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Construction of Rank-One Solvable Rigid Lie Algebras with Nilradicals of a Decreasing Nilpotence Index

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Abstract: It is shown that for any integers $k \geq 2$, $q \geq 2k$ and $N \geq k+q+2$, there exists a real solvable Lie algebra of the first rank with a maximal torus of derivations $\mathfrak{t}$ possessing the eigenvalue spectrum $\mathrm{spec}(\mathfrak{t}) = (1, 2, \ldots, k, q, q+1 \ldots, N)$, a nilradical of the nilpotence index $N-k$ and a characteristic sequence $(N-k, 1^k)$.

Keywords: Lie algebras; solvability; rigid; cohomology; Jacobi scheme

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1. Introduction

In spite of the fact that the work of R. Carles on rigid Lie algebras presents a clear picture concerning their generic structural properties [1–4], unifying previous approaches [5,6] and establishing a subdivision of rigid algebras into six principal types [2], the problem of classifying and characterizing rigid Lie algebras is far from being solved in a satisfactory manner. Although the cohomological tools have been shown to be an effective alternative [7], the existence of a purely geometrical notion of rigidity shows that other procedures, such as the Jacobi schemes [8], must be further developed and refined in order to obtain reliable classifications, even in comparatively low dimensions. Solvable Lie algebras are of special relevance among the rigid ones, as they correspond to a class of algebras that cannot be fully classified beyond low dimensions. In this context, the study of the weight systems of maximal tori of derivations [9–11] is a powerful technique for analyzing the rigidity independently of cohomological tools, and several algorithmic procedures for determining rigid Lie algebras and constructing them systematically from the eigenvalue spectra of maximal tori have been developed [12,13], eventually leading to a classification of low dimensional solvable rigid algebras [14–17] as well as the discovery of various rigid hierarchies in an arbitrary dimension in both the cohomological and geometrically rigid cases [18–20]. With the application of symbolic computer packages, further generalizations of some of the previous results have been made possible, as well as the determination of new series of geometrically rigid Lie algebras or the explicit computation of the integrability obstructions that appear in the cohomological approach [21–23]. In this context, recently, various works have been devoted to the systematic analysis and classification of solvable rigid first-rank Lie algebras associated in various types of eigenvalue spectra (see [24,25] and the references therein), showing the possibility of a unified description of ample classes of spectra in dependence of one or more parameters by means of generating functions.

In this work, we proceed with the study of eigenvalue spectra of one-dimensional tori while focusing on the construction of rank-one solvable, cohomologically rigid Lie algebras such that the nilradical $\mathfrak{n}$ has a nilpotence index $\dim \mathfrak{n} - k$ for $k \geq 2$, hence enlarging for lower nilpotent indices some of the constructions and results already known for the filiform case. In particular, we show that for the arbitrary integers $k \geq 2$, $q \geq 2k$ and $N \geq k+q+2$, there exists a real, solvable rank-one Lie algebra with a maximal torus of derivations $\mathfrak{t}$ possessing the eigenvalue spectrum $\mathrm{spec}(\mathfrak{t}) = (1, 2, \ldots, k, q, q+1 \ldots, N+q-k-1)$ such that the nilradical has a nilpotence index $N-k$ and a characteristic sequence $(N-k, 1^k)$. Some possible generalizations of this spectrum analysis are outlined, as well as some comments on the possibility of obtaining geometrically rigid Lie algebras based on them.

Unless otherwise stated, any Lie algebra in this work is finite dimensional and defined over the field of real numbers $\mathbb{R}$.

1.1. General Properties of Nilpotent Lie Algebras

Let $\mathfrak{n}$ be a nilpotent Lie algebra. For any $X \in \mathfrak{n} \setminus [\mathfrak{n}, \mathfrak{n}]$, we consider the decreasing sequence of dimensions of the Jordan blocks of the adjoint operator $\mathrm{ad}(X)$:

$$c(X) = (c_1(X), c_2(X), \cdots, c_k(X), 1), \quad c_i(X) \geq c_{i+1}(X) \geq 1. \tag{1}$$

As $c(X)$ constitutes a similarity invariant, it determines an invariant $c(\mathfrak{n})$ defined as

$$c(\mathfrak{n}) = \sup \{c(X) \mid X \in \mathfrak{n} \setminus [\mathfrak{n}, \mathfrak{n}]\}. \tag{2}$$

and called the characteristic sequence of $\mathfrak{n}$ (see, for example, [18] and the references therein). A vector X is such that $c(X) = c(\mathfrak{n})$ will be called a characteristic vector of $\mathfrak{n}$. Another invariant is given by the dimensions of the central descending sequence, given recursively by

$$C^0(\mathfrak{n}) = \mathfrak{n}, \; C^k(\mathfrak{n}) = \left[\mathfrak{n}, C^{k-1}(\mathfrak{n})\right], \; k \geq 1. \tag{3}$$

This sequence further determines the so-called associated graded Lie algebra $\mathfrak{gr}(\mathfrak{n}) = \mathfrak{g}_1(\mathfrak{n}) \oplus \cdots \oplus \mathfrak{g}_r(\mathfrak{n})$ with

$$\mathfrak{g}_k(\mathfrak{n}) = C^{k-1}(\mathfrak{n}) / C^k(\mathfrak{n}), \; k \geq 1. \tag{4}$$

The Lie algebra $\mathfrak{n}$ is called naturally graded if the isomorphism of the Lie algebras $\mathfrak{n} \simeq \mathfrak{gr}(\mathfrak{n})$ holds.

We denote with $Der(\mathfrak{n})$ the Lie algebra of the derivations of $\mathfrak{n}$ (i.e., the space of the linear maps $D : \mathfrak{n} \to \mathfrak{n}$ satisfying the following condition):

$$D([X, Y]) = [D(X), Y] + [X, D(Y)], \quad X, Y \in \mathfrak{n}. \tag{5}$$

Definition 1. *Let $\mathfrak{g}$ be a Lie algebra of a dimension n. An external torus of derivations is any Abelian subalgebra of $Der(\mathfrak{g})$, the generators of which are semisimple.*

The elements in a (maximal) torus are simultaneously diagonalizable in the complex extension of the base field (i.e., $f \otimes_{\mathbb{R}} \mathrm{Id} \in End(\mathfrak{n} \otimes_{\mathbb{R}} \mathbb{C})$ admit a diagonal matrix over $\mathbb{C}$ for some basis). As shown in [26], the maximal tori of the complexified Lie algebra $\mathfrak{n} \otimes_{\mathbb{R}} \mathbb{C}$ are conjugated by an inner automorphism, which implies that their dimension is a scalar invariant of the Lie algebra, commonly referred to as the rank of $\mathfrak{n}$ and denoted by $\mathrm{r}(\mathfrak{n})$.

According to the general structure theory, a real or complex solvable Lie algebra $\mathfrak{r}$ admits the decomposition as semidirect sum

$$\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}, \tag{6}$$

satisfying the relations

$$[\mathfrak{t}, \mathfrak{n}] \subset \mathfrak{n}, \; [\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}, \; [\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{n}, \tag{7}$$

where $\mathfrak{n}$ is the maximal nilpotent ideal of $\mathfrak{r}$ (the nilradical) and $\overrightarrow{\oplus}$ denotes the action of $\mathfrak{t}$ on $\mathfrak{n}$ by linearly nil-independent outer derivations. The dimension of $\mathfrak{t}$ has a further upper bound expressed by the following inequality:

$$\dim \mathfrak{n} - \dim[\mathfrak{n}, \mathfrak{n}] \geq \dim \mathfrak{t}. \tag{8}$$

1.2. Solvable Rigid Lie Algebras

Let $\mathcal{L}^n$ denote the variety of n-dimensional Lie algebras $\mathfrak{g} = \left(\mathbb{K}^n, [\,,\,]_{\mathfrak{g}}\right)$ over $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The general linear group $GL(n, \mathbb{K})$ acts naturally on $\mathcal{L}^n$ under the following changes of basis:

$$(f \star \mathfrak{g})(X, Y) = f^{-1}\left([f(X), f(Y)]_{\mathfrak{g}}\right),\ f \in GL(n, \mathbb{K}),\ X, Y \in \mathfrak{g}. \tag{9}$$

The orbit $\mathcal{O}(\mathfrak{g})$ of $\mathfrak{g}$ is therefore identified with the Lie algebras that are isomorphic to $\mathfrak{g}$.

Definition 2. *A Lie algebra $\mathfrak{g}$ is rigid if the orbit $\mathcal{O}(\mathfrak{g})$ is an open set of $\mathcal{L}^n$ with respect to the Euclidean topology.*

This definition of rigidity, although mainly topological, admits various equivalent reformulations in analytical or algebraic terms (see, for example, [1,5,16]). In this context, using the adjoint cohomology of Lie algebras [7,27,28], several criteria to ensure rigidity have been proposed [15,29,30]:

Proposition 1. *Let $\mathfrak{g}$ be a Lie algebra. If the condition* $\dim H^2(\mathfrak{g}, \mathfrak{g}) = 0$ *holds, then $\mathfrak{g}$ is rigid.*

According to this result, we say that a Lie algebra $\mathfrak{g}$ is cohomologically rigid if $H^2(\mathfrak{g}, \mathfrak{g}) = 0$. This criterion, albeit not necessary for rigidity, has been extremely useful in the analysis of large classes of rigid Lie algebras and has further allowed a detailed comparison with rigid algebras whose cohomology is not zero. Using the quadratic Rim map $\mathrm{Sq} : H^2(\mathfrak{g}, \mathfrak{g}) \to H^3(\mathfrak{g}, \mathfrak{g})$ defined by

$$\mathrm{Sq}(\psi)(X_i, X_j, X_k) := \psi\big(\psi(X_i, X_j), X_k\big) + \psi\big(\psi(X_j, X_k), X_i\big) + \psi\big(\psi(X_k, X_i), X_j\big), \tag{10}$$

another sufficiency criterion for rigidity was proven in [31,32]. This criterion states that if Sq is an injective map, then $\mathfrak{g}$ is a rigid Lie algebra.

We also recall briefly the Hochschild–Serre factorization theorem [27,33], which provides a practical procedure for explicitly computing the cohomology spaces of the semidirect sums of Lie algebras. Let $\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$ denote a solvable Lie algebra such that $\mathfrak{t}$ is Abelian and the operators $\mathrm{ad}_{\mathfrak{r}} T$ $(T \in \mathfrak{t})$ are diagonal. Then, the adjoint cohomology $H^p(\mathfrak{r}, \mathfrak{r})$ satisfies the following isomorphism:

$$H^p(\mathfrak{r}, \mathfrak{r}) \simeq \sum_{a+b=p} H^a(\mathfrak{t}, \mathbb{R}) \otimes H^b(\mathfrak{n}, \mathfrak{r})^{\mathfrak{t}}, \tag{11}$$

where

$$H^b(\mathfrak{n}, \mathfrak{r})^{\mathfrak{t}} = \left\{[\varphi] \in H^b(\mathfrak{n}, \mathfrak{r}) \mid (T.\varphi) = 0,\ T \in \mathfrak{t}\right\} \tag{12}$$

is the space of $\mathfrak{t}$-invariant cocycle classes of $\mathfrak{n}$ with values in $\mathfrak{r}$. The invariance of the cocycles is determined by the condition

$$(T.\varphi)(Z_1, \cdots, Z_b) = [T, \varphi(Z_1, \cdots, Z_b)] - \sum_{s=1}^{b} \varphi(Z_1, \cdots, [T, Z_s], \cdots, Z_b). \tag{13}$$

Observing that $H^p(\mathfrak{t}, \mathbb{R}) = \bigwedge^p \mathfrak{t}$, it can easily be justified that $H^p(\mathfrak{r}, \mathfrak{r}) = 0$ is equivalent to the identities $H^b(\mathfrak{n}, \mathfrak{r}) = 0$ for $0 \leq b \leq p$. If, in addition, $\mathfrak{r}$ is a complex, solvable rigid

Lie algebra, then the decomposition theorem of Carles implies that the torus $\mathfrak{t}$ is indeed a maximal external torus of derivations of the nilradical $\mathfrak{n}$ [2].

2. Structural Properties of the Nilpotent Lie Algebra $\mathfrak{n}_{N,k}^0$

For any $k \geq 1$ and $N \geq 2k+1$, let $\mathfrak{n}_{N,k}^0$ be the Lie algebra with nonvanishing commutators

$$\begin{aligned} [X_1, X_j] &= X_{j+1}, \quad k+1 \leq j \leq N-1, \\ [X_2, X_j] &= X_{j+2}, \quad k+1 \leq j \leq N-2, \\ &\dots \\ [X_k, X_j] &= X_{j+k}, \quad k+1 \leq j \leq N-k, \end{aligned} \tag{14}$$

over the basis $\mathcal{B} = \{X_1, \dots, X_N\}$. The central descending sequence is given by

$$C^s(\mathfrak{n}_{N,k}^0) = \langle X_{k+s}, \dots, X_N \rangle, \quad 2 \leq s \leq N-k; \quad C^{N+1-k}(\mathfrak{n}_{N,k}^0) = 0,$$

showing that $\mathfrak{n}_{N,k}^0$ is nilpotent with a nilpotence index $N-k$. It is straightforward to verify that the characteristic sequence of the Lie algebra is given by $c(\mathfrak{n}_{N,k}^0) = (N-k, 1^k)$. We further observe that $\mathfrak{n}_{N,k}^0$ is naturally graded only for $k=1$, in which case $\mathfrak{n}_{N,k}^0$ is isomorphic to the model filiform Lie algebra L_N [14]. In a certain sense, the algebras defined by Equation (14) constitute an extension of the models of the Bratzlavsky type (see [10,23]) to lower characteristic sequences.

For later use, it is convenient to consider the Maurer–Cartan equations of $\mathfrak{n}_{N,k}^0$. If $\{\omega^1, \dots, \omega^N\}$ denotes the dual basis of $\mathcal{B}$, then these are given by

$$\begin{aligned} d\omega^p &= 0, \quad 1 \leq p \leq k+1, \\ d\omega^r &= \sum_{a=1}^{r-k-1} \omega^a \wedge \omega^{r-a}, \quad k+2 \leq r \leq 2k \\ d\omega^s &= \sum_{a=1}^{k} \omega^a \wedge \omega^{s-a}, \quad 2k+1 \leq s \leq N. \end{aligned} \tag{15}$$

If $\theta = \sum_{\ell=1}^{N} a_\ell d\omega^\ell \in \mathcal{L}(\mathfrak{n}_{N,k}^0) = \mathbb{R}\{d\omega_i\}_{1 \leq i \leq N}$ is now a generic linear combination of the 2-forms in Equation (15), then it is straightforward to verify that

$$\bigwedge^k \theta \equiv 0, \quad \bigwedge^{k-1} \theta \neq 0.$$

The quantity

$$j_0\left(\mathfrak{n}_{N,k}^0\right) = \max\left\{ j_0(\omega) \mid \omega \in \mathcal{L}(\mathfrak{n}_{N,k}^0) \right\} = k \tag{16}$$

depends only on the structure of $\mathfrak{n}_{N,k}^0$ and constitutes a numerical invariant of the Lie algebra. (This actually means that $\mathfrak{n}_{N,k}^0$ possesses $N-2k$ functionally independent invariants for the coadjoint representation.)

Lemma 1. *For any $k \geq 1$ and $N \geq 2k+1$, the rank of $\mathfrak{n}_{N,k}^0$ is two.*

Let $f(X_\ell) = \sum_{s=1}^{N} f_\ell^s X_s$ be the expression of a derivation of $\mathfrak{n}_{N,k}^0$. As the center is generated by X_N, it follows immediately that $f(X_N) = f_N^N X_N$. Evaluation of the derivation condition in Equation (5) for $X = X_1$, $Y = X_{N-1}$ shows in particular that

$$f(X_{N-1}) = \sum_{s=1}^{k} f_{N-1}^s X_s + f_{N-1}^{N-1} X_{N-1} + f_{N-1}^N X_N.$$

Now, computation for the pair $X = X_{k+1}$, $Y = X_{N-1}$ implies that

$$f_{N-1}^s = 0, \quad 1 \leq s \leq k; \quad f_{k+1}^1 = 0.$$

Iterating the computation for the pair $X = X_1$, $Y = X_{N-p}$ (for $N - p \geq k+1$) first shows that $f(X_{N-p}) = \sum_{s=1}^{k} f_{N-p}^s X_s + \sum_{q=N-p}^{N} f_{N-p}^q X_q$, while evaluation of Equation (5) for $X = X_{k+1}$, $Y = X_{N-p}$ successively leads to the conditions

$$f_{N-p}^s = 0, \quad 1 \leq s \leq k; \quad f_{k+1}^p = 0.$$

From these identities, we conclude that $f(X_q) = \sum_{s=q}^{N} f_q^s X_s$ for $q \geq k+1$. Now, considering the pair $X = X_m$ and $Y = X_{k+1}$ for $m \leq k$, we obtain

$$\sum_{s=k+1}^{N-m} f_{k+1}^s X_s - \sum_{s=1}^{m} f_m^s X_s = \sum_{s=m+k+1}^{N} f_{m+k+1}^s X_s,$$

from which it follows by iteration on the value of m that $f(X_m) = \sum_{s=m}^{N} f_m^s X_s$, showing that the matrix of f is triangular. In order to compute the semisimple derivations, it therefore suffices to consider a generic diagonal derivation $\Phi(X_i) = \lambda_i X_i$. From the commutators in Equation (14), the following relations are easily obtained:

$$\lambda_i + \lambda_j = \lambda_{i+j}, \quad 1 \leq i \leq k,\ k+1 \leq j \leq N-i. \tag{17}$$

Considering $i = 1$, it follows for $s \geq 2$ that

$$\lambda_{k+s} = \lambda_1 + \lambda_{k+s-1} = 2\lambda_1 + \lambda_{k+s-2} \cdots = (s-1)\lambda_1 + \lambda_{k+1}.$$

On the other hand, for $1 < i \leq k$, the relation

$$\lambda_i + \lambda_{k+1} = \lambda_{i+k+1} = i\lambda_1 + \lambda_{k+1} \tag{18}$$

implies that $\lambda_i = i\lambda_1$. It follows that there exist two diagonalizable derivations F_1 and F_2 with eigenvalues

$$\begin{aligned} \mathrm{spec}(F_1) =& (1, 2, \ldots, k, 0, 1, 2, \ldots, (N-k-1)), \\ \mathrm{spec}(F_2) =& (0, 0, \ldots, 0, 1, 1, 1, \ldots, 1), \end{aligned} \tag{19}$$

from which we conclude that the rank of $\mathfrak{n}_{N,k}^0$ is two. We denote a maximal torus of $\mathfrak{n}_{N,k}^0$ with $\mathfrak{t}_0$.

Let $\mathfrak{r}_0 = \mathfrak{t}_0 \overrightarrow{\oplus} \mathfrak{n}$ be a solvable Lie algebra such that the torus $\mathfrak{t}_0$ is generated by two diagonalizable derivations T_1, T_2 with the eigenvalues given in Equation (19). Although it is not essential for the following, using the properties of the root system associated with solvable Lie algebras [13], the rigidity of $\mathfrak{r}_0$ can be shown directly without applying cohomological methods. Thus, $\mathfrak{r}_0$ defines a series of rank-two solvable rigid Lie algebras for any $k \geq 2$ and $q \geq 2k$. In particular, for $k = 1$, we recover the rigid Lie algebra associated with the model filiform Lie algebra L_n [34]. For higher values of k, the algebra can be seen as the counterpart of the model algebra for characteristic sequences $c(\mathfrak{n}_{N,k}^0) = (N-k, 1^k)$. (Incidentally, the algebras $\mathfrak{r}_0$ are actually cohomologically rigid.)

Generation of Rank-One Solvable Lie Algebras

In this section, we analyze how to derive nilpotent Lie algebras of rank one using the Lie algebra $\mathfrak{n}_{N,k}^0$ such that the eigenvalues of a maximal torus are given in terms of Equation (19). Considering a linear combination $F_1 + qF_2$, we obtain a diagonal derivation with eigenvalues

$$\mathrm{spec}(F_1 + qF_2) = (1, 2, \ldots, k, q, q+1, q+2, \ldots, (N+q-k-1)), \quad q \neq 0. \tag{20}$$

In this context, it can be asked whether, when starting from the nilpotent Lie algebra $\mathfrak{n}_{N,k}^0$, we can obtain another nilpotent Lie algebra that is isomorphic to a nontrivial deformation of $\mathfrak{n}_{N,k}^0$ and such that it is of the first rank, with a torus $\mathfrak{t}$ whose eigenvalues are given by Equation (20). A first example in this direction was already given in [18] for a fixed dimension, where the cohomological rigidity of the solvable Lie algebra $\mathfrak{r}_{q+4,q}$ with commutators

$$\begin{aligned} [T, X_j] &= \mu_j X_j, \\ [X_1, X_j] &= X_{j+1}, \quad 3 \leq j \leq q+3, \\ [X_2, X_j] &= X_{j+2}, \quad 3 \leq j \leq q+2, \\ [X_3, X_4] &= X_{q+4}, \end{aligned} \tag{21}$$

where $q \geq 4$, $\mu_1 = 1, \mu_2 = 2$ and $\mu_s = q + s - 3$ for $3 \leq s \leq q+4$ was proven. In this case, the spectrum of the torus generated by T is given by

$$\mathrm{spec}(T) = (1, 2, q, q+1, q+2, \ldots, 2q+1), \quad q \geq 4, \tag{22}$$

thus belonging to the type in Equation (20), with $k = 2$ and $N = q + 4$. We further observe that the nilradical is isomorphic to the deformation $\mathfrak{n}_{q+4,2}^0 + \varphi$, where $\varphi(X_3, X_4) = X_{q+4}$ defines a nontrivial cocycle. The addition of this cocycle particularly implies that F_2 cannot be a derivation of the deformed algebra, from which the rank reduction follows.

The Lie algebra in Equation (21) can actually be seen as the first element in a series of solvable Lie algebras of the first rank with vanishing cohomology. To this extent, consider $N \geq q + 4$ and the skew-symmetric 2-form φ on $\mathfrak{n}_{N,2}^0$ defined by

$$\varphi(X_3, X_j) = X_{q+j}, \quad 4 \leq j \leq N - q. \tag{23}$$

It can immediately be verified that φ is a 2-cocycle of $\mathfrak{n}_{N,2}^0$. In order to prove that the cohomology class of φ is nonzero, we consider the following 2-form on the (linearly) deformed Lie algebra $\mathfrak{n}_{N,2}^0 + \varepsilon\varphi$:

$$\theta = \mathrm{d}\omega^N = \omega^1 \wedge \omega^{N-1} + \omega^2 \wedge \omega^{N-2} + \varepsilon\, \omega^3 \wedge \omega^{N-3}.$$

For any $\varepsilon \neq 0$, we have $\theta \wedge \theta \wedge \theta \neq 0$, while for $\varepsilon = 0$, the index of a generic 2-form over $\mathfrak{n}_{N,2}^0$ is 2 (see Equation (16)), showing that both algebras are not isomorphic and hence implying that $[\varphi] \neq 0$.

Let $\mathfrak{n}_{2,q,N} = \mathfrak{n}_{N,2}^0 + \varphi$. Repeating the argumentation of Lemma 1, it follows at once that any derivation f of $\mathfrak{n}_{2,q,N}$ is triangular. Assuming that f is a diagonal derivation, it particularly satisfies the conditions in Equations (17) and (18) for $k = 2$ such that

$$\begin{aligned} f(X_1) =& \lambda_1 X_1,\ f(X_2) = 2\lambda_1 X_2,\ f(X_3) = \lambda_3 X_3, \\ f(X_j) =& ((j-3)\lambda_1 + \lambda_3) X_j, \quad 4 \leq j \leq N. \end{aligned}$$

In addition to these constraints, the condition $f([X_3, X_j]) = [f(X_3), X_j] + [X_3, f(X_j)]$ must be fulfilled, leading to the eigenvalue identities

$$\lambda_3 + (j-3)\lambda_1 + \lambda_3 = (q+j-3)\lambda_1 + \lambda_3, \quad j \geq 4, \tag{24}$$

from which $\lambda_3 = q$ follows at once. We conclude that $\mathfrak{n}_{2,q,N}$ has the first rank with a maximal torus $\mathfrak{t}$ having eigenvalues as given in Equation (20) for $k = 2$.

Proposition 2. *For any $q \geq 4$ and $N \geq q + 4$, the solvable Lie algebra $\mathfrak{r}_{2,q,N} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}_{2,q,N}$ is rigid with a vanishing cohomology $H^2(\mathfrak{r}_{2,q,N}, \mathfrak{r}_{2,q,N})$.*

The proof follows by application of the Hochschild–Serre factorization theorem [27]. It is straightforward to verify that any invariant 1-cochain $\varphi \in C^1(\mathfrak{r}_{2,q,N}, \mathfrak{r}_{2,q,N})$ has the form

$$\varphi(X_i) = a_i^i X_i, \quad 1 \leq i \leq N.$$

For the coboundary operator, we have the nonvanishing entries

$$\begin{aligned}
\mathrm{d}\varphi(X_1, X_j) &= (a_1^1 + a_j^j - a_{1+j}^{j+1}) X_{1+j}, \quad j \geq 3,\\
\mathrm{d}\varphi(X_2, X_j) &= (a_2^2 + a_j^j - a_{2+j}^{j+1}) X_{2+j}, \quad j \geq 3,\\
\mathrm{d}\varphi(X_3, X_j) &= (a_3^3 + a_j^j - a_{q+j}^{q+j}) X_{q+j}, \quad j \geq 4,
\end{aligned}$$

from which it follows at once that $\mathrm{d}\varphi = 0$ only if

$$a_2 = 2a_1, \quad a_j = (q + j - 3)a_1, \quad j \geq 3,$$

further showing that $\dim B^2(\mathfrak{n}_{2,q,N}, \mathfrak{r}_{2,q,N})^{\mathfrak{t}} = N - 1$. On the other hand, a $\mathfrak{t}$-invariant 2-form has the shape

$$\begin{aligned}
\mathrm{d}\varphi(X_1, X_j) =& b_{1,j}^{j+1} X_{1+j}, \quad j \geq 3,\\
\mathrm{d}\varphi(X_2, X_j) =& b_{2,j}^{2+j} X_{2+j}, \quad j \geq 3,\\
\mathrm{d}\varphi(X_i, X_j) =& b_{i,j}^{i+j+q-3} X_{i+j+q-3}, \quad j \geq 4,
\end{aligned}$$

Imposing the condition $\mathrm{d}\varphi = 0$ leads to the system of coefficients

$$\begin{aligned}
b_{2,j}^{j+2} - b_{1,j}^{1+j} - b_{2,j+1}^{j+3} + b_{1,j+2}^{j+3} &= 0, \quad j \geq 4\\
b_{3,j}^{j+q} - b_{2,j}^{j+2} - b_{3,j+2}^{j+2+q} + b_{2,j+q}^{j+q+2} &= 0, \quad j \geq 4\\
b_{3,j}^{j+q} - b_{1,j}^{j+1} &= 0, \quad j \geq 5.
\end{aligned}$$

The system can be solved recursively. A routine computation shows that, as a basis of independent coefficients, we can choose

$$b_{3,4}^{4+q},\ b_{2,3}^{5},\ b_{1,j}^{j+1} \quad \text{for} \quad 3 \leq j \leq N-1,$$

implying that $\dim Z^2(\mathfrak{n}_{2,q,N}, \mathfrak{r}_{2,q,N})^{\mathfrak{t}} = N - 1$. It follows at once from this identity that $\dim H^2(\mathfrak{n}_{2,q,N}, \mathfrak{r}_{2,q,N})^{\mathfrak{t}} = 0$, showing that the algebra is cohomologically rigid.

3. The Solvable Lie Algebras $\mathfrak{r}_{k,q,N}$

As the preceding proof does not essentially depend on the value of k, it is naturally suggested that the result can be easily generalized to nilradicals with a characteristic sequence $c(\mathfrak{n}_{N,k}^0) = (N-k, 1^k)$ for arbitrary values $k \geq 3$ by considering the 2-cocycle class of $\mathfrak{n}_{N,k}^0$, defined by

$$\varphi(X_{k+1}, X_j) = X_{k+q}, \quad k+1 \leq j \leq N-q. \tag{25}$$

Consider the Maurer–Cartan equations of $\mathfrak{n}_{N,k}^0 + \varepsilon\varphi$. It is immediate to verify that the 2-form

$$\theta = \mathrm{d}\omega^N = \sum_{p=1}^{k} \omega^p \wedge \omega^{N-p} + \varepsilon\, \omega^{k+1} \wedge \omega^{N-k-1}$$

satisfies the identity $\bigwedge^{k+1} \theta \neq 0$ for any $\varepsilon \neq 0$, showing that the deformation $\mathfrak{n}_{N,k}^0 + \varepsilon\varphi$ is not isomorphic to $\mathfrak{n}_{N,k}^0$. Furthermore, using Lemma 1, the same reasoning as that used in Equation (24) shows that $\mathfrak{n}_{k,q,N} = \mathfrak{n}_{N,k}^0 + \varphi$ has the first rank with a maximal torus $\mathfrak{t}$ possessing the eigenvalues

$$\operatorname{spec}(\mathfrak{t}) = (1, 2, \ldots, k, q, q+1, q+2, \ldots, q+N-k-1), \quad q \geq 2k. \tag{26}$$

In analogy with the previous case, we define the solvable real Lie algebra of the first rank $\mathfrak{r}_{k,q,N} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}_{k,q,N}$. Over a basis $\{T, X_1, \ldots, X_N\}$ with $N \geq 2q + 2 - k$, the precise brackets are given by

$$\begin{aligned} [T, X_i] &= i\, X_i, \quad 1 \leq i \leq k \\ [T, X_j] &= (q + j - k - 1)X_j, \quad k + 1 \leq j \leq N \\ [X_a, X_j] &= X_{j+a}, \quad 1 \leq a \leq k,\ k + 1 \leq j \leq N - a, \\ [X_{k+1}, X_j] &= X_{q+j}, \quad k + 2 \leq j \leq N - q. \end{aligned} \tag{27}$$

Proposition 3. *For any $k \geq 2$, $q \geq 2k$ and $N \geq 2q + 2 - k$, the solvable Lie algebra $\mathfrak{r}_{k,q,N}$ is cohomologically rigid.*

The proof is completely analogous to that of Proposition 1, for which reason we omitted the detailed computations. The results above show that for any $k \geq 2$ and any dimension $N \geq 3k + 2$, such Lie algebras exist, with $N = 8$ (for $k = 2$) being the lowest dimension for which an eigenvalue spectrum as given in Equation (26) appears. The series $\mathfrak{r}_{k,q,N}$ hence gives a partial answer to a question formulated in [19], namely finding the conditions for the existence of rank-one rigid Lie algebras such that the nilradical has a given characteristic sequence.

The results above can be slightly refined. While still considering the eigenvalue spectrum in Equation (20), the requirement $q \geq 2k$ can be relaxed under certain circumstances. Indeed, for the values $k + 2 < q < 2k$ and dimension $N = k + q + 2$, the nilpotent Lie algebra given by the commutators

$$\begin{aligned} [X_i, X_j] &= X_{i+j}, \quad 1 \leq i \leq k, \quad k + 1 \leq j \leq N - i, \\ [X_{k+1}, X_{k+2}] &= X_N \end{aligned} \tag{28}$$

with $k \geq 3$ also leads to rank-one solvable Lie algebras with a vanishing cohomology and hence to a rigid Lie algebra. It is straightforward to verify that this case also arises as a deformation of the nilpotent algebras $\mathfrak{n}_{k,q,k+q+2}$. Observe that for $k = 2$, it coincides with the case where $q = 2k$ and hence does not provides any additional solution for this value.

In Table 1, we enumerated the spectra of the type in Equation (20) and with $k \geq 3$ and the range of values $k + 2 \leq q \leq 2k$, which give rise to a rank-one solvable rigid Lie algebra for dimensions $10 \leq N \leq 20$. A comparison of the dimensions shows that for any dimension $N \geq 10$, there exists at least one solvable rigid Lie algebra satisfying the above constraints. When adding those solutions for which $q \geq 2k$ holds, we conclude that for any fixed dimension $N \geq 10$, there are several nonisomorphic solvable rigid Lie algebras of the first rank and an eigenvalue spectrum of the type in Equation (20). In this context, a question that is still unanswered is whether any cohomologically rigid solvable Lie algebra of the first rank with the eigenvalue spectrum in Equation (20) is characterized by the fact that its nilradical is isomorphic to a deformation of the nilpotent Lie algebra $\mathfrak{n}_{N,k}^0$.

Table 1. Cohomologically rigid Lie algebras with spectrum in Equation (26) and $k+2 \leq q \leq 2k$.

spec $(\mathfrak{t})$	k	q	N
$(1,2,3,5,6,7,8,9,10,11)$	3	5	10
$(1,2,3,6,7,8,9,10,11,12,13)$	3	6	11
$(1,2,3,4,6,7,8,9,10,11,12,13)$	4	6	12
$(1,2,3,4,7,8,9,10,11,12,13,14,15)$	4	7	13
$(1,2,3,4,8,9,10,11,12,13,14,15,16,17)$	4	8	14
$(1,2,3,4,5,7,8,9,10,11,12,13,14,15)$	5	7	14
$(1,2,3,4,5,8,9,10,11,12,13,14,15,16,17)$	5	8	15
$(1,2,3,4,5,9,10,11,12,13,14,15,16,17,18,19)$	5	9	16
$(1,2,3,4,5,6,8,9,10,11,12,13,14,15,16,17)$	6	8	16
$(1,2,3,4,5,10,11,12,13,14,15,16,17,18,19,20,21)$	5	10	17
$(1,2,3,4,5,6,9,10,11,12,13,14,15,16,17,18,19)$	6	9	17
$(1,2,3,4,5,6,10,11,12,13,14,15,16,17,18,19,20,21)$	6	10	18
$(1,2,3,4,5,6,7,9,10,11,12,13,14,15,16,17,18,19)$	7	9	18
$(1,2,3,4,5,6,11,12,13,14,15,16,17,18,19,20,21,22,23)$	6	11	19
$(1,2,3,4,5,6,7,10,11,12,13,14,15,16,17,18,19,20,21)$	7	10	19
$(1,2,3,4,5,6,12,13,14,15,16,17,18,19,20,21,22,23,24,25)$	6	12	20
$(1,2,3,4,5,6,7,11,12,13,14,15,16,17,18,19,20,21,22,23)$	7	11	20
$(1,2,3,4,5,6,7,8,10,11,12,13,14,15,16,17,18,19,20,21)$	8	10	20

4. Conclusions

In this work, certain results from [10,18] concerning rank-one solvable rigid Lie algebras were extended to the case of nilradicals having a characteristic sequence $c(\mathfrak{n}) = (n-k, 1^k)$ for an arbitrary $k \geq 2$, a one-dimensional torus of derivations with eigenvalues from Equation (26) and dimensions $N \geq k+q+2$. This also solves a subsidiary question formulated in [19], providing the minimal dimensions for which a rank-one rigid Lie algebra with a certain characteristic sequence can appear. The guiding principle has been to consider certain deformations of the nilpotent Lie algebra $\mathfrak{n}_{N,k}^0$ that imply the existence of a unique diagonal derivation, hence guaranteeing that the rank is one. However, this approach merely constitutes one of the multiple possibilities that are conceivable. Rigid algebras structurally analogous but not related to $\mathfrak{n}_{N,k}^0$ can also be constructed along similar lines. We started by considering the eigenvalue sequence $\Lambda = (1,2,4,\ldots,9,18,19,\ldots 37)$. A routine computation shows that the 28-dimensional nilpotent algebra defined by

$$\begin{aligned}
[X_a, X_j] &= X_{j+a}, \; 1 \leq a \leq 2; \quad 9 \leq j \leq 28-a, \\
[X_a, X_j] &= X_{j+a+1}, \; 3 \leq a \leq 8; \quad 9 \leq j \leq 27-a, \\
[X_9, X_{10}] &= X_{28}
\end{aligned}$$

has a rank of one, with a maximal torus having the eigenvalues Λ. The corresponding extension of the nilradical by the torus determines a rank-one solvable rigid Lie algebra with a vanishing cohomology. In contrast to the series derived from $\mathfrak{n}_{N,k}^0$, the eigenvalues of Λ are not obtainable as a linear combination of the elements in Equation (19), as there are two jumps (i.e., discontinuities) in the eigenvalue sequence.

The preceding example can be generalized to arbitrary dimensions considering the eigenvalue sequence

$$\mathrm{spec}(\mathfrak{t}) = (1,2,\ldots,k,q,q+1,\ldots,2q+1,p,p+1,\ldots,2p+1) \tag{29}$$

with $N = k + q + p + 4$, $k \geq 2$, $q \geq 2k$ and $p \geq 2q$. The nilpotent Lie algebra defined by the brackets

$$\begin{aligned}
&[X_i, X_j] = X_{i+j}, && 1 \leq i \leq k, && q+k+3 \leq j \leq N-i, \\
&[X_i, X_j] = X_{i+j+q-k-1}, && k+1 \leq i \leq q+k+2, && q+k+3 \leq j \leq N-i-j+q-k, \\
&\left[X_{q+k+3}, X_{q+k+4}\right] = X_N
\end{aligned}$$

can be easily verified to be of a rank of one and to admit the eigenvalues in Equation (29). The corresponding solvable extension again leads to solvable rigid Lie algebras.

A larger number of jumps in the eigenvalue sequence can be introduced along the same lines. For an example with three jumps, the lowest possible spectrum is given by

$$\mathrm{spec}(\mathfrak{t}) = (1, 2, 4, 5, \ldots, 9, 11, 12, \ldots, 23, 25, 26, \ldots, 51). \tag{30}$$

This corresponds to the eigenvalues of a maximal torus of derivations of the 48-dimensional nilpotent Lie algebra

$$\begin{aligned}
[X_i, X_j] =& X_{i+j}, \quad 1 \leq i \leq 2,\ 22 \leq j \leq 48 - i, \\
[X_i, X_j] =& X_{i+j+1}, \quad 3 \leq i \leq 8,\ 22 \leq j \leq 47 - i \\
[X_i, X_j] =& X_{i+j+1}, \quad 9 \leq i \leq 21,\ 22 \leq j \leq 46 - i \\
[X_{22}, X_{23}] =& X_{48},
\end{aligned}$$

and the solvable extension is again cohomologically rigid. Clearly, this case can also be generalized to an arbitrary dimension, leading to another family of rank-one solvable rigid Lie algebras.

A direct extrapolation to a sequence with $s \geq 2$ jumps would lead to eigenvalue spectra of the type

$$\mathrm{spec}(\mathfrak{t}) = (1, \ldots, k_1, k_2, \ldots, 2k_2 + 1, k_3, \ldots, 2k_3 + 1, \ldots, k_{s+1}, \ldots, 2k_{s+1} + 1) \tag{31}$$

with $k_{m+1} \geq 2k_m$ for $m \geq 1$ and $k_1 \geq 2$. A nilpotent algebra admitting the preceding spectrum would have a dimension $N = \sum_{i=1}^{s+1} k_i + 2s$, although it is not entirely obvious that the rank is still one or that the corresponding extension does indeed have a vanishing cohomology. The problem, which is certainly worthy of being inspected in detail, would require finding a generic nilpotent Lie algebra $\mathfrak{N}$ of a rank s that plays the analog role of $\mathfrak{n}_{N,k}^0$ such that the spectrum in Equation (31) could be obtained as a linear combination (in analogy to Equation (20)) of the corresponding eigenvalues of the torus generators and nilpotent algebras admitting these eigenvalues as a deformation of $\mathfrak{N}$. Appropriate algorithmic methods are currently being developed to tackle this problem computationally.

On the other hand, from the Jacobi scheme associated with the eigenvalue spectrum in Equation (26), it follows that a decreasing nilpotence index allows the existence of different characteristic sequences, with the rigidity type (cohomological or geometrical) being deeply related to the particular structure of the characteristic sequence. (This phenomenon cannot occur for filiform algebras, as these correspond to the maximal possible nilpotence index.) In other words, the eigenvalue spectrum in Equation (26) does not uniquely determine the nilradical. Consider for instance the spectrum

$$\mathrm{spec}(\mathfrak{t}) = (1, 2, 4, 5, 6, 7, 8, 9, 10, 11)$$

in a dimension $N = 10$. From the Jacobi equations, we deduce that there exist two nilpotent Lie algebras admitting these eigenvalues. One leads to the rigid solvable algebra $\mathfrak{r}_{2,4,11}$, while the second is given by

$$\begin{aligned} &[X_1, X_i] = X_{i+1}, && 4 \leq i \leq 9, \\ &[X_3, X_i] = C_{3,4}^8 X_{i+4}, && 4 \leq i \leq 5, \\ &[X_3, X_6] = C_{3,6}^{10} X_{10}, && \\ &[X_4, X_5] = (C_{3,4}^8 - C_{3,6}^{10}) X_{10}, && \end{aligned} \tag{32}$$

where $C_{3,4}^8$ and $C_{3,6}^{10}$ are free. Moreover, for any values of the parameters, the latter nilpotent Lie algebra admits a second diagonal derivation, implying that the rank-one solvable extension cannot be rigid.

This shows that, aside from considering deformations of a given nilpotent algebra of an appropriate rank, there is another more systematic approach, namely studying all nilpotent algebras that admit a certain one-dimensional torus. This is essentially the same as studying the Jacobi scheme, and a systematic analysis of the Jacobi equations would lead to a classification of all algebras admitting a diagonal derivation of a specific type. The drawback of this ansatz is that for each of the obtained solutions, it must be analyzed separately whether the rank is one or higher, and in the former case, the potential rigidity (either cohomological or geometrical) must also be considered case by case. This problem is of interest, but it demands the implementation of adequate algorithms to appropriately separate the solutions.

We already mentioned that the spectrum may lead to Lie algebras that, while rigid, are not cohomologically rigid. As an example that illustrates how geometrically rigid Lie algebras arise in this context, consider $k = 3$, $q = 8$, $N = 13$ and the torus $\mathfrak{t}$ with the eigenvalue spectrum $(1, 2, 3, 8, 9, \dots, 17)$. The nilpotent Lie algebra $\mathfrak{m}$ given by

$$\begin{aligned} &[X_1, X_2] = X_3, && \\ &[X_1, X_j] = X_{j+1}, && 4 \leq j \leq 12, \\ &[X_2, X_j] = X_{j+2}, && 4 \leq j \leq 11, \\ &[X_4, X_5] = X_{13} && \end{aligned} \tag{33}$$

admits $\mathfrak{t}$ as a maximal torus of derivations. The corresponding solvable extension $\mathfrak{R} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{m}$ has a one-dimensional adjoint cohomology space generated by the cocycle class ψ, which is defined by

$$\begin{aligned} &\psi(X_2, X_j) = (j-4) X_{j+2}, && 5 \leq j \leq 11, \\ &\psi(X_3, X_j) = -X_{j+3}, && 4 \leq j \leq 10. \end{aligned} \tag{34}$$

Although this cocycle is not integrable, using the Rim map in Equation (10), it can be easily verified that

$$\mathrm{Sq}(\psi)(X_2, X_3, X_5) = 3X_{10} \neq 0,$$

from which we deduce that $\mathrm{Sq} : H^2(\mathfrak{R}, \mathfrak{R}) \to H^3(\mathfrak{R}, \mathfrak{R})$ is injective. Following the criterion in [32], $\mathfrak{R}$ is rigid with a nonvanishing cohomology. It is worth observing that, as happened for the filiform case, the same eigenvalue spectrum can lead to either cohomologically or geometrically rigid Lie algebras, depending on the dimension of the nilradical (see, for example, [4,12,23,25]). The interesting fact that distinguishes this type of eigenvalue spectrum from those associated with filiform algebras is that $\mathfrak{m}$ has a characteristic sequence $c(\mathfrak{m}) = (10, 2, 1)$, and the natural question that arises is whether it is the lowest dimensional hierarchy of a series that generalizes recent constructions of geometrically rigid algebras (such as that proposed in [23]) to characteristic sequences of the type $c(\mathfrak{m}) = (c_1, c_2, 1^{c_1+c+2})$. In a wider context, it can be asked what conditions must be satisfied by the elements of a sequence of integers $\{c_1, \dots, c_s\}$ in order to imply the existence of a nilradical with a characteristic sequence $(c_1, \dots, c_s, 1^{s+1})$ associated with a rigid Lie algebra of the first rank. A complete answer to this question will probably require the use of symbolic computer packages, due to the relatively high dimensions and the number of solutions of the Jacobi equations involved.

To summarize, there are various potential ways to generalize the results of this work to wider classes of spectra, leading to solvable Lie algebras with nilradicals of varying characteristic sequences either by searching for appropriate nilpotent algebras that serve as a "model" and studying their derivations or through a direct approach using the Jacobi equations, given an eigenvalue spectrum. Although computationally cumbersome, a complete classification of rank-one solvable Lie algebras of this type, up to a given dimension, is conceivable and may lead to a further understanding of the rigidity for rank-one Lie algebras. Work in these various directions is currently in progress.

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