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Passivity Analysis and Complete Synchronization of Fractional Order for Both Delayed and Non-Delayed Complex Dynamical Networks with Couplings in the Derivative

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Abstract: This manuscript explores the analysis of passivity and synchronization criteria for a complex fractional-order dynamical network model with derivative couplings and time-varying delays. The passivity problem of the proposed network model is deduced using various inequality methods and presented as a linear matrix inequality. To ensure complete synchronization for a fractional-order complex dynamical network with derivative couplings (CDNMDC), we derive suitable criteria using an adaptive feedback control method. Additionally, we investigate the synchronization criterion of these complex networks while accounting for parameter uncertainties. Finally, we provide an example to demonstrate the effectiveness of the proposed solutions.

Keywords: passivity analysis; asymptotic synchronization; adaptive feedback control; parameter uncertainty; linear matrix inequality

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1. Introduction

In the sciences and engineering, complex network research is extremely important. A complex network is a significant collection of nodes that are all connected to one another, with each node often being a nonlinear system. Access to the virtual in many forms, including social, biological, and technical networks, has been used to simulate many real-world systems. An extensive network of interconnected neurons, for instance, makes up the brain. The result of intricate interactions between genes, proteins, and other molecules is the structure and operation of the cell. Graphs depicting the interactions of individuals can be used to illustrate social systems. In food webs, the interconnectedness of the species that make up eco-systems may be seen. In science and engineering, synchronized behavior is crucial and has multiple uses in a wide range of disciplines, including mechanics, electronics, physics, chemistry, biology, and even economics. In addition, synchronization occurs constantly in both nature and human existence. The literature [1–3] provides some instances of synchronous motion that are seen in real-world systems. In the literature, authors have studied a number of synchronization types, including complete synchronization [4], lag synchronization [5], asymptotic synchronization [6], projective synchronization [7], etc.

In reality, nodes frequently are unable to achieve synchronization on their own because of their complex design, network topology, circumstances around CDNs, and connection

quality. The main goal of control is to provide the designers of engineering systems with the concepts and strategies that they need to safeguard visually pleasing execution by naturally adjusting to environmental changes. By weighing the assessment of plant work and a chosen direction reference, the self-controller might choose how to adjust the available control factors (controlled data sources) such that the system reacts appropriately regardless of whether the state over the plant changes unexpectedly. This is a fundamental aspect of engineering systems because they have to work effectively and dependably in a range of environments, incorporating pinning control, non-fragile control, adaptive control, state feed-back control, and control. Due to the finite switching speeds of amplifiers and traffic congestion, temporal delays frequently occur in the process of information storage and transmission in complex dynamic systems in biological and physical systems. However, the utilization of complex networks is impacted by several unanticipated performance factors and features brought about by the presence of periodic delays, such as instability, oscillation, bifurcation, and chaos. For the dynamic analysis of complex systems, it is crucial to research the impacts of temporal delays. The literature [8–14] has provided a large amount of data on the passivity analysis of dynamic systems with time delays.

The basic claim of passivity theory is that a system's passive characteristics can maintain the system's internal stability. Passivity was initially introduced in the context of circuit analysis from an energy perspective. In most cases, passivity is a useful technique in assessing the stability of complicated dynamic systems. The passivity problem has also found success in a few domains, including stability, chaotic control and synchronization, signal processing, complexity, and fuzzy control. There is a wealth of literature available examining the dynamic features of practical systems, and recent years have seen a resurgence in interest in the passivity difficulties for several practical systems.

Each dynamical node in the earlier research on complex networks with coupling delays is determined by the derivatives of the present state variables of every node. However, in some circumstances, the more accurate network model should also include historical data on the rates of change of the network's state variables. Examples of such networks include the stock market, the population ecological system, the biological system, and the ecosystem, where the current and historical data on rate fluctuations define each node's state. Investigating fractional-order complex dynamical networks brings numerous advantages. By incorporating fractional derivatives and integrals, these networks offer enhanced modeling accuracy, capturing intricate behaviors and phenomena beyond traditional integer-order models. They exhibit flexibility in capturing complex dynamics, including memory effects, nonlocal interactions, and anomalous diffusion, prevalent in various fields. Fractional-order networks demonstrate resilience to noise and disturbances, improved stability analysis, and the uncovering of emergent properties, shedding light on collective phenomena and self-organization. They provide a realistic representation of real-world systems, enabling efficient control and optimization strategies.

As a result, an increasing number of authors have investigated the dynamical behaviors of complex dynamical networks, including passivity. In [15], the authors investigated the finite time passivity analysis of complex dynamical networks with multi-weights in different dimensional nodes. In [16], the authors studied the synchronization of coupled reaction diffusion neural networks with multiple derivative couplings and its passivity. The authors investigated fractional-order neural networks with interval uncertainties and analyzed their passivity in [17]. In [18], the authors studied coupled reaction–diffusion neural networks by considering state and diffusion couplings with proportional derivative control and studied the passivity. Moreover, in [19], they investigated the passivity analysis of fractional-order neural networks using a matrix polytope approach. Meanwhile, in [20], the authors examined the passivity analysis of complex dynamical networks with spatial diffusive couplings using pinning controllers. In [21], based on passivity theory, the authors studied the synchronization of complex dynamical networks incorporating the uncertainty parameter in the inner couplings. Moreover, most of the results based on passivity and synchronization with derivative couplings are mainly focused only on integer-order cases.

However, some authors have investigated fractional-order cases, but without considering any type of delay or uncertainty. Very recently, authors studied and analyzed the passivity of fractional-order neural networks with multiple derivative couplings in [22]. In [23], the authors studied H-infinity-based mixed stability for genetic regulatory networks considering variable delays. Additionally, authors investigated Markovian switching complex dynamical networks considering multiple time-varying delays with stochastic perturbation and analyzed the passivity in [24]. To date, there are no results noted for fractional-order delayed complex dynamical networks with derivative couplings including parameter uncertainty, as well as passivity and synchronization analysis. Motivated by the above discussion, the main contributions of the paper are given as follows.

- We have constructed general fractional-order complex dynamical networks with coupling delays including parameter uncertainty and performed a passivity analysis.
- We have analyzed the complete synchronization of the fractional-order complex dynamical networks with derivative couplings including the parameter uncertainty for the first time in the literature.
- Using Lyapnouv stability theory and fractional-order inequalities, we have given some sufficient conditions to prove the passivity and synchronization analysis in terms of linear matrix inequalities using an adaptive controller.
- The conditions of the complete synchronization have been derived in terms of LMI, and the viability of obtaining results using the LMI MATLAB control toolbox has been checked. Finally, numerical simulations have been given.

2. Preliminaries and Lemmas

The following definitions, lemmas, and properties are very useful to prove our results effectively.

Definition 1 ([25]). *Let us define a function $\mathcal{Z}(t) \in \mathbb{C}^n([t_0, +\infty))$ of fractional order ς with the Caputo derivative given by*

$$D^{\varsigma}\mathcal{Z}(t) = \frac{1}{\Gamma(n-\varsigma)} \int_{t_0}^{t} \frac{\mathcal{Z}^n(\phi)}{(t-\phi)^{\varsigma-n+1}} \, \mathrm{d}\phi,$$

Here, $t \geq t_0$ and n is the non-negative integer such that $n-1 < \varsigma < n$. Specifically, in this paper, we consider $\varsigma \in (0,1)$, which can be given by

$$D^{\varsigma}\mathcal{Z}(t) = \frac{1}{\Gamma(1-\varsigma)} \int_{0}^{t} \frac{\mathcal{Z}'(\phi)}{(t-\phi)^{\varsigma}} \, \mathrm{d}\phi.$$

Definition 2 ([25]). *The fractional integral of order $\varsigma > 0$ for a function $y(t)$ is given by*

$$I^{\varsigma}y(t) = \frac{1}{\Gamma(1-\varsigma)} \int_{t_o}^{t} \frac{y(\phi)}{(t-\phi)^{1-\varsigma}} \mathrm{d}\phi.$$

Definition 3. *For a function $f(t) \in \mathbb{C}^n([0, +\infty))$, we define the Caputo fractional derivative by*

$$({}^{c}D_{0^+}^{\varsigma} f)(t) = (I_{0^+}^{n-\varsigma} D^n f)(t), \text{where } (I_{0^+}^{\varsigma} f)(t) = \frac{1}{\Gamma(\varsigma)} \int_{0}^{t} (t-s)^{\varsigma-1} f(s) \mathrm{d}s,$$

where $\varsigma > 0$ with $n-1 < \varsigma \leq n, n \in \mathbb{N}$ and Γ denotes the usual gamma function.

For notational convenience, we set D^{ς} for ${}^{c}D_{0^+}^{\varsigma}$.

Definition 4 ([22]). *Let $V(t)$ be the non-negative function; then, the system is said to be*

1. *Passive if*

$$D^{\varsigma}V(t) \leq \pi^T(t) W \rho(t), \; t \geq 0.$$

2. *Strictly input-passive if*

$$D^{\varsigma}V(t) \leq \pi^{T}(t)W\rho(t) - \rho^{T}(t)W_{1}\rho(t),\ t \geq 0.$$

3. *Strictly output-passive if*

$$D^{\varsigma}V(t) \quad \leq \quad \pi(t)W\rho(t) - \pi^{T}(t)W_{2}\rho(t),\ t \geq 0.$$

where $\varsigma \in (0,1)$, $W \in \mathbb{R}^{k\times n}$, $0 < W_1 \in \mathbb{R}^{n\times n}$, $0 < W_2 \in \mathbb{R}^{k\times k}$, $\rho(t)$ *and* $\pi(t)$ *are the output and input of the system, respectively.*

Lemma 1 ([22]). *Let* $\xi(t)$ *be a continuous and differentiable vector-valued function and* $P \in \mathbb{R}^{n\times n}$ *be a positive definite matrix; then, we have*

$$D^{\varsigma}[\xi^{T}(t)P\xi(t)] \quad \leq \quad 2\xi^{T}(t)PD^{\varsigma}\xi(t).$$

Lemma 2 ([26]). *If* $\vartheta > 0$, *as a given positive scalar, and* $\mathcal{A}, \mathcal{C} \in \mathbb{R}^{n}$ *and matrix* $\mathcal{B}$ *with suitable dimension, then*

$$2\mathcal{A}^{T}\mathcal{B}\mathcal{C} \quad = \quad \vartheta^{-1}\mathcal{A}^{T}\mathcal{B}\mathcal{B}^{T}\mathcal{A} + \vartheta\mathcal{C}^{T}\mathcal{C}.$$

Lemma 3 ([26]). *Let* $\mathcal{M} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ * & \mathcal{M}_{22} \end{pmatrix}$ *be a given matrix with* $\mathcal{M}_{11} = \mathcal{M}_{11}^{T}$ *and* $\mathcal{M}_{22} = \mathcal{M}_{22}^{T}$, *and the following conditions are equivalent:*

1. $\mathcal{M} < 0$,
2. $\mathcal{M}_{11} < 0,\ \mathcal{M}_{22} - \mathcal{M}_{12}^{T}\mathcal{M}_{11}^{-1}\mathcal{M}_{12} < 0$,
3. $\mathcal{M}_{22} < 0,\ \mathcal{M}_{11} - \mathcal{M}_{12}\mathcal{M}_{22}^{-1}\mathcal{M}_{12}^{T} < 0$.

3. Problem Formulation

Consider fractional-order delayed and non-delayed complex dynamical networks composed of N numbers of nodes, where the n-dimensional subsystem can be given by

$$\begin{aligned} D^{\varsigma}\mu_{u}(t) \quad = \quad & A\mu_{u}(t) + B\mathcal{F}(\mu_{u}(t)) + C\mathcal{F}(\mu_{u}(t-\kappa)) + a_{1}\sum_{v=1}^{K} g_{uv}\Delta_{1}\mu_{v}(t) + a_{2}\sum_{v=1}^{K} h_{uv}\Delta_{2}\mu_{v}(t-\kappa) \\ & + E\rho_{u}(t). \end{aligned} \tag{1}$$

where D^{ς} denotes the Caputo derivative operator. $\varsigma \in (0,1)$. $\mu_{u}(t) = \{\mu_{u1}(t), \mu_{u2}(t), \ldots, \mu_{un}(t)\} \in \mathbb{R}^{n}$ represents the state vector of the u^{th} node. A, B and C denote the weight connection matrices of the state vector. $\mathcal{F}(\mu_{u}(t) = \{\mathcal{F}_{1}(\mu_{u1}(t)), \mathcal{F}_{2}(\mu_{u2}(t)), \ldots, \mathcal{F}_{n}(\mu_{un}(t))\}$ and $\mathcal{F}(\mu_{u}(t-\kappa) = \{\mathcal{F}_{1}(\mu_{u1}(t-\kappa)), \mathcal{F}_{2}(\mu_{u2}(t-\kappa)), \ldots, \mathcal{F}_{n}(\mu_{un}(t-\kappa))\}$ denotes the nonlinear function, which describes the dynamical behavior of the network, and it satisfies the Lipschitz condition. a_1 and a_2 represent the coupling strength of the delayed and non-delayed connection matrices. Δ_1 and Δ_2 denote the inner coupling matrices. $\rho(t)$ denotes the external input vector. The outer coupling matrix can be represented by

$$g_{uv} \quad = \quad \begin{cases} g_{uv} > 0, & \text{if } \textit{there } \exists \textit{ a link between node } u \textit{ and } v; \\ \sum_{m=1,m\neq u}^{K} g_{um}, & \text{if } u = v \\ 0, & \text{otherwise.} \end{cases}$$

$$h_{uv} \quad = \quad \begin{cases} h_{uv} > 0, & \text{if } \textit{there } \exists \textit{ a link between node } u \textit{ and } v; \\ \sum_{m=1,m\neq u}^{K} h_{um}, & \text{if } u = v \\ 0, & \text{otherwise.} \end{cases}$$

Now, let $\bar{\mu}(t) = \frac{1}{K}\sum_{m=1}^{K} D^{\varsigma}\mu_m(t)$

$$\begin{aligned} D^{\varsigma}\bar{\mu}(t) &= \frac{1}{K}\sum_{m=1}^{K} D^{\varsigma}\mu_m(t) \\ &= \frac{1}{K}\sum_{m=1}^{K} A\mu_u(t) + \frac{1}{K}\sum_{m=1}^{K} B\mathcal{F}(\mu_m(t)) + \frac{1}{K}\sum_{m=1}^{K} C\mathcal{F}(\mu_m(t-\kappa)) + \frac{1}{K}\sum_{v=1}^{K} a_1(\sum_{m=1}^{K} g_{mv})\Delta_1\mu_v(t) \\ &\quad + \frac{1}{K}\sum_{v=1}^{K} a_2(\sum_{m=1}^{K} h_{mv})\Delta_2\mu_v(t-\kappa) + \frac{1}{K}\sum_{m=1}^{K} E\rho_m(t) \end{aligned}$$

$$D^{\varsigma}\bar{\mu}(t) = \frac{1}{K}\sum_{m=1}^{K} A\mu_u(t) + \frac{1}{K}\sum_{m=1}^{K} B\mathcal{F}(\mu_m(t)) + \frac{1}{K}\sum_{m=1}^{K} C\mathcal{F}(\mu_m(t-\kappa)) + \frac{1}{K}\sum_{m=1}^{K} E\rho_m(t).$$

Consider the error vector as $\xi_u(t) = \{\xi_{u1}(t), \xi_{u2}(t), \ldots, \xi_{un}(t)\} \in \mathbb{R}^n$, and $\xi_u(t) = \mu_u(t) - \bar{\mu}(t)$; then, we can write the error system as

$$\begin{aligned} D^{\varsigma}\xi_u(t) &= A\xi_u(t) + B\mathcal{F}(\mu_u(t)) - \frac{1}{K}\sum_{m=1}^{K} B\mathcal{F}(\mu_m(t)) + C\mathcal{F}(\mu_u(t-\kappa)) - \frac{1}{K}\sum_{m=1}^{K} C\mathcal{F}(\mu_m(t-\kappa)) \\ &\quad + E\rho_u(t) - \frac{1}{K}\sum_{m=1}^{K} E\rho_m(t) + a_1\sum_{v=1}^{K} g_{uv}\Delta_1\xi_v(t) + a_2\sum_{v=1}^{K} h_{uv}\Delta_2\xi_v(t-\kappa). \end{aligned}$$

where

$$\begin{aligned} \phi(\xi_u(t)) &= \mathcal{F}(\mu_u(t)) - \frac{1}{K}\sum_{m=1}^{K}\mathcal{F}(\mu_m(t)) \\ \psi(\xi_u(t-\kappa)) &= \mathcal{F}(\mu_u(t-\kappa)) - \frac{1}{K}\sum_{m=1}^{K}\mathcal{F}(\mu_m(t-\kappa)). \end{aligned}$$

Then, the above equation can be written as

$$\begin{aligned} D^{\varsigma}\xi_u(t) &= A\xi_u(t) + B\phi(\xi_u(t)) + C\psi(\xi_u(t-\kappa)) + E\rho_u(t) - \frac{1}{K}\sum_{m=1}^{K} E\rho_m(t) + a_1\sum_{v=1}^{K} g_{uv}\Delta_1\xi_v(t) \\ &\quad + a_2\sum_{v=1}^{K} h_{uv}\Delta_2\xi_v(t-\kappa). \end{aligned} \tag{2}$$

Let $\pi_u(t) \in \mathbb{R}^s, (1 \le s < n)$, where $Z_1, Z_2, Z_3 \in \mathbb{R}^{s\times n}$ are the output vectors for the considered system, and it is in the form given by

$$\pi_u(t) = Z_1\xi_u(t) + Z_2\xi_u(t-\kappa) + Z_3\rho_u(t). \tag{3}$$

Let us denote

$$\begin{aligned} \xi(t) &= (\xi_1^T(t), \xi_2^T(t), \ldots, \xi_K^T(t)) \\ \xi(t-\kappa) &= (\xi_1^T(t-\kappa, \xi_2^T(t-\kappa), \ldots, \xi_K^T(t-\kappa) \\ \pi(t) &= (\pi_1^T(t), \pi_2^T(t), \ldots, \pi_K^T(t)) \\ \rho(t) &= (\rho_1^T(t), \rho_2^T(t), \ldots, \rho_K^T(t)) \end{aligned}$$

By using the Kronecker product, the following error system (2) can be written in vector form as

$$\begin{aligned} D^{\varsigma}\xi(t) &= (I_k \otimes A)\xi(t) + (I_k \otimes B)\phi(\xi(t)) + (I_k \otimes C)\psi(\xi(t-\kappa)) + (I_k \otimes E)\rho(t) \\ &\quad -\frac{1}{K}(I_k \otimes E)\sum_{m=1}^{K}\rho_m(t) + a_1(G \otimes \Delta_1)\xi(t) + a_2(H \otimes \Delta_2)\xi(t-\kappa). \end{aligned} \tag{4}$$

The upcoming assumptions are very useful to prove our theoretical results.

Hypothesis 1. *Let us assume that there exists a constant $N > 0$, for a nonlinear function $\xi(\mu(t))$, such that, for μ_1 and μ_2,*

$$|\xi(\mu_1(t)) - \xi(\mu_2(t))| \leq N|\mu_1(t) - \mu_2(t)|. \tag{5}$$

4. Passivity Analysis of Fractional-Order Complex Dynamical Networks

In this section, we investigate the passivity analysis for FOCDN and some sufficient conditions are derived.

Theorem 1. *Let Assumption H1 hold. If there exists a positive definite matrix $0 < P \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{sK\times nK}$, such that*

$$\begin{pmatrix} \Omega_1 & \Omega_2 & \Pi_1 \\ \Omega_2^T & \Omega_3 & \Pi_2 \\ \Pi_1^T & \Pi_2^T & \Omega_4 \end{pmatrix} \leq 0, \tag{6}$$

where
$\Omega_1 = I_K \otimes (PA + AP) + I_K \otimes (PBJ_1^{-1}B^TP) + (I_K \otimes N_1J_1N_1) + I_K \otimes (PCJ_2^{-1}C^TP) + a_1(G \otimes P\Delta_1) + a_1(G^T \otimes P\Delta_1^T)$: $\Omega_2 = a_2[H \otimes P\Delta_2]$; $\Omega_3 = (I_K \otimes N_2J_2N_2)$; $\Omega_4 = \frac{(I_N \otimes Z_3^T)W + W^T(I_K \otimes Z_3)}{2}$; $\Pi_1 = (I_k \otimes PE) - \frac{1}{2}(I_K \otimes Z_1)W$; $\Pi_2 = -\frac{1}{2}(I_K \otimes Z_2)W$, *then the FOCDN (4) is passive.*

Proof. Let us define the Lyapunov function by

$$V(t) = \xi^T(t)(I_K \otimes P)\xi(t). \tag{7}$$

Now, by using Lemma (1) in system (7) along with system (4), we obtain

$$D^{\varsigma}V(t) = 2\xi^T(t)(I_K \otimes P)D^{\varsigma}\xi(t)$$

$$\begin{aligned} D^{\varsigma}V(t) &= 2\xi^T(t)(I_K \otimes P)\Big[(I_k \otimes A)\xi(t) + (I_k \otimes B)\phi(\xi(t)) + (I_k \otimes C)\psi(\xi(t-\kappa)) + (I_k \otimes E)\rho(t) \\ &\quad -\frac{1}{K}(I_k \otimes E)\sum_{m=1}^{K}\rho_m(t) + a_1(G \otimes \Delta_1)\xi(t) + a_2(H \otimes \Delta_2)\xi(t-\kappa)\Big]. \end{aligned}$$

We know that

$$\begin{aligned} \sum_{u=1}^{K}\xi_u(t) &= \sum_{u=1}^{K}(\mu_u(t) - \frac{1}{K}\sum_{m=1}^{K}\mu_m(t)) \\ &= \sum_{u=1}^{K}\mu_u(t) - \sum_{m=1}^{K}\mu_m(t) \\ &= 0. \end{aligned}$$

Therefore,

$$\sum_{u=1}^{K} \xi_u^T(t)(I_K \otimes P)(-\frac{1}{K}(I_k \otimes E) \sum_{m=1}^{K} \rho_m(t)) = 0$$

$$\begin{aligned} D^{\varsigma}V(t) &= 2\xi(t)(I_K \otimes P)\{(I_k \otimes A)\xi(t) + (I_k \otimes B)\phi(\xi(t)) + (I_k \otimes C)\psi(\xi(t-\kappa)) + (I_k \otimes E)\rho(t) \\ &\quad + a_1(G \otimes \Delta_1)\xi(t) + a_2(H \otimes \Delta_2)\xi(t-\kappa)\} \\ &= 2\xi^T(t)(I_K \otimes PA)\xi(t) + 2\xi^T(t)(I_k \otimes PB)\phi(\xi(t)) + 2\xi^T(t)(I_k \otimes C)\psi(\xi(t-\kappa)) \\ &\quad + 2\xi^T(t)(I_k \otimes PE)\rho(t) + 2\xi^T(t)a_1(G \otimes (P\Delta_1))\xi(t) + 2\xi^T(t)a_2(H \otimes (P\Delta_2))\xi(t-\kappa). \end{aligned}$$

By applying Assumption H1 and Lemma (2), the following equation can be deduced:

$$\begin{aligned} 2\xi^T(t)(I_k \otimes PB)\phi(\xi(t)) &\leq \xi^T(t)(I_K \otimes (PB))(I_K \otimes J_1)^{-1}(I_K \otimes (PB)^T)\xi(t) \\ &\quad + \phi^T(\xi(t))(I_K \otimes J_1)\phi(\xi(t)) \\ &= \xi^T(t)\{I_K \otimes (PBJ_1^{-1}B^TP) + (I_K \otimes N_1J_1N_1)\}\xi(t) \quad (8) \\ 2\xi^T(t)(I_k \otimes PC)\psi(\xi(t-\kappa)) &\leq \xi^T(t)(I_K \otimes (PC))(I_K \otimes J_2)^{-1}(I_K \otimes (PC)^T)\xi(t) \\ &\quad + \psi^T(\xi(t-\kappa)) \times (I_K \otimes J_2)\psi(\xi(t-\kappa)) \end{aligned}$$

$$2\xi^T(t)(I_k \otimes PC)\psi(\xi(t-\kappa)) \leq \xi^T(t)\{I_K \otimes (PCJ_2^{-1}C^TP)\} + \xi^T(t-\kappa)\{I_K \otimes (N_2J_2N_2)\}\xi(t-\kappa). \quad (9)$$

Combining the above Equations (8) and (9), we have

$$\begin{aligned} D^{\varsigma}V(t) &= \xi^T(t)\Bigg\{I_K \otimes (PA + AP) + I_K \otimes (PBJ_1^{-1}B^TP) + (I_K \otimes N_1J_1N_1) + I_K \otimes (PCJ_2^{-1}C^TP) \\ &\quad + a_1(G \otimes P\Delta_1) + a_1(G^T \otimes P\Delta_1^T)\Bigg\} + \xi^T(t-\kappa)\Bigg\{(I_K \otimes N_2J_2N_2)\Bigg\}\xi(t-\kappa) \\ &\quad + a_2\xi^T(t)[H \otimes P\Delta_2]\xi(t-\kappa) + a_2\xi^T(t-\kappa)[H^T \otimes \Delta_2^TP]\xi(t) + 2\xi^T(t)(I_k \otimes PE)\rho(t). \end{aligned}$$

Now, by Definition 4, we can write

$$\begin{aligned} D^{\varsigma}V(t) - \pi(t)W\rho(t) &\leq \xi^T(t)\Bigg\{I_K \otimes (PA + AP) + I_K \otimes (PBJ_1^{-1}B^TP) + (I_K \otimes N_1J_1N_1) \\ &\quad + I_K \otimes (PCJ_2^{-1}C^TP) + a_1(G \otimes P\Delta_1) \\ &\quad + a_1(G^T \otimes P\Delta_1^T)\Bigg\} + \xi^T(t-\kappa)\Bigg\{(I_K \otimes N_2J_2N_2)\Bigg\}\xi(t-\kappa) \\ &\quad + a_2\xi^T(t)[H \otimes P\Delta_2]\xi(t-\kappa) + a_2\xi^T(t-\kappa) \times [H^T \otimes \Delta_2^TP]\xi(t) \\ &\quad + 2\xi^T(t)(I_k \otimes PE)\rho(t) - \Bigg\{Z_1\xi_u(t) + Z_2\xi_u(t-\kappa) + Z_3\rho_u(t)\Bigg\}W\rho(t) \\ &= \xi^T(t)\Bigg\{I_K \otimes (PA + AP) + I_K \otimes (PBJ_1^{-1}B^TP) + (I_K \otimes N_1J_1N_1) \\ &\quad + I_K \otimes (PCJ_2^{-1}C^TP) + a_1(G \otimes P\Delta_1) + a_1(G^T \otimes P\Delta_1^T)\Bigg\} \\ &\quad + \xi^T(t-\kappa)\Bigg\{(I_K \otimes N_2J_2N_2)\Bigg\}\xi(t-\kappa) + a_2\xi^T(t)[H \otimes P\Delta_2]\xi(t-\kappa) \\ &\quad + a_2\xi^T(t-\kappa) \times [H^T \otimes \Delta_2^TP]\xi(t) + 2\xi^T(t)(I_k \otimes PE)\rho(t) - (I_K \otimes Z_1)\xi_u(t)^TW\rho(t) \\ &\quad - (I_K \otimes Z_2)\xi_u^T(t-\kappa)W\rho(t) - (I_K \otimes Z_3)\rho_u^T(t)W\rho(t) \end{aligned}$$

$$
\begin{aligned}
D^{\varsigma}V(t)-\pi(t)W\rho(t) &= \xi^T(t)\Big\{I_K\otimes(PA+AP)+I_K\otimes(PBJ_1^{-1}B^TP)+(I_K\otimes N_1J_1N_1)\\
&\quad +I_K\otimes(PCJ_2^{-1}C^TP)+a_1(G\otimes P\Delta_1)+a_1(G^T\otimes P\Delta_1^T)\Big\}\\
&\quad +\xi^T(t-\kappa)\Big\{(I_K\otimes N_2J_2N_2)\Big\}\xi(t-\kappa)+a_2\xi^T(t)[H\otimes P\Delta_2]\xi(t-\kappa)\\
&\quad +a_2\xi^T(t-\kappa)[H^T\otimes\Delta_2^TP]\xi(t)+2\xi^T(t)(I_k\otimes PE)\rho(t)-\frac{1}{2}(I_K\otimes Z_1)\xi(t)^TW\rho(t)\\
&\quad -\frac{1}{2}(I_K\otimes Z_1)\rho(t)^TW\xi(t)-\frac{1}{2}(I_K\otimes Z_2)\xi^T(t-\kappa)W\rho(t)-\frac{1}{2}(I_K\otimes Z_2)\rho^TW\xi(t-\kappa)\\
&\quad -\rho^T(t)\frac{(I_N\otimes Z_3^T)W+W^T(I_K\otimes Z_3)}{2}\rho(t) \qquad (10)
\end{aligned}
$$

$$
D^{\varsigma}V(t)-\pi(t)W\rho(t) = \zeta^T(t)\begin{pmatrix}\Omega_1 & \Omega_2 & \Pi_1\\ \Omega_2^T & \Omega_3 & \Pi_2\\ \Pi_1^T & \Pi_2^T & \Omega_4\end{pmatrix}\zeta^T(t).
$$

By inequality (6) and by Definition 4, we can conclude that

$$
\pi(t)W\rho(t) \geq D^{\varsigma}V(t). \qquad (11)
$$

Therefore, we conclude that the FOCDN (4) is passive. □

Remark 1. *Previous work on the passivity of complex networks mainly deals with scenarios in which the input and output vectors have equal dimensions, whereas real-world problems often feature different dimensional vector values. Some of the existing works consider passivity in complex dynamical systems with different input and output dimensions, such as [16–18,22,23]. In this work, we analyze the passivity of a fractional-order complex dynamical network with different input and output dimensions with internal and constant coupling delays for the first time in the literature. In the upcoming theorems, we show that the fractional-order complex dynamical network is strictly input- and strictly output-passive.*

Theorem 2. *Let Assumption H1 hold. If there exists a positive definite matrix* $0 < P \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{sK\times nK}$, $W_1 \in \mathbb{R}^{nK\times nK}$ *such that*

$$
\begin{pmatrix}\Omega_5 & \Omega_6 & \Pi_3\\ \Omega_6^T & \Omega_7 & \Pi_4\\ \Pi_3^T & \Pi_4^T & \Omega_8\end{pmatrix} \leq 0, \qquad (12)
$$

where $\Omega_5 = I_K\otimes(PA+AP)+I_K\otimes(PBJ_1^{-1}B^TP)+(I_K\otimes N_1J_1N_1)+I_K\otimes(PCJ_2^{-1}C^TP)+a_1(G\otimes P\Delta_1)+a_1(G^T\otimes P\Delta_1^T)$; $\Omega_6 = a_2[H\otimes P\Delta_2]$; $\Omega_7 = (I_K\otimes N_2J_2N_2)$; $\Omega_8 = W_1-\frac{(I_N\otimes Z_3^T)W+W^T(I_K\otimes Z_3)}{2}$; $\Pi_3 = (I_k\otimes PE)-\frac{1}{2}(I_K\otimes Z_1)W$; $\Pi_4 = -\frac{1}{2}(I_K\otimes Z_2)W$, *then the FOCDN (4) is input strictly passive (ISP).*

Proof. By (10), one obtains

$$
\begin{aligned}
D^{\varsigma}V(t)-\pi(t)W\rho(t)+\rho^{T}(t)W_1\rho(t) \quad \leq \quad & \xi^{T}(t)\Big\{I_K\otimes(PA+AP)+I_K\otimes(PBJ_1^{-1}B^{T}P)+(I_K\otimes N_1J_1N_1)\\
& +I_K\otimes(PCJ_2^{-1}C^{T}P)+a_1(G\otimes P\Delta_1)+a_1(G^{T}\otimes P\Delta_1^{T})\Big\}\\
& +\xi^{T}(t-\kappa)\Big\{(I_K\otimes N_2J_2N_2)\Big\}\xi(t-\kappa)+a_2\xi^{T}(t)[H\otimes P\Delta_2]\xi(t-\kappa)\\
& +a_2\xi^{T}(t-\kappa)\times[H^{T}\otimes\Delta_2^{T}P]\xi(t)+2\xi^{T}(t)(I_k\otimes PE)\rho(t)-\Big\{Z_1\xi_u(t)\\
& +Z_2\xi_u(t-\kappa)+Z_3\rho_u(t)\Big\}W\rho(t)+\rho^{T}(t)W_1\rho(t)\\
D^{\varsigma}V(t)-\pi(t)W\rho(t)+\rho^{T}(t)W_1\rho(t) \quad \leq \quad & \xi^{T}(t)\Big\{I_K\otimes(PA+AP)+I_K\otimes(PBJ_1^{-1}B^{T}P)+(I_K\otimes N_1J_1N_1)\\
& +I_K\otimes(PCJ_2^{-1}C^{T}P)+a_1(G\otimes P\Delta_1)+a_1(G^{T}\otimes P\Delta_1^{T})\Big\}\\
& +\xi^{T}(t-\kappa)\Big\{(I_K\otimes N_2J_2N_2)\Big\}\xi(t-\kappa)+a_2\xi^{T}(t)[H\otimes P\Delta_2]\xi(t-\kappa)\\
& +a_2\xi^{T}(t-\kappa)[H^{T}\otimes\Delta_2^{T}P]\xi(t)+2\xi^{T}(t)(I_k\otimes PE)\rho(t)\\
& -\frac{1}{2}(I_K\otimes Z_1)\xi(t)^{T}W\rho(t)-\frac{1}{2}(I_K\otimes Z_1)\rho(t)^{T}W\xi(t)\\
& -\frac{1}{2}(I_K\otimes Z_2)\xi^{T}(t-\kappa)W\rho(t)-\frac{1}{2}(I_K\otimes Z_2)\rho^{T}W\xi(t-\kappa)\\
& -\rho^{T}(t)\frac{(I_N\otimes Z_3^{T})W+W^{T}(I_K\otimes Z_3)}{2}\rho(t)+\rho^{T}(t)W_1\rho(t)
\end{aligned}
$$

$$
D^{\varsigma}V(t)-\pi(t)W\rho(t)+\rho^{T}(t)W_1\rho(t) \quad = \quad \zeta^{T}(t)\begin{pmatrix}\Omega_5 & \Omega_6 & \Pi_3\\ \Omega_6^{T} & \Omega_7 & \Pi_4\\ \Pi_3^{T} & \Pi_4^{T} & \Omega_8\end{pmatrix}\zeta^{T}(t).
$$

By inequality (12) and by Definition 4, one has

$$
\pi(t)W\rho(t)-\rho^{T}(t)W_1\rho(t) \quad \geq \quad D^{\varsigma}V(t). \tag{13}
$$

Therefore, we conclude that the FOCDN (4) is input strictly passive (ISP). □

Theorem 3. *The FOCDN (4) is said to be output strictly passive (OSP) if there exists a positive definite matrix* $0<P\in\mathbb{R}^{n\times n}$, $W_2\in\mathbb{R}^{sK\times sK}$, *such that*

$$
\begin{pmatrix}\Omega_9 & \Omega_{10} & \Pi_5\\ \Omega_{10}^{T} & \Omega_{11} & \Pi_6\\ \Pi_5^{T} & \Pi_6^{T} & \Omega_{12}\end{pmatrix}\leq 0, \tag{14}
$$

where
$\Omega_9=I_K\otimes(PA+AP)+I_K\otimes(PBJ_1^{-1}B^{T}P)+(I_K\otimes N_1J_1N_1)+I_K\otimes(PCJ_2^{-1}C^{T}P)+a_1(G\otimes P\Delta_1)+a_1(G^{T}\otimes P\Delta_1^{T})+[I_K\otimes Z_1^{T}]W_2[I_K\otimes Z_1]$; $\Omega_{10}=a_2[H\otimes P\Delta_2]+[I_K\otimes Z_1^{T}]W_2[I_K\otimes Z_2]$; $\Omega_{11}=(I_K\otimes N_2J_2N_2)+[I_K\otimes Z_2^{T}]W_2[I_K\otimes Z_2]$; $\Omega_{12}=\frac{(I_N\otimes Z_3^{T})W+W^{T}(I_K\otimes Z_3)}{2}+[I_K\otimes Z_3^{T}]W_2[I_K\otimes Z_3]$; $\pi_5=(I_k\otimes PE)-\frac{1}{2}(I_K\otimes Z_1)W+[I_K\otimes Z_1^{T}]W_2[I_K\otimes Z_3]$; $\Pi_6=-\frac{1}{2}(I_K\otimes Z_2)W+[I_K\otimes Z_2^{T}]W_2[I_K\otimes Z_3]$.

Proof. By (10), one obtains

$$
\begin{aligned}
D^{\varsigma}V(t)-\pi^{T}(t)W\rho(t)+\pi^{T}(t)W_{2}\pi(t) \quad \leq \quad & \xi^{T}(t)\Bigg\{I_{K}\otimes(PA+AP)+I_{K}\otimes(PBJ_{1}^{-1}B^{T}P)+(I_{K}\otimes N_{1}J_{1}N_{1}) \\
& +I_{K}\otimes(PCJ_{2}^{-1}C^{T}P)+a_{1}(G\otimes P\Delta_{1})+a_{1}(G^{T}\otimes P\Delta_{1}^{T})\Bigg\} \\
& +\xi^{T}(t-\kappa)\Bigg\{(I_{K}\otimes N_{2}J_{2}N_{2})\Bigg\}\xi(t-\kappa)+a_{2}\xi^{T}(t)[H\otimes P\Delta_{2}]\xi(t-\kappa) \\
& +a_{2}\xi^{T}(t-\kappa)[H^{T}\otimes\Delta_{2}^{T}P]\xi(t)+2\xi^{T}(t)(I_{k}\otimes PE)\rho(t)-\Bigg\{Z_{1}\xi_{u}(t) \\
& +Z_{2}\xi_{u}(t-\kappa)+Z_{3}\rho_{u}(t)\Bigg\}W\rho(t)+\pi^{T}(t)W_{2}\pi(t)
\end{aligned}
$$

$$
\begin{aligned}
&D^{\varsigma}V(t)-\pi^{T}(t)W\rho(t)+\pi^{T}(t)W_{2}\pi(t)\leq \\
&\xi^{T}(t)\Bigg\{I_{K}\otimes(PA+AP)+I_{K}\otimes(PBJ_{1}^{-1}B^{T}P)+(I_{K}\otimes N_{1}J_{1}N_{1})+I_{K}\otimes(PCJ_{2}^{-1}C^{T}P)+a_{1}(G\otimes P\Delta_{1}) \\
&+a_{1}(G^{T}\otimes P\Delta_{1}^{T})\Bigg\}+\xi^{T}(t-\kappa)\Bigg\{(I_{K}\otimes N_{2}J_{2}N_{2})\Bigg\}\xi(t-\kappa)+a_{2}\xi^{T}(t)[H\otimes P\Delta_{2}]\xi(t-\kappa) \\
&+a_{2}\xi^{T}(t-\kappa)[H^{T}\otimes\Delta_{2}^{T}P]\xi(t)+2\xi^{T}(t)(I_{k}\otimes PE)\rho(t)-\frac{1}{2}(I_{K}\otimes Z_{1})\xi(t)^{T}W\rho(t)-\frac{1}{2}(I_{K}\otimes Z_{1})\rho(t)^{T}W\xi(t) \\
&-\frac{1}{2}(I_{K}\otimes Z_{2})\xi^{T}(t-\kappa)W\rho(t)-\frac{1}{2}(I_{K}\otimes Z_{2})\rho^{T}W\xi(t-\kappa)-\rho^{T}(t)\frac{(I_{N}\otimes Z_{3}^{T})W+W^{T}(I_{K}\otimes Z_{3})}{2}\rho(t) \\
&+\xi^{T}(t)[I_{K}\otimes Z_{1}^{T}]W_{2}[I_{K}\otimes Z_{1}]\xi(t)+\xi^{T}(t)[I_{K}\otimes Z_{1}^{T}]W_{2}[I_{K}\otimes Z_{2}]\xi(t-\kappa)+\xi^{T}(t)[I_{K}\otimes Z_{1}^{T}]W_{2}[I_{K}\otimes Z_{3}]\rho(t) \\
&+\xi^{T}(t-\kappa)[I_{K}\otimes Z_{2}^{T}]W_{2}[I_{K}\otimes Z_{1}]\xi(t)+\xi^{T}(t-\kappa)[I_{K}\otimes Z_{2}^{T}]W_{2}[I_{K}\otimes Z_{2}]\xi(t-\kappa)+\xi^{T}(t-\kappa)[I_{K}\otimes Z_{2}^{T}] \\
&\times W_{2}[I_{K}\otimes Z_{3}]\rho(t)+\rho^{T}(t)[I_{K}\otimes Z_{3}^{T}]W_{2}[I_{K}\otimes Z_{1}]\xi(t)+\rho^{T}(t)[I_{K}\otimes Z_{3}^{T}]W_{2}[I_{K}\otimes Z_{2}]\xi(t-\kappa) \\
&+\rho^{T}(t)[I_{K}\otimes Z_{3}^{T}]W_{2}[I_{K}\otimes Z_{3}]\rho(t)
\end{aligned}
$$

$$
D^{\varsigma}V(t)-\pi(t)W\rho(t)+\rho^{T}(t)W_{1}\rho(t) \quad = \quad \zeta^{T}(t)\begin{pmatrix}\Omega_{9} & \Omega_{10} & \Pi_{5}\\ \Omega_{10}^{T} & \Omega_{11} & \Pi_{6}\\ \Pi_{5}^{T} & \Pi_{6}^{T} & \Omega_{12}\end{pmatrix}\zeta^{T}(t).
$$

By inequality (14) and by Definition 4, we can write

$$
\pi(t)W\rho(t)-\pi^{T}(t)W_{2}\rho(t) \quad \geq \quad D^{\varsigma}V(t). \tag{15}
$$

Therefore, we conclude that the FOCDN (4) is output strictly passive (OSP). □

5. Asymptotic Synchronization Analysis of Fractional-Order Complex Dynamical Networks with Derivative Couplings and Parameter Uncertainties

In this section, we study and discuss the asymptotic synchronization for fractional-order complex dynamical networks with time-varying delays via a state feedback controller, letting $\rho(t)=0$. Consider the fractional-order delayed and non-delayed complex dynamical networks with derivative couplings composed of N numbers of nodes, where the n-dimensional subsystem can be given by

$$
\begin{aligned}
D^{\varsigma}\mu_{u}(t) \quad = \quad & A\mu_{u}(t)+B\mathcal{F}(\mu_{u}(t))+C\mathcal{F}(\mu_{u}(t-\kappa))+a_{1}\sum_{v=1}^{K}g_{uv}\Delta_{1}D^{\varsigma}\mu_{v}(t)+a_{2}\sum_{v=1}^{K}h_{uv}\Delta_{2}D^{\varsigma}\mu_{v}(t-\kappa) \\
& +E\rho_{u}(t).
\end{aligned} \tag{16}
$$

Letting $\rho(t)=0$, we have $\bar{\mu}(t)=\frac{1}{K}\sum_{m=1}^{K}D^{\varsigma}\mu_{m}(t)$

$$\begin{aligned} D^{\varsigma}\bar{\mu}(t) &= \frac{1}{K}\sum_{m=1}^{K} D^{\varsigma}\mu_m(t) \\ &= \frac{1}{K}\sum_{m=1}^{K} A\mu_u(t) + \frac{1}{K}\sum_{m=1}^{K} B\mathcal{F}(\mu_m(t)) + \frac{1}{K}\sum_{m=1}^{K} C\mathcal{F}(\mu_m(t-\kappa)) + \frac{1}{K}\sum_{v=1}^{K} a_1(\sum_{m=1}^{K} g_{mv})\Delta_1 D^{\varsigma}\mu_v(t) \\ &\quad + \frac{1}{K}\sum_{v=1}^{K} a_2(\sum_{m=1}^{K} h_{mv})\Delta_2 D^{\varsigma}\mu_v(t-\kappa). \end{aligned}$$

Consider the error vector as $\xi_u(t) = \{\xi_{u1}(t), \xi_{u2}(t), \ldots, \xi_{un}(t)\} \in \mathbb{R}^n$, and $\xi_u(t) = \mu_u(t) - \bar{\mu}(t)$, and we can write the error system as

$$\begin{aligned} D^{\varsigma}\xi_u(t) &= A\xi_u(t) + B\phi(\xi_u(t)) + C\psi(\xi_u(t-\kappa)) + a_1\sum_{v=1}^{K} g_{uv}\Delta_1 D^{\varsigma}\xi_v(t) \\ &\quad + a_2\sum_{v=1}^{K} h_{uv}\Delta_2 D^{\varsigma}\xi_v(t-\kappa). \end{aligned}$$

Then, the controlled error system can be given by

$$\begin{aligned} D^{\varsigma}\xi_u(t) &= A\xi_u(t) + B\phi(\xi_u(t)) + C\psi(\xi_u(t-\kappa)) + a_1\sum_{v=1}^{K} g_{uv}\Delta_1 D^{\varsigma}\xi_v(t) \\ &\quad + a_2\sum_{v=1}^{K} h_{uv}\Delta_2 D^{\varsigma}\xi_v(t-\kappa) + U_u(t). \end{aligned}$$

The above equation can be written in vector form as

$$\begin{aligned} D^{\varsigma}\xi(t) &= (I_k \otimes A)\xi(t) + (I_k \otimes B)\phi(\xi(t)) + (I_k \otimes C)\psi(\xi(t-\kappa)) + a_1(G \otimes \Delta_1)D^{\varsigma}\xi(t) \\ &\quad + a_2(H \otimes \Delta_2)D^{\varsigma}\xi(t-\kappa) + U(t). \end{aligned} \tag{17}$$

The time-varying uncertainty parameter can be described in the form $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$, satisfying the following conditions:

$$\Delta A(t) = T_a Q(t)\mathcal{S}_a,\ \Delta B(t) = T_b Q(t) S_b,\ \Delta C(t) = T_c Q(t) S_c. \tag{18}$$

where T, S_a, S_b, S_c are known matrices, and $Q(t)$ is an unknown time-varying matrix, satisfying $Q^T(t)Q(t) \leq I$.

Now, the system can be written after injecting the time-varying parameter uncertainties, and we obtain

$$\begin{aligned} D^{\varsigma}\xi(t) &= (I_k \otimes A)\xi(t) + (I_k \otimes \Delta A)\xi(t) + (I_k \otimes B)\phi(\xi(t)) + (I_k \otimes \Delta B)\phi(\xi(t)) + (I_k \otimes C)\psi(\xi(t-\kappa)) \\ &\quad + (I_k \otimes \Delta C)\psi(\xi(t-\kappa)) + a_1(G \otimes \Xi_1)D^{\varsigma}\xi(t) + a_2(H \otimes \Xi_2)D^{\varsigma}\xi(t-\kappa) + U(t). \end{aligned} \tag{19}$$

Theorem 4. *If Assumption 1 holds, suppose that there exist positive definite matrices* Φ *and* N *and scalars* ϑ_1, ϑ_2, α_1, α_2 *and* α_3 *and positive constant* $k^* > 0$ *such that the following conditions are satisfied:*

$$\begin{cases} U(t) = -\hat{K}\xi(t) - a_2(H \otimes \Xi_2)D^{\varsigma}\xi(t-\kappa) \\ \Psi_1 = \Psi_2 + \vartheta_1^{-1} + (I_K \otimes (\Phi B B^T \Phi^T)) + \vartheta_2^{-1}(I_K \otimes (\Phi C C^T \Phi^T)) + \alpha_1^{-1}(I_k \otimes \Phi T_1 T_1^T \Phi_1^T) \\ \qquad + \alpha_2^{-1}(I_k \otimes \Phi T_2 T_2^T \Phi^T) + \alpha_3^{-1}(I_k \otimes \Phi T_3 T_3^T \Phi^T) < 0, \end{cases}$$

where $D^{\varsigma}\hat{K} = r_1\xi^T(t)\xi(t)$, $\Psi_2 = I_K \otimes (\Phi A + \Phi A)) + \vartheta_1(I_k \otimes N^T N) + \alpha_1(I_k \otimes S_1^T S_1) + \alpha_2(I_k \otimes N^T S_2^T S_2 N) - 2(I_k \otimes K^*)$. *Then, the error dynamical system* (19) *is globally completely synchronized based on the adaptive controller and laws.*

Proof. According to the error system (19), the Lyapunov function be chosen as

$$\begin{aligned} V(t) &= \xi^T(t)(I_K \otimes \Phi)\xi(t) + \frac{1}{\Gamma(\varsigma)}\int_{t-\kappa}^{0}(t-\kappa-\chi)\xi^T(s)(I_K \otimes N)\xi(s)\mathrm{d}s + \frac{1}{r_1}(\hat{K}-K^*)^2(I_K \otimes \Phi) \\ &\quad -a_1(G \otimes \Xi_1)(I_K \otimes \Phi)\xi^T(t)\xi(t). \end{aligned} \tag{20}$$

Taking the Lyapnouv derivative for (20), we obtain

$$\begin{aligned} D^{\varsigma}V(t) &= 2\xi^T(t)(I_K \otimes \Phi)D^{\varsigma}\xi(t) + \xi^T(t-\kappa)(I_k \otimes N)\xi(t-\kappa) + \frac{2}{r_1}(\hat{K}-K^*)(I_K \otimes \Phi)D^{\varsigma}\hat{K} \\ &\quad -2a_1(G \otimes \Xi_1\Phi)\xi^T(t)D^{\varsigma}\xi(t) \end{aligned}$$

$$\begin{aligned} D^{\varsigma}V(t) &= 2\xi^T(t)(I_K \otimes \Phi)\Big\{(I_k \otimes A)\xi(t) + (I_k \otimes \Delta A)\xi(t) + (I_k \otimes B)\phi(\xi(t)) + (I_k \otimes \Delta B)\phi(\xi(t)) \\ &\quad +(I_k \otimes C)\psi(\xi(t-\kappa)) + (I_k \otimes \Delta C)\psi(\xi(t-\kappa)) + a_1(G \otimes \Xi_1)D^{\varsigma}\xi(t) + a_2(H \otimes \Xi_2)D^{\varsigma}\xi(t-\kappa) \\ &\quad +U(t)\Big\} + \frac{2}{r_1}(\hat{K}-K^*)(I_K \otimes \Phi)D^{\varsigma}\hat{K} - 2a_1(G \otimes \Xi_1\Phi)\xi^T(t)D^{\varsigma}\xi(t) - \xi^T(t-\kappa)(I_k \otimes N)\xi(t-\kappa) \\ D^{\varsigma}V(t) &= 2\xi^T(t)(I_K \otimes \Phi)\Big\{((I_k \otimes A)\xi(t) + (I_k \otimes \Delta A)\xi(t) + (I_k \otimes B)\phi(\xi(t)) + (I_k \otimes \Delta B)\phi(\xi(t)) \\ &\quad +(I_k \otimes C)\psi(\xi(t-\kappa)) + (I_k \otimes \Delta C)\psi(\xi(t-\kappa)) + a_1(G \otimes \Xi_1)D^{\varsigma}\xi(t) + a_2(H \otimes \Xi_2)D^{\varsigma}\xi(t-\kappa) \\ &\quad -\hat{K}\xi(t) - a_2(H \otimes \Xi_2)D^{\varsigma}\xi(t-\kappa)\Big\} + \frac{2}{r_1}(\hat{K}-K^*)(I_K \otimes \Phi)D^{\varsigma}\hat{K} - 2a_1(G \otimes \Xi_1\Phi)\xi^T(t)D^{\varsigma}\xi(t) \\ &\quad -\xi^T(t-\kappa)(I_k \otimes N)\xi(t-\kappa) \end{aligned}$$

$$\begin{aligned} D^{\varsigma}V(t) &= 2\xi^T(t)(I_K \otimes \Phi)\Big\{(((I_k \otimes A)\xi(t) + (I_k \otimes \Delta A)\xi(t) + (I_k \otimes B)\phi(\xi(t)) + (I_k \otimes \Delta B)\phi(\xi(t)) \\ &\quad +(I_k \otimes C)\psi(\xi(t-\kappa)) + (I_k \otimes \Delta C)\psi(\xi(t-\kappa)) + a_1(G \otimes \Xi_1)D^{\varsigma}\xi(t) - \hat{K}\xi(t)\Big\} \\ &\quad +\frac{2}{r_1}(\hat{K}-K^*)(I_K \otimes \Phi)D^{\varsigma}\hat{K} - 2a_1(G \otimes \Xi_1\Phi)\xi^T(t)D^{\varsigma}\xi(t) - \xi^T(t-\kappa)(I_k \otimes N)\xi(t-\kappa) \\ D^{\varsigma}V(t) &= 2\xi^T(t)(I_K \otimes \Phi A)\xi(t) + 2\xi^T(t)(I_K \otimes \Phi\Delta A) + 2\xi^T(t)(I_k \otimes \Phi B)\phi(\xi(t)) \\ &\quad +2\xi^T(t)(I_k \otimes \Phi\Delta B)\phi(\xi(t)) + 2\xi^T(t)(I_k \otimes \Phi C)\psi(\xi(t-\kappa)) \\ &\quad +2\xi^T(t)(I_k \otimes \Phi\Delta C)\psi(\xi(t-\kappa) + 2\xi^T(t)a_1(G \otimes \Xi_1\Phi)D^{\varsigma}\xi(t) \\ &\quad +2\xi^T(t)a_2(H \otimes \Xi_2\Phi)D^{\varsigma}\xi(t-\kappa) - 2\hat{K}\xi^T(t)\xi(t) \\ &\quad -2\xi^T(t)a_2(H \otimes \Xi_2\Phi)D^{\varsigma}\xi(t-\kappa) + \frac{2}{r_1}(\hat{K}-K^*)(I_K \otimes \Phi)D^{\varsigma}\hat{K} \\ &\quad -2a_1(G \otimes \Xi_1\Phi)\xi^T(t)D^{\varsigma}\xi(t) - \xi^T(t-\kappa)(I_k \otimes N)\xi(t-\kappa) \\ D^{\varsigma}V(t) &= 2\xi^T(t)(I_K \otimes \Phi A)\xi(t) + 2\xi^T(t)(I_K \otimes \Phi\Delta A)\xi(t) + 2\xi^T(t)(I_k \otimes \Phi B)\phi(\xi(t)) \\ &\quad +2\xi^T(t)(I_k \otimes \Phi\Delta B)\phi(\xi(t)) + 2\xi^T(t)(I_k \otimes \Phi C)\psi(\xi(t-\kappa)) \\ &\quad +2\xi^T(t)(I_k \otimes \Phi\Delta C)\psi(\xi(t-\kappa) - 2(I_k \otimes K^*)\xi^T(t)\xi(t) - \xi^T(t-\kappa)(I_k \otimes N)\xi(t-\kappa). \end{aligned} \tag{21}$$

By applying Assumption (5) and Lemma (2), we have

$$
\begin{aligned}
2\xi(t)(I_K \otimes \Phi B)\phi(\xi(t)) &= \vartheta_1^{-1}\xi^T(t)\{I_K \otimes (\Phi BB^T\Phi^T)\} + \vartheta_1\xi^T(t)\{(I_k \otimes N^TN)\}\xi(t) && (22)\\
2\xi^T(t)(I_k \otimes \Phi C)\psi(\xi(t-\kappa)) &\leq \vartheta_2^{-1}\xi^T(t)\{I_K \otimes (\Phi CC^T\Phi^T)\}\xi(t) + \xi^T(t-\kappa)\{I_k \otimes N^TN\}\xi(t-\kappa) && (23)\\
2\xi^T(t)(I_K \otimes \Phi\Delta A)\xi(t) &\leq 2\xi^T(t)(I_k \otimes T_1Q(t)S_1)\xi(t)\\
&= \alpha_1^{-1}(I_k \otimes \Phi T_1T_1^T\Phi_1^T)\xi(t) + \alpha_1\xi^T(t)(I_k \otimes S_1^TS_1)\xi(t) && (24)
\end{aligned}
$$

$$
\begin{aligned}
2\xi^T(t)(I_k \otimes \Phi\Delta B)\phi(\xi(t)) &\leq 2\xi^T(t)(I_k \otimes \Phi T_2Q(t)S_2)\phi(\xi(t))\\
&= \alpha_2^{-1}(I_k \otimes \Phi T_2T_2^T\Phi^T)\xi(t) + \alpha_2\xi^T(t)(I_k \otimes N^TS_2^TS_2N)\xi(t) && (25)\\
2\xi^T(t)(I_k \otimes \Phi\Delta C)\psi(\xi(t-\kappa)) &\leq 2\xi^T(t)(I_k \otimes \Phi T_3Q(t)S_3)\psi(\xi(t-\kappa))\\
&= \alpha_3^{-1}(I_k \otimes \Phi T_3T_3^T\Phi^T)\xi(t) + \alpha_3\xi^T(t-\kappa)(I_k \otimes N^TS_3^TS_3N)\xi(t-\kappa). && (26)
\end{aligned}
$$

Combining Equations (21)–(26), we obtain

$$
\begin{aligned}
D^{\varsigma}V(t) \leq\ & \xi^T(t)(I_K \otimes (\Phi A + \Phi A))\xi(t) + \vartheta_1^{-1}\xi^T(t)\{I_K \otimes (\Phi BB^T\Phi^T) + \vartheta_1(I_k \otimes N^TN)\}\xi(t)\\
&+\vartheta_2^{-1}\xi^T(t)\{I_K \otimes (\Phi CC^T\Phi^T)\}\xi(t) + \xi^T(t-\kappa)\{I_k \otimes N^TN\}\xi(t-\kappa)\\
&+\alpha_1^{-1}(I_k \otimes \Phi T_1T_1^T\Phi_1^T)\xi(t) + \alpha_1\xi^T(t)(I_k \otimes S_1^TS_1)\xi(t)\\
&+\alpha_2^{-1}(I_k \otimes \Phi T_2T_2^T\Phi^T)\xi(t) + \alpha_2\xi^T(t)(I_k \otimes N^TS_2^TS_2N)\xi(t)\\
&+\alpha_3^{-1}(I_k \otimes \Phi T_3T_3^T\Phi^T)\xi(t) + \alpha_3\xi^T(t-\kappa)(I_k \otimes N^TS_3^TS_3N)\xi(t-\kappa)\\
&-2(I_k \otimes K^*)\xi^T(t)\xi(t) - \xi^T(t-\kappa)(I_k \otimes N)\xi(t-\kappa)
\end{aligned}
$$

$$
\begin{aligned}
D^{\varsigma}V(t) \leq\ & \xi^T(t)(I_K \otimes (\Phi A + \Phi A))\xi(t) + \vartheta_1^{-1}\xi^T(t)\{I_K \otimes (\Phi BB^T\Phi^T) + \vartheta_1(I_k \otimes N^TN)\}\xi(t)\\
&+\vartheta_2^{-1}\xi^T(t)\{I_K \otimes (\Phi CC^T\Phi^T)\}\xi(t) + \alpha_1^{-1}(I_k \otimes \Phi T_1T_1^T\Phi_1^T)\xi(t)\\
&+\alpha_1\xi^T(t)(I_k \otimes S_1^TS_1)\xi(t) + \alpha_2^{-1}(I_k \otimes \Phi T_2T_2^T\Phi^T)\xi(t)\\
&+\alpha_2\xi^T(t)(I_k \otimes N^TS_2^TS_2N)\xi(t) + \alpha_3^{-1}(I_k \otimes \Phi T_3T_3^T\Phi^T)\xi(t)\\
&-2(I_k \otimes K^*)\xi^T(t)\xi(t) + \alpha_3\xi^T(t-\kappa)(I_k \otimes N^TS_3^TS_3N)\xi(t-\kappa)\\
&+\xi^T(t-\kappa)\{I_k \otimes N^TN\}\xi(t-\kappa) - \xi^T(t-\kappa)(I_k \otimes N)\xi(t-\kappa)
\end{aligned}
$$

$$
\begin{aligned}
D^{\varsigma}V(t) \leq\ & \xi^T(t)\Big\{(I_K \otimes (\Phi A + \Phi A)) + \vartheta_1^{-1} + (I_K \otimes (\Phi BB^T\Phi^T)) + \vartheta_1(I_k \otimes N^TN) + \vartheta_2^{-1}(I_K \otimes (\Phi CC^T\Phi^T))\\
&+\alpha_1^{-1}(I_k \otimes \Phi T_1T_1^T\Phi_1^T) + \alpha_1(I_k \otimes S_1^TS_1) + \alpha_2^{-1}(I_k \otimes \Phi T_2T_2^T\Phi^T) + \alpha_2(I_k \otimes N^TS_2^TS_2N)\\
&+\alpha_3^{-1}(I_k \otimes \Phi T_3T_3^T\Phi^T) - 2(I_k \otimes K^*)\Big\}\xi(t) + \alpha_3\xi^T(t-\kappa)(I_k \otimes N^TS_3^TS_3N)\xi(t-\kappa)\\
&+\xi^T(t-\kappa)\{I_k \otimes N^TN\}\xi(t-\kappa) - \xi^T(t-\kappa)(I_k \otimes N)\xi(t-\kappa)
\end{aligned}
$$

$$
D^{\varsigma}V(t) \leq \xi^T(t)\Big\{\Psi_1\Big\}\xi(t) + \xi^T(t-\kappa)\Big\{\alpha_3(I_k \otimes N^TS_3^TS_3N) + (I_k \otimes N^TN) - N\Big\}\xi(t-\kappa).
$$

Now, letting $N = \alpha_3(I_k \otimes N^TS_3^TS_3N) + (I_k \otimes N^TN)$, the following equation becomes

$$
D^{\varsigma}V(t) \leq \xi^T(t)\Psi_1\xi(t) \qquad (27)
$$

where

$$\begin{cases} \Psi_1 = I_K \otimes (\Phi A + \Phi A)) + \vartheta_1^{-1} + (I_K \otimes (\Phi B B^T \Phi^T)) + \vartheta_1 (I_k \otimes N^T N) + \vartheta_2^{-1} (I_K \otimes (\Phi C C^T \Phi^T)) \\ \quad +\alpha_1^{-1} (I_k \otimes \Phi T_1 T_1^T \Phi_1^T) + \alpha_1 (I_k \otimes S_1^T S_1) + \alpha_2^{-1} (I_k \otimes \Phi T_2 T_2^T \Phi^T) + \alpha_2 (I_k \otimes N^T S_2^T S_2 N) \\ \quad +\alpha_3^{-1} (I_k \otimes \Phi T_3 T_3^T \Phi^T) - 2(I_k \otimes K^*) \end{cases}$$

The above inequality can be written as

$$\begin{cases} \Psi_1 = \Psi_2 + \vartheta_1^{-1} + (I_K \otimes (\Phi B B^T \Phi^T)) + \vartheta_2^{-1} (I_K \otimes (\Phi C C^T \Phi^T)) + \alpha_1^{-1} (I_k \otimes \Phi T_1 T_1^T \Phi_1^T) \\ \quad +\alpha_2^{-1} (I_k \otimes \Phi T_2 T_2^T \Phi^T) + \alpha_3^{-1} (I_k \otimes \Phi T_3 T_3^T \Phi^T). \end{cases}$$

where $\Psi_2 = I_K \otimes (\Phi A + \Phi A)) + \vartheta_1 (I_k \otimes N^T N) + \alpha_1 (I_k \otimes S_1^T S_1) + \alpha_2 (I_k \otimes N^T S_2^T S_2 N) - 2(I_k \otimes K^*)$. By virtue of Lemma (3), the above inequality can be written in LMI form as

$$\begin{pmatrix} \Psi_1 & \Phi B & \Phi C & \Phi T_1 & \Phi T_2 & \Phi T_3 \\ \maltese & -\vartheta_1 I & 0 & 0 & 0 & 0 \\ \maltese & 0 & -\vartheta_2 I & 0 & 0 & 0 \\ \maltese & 0 & 0 & \alpha_1 I & 0 & 0 \\ \maltese & 0 & 0 & 0 & -\alpha_2 I & 0 \\ \maltese & 0 & 0 & 0 & 0 & -\alpha_3 I \end{pmatrix} < 0. \tag{28}$$

Based on the above inequality and Lyapunov stability theory, the fractional-order complex dynamical network with derivative couplings is synchronized asymptotically when the LMI condition (28) holds under the adaptive controller and laws. This completes the proof. □

6. Numerical Simulation

In this section, to validate our theoretical results, we provide numerical simulations with uncertainty. Consider the following complex dynamical network with derivative couplings given by

$$\begin{aligned} D^{\varsigma} \mu_u(t) &= (A + \Delta A)\mu_u(t) + (B + \Delta B)\mathcal{F}(\mu_u(t)) + (C + \Delta C)\mathcal{F}(\mu_u(t - \kappa)) + a_1 \sum_{v=1}^{K} g_{uv} \Delta_1 D^{\varsigma} \mu_v(t) \\ &\quad + a_2 \sum_{v=1}^{K} h_{uv} \Delta_2 D^{\varsigma} \mu_v(t - \kappa) + E\rho_u(t). \end{aligned}$$

Here, $u = 1, 2, 3$. $\varsigma = 0.98$ $a_1 = 0.02$, $a_2 = 0.03$; $\mathcal{F}(\mu_u(t)) = tanh(\mu_u(t))$, $\kappa = 0.52$. The state matrices A, B and C are given by

$$A = \begin{pmatrix} 40.830 & \\ 0 & 35.12 \end{pmatrix} \quad B = \begin{pmatrix} -13.45 & 0 \\ 0 & 13.54 \end{pmatrix} \quad C = \begin{pmatrix} 10.53 & 0 \\ 0 & 15.15 \end{pmatrix}$$

The inner and outer coupling matrices are given by

$$G = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad H = \begin{pmatrix} 2 & -1 & -1 \\ -1 & -3 & 4 \\ 1 & 4 & -3 \end{pmatrix} \quad \Delta_1 = \begin{pmatrix} 0.35 & \\ 0 & 0.35 \end{pmatrix} \quad \Delta_2 = \begin{pmatrix} 0.45 & \\ 0 & 0.45 \end{pmatrix}$$

T1 = [0.25 0; 0 0.25]; T2 = [0.35 0; 0 0.35]; T3 = [0.5 0; 0 0.5]; S1 = [0.13 0; 0 0.13]; S2 = [0.21 0; 0 0.21];

$$T_1 = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.25 \end{pmatrix} \quad T_2 = \begin{pmatrix} 0.35 & 0 \\ 0 & 0.35 \end{pmatrix} \quad T_3 = \begin{pmatrix} 0.50 & 0 \\ 0 & 0.50 \end{pmatrix}$$

$$S_1 = \begin{pmatrix} 0.15 & 0 \\ 0 & 0.15 \end{pmatrix} \quad S_2 = \begin{pmatrix} 0.30 & 0 \\ 0 & 0.30 \end{pmatrix} \quad S_3 = \begin{pmatrix} 0.45 & 0 \\ 0 & 0.45 \end{pmatrix}$$

Let us choose the output vector $\pi_u(t)$ as follows:

$$\pi_u(t) \quad = \quad Z_1\xi_u(t) + Z_2\xi_u(t-\kappa) + Z_3\rho_u(t)$$

where

$$Z_1 = \begin{pmatrix} 0.1 & 0.5 \\ 0.2 & 0.1 \\ 0.3 & 0.2 \end{pmatrix} \quad Z_2 = \begin{pmatrix} 0.3 & 0.4 \\ 0.1 & 0.3 \\ 0.2 & 0.1 \end{pmatrix} \quad Z_3 = \begin{pmatrix} 0.1 & 0.2 \\ 0.5 & 0.7 \\ 0.2 & 0.4 \end{pmatrix}$$

By using the toolbox for optimization in MATLAB, we obtain the following positive definite matrices, which satisfy Theorem 1, i.e., the network is passive.

$$P = \begin{pmatrix} 0.3514 & 0.2485 \\ 0.2485 & 6.1742 \end{pmatrix} \quad N1 = \begin{pmatrix} 1.4902 & -0.0103 \\ -0.0103 & 1.4634 \end{pmatrix} \quad N2 = \begin{pmatrix} 2.3881 & -0.3110 \\ 0.3110 & 3.2058 \end{pmatrix}$$

$$W = I_3 \otimes \begin{pmatrix} 1.9933 & -0.1216 \\ -2.0113 & -1.3109 \\ 0.4135 & -1.3464 \end{pmatrix}$$

By using the toolbox for optimization in MATLAB, we obtain the following positive definite matrices, which satisfy Theorem 2, i.e., the network is input strictly passive.

$$P = \begin{pmatrix} 0.0267 & 0.2485 \\ -0.0228 & 0.5905 \end{pmatrix} \quad N1 = \begin{pmatrix} 1.3540 & -0.0035 \\ -0.0035 & 1.0196 \end{pmatrix} \quad N2 = \begin{pmatrix} 3.0297 & 0.9838 \\ 0.9838 & 2.4894 \end{pmatrix}$$

$$W = I_3 \otimes \begin{pmatrix} 3.5234 & -1.1643 \\ -2.1222 & 0.4623 \\ -1.0132 & -3.1321 \end{pmatrix} \quad W1 = I_3 \otimes \begin{pmatrix} -2.3140 & 0.9208 \\ -1.5350 & -1.5350 \\ -0.2120 & 1.1342 \end{pmatrix}$$

According to Theorem 3, the network is output strictly passive for the following positive definite matrices:

$$P = \begin{pmatrix} 0.0982 & -0.0603 \\ -0.0603 & 1.8075 \end{pmatrix} \quad N1 = \begin{pmatrix} 3.7641 & -0.0018 \\ -0.0018 & 2.9692 \end{pmatrix} \quad N2 = \begin{pmatrix} 7.6926 & 3.5428 \\ 3.5428 & 8.8839 \end{pmatrix}$$

$$W = I_3 \otimes \begin{pmatrix} 4.1098 & -2.7040 \\ 1.6606 & -1.5610 \\ 0.1019 & -3.8012 \end{pmatrix} \quad W2 = I_3 \otimes \begin{pmatrix} 0.8713 & -0.2699 \\ 1.9610 & -2.1093 \\ -3.0181 & 2.3583 \end{pmatrix}$$

According to Theorem 4, the complex dynamical network achieves complete synchronization for the following positive definite matrices and parameters

$$\Phi = \begin{pmatrix} 0.1124 & 1.1257 \\ 0.0451 & 0.0095 \end{pmatrix}$$

and we have a feasible solution $t_{min} = -0.0091$ with the following parameters: $\vartheta_1 = 0.4761$, $\vartheta_2 = 0.2942$, $\alpha_1 = 1.0295$, $\alpha_2 = 0.4778$ and $\alpha_3 = 0.2492$ and positive constant $k^* = 3.2700$. The simulation results in Figures 1 and 2 represent the state trajectories of μ_{11} synchronized with $\bar{\mu}(t)$ with and without a controller. Moreover, the state trajectories of μ_{12}, μ_{22}, μ_{31}, μ_{32} synchronized with $\bar{\mu}(t)$ with and without adaptive state feedback controllers are depicted in Figures 3–12. In Figures 13 and 14, we depict the error trajectories, which completely converge to zero in a short time using the adaptive state feedback controller.

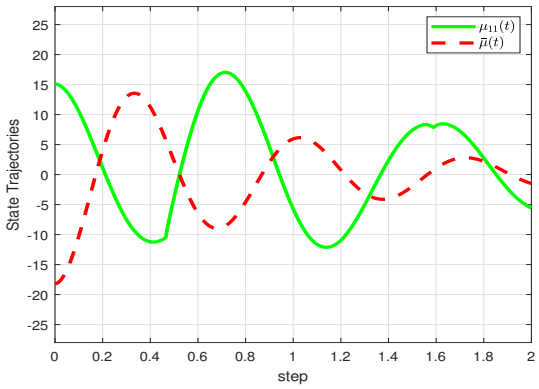


Figure 1. Evolution of synchronization states of μ_{11} and $\bar{\mu}(t)$ without controller.

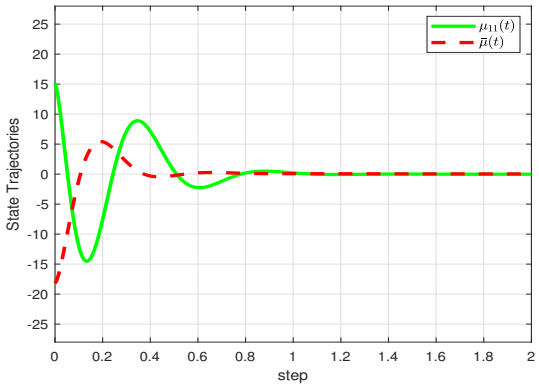


Figure 2. Evolution of synchronization states of μ_{11} and $\bar{\mu}(t)$ with controller.

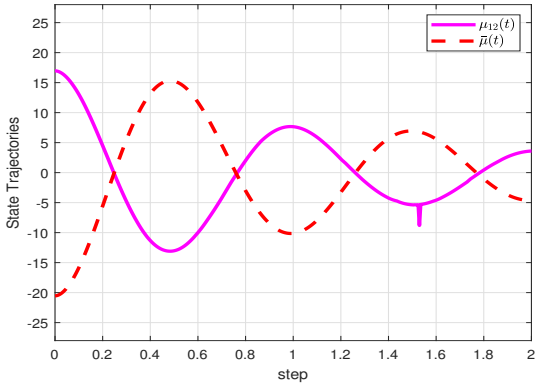


Figure 3. Evolution of synchronization states of μ_{12} and $\bar{\mu}(t)$ without controller.

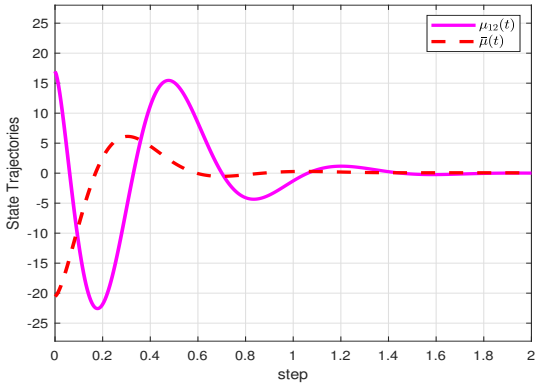


Figure 4. Evolution of synchronization states of μ_{12} and $\bar{\mu}(t)$ with controller.

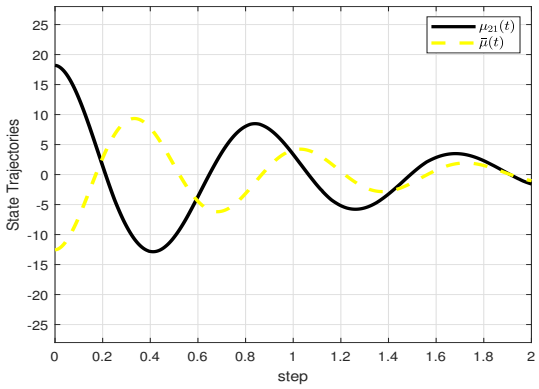


Figure 5. Evolution of synchronization states of μ_{21} and $\bar{\mu}(t)$ without controller.

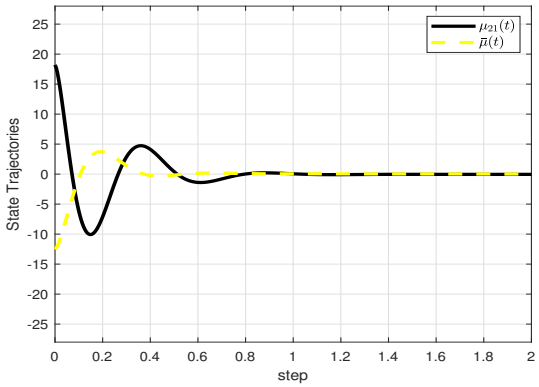


Figure 6. Evolution of synchronization states of μ_{21} and $\bar{\mu}(t)$ with controller.

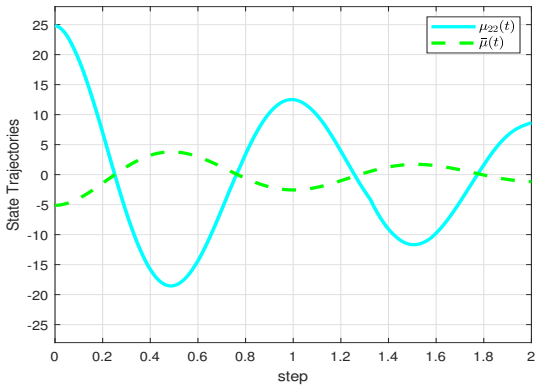


Figure 7. Evolution of synchronization states of μ_{22} and $\bar{\mu}(t)$ without controller.

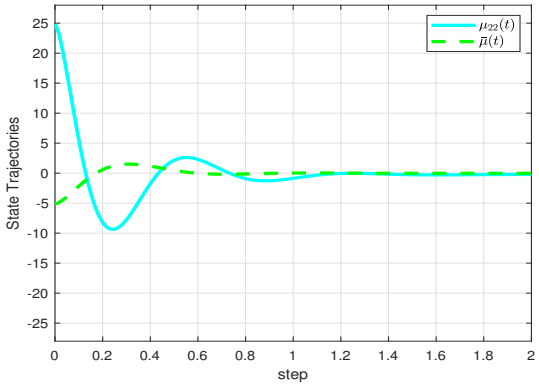


Figure 8. Evolution of synchronization states of μ_{22} and $\bar{\mu}(t)$ with controller.

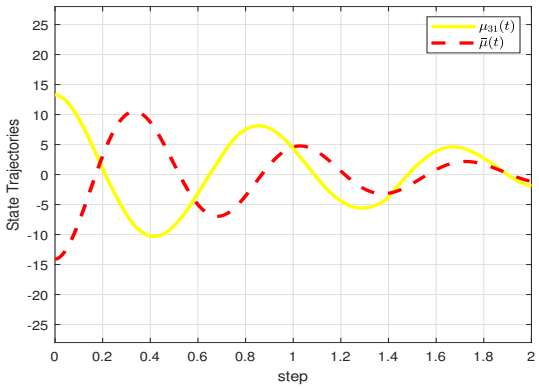


Figure 9. Evolution of synchronization states of μ_{31} and $\bar{\mu}(t)$ without controller.

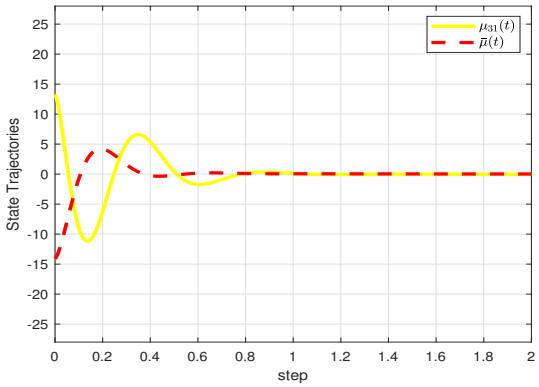


Figure 10. Evolution of synchronization states of μ_{31} and $\bar{\mu}(t)$ with controller.

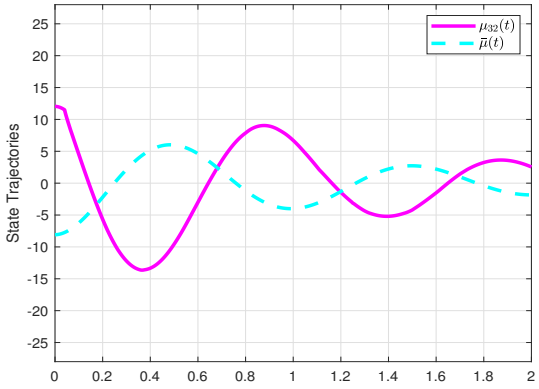


Figure 11. Evolution of synchronization states of μ_{32} and $\bar{\mu}(t)$ without controller.

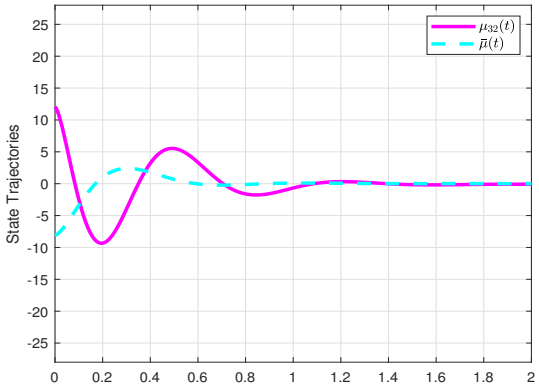


Figure 12. Evolution of synchronization states of μ_{32} and $\bar{\mu}(t)$ with controller.

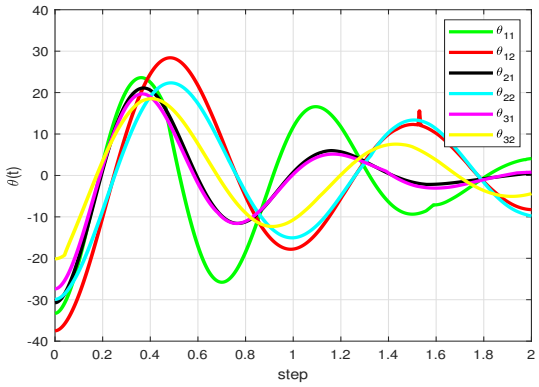


Figure 13. Evolution of synchronization error $\xi(t)$ without controller.

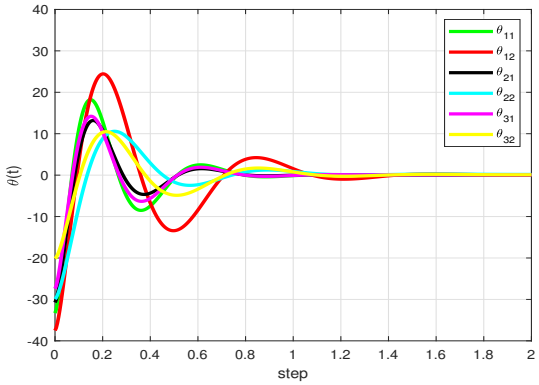


Figure 14. Evolution of synchronization error $\xi(t)$ with controller.

7. Conclusions

We investigated the passivity analysis and synchronization criteria of a fractional-order complex dynamical network model with derivative connections and time-varying delays in this paper. We deduced the passivity problem for the proposed network model in terms of a linear matrix inequality using different inequality methods. Furthermore, using an adaptive feedback control technique, some adequate criteria to ensure complete synchronization for a fractional-order complex dynamical network with derivative couplings (CDNMDC) were obtained. We also examined the synchronization criterion of these complex networks, which included accounting for parameter uncertainty. The passivity analysis of fractional-order complex dynamical networks with derivative coupling holds significant real-life application value in various fields. Its utilization spans electrical engineering, control systems, communication networks, biomedical engineering, social networks, economics, and finance. This analysis enables an understanding of the stability, performance, and dynamics of complex interconnected systems, leading to advancements in system design, control strategies, information flow, and optimization in diverse domains. Finally, we provide an example to demonstrate that the proposed methods work. In the future, we will attempt to investigate these complex dynamical networks by incorporating impulsive perturbations and synchronization analysis.

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Conflicts of Interest: The authors declare that they have no conflict of interest.

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