $n = 0, m > 0, b < 0$ in Figure 11(20);
- $n = 0, m > 0, b = 0,$ and $d > 0$ in Figure 11(21);
- $n = 0, m > 0, b = 0,$ and $d < 0;$ this case has a symmetric phase portrait with respect to $y = 0$ in the previous case; Phase portraits of cases $n = 0$ and $m < 0$ are symmetric with respect to the straight line $x = 0$ of the phase portraits of cases $n = 0$ and $m > 0;$
- $n = 0, m = 0, b > 2 + \ell,$ and $\ell > 0$ in Figure 11(22);
- $n = 0, m = 0, b > 2 + \ell,$ and $\ell < 0;$ this case has a symmetric phase portrait with respect to the $y = 0$ axis;
- $n = 0, m = 0, 0 < b \leq 2 + \ell,$ and $\ell \neq 0$ in Figure 11(23);
- $n = 0, m = 0, b < 0,$ and $\ell > 0$ in Figure 11(24);
- $n = 0, m = 0, b < 0,$ and $\ell < 0;$ the phase portrait of this case is symmetric with respect to the straight line $y = 0$ in the previous phase portrait;
- $n = 0, m = 0, b = 0, \ell > 0,$ and $d < 0$ in Figure 11(25);
- $n = 0, m = 0, b = 0, \ell < 0,$ and $d > 0;$ the phase portrait of this case is symmetric with respect to the straight line $y = 0$ in the previous phase portrait;
- $n = 0, m = 0, b = 0, \ell > 0,$ and $d > 0;$ this case has the same phase portrait as Figure 11(8);
- $n = 0, m = 0, b = 0, \ell < 0,$ and $d < 0;$ this case has the symmetric phase portrait with respect to the straight line $y = 0$ in the phase portrait of Figure 11(8).

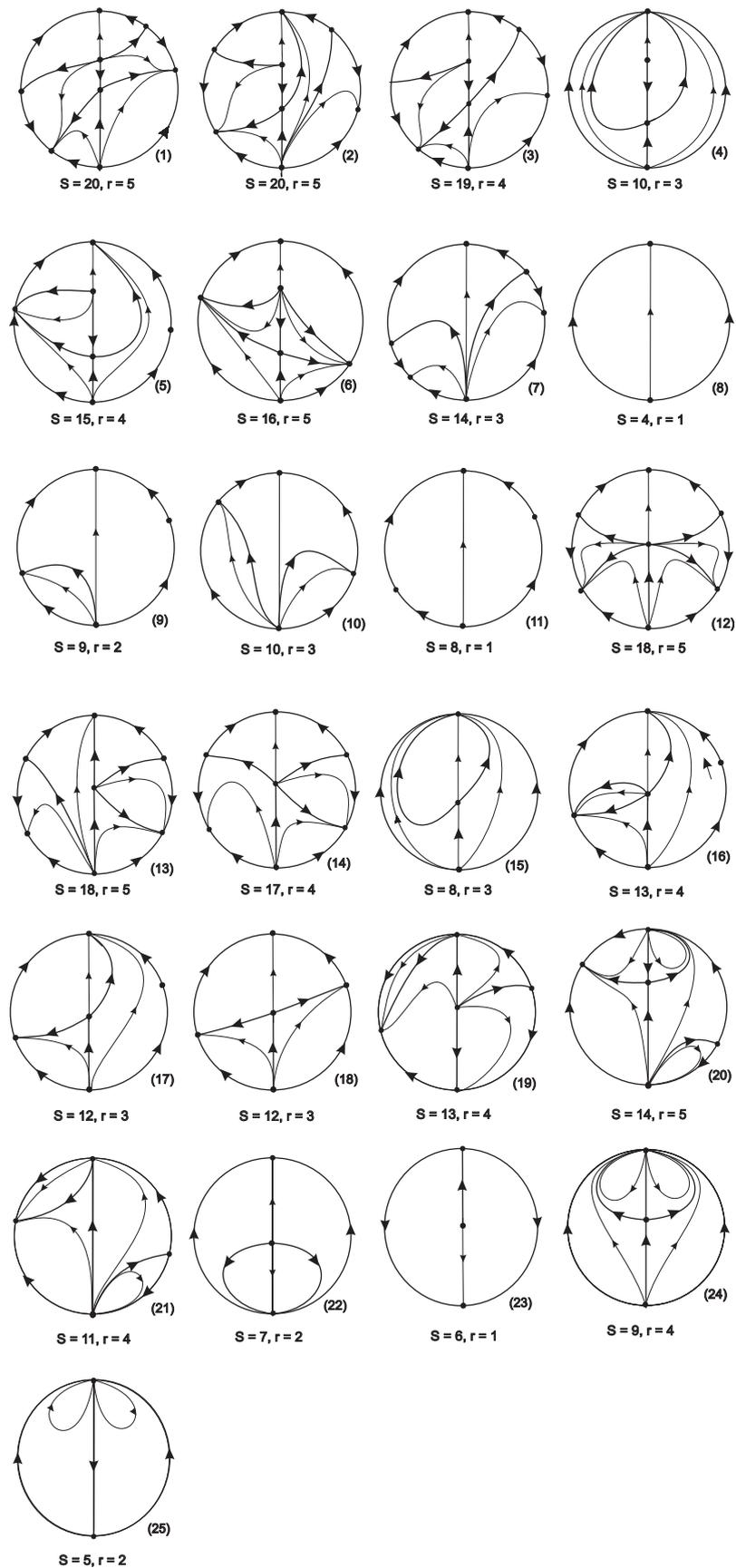


Figure 11. All distinct topological phase portraits of quadratic system VIII. Here, s (respectively, r) denotes the number of separatrices of a phase portrait in the Poincaré disc (respectively, canonical regions).

Of course, from Table 2, the phase portraits with different numbers of separatrices and canonical regions are topologically distinct. Now, we shall see that the phase portraits with the same numbers of separatrices and canonical regions in Table 2 are topologically different.

Table 2. Here, p.p. denotes the phase portrait in the Poincaré disc, s denotes the number of separatrices of the phase portrait, and r denotes the number of canonical regions of the phase portrait.

s	4	5	6	7	8	8	9	9	10	11
r	1	2	1	2	1	3	2	4	3	4
p.p.	8	25	23	22	11	15	9	24	4, 10	21
s	12	13	14	14	15	16	17	18	19	20
r	3	4	3	5	4	5	4	5	4	5
p.p.	17, 18	16, 19	7	20	5	6	14	12, 13	3	1, 2

Phase portraits 4 and 10 of Figure 11 are topologically different because phase portrait 4 has two finite equilibrium points and phase portrait 10 has no finite equilibrium points.

Phase portraits 17 and 18 (respectively, 16 and 19) of Figure 11 are topologically different because phase portrait 17 (respectively, 16) has two orbits going toward the origin of chart U_2 , and such orbits do not exist in phase portrait 18 (respectively, 19).

Phase portrait 14 of Figure 11 has three pairs of infinite equilibrium points, while phase portraits 16 and 19 only have two pairs of infinite equilibrium points, so phase portrait 14 is different from phase portraits 16 and 19.

We note that phase portrait 13 in Figure 11 has three pairs of infinite equilibrium points, while phase portrait 20 only has two pairs, so these two phase portraits are topologically distinct.

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