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On Orthogonal Fuzzy Interpolative Contractions with Applications to Volterra Type Integral Equations and Fractional Differential Equations

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Abstract: In this paper, orthogonal fuzzy versions are reported for some celebrated iterative mappings. We provide various concrete conditions on the real valued functions $\mathcal{J}, \mathcal{S} : (0,1] \rightarrow (-\infty, \infty)$ for the existence of fixed-points of $(\mathcal{J}, \mathcal{S})$-fuzzy interpolative contractions. This way, many fixed point theorems are developed in orthogonal fuzzy metric spaces. We apply the $(\mathcal{J}, \mathcal{S})$-fuzzy version of Banach fixed point theorem to demonstrate the existence and uniqueness of the solution. These results are supported with several non-trivial examples and applications to Volterra-type integral equations and fractional differential equations.

Keywords: fixed point; fuzzy metric spaces; $(\mathcal{J}, \mathcal{S})$-fuzzy iterative mappings; fractional differential equations

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1. Introduction

A self-mapping $L : \mathcal{B} \rightarrow \mathcal{B}$ has a fixed point if $L(\sigma) = \sigma$ for $\sigma \in \mathcal{B}$. It is a great achievement to find a unique solution of nonlinear equations. In 1960, Schweizer and Sklar [1] initiated the concept of continuous t-norm (in short ctn), which is a binary relation. In 1965, Zadeh [2] initiated the concept of a fuzzy set (FS) and its properties. Then, in 1975, Kramosil and Michalek [3] initiated the notion of the fuzzy metric space (in short, FMS) by using the concepts of ctn and FSs. In 1994, George and Veeramani [4] presented the further modified version of FMSs. After that, Grabeic [5] initiated and improved the well known Banach's fixed point theorem (FPT) in the framework of FMSs in the context of Kramosil and Michalek [3]. By following the concepts of Grabeic [5], Gregori and Sapena [6] provided an addition to Banach's contraction theorem by using FMSs.

In 1968, Kannan [7] provided a new type of contraction and proved some fixed point (in short, FP) results for discontinuous mappings. Karapinar [8] established a new type of contraction via interpolative contraction and proved some FP results on it. Thus, he provided a new way of research, and many authors worked on it and proved different FP results on it (see [9,10]). Hierro et al. [11] proved the FP result in FMSs. Then, Zhou et al. [12] generalized the result of Hierro et al. [11] in the framework of FMSs. Nazam et al. [13] proved some FP results in orthogonal (Ψ, Φ) complete metric spaces. Hezarjaribi [14] established several FP results in a newly introduced concept, named the orthogonal fuzzy metric space (in short, OFMS). Important results and applications can be found in the following literature [15–21]. Uddin et al. [22] proved several fixed point results for contraction mappings in the context of orthogonal controlled FMSs. Ishtiaq et al. [23] extended the results proved in [22] in a more generalized framework named orthogonal neutrosophic metric spaces.

Inspired by the results in [8,11–14], we aim to establish FP results in the framework of an OFMS. We divide this paper into four main parts. The first part is based on the introduction. In the second part, we revise some basic concepts for understanding our main results. In the third part, we provide some FP results in OFMS and some examples to illustrate our results. In the fourth part, we provide an application involving Voltera-type integral equations and fractional differential equations.

2. Preliminaries

In this section, we provided several basic definitions and results.

Definition 1 ([12])**.** *A binary operation $* : H \times H \to H$ (where $H = [0,1]$) is called a ctn if it is verifying the below axioms:*

(1) $\sigma * \theta = \theta * \sigma$ *and* $\sigma * (\theta * \omega) = (\sigma * \theta) * \omega$ *for all* $\sigma, \theta, \omega \in H$;
(2) $*$ *is continuous;*
(3) $\sigma * 1 = \sigma$ *for all* $\sigma \in H$;
(4) $\sigma * \theta \leq \omega * \varpi$, *if* $\sigma \leq \omega$ *and* $\theta \leq \varpi$, *with* $\sigma, \omega, \varpi \in H$.

Definition 2 ([12])**.** *A triplet $(\mathcal{B}, \vartheta, *)$ is termed as FMS if $*$ is ctn, $\mathcal{B}$ is an arbitrary set, and ϑ is FS on $\mathcal{B} \times \mathcal{B} \times (0, \infty)$ fulfilling the accompanying conditions for all $\sigma, \theta, \omega \in \mathcal{B}$ and $\varsigma, \varpi > 0$.*

(i) $\vartheta(\sigma, \theta, \varsigma) > 0$;
(ii) $\vartheta(\sigma, \theta, \varsigma) = 0$ *if and only if* $\sigma = \theta$;
(iii) $\vartheta(\sigma, \theta, \varsigma) = \vartheta(\theta, \sigma, \varsigma)$;
(iv) $\vartheta(\sigma, \omega, \varsigma + \varpi) \geq \vartheta(\sigma, \theta, \varsigma) * \vartheta(\theta, \omega, \varpi)$;
(v) $\vartheta(\sigma, \theta, .) : (0, \infty) \to [0,1]$.

Example 1. *Let $\mathcal{B} = \mathbb{R}^+$ and $(\mathcal{B}, L^*)$ denote a metric space. Set $\vartheta(\sigma, \theta, \varsigma) = \frac{\varsigma}{\varsigma + L^*(\sigma, \theta)}$ and define an ctn as $m * n = mn$. Then, $\mathcal{B}$ is FMS.*

Definition 3 ([5])**.** *A mapping $L : \mathcal{B} \to \mathcal{B}$ satisfying the following inequality,*

$$\vartheta(L\sigma, L\theta, k\varsigma) \geq \vartheta(\sigma, \theta, \varsigma) \text{ for all } \sigma, \theta \in \mathcal{B},$$

is called a fuzzy contraction with $k \in [0, 1)$.

Definition 4 ([14])**.** *Let $(\mathcal{B}, \vartheta, *)$ be a FMS and $\perp \in \mathcal{B} \times \mathcal{B}$ be a binary relation. Suppose there exists $\sigma_0 \in \mathcal{B}$ such that $\sigma_0 \perp \sigma$ or $\sigma \perp \sigma_0$ for all $\sigma \in \mathcal{B}$. Then, $\mathcal{B}$ is an OFMS. We denote OFMS by $(\mathcal{B}, \vartheta, *, \perp)$.*

Definition 5 ([14])**.** *A mapping $L : \mathcal{B} \to \mathcal{B}$ verifying the below inequality,*

$$\vartheta(L\sigma, L\theta, k\varsigma) \geq \vartheta(\sigma, \theta, \varsigma) \text{ for all } \sigma, \theta \in \mathcal{B}, \text{ with } \sigma \perp \theta,$$

is called an orthogonal fuzzy contraction(in short, OFC) where $(\mathcal{B}, \vartheta, *, \perp)$ is an OFMS, and $k \in [0, 1)$.

Theorem 1 ([14])**.** *Suppose $(\mathcal{B}, \vartheta, *, \perp)$ is an OFMS. Consider a mapping $L : \mathcal{B} \to \mathcal{B}$ be $\perp$-continuous, OFC, and $\perp$-preserving. Then, L has a unique FP, namely $u \in \mathcal{B}$. Furthermore,*

$$\lim_{n \to \infty} \vartheta(L^n \sigma, L\theta, \varsigma) = 1,$$

for each $u \in \mathcal{B}$.

Remark 1. *The fuzzy contraction is an orthogonal fuzzy contraction but the converse may not be true in general.*

Example 2. *Suppose* $\mathcal{B} = [0, 10)$ *with FMS* ϑ *as defined as in Example 1, then the* $(\mathcal{B}, \vartheta, *, \perp)$ *represents an FMS. Define* $\perp \subseteq \mathcal{B} \times \mathcal{B}$ *by*

$$\sigma \perp \theta \text{ if } \sigma\theta \leq \sigma \vee \theta.$$

Then, $(\mathcal{B}, \vartheta, *, \perp)$ *is an OFMS with ctn* $\sigma * \theta = \sigma\theta$. *Let the mapping* $L : \mathcal{B} \to \mathcal{B}$ *be given by*

$$L(\sigma) = \left\{ \begin{array}{l} \frac{\sigma}{3} \text{ for } \sigma \leq 3 \\ 0 \text{ for } \sigma > 3 \end{array} \right\}.$$

We note that

$$\begin{array}{rcl} \vartheta(L(4), L(3), (0.4)1) & \geq & \vartheta(4, 3, 1) \\ \vartheta(0, 1, (0.4)1) & \geq & \vartheta(4, 3, 1) \\ 0.2857 & \geq & 0.5. \end{array}$$

This is a contradiction. Thus, L *is not a fuzzy contraction. However,* L *is an orthogonal fuzzy contraction.*

Lemma 1. *Let* $(\mathcal{B}, \vartheta, *)$ *be an FMS and* $\{a_n\} \subset \mathcal{B}$ *be a sequence satisfying* $\lim_{n\to\infty} \vartheta(a_n, a_{n+1}, \varsigma) = 1$. *If the sequence* $\{a_n\}$ *is not Cauchy, then there exists* $\{a_{n_k}\}, \{a_{m_k}\}$ *and* $\varepsilon \geq 0$, *such that*

$$\lim_{k\to\infty} \vartheta\big(a_{n_k+1}, a_{m_k+1}, \varsigma\big) = 1 + \varepsilon, \tag{1}$$

and

$$\lim_{k\to\infty} \vartheta\big(a_{n_k}, a_{m_k}, \varsigma\big) = \lim_{k\to\infty} \vartheta\big(a_{n_k+1}, a_{m_k}, \varsigma\big) = \lim_{k\to\infty} \vartheta\big(a_{n_k}, a_{m_k+1}, \varsigma\big) = 1 + \varepsilon. \tag{2}$$

Proof. Let $(\mathcal{B}, \vartheta, *)$ be an FMS. Given $\{a_n\}$ is not Cauchy and $\lim_{n\to\infty} \vartheta(a_n, a_{n+1}, \varsigma) = 1$. Thus, for every $\varepsilon > 0$, there exists a natural number k_0, such that for the smallest $m \geq n$

$$\vartheta(a_{n+1}, a_m, \varsigma) \geq 1 + \varepsilon \text{ and } \vartheta(a_{n+1}, a_m, \varsigma) < 1 + \varepsilon \text{ for all } n, m \geq k_0.$$

As a result, we construct two subsequences of $\{a_n\}$; $\{a_{n_k}\}$ and $\{a_{m_k}\}$, verifying the following inequalities

$$\vartheta\big(a_{n_k+1}, a_{m_k}, \varsigma\big) \geq 1 + \varepsilon \text{ and } \vartheta\big(a_{n_k+1}, a_{m_k+1}, \varsigma\big) < 1 + \varepsilon \text{ for each } n_k, m_k > k_0.$$

Using (iv) of the OFMS, we have the following information:

$$\begin{array}{rcl} 1 + \varepsilon & > & \vartheta\big(a_{n_k+1}, a_{m_k+1}, \varsigma\big) \\ & \geq & \vartheta\big(a_{n_k+1}, a_{m_k}, \varsigma\big) * \vartheta\big(a_{m_k}, a_{m_k+1}, \varsigma\big) \\ & \geq & 1 + \varepsilon . \vartheta\big(a_{m_k}, a_{m_k+1}, \varsigma\big). \end{array}$$

This implies that,

$$\lim_{k\to\infty} \vartheta\big(a_{n_k+1}, a_{m_k+1}, \varsigma\big) = 1 + \varepsilon.$$

Again, by utilizing axiom (iv) of the FMS, we have

$$\frac{\vartheta\big(a_{n_k+1}, a_{m_k+1}, \varsigma\big)}{\vartheta\big(a_{m_k}, a_{m_k+1}, \varsigma\big)} \geq \vartheta\big(a_{n_k+1}, a_{m_k}, \varsigma\big) \geq 1 + \varepsilon.$$

We obtain

$$\lim_{k\to\infty} \vartheta\big(a_{n_k+1}, a_{m_k}, \varsigma\big) = 1 + \varepsilon.$$

Since,

$$\vartheta\big(a_{n_k+1}, a_{m_k}, \varsigma\big) \geq \vartheta\big(a_{n_k+1}, a_{n_k}, \varsigma\big) * \vartheta\big(a_{n_k}, a_{m_k}, \varsigma\big),$$

we have the following inequality:

$$\begin{aligned} \frac{\vartheta\big(a_{n_k+1}, a_{m_k}, \varsigma\big)}{\vartheta\big(a_{n_k+1}, a_{n_k}, \varsigma\big)} &\geq \vartheta\big(a_{n_k}, a_{m_k}, \varsigma\big) \\ &\geq \vartheta\big(a_{m_k}, a_{n_k+1}, \varsigma\big) * \vartheta\big(a_{n_k+1}, a_{n_k}, \varsigma\big). \end{aligned}$$

That is

$$\lim_{k\to\infty} \vartheta\big(a_{n_k+1}, a_{m_k+1}, \varsigma\big) = 1 + \varepsilon.$$

Since,

$$1 + \varepsilon > \vartheta\big(a_{n_k+1}, a_{m_k+1}, \varsigma\big) \geq \vartheta\big(a_{n_k+1}, a_{n_k}, \varsigma\big) * \vartheta\big(a_{n_k}, a_{m_k+1}, \varsigma\big)$$

$$\begin{aligned} \frac{1+\varepsilon}{\vartheta\big(a_{n_k+1}, a_{n_k}, \varsigma\big)} &\geq \vartheta\big(a_{n_k}, a_{m_k+1}, \varsigma\big) \\ &\geq \vartheta\big(a_{n_k}, a_{n_k+1}, \varsigma\big) * \vartheta\big(a_{n_k+1}, a_{m_k+1}, \varsigma\big) \end{aligned}$$

That is

$$\vartheta\big(a_{n_k}, a_{m_k}, \varsigma\big) = 1 + \varepsilon.$$

This completes the proof. □

Definition 6 ([14])**.** *The OFMS $(\mathcal{B}, \vartheta, *, \perp)$ verifying the property (R) is called $\perp$-regular.*

(R) For any O-sequence $\{\sigma_n\} \subseteq \mathcal{B}$ converging to σ, we have either $\sigma \perp \sigma_n$, or $\sigma_n \perp \sigma$ for each $n \in \mathbb{N}$.

3. Main Results

3.1. Banach Type $(\mathcal{J}, \mathcal{S})$-Orthogonal Fuzzy Interpolative Contraction

In this section, we present the new results for orthogonal fuzzy interpolative contractions (OFIPC) involving the functions $\mathcal{J}, \mathcal{S} : (0,1] \to \mathbb{R}$.

Definition 7. *Let $\mathcal{J}, \mathcal{S} : (0,1] \to \mathbb{R}$ be two functions. A mapping $L : \mathcal{B} \to \mathcal{B}$ defined on OFMS $(\mathcal{B}, \vartheta, *, \perp)$ will be called a Banach-type $(\mathcal{J}, \mathcal{S})$-OFIPC, if there exists $\nu \in (0,1]$ verifying*

$$\mathcal{J}(\vartheta(L\sigma, L\theta, \varsigma)) \geq \mathcal{S}\big((\vartheta(\sigma, \theta, \varsigma))^{v}\big), \tag{3}$$

for each $(\sigma, \theta) \in \mathcal{B}$, $\vartheta(L\sigma, L\theta, \varsigma) > 0$.

Example 3. *Let $\mathcal{B} = [1,7)$ and define the FMS $\vartheta(\sigma, \theta, \varsigma) = e^{-\frac{|\sigma-\theta|}{\varsigma}}$. Let $\perp \subset \mathcal{B} \times \mathcal{B}$ be defined by*

$$\sigma \perp \theta \text{ if } \sigma\theta \leq \{\sigma, \theta\}.$$

Then $(\mathcal{B}, \vartheta, *, \perp)$ is OFMS with $m * n = mn$. Define $L : \mathcal{B} \to \mathcal{B}$ by

$$L(\sigma) = \left\{ \begin{array}{c} 5 \text{ if } 1 \leq \sigma < 2, \\ 3.1 \text{ if } 2 \leq \sigma < 3, \\ 1.8 \text{ if } 3 \leq \sigma < 7. \end{array} \right\}$$

Define $\mathcal{J}, \mathcal{S} : (0,1] \to \mathbb{R}$ by

$$\mathcal{J}(t) = \left\{ \begin{array}{c} \frac{1}{\ln t} \text{ if } 0 < t < 1 \\ 1 \text{ if } t = 1 \end{array} \right\} \text{ and } \mathcal{S} = \left\{ \begin{array}{c} \frac{1}{\ln t^2} \text{ if } 0 < t < 1 \\ 2 \text{ if } t = 1 \end{array} \right\}.$$

Case 1: Let, L be a Banach-type $(\mathcal{J}, \mathcal{S})$-OFIPC. Then,

$$\begin{array}{rcl} (\vartheta(L1, L2, k1)) & \geq & (\vartheta(1,2,1))^{\frac{1}{2}} \\ \vartheta\left(5, 3.1, \left(\frac{1}{2}\right)1\right) & \geq & (\vartheta(1,2,1))^{\frac{1}{2}} \\ e^{-\frac{|5-3.1|}{0.5}} & \geq & \left(e^{-\frac{|1-2|}{1}}\right)^{\frac{1}{2}} \\ 0.0224 & \geq & 0.6065. \end{array}$$

However, this is a contradiction. Therefore, L is not a Banach-type FIPC.

Case 2: Let L be a Banach-type $(\mathcal{J}, \mathcal{S})$-OFIPC. Then,

$$\begin{array}{rcl} \vartheta(L1, L3, 1) & \geq & (\vartheta(1,3,1))^{\frac{1}{2}} \\ \vartheta\left(5, 1.8, \frac{1}{2}1\right) & \geq & (\vartheta(1,3,1))^{\frac{1}{2}} \\ e^{-\frac{|5-1.8|}{0.5}} & \geq & \left(e^{-\frac{|1-3|}{1}}\right)^{\frac{1}{2}} \\ 0.0017 & \geq & 0.3678. \end{array}$$

This is a contradiction. Thus, L is not a Banach-type OFIPC.

Case 3: Let L be a Banach-type $(\mathcal{J}, \mathcal{S})$-OFIPC. Then,

$$\begin{array}{rcl} \vartheta(L1, L4, k1) & \geq & (\vartheta(1,4,1))^{\frac{1}{2}} \\ \vartheta\left(5, 1.8, \frac{1}{2}1\right) & \geq & (\vartheta(1,4,1))^{\frac{1}{2}} \\ e^{-\frac{|5-1.8|}{0.5}} & \geq & \left(e^{-\frac{|1-4|}{1}}\right)^{\frac{1}{2}} \\ 0.0017 & \geq & 0.2231. \end{array}$$

This is a contradiction. Thus, L is not a Banach-type OFIPC.

Hence, in general, let $\sigma, \theta \in \mathcal{B}$, such that $\sigma \perp \theta$ or $\theta \perp \sigma$

$$\begin{array}{rcl} \mathcal{J}(\vartheta(L\sigma, L\theta, \varsigma)) & = & -\frac{\varsigma}{\mid L\sigma - L\theta \mid} = -\frac{\varsigma}{L \mid \sigma - \theta \mid} \\ & \geq & -\frac{\varsigma}{\mid \sigma - \theta \mid} = \mathcal{S}(\vartheta(\sigma, \theta, \varsigma))^{\frac{1}{2}} \end{array}$$

Therefore, the Banach contraction is fulfilled.

For the $\perp$ (orthogonal relation), two functions $(\mathcal{J}, \mathcal{S}) : (0,1] \to \mathbb{R}$, and the self-mapping L, we write the below properties:

(i) For every $\sigma_0 \in \mathcal{B}$, there is $\sigma_1 = L(\sigma_0)$ such that $\sigma_1 \perp \sigma_0$ or $\sigma_0 \perp \sigma_1$;
(ii) $\mathcal{J}$ is non-decreasing and for every $1 > r \geq t > 0$, one has $\mathcal{S}(r) > \mathcal{J}(t)$;
(iii) $\lim_{s \to L^-} \inf \mathcal{S}(s) > \lim_{s \to L^-} \sup(\mathcal{J}(s))$;
(iv) If $t \in (0,1]$ such that $\mathcal{J}(t) \geq \mathcal{S}(1)$;
(v) $\sup_{\sigma_a > \varepsilon} \mathcal{J}(\sigma_a) > -\infty$;
(vi) $\lim_{\sigma_a \to \delta} \inf \mathcal{S}(\sigma_a) > \mathcal{J}(\delta)$ for all $\delta \in (0,1)$;
(vii) If $\{\mathcal{J}(y_n)\}$ and $\{\mathcal{S}(y_n)\}$ are converging to same limit and $\{\mathcal{J}(y_n)\}$ is strictly increasing, then $\lim_{n \to \infty} y_n = 1$;
(viii) $\mathcal{J}\left(\sigma_a^v \sigma_b^\eta\right) \geq \mathcal{J}(\sigma_a)$ and $\mathcal{S}(\sigma_a) > \mathcal{J}(\sigma_a)$ for each $\sigma_a \in (0,1)$ and $v, \eta \in (0,1]$.

The next two theorems deal with the Banach-type $(\mathcal{J}, \mathcal{S})$-OFIPC.

Theorem 2. *Suppose $\perp$ is a transitive orthogonal relation (in short, TOR) (i.e., if $\sigma_0 \perp \sigma_1$ and $\sigma_1 \perp \sigma_2$. Then, $\sigma_0 \perp \sigma_2$ for each $\sigma_0, \sigma_1, \sigma_2 \in \mathcal{B}$). Moreover, each $\perp$-preserving self-mapping (in short, PSM) on a $\perp$-regular OCFMS $(\mathcal{B}, \vartheta, *, \perp)$ satisfying (3) and (i)–(iv), have a FP in $\mathcal{B}$.*

Proof. Pick an initial guess $\sigma_0 \in \mathcal{B}$ so that $\sigma_0 \perp \sigma_1$ or $\sigma_1 \perp \sigma_0$ for every $\sigma_1 \in \mathcal{B}$, then, by utilizing the $\perp$-preservation of L, we build an OS $\{\sigma_n\}$ such that $\sigma_n = L(\sigma_{n-1}) = L^n(\sigma_0)$ and $\sigma_{n-1} \perp \sigma_n$ for every $n \in \mathbb{N}$. Note that, if $\sigma_n = L(\sigma_n)$ then σ_n is FP of L for each $n \geq 0$. We let that $\sigma_n \neq \sigma_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Let $y_n = \vartheta(\sigma_n, \sigma_{n+1}, \varsigma)$ for all $n \geq 0$. By the first part of (ii) and (3), we have

$$\mathcal{J}(y_n) \geq \mathcal{J}(\vartheta(L\sigma_{n-1}, L\sigma_n, \varsigma)) \geq \mathcal{S}\big((\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^v\big).$$

By utilizing (ii), we obtain

$$\mathcal{J}(y_n) \geq \mathcal{S}\big((y_{n-1})^v\big) > \mathcal{J}\big((y_{n-1})^v\big). \tag{4}$$

Since $\mathcal{J}$ is non decreasing, one obtains $y_n > y_{n-1}$; for every $n \geq 1$, we have $L < 1$, that is $\lim_{n\to\infty} y_n = L+$. If $L < 1$, by (4), we obtain the following information:

$$\mathcal{J}(L+) = \lim_{n\to\infty} \mathcal{J}(y_n) \geq \lim_{n\to\infty} \inf \mathcal{S}\big((y_{n-1})^v\big) \geq \lim_{\sigma_a \to L+} \inf \mathcal{S}(\sigma_a).$$

Thus, this contradicts (iii), so $L = 1$.

The sequence $\{\sigma_n\}$is Cauchy: Let $\{\sigma_n\}$ not equal OCS, so that in the following Lemma 1, there exists two subsequences $\{\sigma_{n_k}\}$, $\{\sigma_{m_k}\}$ of $\{\sigma_n\}$ and $\varepsilon > 0$, such that (1) and (2) are satisfied. From (1), we deduce

$$\vartheta(\sigma_{n_k+1}, \sigma_{m_k+1}, \varsigma) > 1 + \varepsilon.$$

Since $\sigma_n \perp \sigma_{n+1}$ for each $n \geq 0$. Hence, by the transitivity of $\perp$, we have $\sigma_{n_k} \perp \sigma_{m_k}$ for some $k \geq 1$,

$$\mathcal{J}\left(\vartheta\left(\sigma_{n_k+1}, \sigma_{m_k+1}, \varsigma\right)\right) \geq \mathcal{J}\left(\vartheta\left(L\sigma_{n_k}, L\sigma_{m_k}, \varsigma\right)\right) \geq \mathcal{S}\left(\left(\vartheta\left(\sigma_{n_k}, \sigma_{m_k}, \varsigma\right)\right)^v\right)$$

If $\sigma_{a_k} = \vartheta\left(\sigma_{n_k+1}, \sigma_{m_k+1}, \varsigma\right)$, $\sigma_{b_k} = \vartheta\left(\sigma_{n_k}, \sigma_{m_k}, \varsigma\right)$, we have

$$\mathcal{J}\left(\sigma_{a_k}\right) \geq \mathcal{S}\left(\left(\sigma_{b_k}\right)^v\right), \text{ for some } k \geq 1. \tag{5}$$

By (1), we have $\lim_{k\to\infty} \sigma_{a_k} = 1 + \varepsilon$ and (5) implies

$$\lim_{\sigma_a \to (1+\varepsilon)} \sup \mathcal{J}\left(\sigma_{a_k}\right) \geq \lim_{k\to\infty} \sup \mathcal{J}\left(\sigma_{a_k}\right) \geq \lim_{k\to\infty} \inf \mathcal{S}\left(\left(\sigma_{b_k}\right)^v\right) \geq \lim_{\sigma_a \to 0} \inf \mathcal{S}(\sigma_a). \tag{6}$$

The information obtained in (6), contradicts the assumption (iii). Thus, the sequence $\{\sigma_n\}$ is OC in the OCFMS $(\mathcal{B}, \vartheta, *, \perp)$. Hence, there is $\sigma_a \in \mathcal{B}$, so that $\sigma_n \to \sigma_a$ as $n \to \infty$. Since $(\mathcal{B}, \vartheta, *, \perp)$ is a $\perp$-regular space, we write $\sigma_a \perp \sigma_n$ or $\sigma_n \perp \sigma_a$. We claim that $\vartheta(\sigma_a, L\sigma_a, \varsigma) = 1$. If $\vartheta(\sigma_{n+1}, L\sigma_a, \varsigma) > 1$, then we have (3)

$$\begin{aligned} \mathcal{J}(\vartheta(\sigma_{n+1}, L\sigma_a, \varsigma)) &\geq \mathcal{J}(\vartheta(L\sigma_n, L\sigma_a, \varsigma)) \geq \mathcal{S}\big((\vartheta(\sigma_n, \sigma_a, \varsigma))^v\big) \\ &> \mathcal{J}\big((\vartheta(\sigma_n, \sigma_a, \varsigma))^v\big). \end{aligned}$$

By the first part of (ii), we obtain

$$\vartheta\left(\sigma_{n+1}, L\sigma_a, \varsigma^k\right) > (\vartheta(\sigma_n, \sigma_a, \varsigma))^v.$$

Applying limit $n \to \infty$, we obtain $\vartheta(\sigma_a, L\sigma_a, \varsigma) \geq 1$. This implies that $\vartheta(\sigma_a, L\sigma_a, \varsigma) = 1$. Hence, $\sigma_a = L\sigma_a$. $\square$

Theorem 3. *Let $\perp$ be a TOR; then, every $\perp$-PSM defined on a $\perp$-regular OCFMS $(\mathcal{B}, \vartheta, *, \perp)$ verifying (3) and (i), (iii), (v)–(viii) admits a fixed point in $\mathcal{B}$.*

Proof. Choose an initial guess $\sigma_0 \in \mathcal{B}$, so that $\sigma_0 \perp \sigma_1$ or $\sigma_1 \perp \sigma_0$ for each $\sigma_1 \in \mathcal{B}$. Then, by utilizing the $\perp$-preservation of L, we build an OS $\{\sigma_n\}$, so that $\sigma_n = L(\sigma_{n-1}) = L^n(\sigma_0)$ and $\sigma_{n-1} \perp \sigma_n$ for every $n \in \mathbb{N}$. Note that, if $\sigma_n = L(\sigma_n)$ then σ_n is the FP of L for each $n \geq 0$. Let $\sigma_n \neq \sigma_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Let $y_n = \vartheta(\sigma_n, \sigma_{n+1}, \varsigma)$ for each $n \geq 0$. By the first part of (ii) and (3), we have

$$\begin{aligned} \mathcal{J}(\vartheta(\sigma_n, \sigma_{n+1}, \varsigma)) &\geq \mathcal{J}(\vartheta(L\sigma_{n-1}, L\sigma_n, \varsigma)) \geq \mathcal{S}\big((\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^v\big) \\ &> \mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^v \\ &\geq \mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma)). \end{aligned} \tag{7}$$

The inequality shows that (7) $\{\mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))\}$ is strictly increasing. If it is not bounded above, then from (v), we obtain $\sup_{\vartheta(\sigma_{n-1}, \sigma_n, \varsigma) > \varepsilon} \mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma)) > -\infty$. This implies that

$$\lim_{\vartheta(\sigma_{n-1}, \sigma_n, \varsigma) \to \varepsilon+} \sup \mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma)) > -\infty.$$

Thus, $\lim_{n\to\infty} \vartheta(\sigma_{n-1}, \sigma_n, \varsigma) = 1$. Otherwise, we have

$$\lim_{\vartheta(\sigma_{n-1}, \sigma_n, \varsigma) \to \varepsilon+} \sup \mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma)) = -\infty$$

(i.e., a contradiction (v)). If it is bounded above, then $\{\mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))\}$ is a convergent sequence and by (7), $\{\mathcal{S}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))\}$ also converges to the same limit point. By using (iii), we have $\lim_{n\to\infty} \vartheta(\sigma_{n-1}, \sigma_n, \varsigma) = 1$. Hence, L is asymptotically regular (in short, AR).

Next, we assert that $\{\sigma_n\}$ is CS. Then, by Lemma 1 there exists $\{\sigma_{n_k}\}$, $\{\sigma_{m_k}\}$ and $\varepsilon > 0$, so that (1) and (2); we deduce $\vartheta\big(\sigma_{n_k+1}, \sigma_{m_k+1}, \varsigma\big) > 1 + \varepsilon$. Since $\sigma_n \perp \sigma_{n+1}$ for each $n \geq 0$, by transitivity of $\perp$, we obtain $\sigma_{n_k} \perp \sigma_{m_k}$. Letting $g = \sigma_{n_k}$ and $e = \sigma_{m_k}$ in (3), one writes for each $k \geq 1$,

$$\mathcal{J}\big(\vartheta\big(\sigma_{n_k+1}, \sigma_{m_k+1}, \varsigma\big)\big) \geq \mathcal{J}\big(\vartheta\big(L\sigma_{n_k}, L\sigma_{m_k}, \varsigma\big)\big) \geq \mathcal{S}\Big(\big(\vartheta\big(\sigma_{n_k}, \sigma_{m_k}, \varsigma\big)\big)^v\Big).$$

If $\sigma_k = \vartheta\big(\sigma_{n_k+1}, \sigma_{m_k+1}, \varsigma\big)$, $\sigma_{bk} = \vartheta\big(\sigma_{n_k}, \sigma_{m_k}, \varsigma\big)$, we have

$$\mathcal{J}(\sigma_k) \geq \mathcal{S}\big((\sigma_{bk})^v\big), \text{ for some } k \geq 1. \tag{8}$$

By (1), we have $\lim_{k\to\infty} \sigma_k = 1 + \varepsilon$, and (8) implies

$$\lim_{\sigma_a \to (1+\varepsilon)} \sup \mathcal{J}(\sigma_a) \geq \lim_{k\to\infty} \sup \mathcal{J}(\sigma_k) \geq \lim_{k\to\infty} \inf \mathcal{S}\big((\sigma_{bk})^v\big) \geq \lim_{\sigma_a \to 0} \inf \mathcal{S}(\sigma_a). \tag{9}$$

The information obtained in (9) contradicts the assumption (viii) and thus stamps the sequence $\{\sigma_n\}$ as OC in the OCFMS $(\mathcal{B}, \vartheta, *, \perp)$. The completeness of the space ensures the convergence of $\{\sigma_n\}$. Let it converge to $i \in \mathcal{B}$.

Case 1. If $\vartheta(\sigma_{n+1}, Li, \varsigma) = 1$ for some $n \geq 0$, then

$$\vartheta(i, Li, \varsigma) \geq \vartheta(i, \sigma_{n+1}, \varsigma) * \vartheta(\sigma_{n+1}, Li, \varsigma)$$

taking limit $n \to \infty$ on both sides, we have $\vartheta(i, Li, \varsigma) \geq 1$. This implies that $\vartheta(i, Li, \varsigma) = 1$. Hence, $i = Li$.

Case 2. For each $n \geq 0$, $\vartheta(\sigma_{n+1}, Li, \varsigma) < 1$. Then, by the $\perp$-regularity of $\mathcal{B}$, we find $\sigma_n \perp i$ or $i \perp \sigma_n$. By (3), one writes

$$\mathcal{J}(\vartheta(\sigma_{n+1}, Li, \varsigma)) \geq \mathcal{J}(\vartheta(L\sigma_n, Li, \varsigma)) \geq \mathcal{S}\big((\vartheta(\sigma_n, i, \varsigma))^v\big) \text{ for all } n \geq 0.$$

By taking $\sigma_n = \vartheta(\sigma_{n+1}, Li, \varsigma)$ and $b_n = \vartheta(\sigma_n, i, \varsigma)$, one writes

$$\mathcal{J}(\sigma_n) \geq \mathcal{S}\left((b_n)^v\right) \text{ for all } n \geq 0. \tag{10}$$

Note that $\sigma_n \to \delta$ and $b_n \to 1$ as $n \to \infty$. By applying limits on (10), we have

$$\limsup_{i\to\delta} \mathcal{J}(i) \geq \limsup_{n\to\infty} \mathcal{J}(\sigma_n) \geq \liminf_{n\to\infty} \mathcal{S}\left((b_n)^v\right) \geq \limsup_{i\to 0} \mathcal{S}(i).$$

This contradicts (v) if $\delta > 1$. Thus, we obtain $\vartheta(i, Li, \varsigma) = 1$. That is, i is a FP of L.
□

Example 4. *Let $\mathcal{B} = \{1,2,3,4,5,6,7,8\}$ and define the FMS $\vartheta(\sigma,\theta,\varsigma) = \frac{\varsigma}{\varsigma+|\sigma-\theta|}$. Let $\perp \subset \mathcal{B}\times\mathcal{B}$ defined by*

$$\sigma \perp \theta \text{ if } \sigma\theta \leq \sigma \vee \theta \text{ for } \sigma \neq \theta.$$

Then, $(\mathcal{B}, \vartheta, *, \perp)$ is OFMS with $\sigma * \theta = \sigma\theta$. Define $L : \mathcal{B} \to \mathcal{B}$ by

$$L(\sigma) = \left\{ \begin{array}{c} 5 \text{ if } \sigma = 5 \\ \sigma - 1 \text{ otherwise} \end{array} \right\}.$$

Define $\mathcal{J}, \mathcal{S} : (0,1] \to \mathbb{R}$ by

$$\mathcal{J}(t) = \left\{ \begin{array}{c} \frac{1}{\ln t} \text{ if } 0 < t < 1 \\ 1 \text{ if } t = 1 \end{array} \right\} \text{and } \mathcal{S} = \left\{ \begin{array}{c} \frac{1}{\ln t^2} \text{ if } 0 < t < 1 \\ 2 \text{ if } t = 1 \end{array} \right\}.$$

Case 1:
Let L be a Banach-type $(\mathcal{J}, \mathcal{S})$-OFIPC. Then,

$$\begin{array}{lcl} \vartheta(L2, L1, 1) & \geq & (\vartheta(2,1,1))^{\frac{1}{2}} \\ \vartheta(1,5,1) & \geq & (\vartheta(2,1,1))^{\frac{1}{2}} \\ \left(\frac{1}{1+|1-5|}\right) & \geq & \left(\frac{1}{1+|2-1|}\right)^{\frac{1}{2}} \\ 0.2 & \geq & 0.7071. \end{array}$$

which is a contradiction. Thus, L is not a Banach-type FIPC.

Case 2:
Let L be a Banach-type $(\mathcal{J}, \mathcal{S})$-OFIPC. Then,

$$\begin{array}{lcl} \vartheta(L3, L1, 1) & \geq & (\vartheta(3,1,1))^{\frac{1}{2}} \\ \vartheta(2,5,1) & \geq & (\vartheta(3,1,1))^{\frac{1}{2}} \\ \left(\frac{1}{1+|2-5|}\right) & \geq & \left(\frac{1}{1+|3-1|}\right)^{\frac{1}{2}} \\ 0.25 & \geq & 0.57771. \end{array}$$

This is a contradiction. Thus, L is not a Banach-type FIPC.

Since the condition of Theorem 2 (ii) is held for every $t \in (0,1)$ $\mathcal{S}(t) > \mathcal{J}(t)$, all the remaining conditions of Theorem 2 are also held.

3.2. *Kannan-Type $(\mathcal{J}, \mathcal{S})$-Orthogonal Fuzzy Interpolative Contraction*

Definition 8. *Let $\mathcal{J}, \mathcal{S} : (0,1] \to \mathbb{R}$ be two functions. A mapping $L : \mathcal{B} \to \mathcal{B}$ defined on OFMS $(\mathcal{B}, \vartheta, *, \perp)$ is called a Kannan-type $(\mathcal{J}, \mathcal{S})$-OFIPC, if there exists $v \in (0,1)$ verifying*

$$\mathcal{J}(\vartheta(L\sigma, L\theta, \varsigma)) \geq \mathcal{S}\Big((\vartheta(\sigma, L\sigma, \varsigma))^v (\vartheta(\theta, L\theta, \varsigma))^{1-v}\Big). \tag{11}$$

for each $(\sigma, \theta) \in \mathcal{B}$, $\vartheta(L\sigma, L\theta, \varsigma) > 0$.

Theorem 4. *Let $\perp$ be a TOR. Then, every $\perp$-PSM defined on a $\perp$-regular OCFMMS $(\mathcal{B}, \vartheta, *, \perp)$ satisfying (11) and (i)–(iv) have a fixed point in $\mathcal{B}$.*

Proof. Choose an initial guess $\sigma_0 \in \mathcal{B}$ so that $\sigma_0 \perp \sigma_1$ or $\sigma_1 \perp \sigma_0$ for every $\sigma_1 \in \mathcal{B}$. Then, by utilizing the $\perp$-preservation of L, we build an OS $\{\sigma_n\}$ such that $\sigma_n = L(\sigma_{n-1}) = L^n(\sigma_0)$ and $\sigma_{n-1} \perp \sigma_n$ for every $n \in \mathbb{N}$. Observe that, if $\sigma_n = L(\sigma_n)$. Then, σ_n is the FP of L for each $n \geq 0$. Let $\sigma_n \neq \sigma_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Let $y_n = \vartheta(\sigma_n, \sigma_{n+1}, \varsigma)$ for each $n \geq 0$. By the first part of (ii) and (11), we obtain

$$\mathcal{J}(y_n) \geq \mathcal{J}(\vartheta(L\sigma_{n-1}, L\sigma_n, \varsigma)) \geq \mathcal{S}\Big((\vartheta(\sigma_{n-1}, L\sigma_{n-1}, \varsigma))^v * (\vartheta(\sigma_n, L\sigma_n, \varsigma))^{1-v}\Big).$$

By applying (ii), we have

$$\mathcal{J}(y_n) \geq \mathcal{S}\Big((y_{n-1})^v (y_n)^{1-v}\Big) > \mathcal{J}\Big((y_{n-1})^v (y_n)^{1-v}\Big). \tag{12}$$

Since $\mathcal{J}$ is non decreasing, one obtains $y_n > y_{n-1}$ for every $n \geq 1$; we have $L < 1$,, that is $\lim_{n\to\infty} y_n = L+$. If $L < 1$, by (12), we obtain the following information:

$$\mathcal{J}(L+) = \lim_{n\to\infty} \mathcal{J}(y_n) \geq \lim_{n\to\infty} \inf \mathcal{S}\Big((y_{n-1})^v (y_n)^{1-v}\Big) \geq \lim_{\sigma_a \to L+} \inf \mathcal{S}(\sigma_a).$$

Thus, this contradicts (iii). Hence, $L = 1$.

The sequence $\{\sigma_n\}$ is Cauchy: Assume that $\{\sigma_n\}$ is not CS, so by the following Lemma 1, there exists two subsequences $\{\sigma_{n_k}\}$, $\{\sigma_{m_k}\}$ of $\{\sigma_n\}$ and $\varepsilon > 0$, such that (1) and (2) are satisfied. We deduce from (1)

$$\vartheta\big(\sigma_{n_k+1}, \sigma_{m_k+1}, \varsigma\big) > 1 + \varepsilon.$$

Since $\sigma_n \perp \sigma_{n+1}$ for each $n \geq 0$. Thus, by the transitive of $\perp$, we obtain $\sigma_{n_k} \perp \sigma_{m_k}$ for some $k \geq 1$,

$$\begin{aligned} \mathcal{J}\big(\vartheta\big(\sigma_{n_k+1}, \sigma_{m_k+1}, \varsigma\big)\big) &\geq \mathcal{J}(\vartheta(L\sigma_{n_k}, L\sigma_{m_k}, \varsigma)) \geq \mathcal{S}\Big((\vartheta(\sigma_{n_k}, L\sigma_{n_k}, \varsigma))^v * (\vartheta(\sigma_{m_k}, L\sigma_{m_k}, \varsigma))^{1-v}\Big) \\ &\geq \mathcal{S}\Big((\vartheta(\sigma_{n_k}, \sigma_{n_k+1}, \varsigma))^v * (\vartheta(\sigma_{m_k}, \sigma_{m_{K+1}}, \varsigma))^{1-v}\Big). \end{aligned}$$

If $\sigma_k = \vartheta\big(\sigma_{n_k+1}, \sigma_{m_k+1}, \varsigma\big), \sigma_{b_k} = \vartheta\big(\sigma_{n_k}, \sigma_{n_k+1}, \varsigma\big), \sigma_{c_k} = \vartheta\big(\sigma_{m_k}, \sigma_{m_k+1}, \varsigma\big)$, we have

$$\mathcal{J}(\sigma_k) \geq \mathcal{S}\Big((\sigma_{b_k})^v (\sigma_{c_k})^{1-v}\Big), \text{ for all } k \geq 1. \tag{13}$$

By (1), we have $\lim_{k\to\infty} \sigma_k = 1 + \varepsilon$, and (13) implies

$$\lim_{\sigma_k \to (1+\varepsilon)} \sup \mathcal{J}(\sigma_k) \geq \lim_{k\to\infty} \sup \mathcal{J}(\sigma_k) \geq \lim_{k\to\infty} \inf \mathcal{S}\Big((\sigma_{b_k})^v (\sigma_{c_k})^{1-v}\Big) \geq \lim_{\sigma_k \to 0} \inf \mathcal{S}(\sigma_k). \tag{14}$$

The information obtained in (14), contradicts the assumption (iii) and thus stamping the sequence $\{\sigma_n\}$ as OC in the OCFMS $(\mathcal{B}, \vartheta, *, \perp)$. Hence, there is $\sigma_{\sigma_a} \in \mathcal{B}$ so that $\sigma_n \to \sigma_a$ as $n \to \infty$. Since $(\mathcal{B}, \vartheta, *, \perp)$ is a $\perp$-regular space, we write $\sigma_a \perp \sigma_n$ or $\sigma_n \perp \sigma_a$. We claim that $\vartheta(\sigma_a, L\sigma_a, \varsigma) = 1$. If $\vartheta(\sigma_{n+1}, L\sigma_a, \varsigma) > 1$. Then, by (11), we have

$$\begin{aligned} \mathcal{J}(\vartheta(\sigma_{n+1}, L\sigma_a, \varsigma)) &\geq \mathcal{J}(\vartheta(L\sigma_n, L\sigma_a, \varsigma)) \geq \mathcal{S}\Big((\vartheta(\sigma_n, L\sigma_n, \varsigma))^v * (\vartheta(\sigma_a, L\sigma_a, \varsigma))^{1-v}\Big) \\ &> \mathcal{J}\Big((\vartheta(\sigma_n, L\sigma_n, \varsigma))^v * (\vartheta(\sigma_a, L\sigma_a, \varsigma))^{1-v}\Big) \\ &\geq \mathcal{J}\Big((\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^v * (\vartheta(\sigma_a, L\sigma_a, \varsigma))^{1-v}\Big). \end{aligned}$$

By the first part of (ii), we obtain

$$\vartheta(\sigma_{n+1}, L\sigma_a, k\varsigma) > (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^v * (\vartheta(\sigma_a, L\sigma_a, \varsigma))^{1-v}.$$

Applying limit $n \to \infty$, we obtain $\vartheta(\sigma_a, L\sigma_a, k\varsigma) \geq 1$. This implies that $\vartheta(\sigma_a, L\sigma_a, \varsigma) = 1$. Hence, $\sigma_a = L\sigma_a$. $\square$

Theorem 5. *Let $\perp$ be a TOR. Then, every $\perp$-PSM defined on a $\perp$-regular OCFMS $(\mathcal{B}, \vartheta, *, \perp)$ satisfying (11) and (i), (iii), (v)–(viii) have a fixed point in $\mathcal{B}$.*

Proof. Choose an initial guess $\sigma_0 \in \mathcal{B}$ so that $\sigma_0 \perp \sigma_1$ or $\sigma_1 \perp \sigma_0$ for every $\sigma_1 \in \mathcal{B}$, then by using the $\perp$-preservation of L, we build an OS $\{\sigma_n\}$, such that $\sigma_n = L(\sigma_{n-1}) = L^n(\sigma_0)$ and $\sigma_{n-1} \perp \sigma_n$ for every $n \in \mathbb{N}$. Note that, if $\sigma_n = L(\sigma_n)$, then σ_n is FP of L for each $n \geq 0$. Let $\sigma_n \neq \sigma_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Let $y_n = \vartheta(\sigma_n, \sigma_{n+1}, \varsigma)$ for all $n \geq 0$. By the first part of (ii) and (11), we have

$$\begin{aligned}
\mathcal{J}(\vartheta(\sigma_n, \sigma_{n+1}, \varsigma)) &\geq \mathcal{J}(\vartheta(L\sigma_{n-1}, L\sigma_n, \varsigma)) \geq \mathcal{S}\Big((\vartheta(\sigma_{n-1}, L\sigma_{n-1}, \varsigma))^{v} * (\vartheta(\sigma_n, L\sigma_n, \varsigma))^{1-v}\Big) \\
&\geq \mathcal{S}\Big((\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{v} * (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{1-v}\Big) \\
&> \mathcal{J}\Big((\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{v} * (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{1-v}\Big) \\
&\geq \mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma)).
\end{aligned} \tag{15}$$

By the inequality (15), $\{\mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))\}$ is strictly increasing. If it is not bounded above, by (v), we obtain $\sup_{\vartheta(\sigma_{n-1}, \sigma_n, \varsigma) > \varepsilon} \mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma)) > -\infty$. This implies that

$$\lim_{\vartheta(\sigma_{n-1}, \sigma_n, \varsigma) \to \varepsilon+} \sup \mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma)) > -\infty.$$

Thus, $\lim_{n\to\infty} \vartheta(\sigma_{n-1}, \sigma_n, \varsigma) = 1$, otherwise, we have

$$\lim_{\vartheta(\sigma_{n-1}, \sigma_n, \varsigma) \to \varepsilon+} \sup \mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma)) = -\infty$$

(i.e., a contradiction (v)). If it is bounded above, then $\{\mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))\}$ is a CS and by (15), $\{\mathcal{S}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))\}$ also converges to the same limit point. Thus, by (iii), we obtain $\lim_{n\to\infty} \vartheta(\sigma_{n-1}, \sigma_n, \varsigma) = 1$. Hence, L is AR.

Now, we assert that $\{\sigma_n\}$ is CS; thus, by Lemma 1, there exists $\{\sigma_{n_k}\}$, $\{\sigma_{m_k}\}$ and $\varepsilon > 0$, such that (1) and (2) examine $\vartheta(\sigma_{n_k+1}, \sigma_{m_k+1}, \varsigma) > 1 + \varepsilon$. Since $\sigma_n \perp \sigma_{n+1}$, for all $n \geq 0$; thus, by transitivity of $\perp$, we have $\sigma_{n_k} \perp \sigma_{m_k}$. Letting $g = \sigma_{n_k}$ and $e = \sigma_{m_k}$ in (11), one writes for some $k \geq 1$,

$$\begin{aligned}
\mathcal{J}\big(\vartheta\big(\sigma_{n_k+1}, \sigma_{m_k+1}, \varsigma\big)\big) &\geq \mathcal{J}(\vartheta(L\sigma_{n_k}, L\sigma_{m_k}, \varsigma)) \geq \mathcal{S}\Big((\vartheta(\sigma_{n_k}, L\sigma_{n_k}, \varsigma))^{v} * (\vartheta(\sigma_{m_k}, L\sigma_{m_k}, \varsigma))^{1-v}\Big) \\
&\geq \mathcal{S}\Big(\big(\vartheta\big(\sigma_{n_k}, \sigma_{n_k+1}, \varsigma\big)\big)^{v} * \big(\vartheta\big(\sigma_{m_k}, \sigma_{m_k+1}, \varsigma\big)\big)^{1-v}\Big).
\end{aligned}$$

If $\sigma_k = \vartheta\big(\sigma_{n_k+1}, \sigma_{m_k+1}, \varsigma\big)$, $\sigma_{b_k} = \vartheta\big(\sigma_{n_k}, \sigma_{n_k+1}, \varsigma\big)$, $\sigma_{c_k} = \vartheta\big(\sigma_{m_k}, \sigma_{m_k+1}, \varsigma\big)$, we have

$$\mathcal{J}(\sigma_k) \geq \mathcal{S}\Big(\big(\sigma_{b_k}\big)^{v}\big(\sigma_{c_k}\big)^{1-v}\Big), \text{ for some } k \geq 1. \tag{16}$$

By (1), we have $\lim_{k\to\infty} \sigma_k = 1 + \varepsilon$ and (16) implies

$$\lim_{\sigma_a \to (1+\varepsilon)} \sup \mathcal{J}(\sigma_a) \geq \lim_{k\to\infty} \sup \mathcal{J}(\sigma_k) \geq \lim_{k\to\infty} \inf \mathcal{S}\Big(\big(\sigma_{b_k}\big)^{v}\big(\sigma_{c_k}\big)^{1-v}\Big) \geq \lim_{\sigma_a \to 0} \inf \mathcal{S}(\sigma_a). \tag{17}$$

The information obtained in (17) contradicts the assumption (viii). Thus, stamping the sequence $\{\sigma_n\}$ as OC in the OCFMS $(\mathcal{B}, \vartheta, *, \perp)$. The completeness of the space ensures the convergence of $\{\sigma_n\}$. Let it converge to $i \in \mathcal{B}$.

Case 1: If $\vartheta(\sigma_{n+1}, Li, \varsigma) = 1$ for some $n \geq 0$, then

$$\vartheta(i, Li, \varsigma) \geq \vartheta(i, \sigma_{n+1}, \varsigma) * \vartheta(\sigma_{n+1}, Li, \varsigma)$$

taking limit $n \to \infty$ on both sides, we have $\vartheta(i, Li, \varsigma) \geq 1$. This implies that $\vartheta(i, Li, \varsigma) = 1$. Hence, $i = Li$.

Case 2: For each $n \geq 0$, $\vartheta(\sigma_{n+1}, Li, \kappa) < 1$, then by the $\perp$-regularity of $\mathcal{A}$, we find $\sigma_n \perp i$ or $i \perp \sigma_n$. By (11), one can write

$$\mathcal{J}(\vartheta(\sigma_{n+1}, Li, \kappa)) \geq \mathcal{J}(\vartheta(L\sigma_n, Li, \kappa)) \geq \mathcal{S}\Big((\vartheta(\sigma_n, L\sigma_n, \kappa))^v * (\vartheta(i, Li, \kappa))^{1-v}\Big)$$

for all n ≥ 0. By taking $\sigma_n = \vartheta(\sigma_{n+1}, Li, \kappa)$ and $b_n = \vartheta(\sigma_n, \sigma_{n+1}, \kappa)$, one writes

$$\mathcal{J}(\sigma_n) \geq \mathcal{S}\Big((b_n)^v (\vartheta(i, Li, \kappa))^{1-v}\Big) \text{ for all } n \geq 0. \tag{18}$$

Take $\delta = \vartheta(i, Li, \kappa)$. Note that $\sigma_n \to \delta$ and $b_n \to 1$ as $n \to \infty$. By limits on (18), it follows

$$\limsup_{i \to \delta} \mathcal{J}(i) \geq \limsup_{n \to \infty} \mathcal{J}(\sigma_n) \geq \lim_{n \to \infty} \inf \mathcal{S}\Big((b_n)^v (\delta)^{1-v}\Big) \geq \limsup_{i \to 0} \mathcal{S}(i).$$

Thus, contradicting (v) if $\delta > 1$. Therefore, we have $\vartheta(i, Li, \kappa) = 1$. That is, i is a fixed point of L.

□

3.3. Chatarjea-Type $(\mathcal{J}, \mathcal{S})$-Orthogonal Fuzzy Interpolative Contraction

Definition 9. *Let $\mathcal{J}, \mathcal{S} : (0,1] \to \mathbb{R}$ be two functions. A mapping $L : \mathcal{B} \to \mathcal{B}$ defined on OFMS $(\mathcal{B}, \vartheta, *, \perp)$ is called a Chatarjea-type $(\mathcal{J}, \mathcal{S})$-OFIPC, verifying*

$$\mathcal{J}(\vartheta(L\sigma, L\theta, \varsigma)) \geq \mathcal{S}\left(\sqrt{(\vartheta(\sigma, L\theta, \varsigma)) * (\vartheta(\theta, L\sigma, \varsigma))}\right), \tag{19}$$

for each $(\sigma, \theta) \in \mathcal{B}$, $\vartheta(L\sigma, L\theta, \varsigma) > 0$.

Theorem 6. *Let $\perp$ be a TOR. Then, every $\perp$-PSM defined on a $\perp$-regular OCFMS $(\mathcal{B}, \vartheta, *, \perp)$ verifying (19) and (i)–(iv), has a fixed point in $\mathcal{B}$.*

Proof. By following the starting steps taken in proof of Theorem 4, we have

$$\begin{aligned}
\mathcal{J}(y_n) &\geq \mathcal{J}(\vartheta(L\sigma_{n-1}, L\sigma_n, \varsigma)) \geq \mathcal{S}\left(\sqrt{(\vartheta(\sigma_{n-1}, L\sigma_n, \varsigma)) * (\vartheta(\sigma_n, L\sigma_{n-1}, \varsigma))}\right) \\
&\geq \mathcal{S}\left(\sqrt{(\vartheta(\sigma_{n-1}, L\sigma_n, \varsigma)) * (\vartheta(\sigma_n, \sigma_n, \varsigma))}\right) \\
&\geq \mathcal{S}\left(\sqrt{(\vartheta(\sigma_{n-1}, L\sigma_n, \varsigma))}\right) \\
&\geq \mathcal{S}\left(\sqrt{\vartheta(\sigma_{n-1}, \sigma_{n+1}, \varsigma)}\right) \qquad (20) \\
&\geq \mathcal{S}\left(\sqrt{(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma)) * (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))}\right). \qquad (21)
\end{aligned}$$

Suppose that $\vartheta(\sigma_{n-1}, \sigma_n, \varsigma) > \vartheta(\sigma_n, \sigma_{n+1}, \varsigma)$ for some $n \geq 1$. Then, by (21) and (ii), we obtain

$$\mathcal{J}(y_n) \geq \mathcal{S}(y_n) > \mathcal{J}(y_n). \tag{22}$$

The information obtained in (22) contradicts the definition of $\mathcal{J}$; therefore, we go with

$$\mathcal{J}(y_n) \geq \mathcal{S}(y_n) > \mathcal{J}(y_n), \text{ for all } n \geq 1.$$

Next, by the proof of Theorem 4, we reach the statement $\sigma_n \to o$ as $n \to \infty$. Then, by taking the support of the $\perp$-regularity of the space $(\mathcal{B}, \vartheta, *, \perp)$, we achieve $\sigma_n \perp o$ or $o \perp \sigma_n$. We must have $\vartheta(o, Lo, \varsigma) = 1$. Letting $\vartheta(\sigma_{n+1}, Lo, \varsigma) < 1$ and using (19), we obtain

$$\begin{aligned}\mathcal{J}(\vartheta(\sigma_{n+1}, Lo, \varsigma)) &\geq \vartheta(L\sigma_n, Lo, \varsigma) \geq \mathcal{S}\left(\sqrt{(\vartheta(\sigma_n, Lo, \varsigma)) * (\vartheta(o, L\sigma_n, \varsigma))}\right) \\ &\geq \mathcal{S}\left(\sqrt{(\vartheta(\sigma_n, Lo, \varsigma)) * (\vartheta(o, \sigma_{n+1}, \varsigma))}\right) \\ &> \mathcal{J}\left(\sqrt{(\vartheta(\sigma_n, Lo, \varsigma)) * (\vartheta(o, \sigma_{n+1}, \varsigma))}\right).\end{aligned}$$

Given that the function $\mathcal{J}$ satisfies assumption (ii), thus

$$\vartheta(\sigma_{n+1}, Lo, \varsigma) > \sqrt{(\vartheta(\sigma_n, Lo, \varsigma)) * (\vartheta(o, \sigma_{n+1}, \varsigma))}.$$

The last inequality implies that $\vartheta(o, Lo, \varsigma) \geq \sqrt{\vartheta(o, Lo, \varsigma)}$ (for large n). Hence, $\vartheta(o, Lo, \varsigma) = 1$, or $o = Lo$. □

Theorem 7. *Let $\perp$ be a TOR. Then, every $\perp$-PSM defined on a $\perp$-regular OCFMS $(\mathcal{B}, \vartheta, *, \perp)$ verifying (19), (i), (iii), and (v)–(viii) has a fixed point in $\mathcal{B}$.*

Proof. By following the steps taken in the proof of Theorems 5 and 6, we achieve the objective. □

3.4. Ciric–Reich–Rus-Type $(\mathcal{J}, \mathcal{S})$-Orthogonal Fuzzy Interpolative Contraction

Definition 10. *Let $\mathcal{J}, \mathcal{S} : (0, 1] \to \mathbb{R}$ be two functions. A mapping $L : \mathcal{B} \to \mathcal{B}$ defined on OFMS $(\mathcal{B}, \vartheta, *, \perp)$ is called a Ciric–Reich–Rus-type $(\mathcal{J}, \mathcal{S})$-OFIPC, if there exists $\nu, \eta \in [0, 1)$ verifying*

$$\mathcal{J}(\vartheta(L\sigma, L\theta, \varsigma)) \geq \mathcal{S}\left((\vartheta(\sigma, \theta, \varsigma))^{v} * (\vartheta(\sigma, L\sigma, \varsigma))^{\eta} * (\vartheta(\theta, L\theta, \varsigma))^{1-v-\eta}\right), \quad (23)$$

for each $(\sigma, \theta) \in \mathcal{B}$, $\vartheta(\sigma, L\sigma, \varsigma) > 0$ where $v + \eta < 1$.

The requirements for the presence of a fixed-point of the Ciric–Reich–Rus-type $(\mathcal{J}, \mathcal{S})$-OFIPC are stated in the following two theorems.

Theorem 8. *Let $\perp$ be a TOR. Then, every $\perp$-PSM defined on a $\perp$-regular OCFMS $(\mathcal{B}, \vartheta, *, \perp)$ verifying (23) and (i)–(iv), admits a fixed point in $\mathcal{B}$.*

Proof. By following the starting steps taken in the proof of Theorem 4, we have

$$\begin{aligned}\mathcal{J}(y_n) &\geq \mathcal{J}(\vartheta(L\sigma_{n-1}, L\sigma_n, \varsigma)) \\ &\geq \mathcal{S}\left((\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{v} * (\vartheta(\sigma_{n-1}, L\sigma_{n-1}, \varsigma))^{\eta} * (\vartheta(\sigma_n, L\sigma_n, \varsigma))^{1-v-\eta}\right) \\ &\geq \mathcal{S}\left((\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{v} * (\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{\eta} * (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{1-v-\eta}\right) \\ &\geq \mathcal{S}\left((\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{v-\eta} * (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{1-v-\eta}\right) \\ &> \mathcal{J}\left((\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{v-\eta} * (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{1-v-\eta}\right).\end{aligned} \quad (24)$$

By (24) and the monotonicity of $\mathcal{J}$, we obtain

$$(y_n)^{v+\eta} \geq (y_{n-1})^{v+\eta}, \text{ for all } n \geq 1.$$

Next, by taking steps as in Theorem 4, we obtain $\sigma_n \to t$ as $n \to \infty$, and with the support of the $\perp$-regularity of $(\mathcal{B}, \vartheta, *, \perp)$, we have $\sigma_n \perp t$ or $t \perp \sigma_n$. We need to prove $\vartheta(t, Lt, \varsigma) = 1$. Letting $\vartheta(\sigma_{n+1}, Lt, \varsigma) < 1$ and using (23), we obtain

$$\begin{aligned}\mathcal{J}(\vartheta(\sigma_{n+1}, Lt, \varsigma)) &\geq \mathcal{J}(\vartheta(L\sigma_n, Lt, \varsigma))\\ &\geq \mathcal{S}\Big((\vartheta(\sigma_n, t, \varsigma))^{v} * (\vartheta(\sigma_n, L\sigma_n, \varsigma))^{\eta} * (\vartheta(t, Lt, \varsigma))^{1-v-\eta}\Big)\\ &\geq \mathcal{S}\Big((\vartheta(\sigma_n, t, \varsigma))^{v} * (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{\eta} * (\vartheta(t, Lt, \varsigma))^{1-v-\eta}\Big)\\ &> \mathcal{J}\Big((\vartheta(\sigma_n, t, \varsigma))^{v} * (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{\eta} * (\vartheta(t, Lt, \varsigma))^{1-v-\eta}\Big).\end{aligned}$$

Using (ii), we obtain

$$\vartheta(\sigma_{n+1}, Lt, \varsigma) > (\vartheta(\sigma_n, t, \varsigma))^{v} * (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{\eta} * (\vartheta(t, Lt, \varsigma))^{1-v-\eta}.$$

Now, for large n, the last inequality implies that $\vartheta(t, Lt, \varsigma) \geq 1$. Hence, $\vartheta(t, Lt, \varsigma) = 1$, or $t = Lt$. □

Theorem 9. *Suppose $\perp$ is a TOR. Then, every $\perp$-PSM defined on a $\perp$-regular OCFMS $(\mathcal{B}, \vartheta, *, \perp)$ verifying (23), (i), (iii), and (v)–(viii), has a fixed point in $\mathcal{B}$.*

Proof. By following the steps taken in the proof of Theorems 5 and 8, we complete the proof of Theorem 9. □

3.5. Hardy–Rogers-Type $(\mathcal{J}, \mathcal{S})$-Orthogonal Fuzzy Interpolative Contraction

Definition 11. *Let $\mathcal{J}, \mathcal{S} : (0, 1] \to \mathbb{R}$ be two functions. A mapping $L : \mathcal{B} \to \mathcal{B}$ defined on OFMS $(\mathcal{B}, \vartheta, *, \perp)$ is called a Hardy–Rogers-type $(\mathcal{J}, \mathcal{S})$-OFIPC, if there exists $v, \eta, \gamma, \delta \in [0, 1)$, verifying*

$$\mathcal{J}(\vartheta(L\sigma_a, L\sigma_b, \varsigma)) \geq \mathcal{S}\left(\begin{array}{c} (\vartheta(\sigma, \theta, \varsigma))^{v} * (\vartheta(\sigma, L\sigma, \varsigma))^{\eta} * (\vartheta(\theta, L\theta, \varsigma))^{\gamma} \\ *(\vartheta(\sigma, L\theta, \varsigma))^{\delta} * (\vartheta(\theta, L\sigma, \varsigma))^{1-v-\eta-\gamma-\delta} \end{array} \right), \tag{25}$$

for each $(\sigma, \theta) \in \mathcal{B}$, $\vartheta(L\sigma, L\theta, \varsigma) > 0$ where $v + \eta + \gamma + \delta < 1$.

Example 5. *Let $\mathcal{B} = [1, 7)$ and define the FMS $\vartheta(\sigma, \theta, \varsigma) = e^{-\frac{|\sigma-\theta|}{\varsigma}}$, where $\perp \subset \mathcal{B} \times \mathcal{B}$ is defined by*

$$\sigma \perp \theta \text{ if } \sigma\theta \leq \sigma \vee \theta \text{ for } \sigma \neq \theta.$$

Then, $(\mathcal{B}, \vartheta, *, \perp)$ is an OFMS with $m * n = mn$. Define $L : \mathcal{B} \to \mathcal{B}$ by

$$L(\sigma_a) = \left\{ \begin{array}{c} 5, \text{if } \sigma = 1 \\ \sigma - 1 \text{ otherwise} \end{array} \right\}.$$

Define $\mathcal{J}, \mathcal{S} : (0, 1] \to \mathbb{R}$ by

$$\mathcal{J}(t) = \left\{ \begin{array}{c} \frac{1}{\ln t} \text{ if } 0 < t < 1 \\ 1 \text{ if } t = 1 \end{array} \right\} \text{and } \mathcal{S} = \left\{ \begin{array}{c} \frac{1}{\ln t^2} \text{ if } 0 < t < 1 \\ 2 \text{ if } t = 1 \end{array} \right\}$$

Case 1: Let, L be a Hardy–Rogers-type $(\mathcal{J}, \mathcal{S})$-OFIPC. Then,

$$\begin{array}{lcc} \vartheta(L2, L1, k1) & \geq & \begin{array}{c}(\vartheta(2,1,1))^{0.01}(\vartheta(2,L2,1))^{0.02}(\vartheta(1,L1,1))^{0.03} \\ (\vartheta(2,L1,1))^{0.04}(\vartheta(1,L2,1))^{1-0.01-0.02-0.03-0.04}\end{array} \\ \vartheta\left(1,5,1\frac{1}{2}\right) & \geq & \begin{array}{c}(\vartheta(2,1,1))^{0.01}(\vartheta(2,1,1))^{0.02}(\vartheta(1,5,1))^{0.03} \\ (\vartheta(2,5,1))^{0.04}(\vartheta(1,1,1))^{0.9}\end{array} \\ \left(e^{-\frac{|1-5|}{0.5}}\right) & \geq & \begin{array}{c}\left(e^{-\frac{|1-2|}{1}}\right)^{0.01}\left(e^{-\frac{|1-2|}{1}}\right)^{0.02}\left(e^{-\frac{|1-5|}{1}}\right)^{0.03} \\ \left(e^{-\frac{|2-5|}{1}}\right)^{0.04}(1)^{0.9}\end{array} \\ 0.0003 & \geq & 0.7632. \end{array}$$

However, this is a contradiction. Thus, L is not a Hardy–Rogers-type OFIPC.

Case 2: Let L be a Hardy–Rogers-type $(\mathcal{J}, \mathcal{S})$-OFIPC. Then,

$$\begin{aligned}
\vartheta(L3, L1, k1) &\geq \left(\begin{array}{c} (\vartheta(3,1,1))^{0.01}(\vartheta(3,L3,1))^{0.02}(\vartheta(1,L1,1))^{0.03} \\ (\vartheta(3,L1,1))^{0.04}(\vartheta(1,L3,1))^{1-0.01-0.02-0.03-0.04} \end{array} \right) \\
e^{-\frac{|2-5|}{0.5}} &\geq \begin{array}{c} \left(e^{-\frac{|1-3|}{1}}\right)^{0.01}\left(e^{-\frac{|1-5|}{1}}\right)^{0.02}\left(e^{-\frac{|2-5|}{1}}\right)^{0.03} \\ \left(e^{-\frac{|3-5|}{1}}\right)^{0.04}\left(e^{-\frac{|1-2|}{1}}\right)^{0.9} \end{array} \\
0.0025 &\geq 0.3104.
\end{aligned}$$

This is a contradiction. Thus, L is not a Hardy–Rogers-type OFIPC.

The requirements for the presence of a fixed-point of the Hardy–Rogers-type $(\mathcal{J}, \mathcal{S})$-OFIPC is stated in the following two theorems.

Theorem 10. *Let $\perp$ be a TOR. Then, every $\perp$-PSM defined on a $\perp$-regular OCFMS $(\mathcal{B}, \vartheta, *, \perp)$ verifying (25) and (i)–(iv) has a fixed point in $\mathcal{B}$.*

Proof. Assume $\sigma_0 \in \mathcal{B}$, such that $\sigma_0 \perp \sigma_1$ or $\sigma_1 \perp \sigma_0$ for every $\sigma_1 \in \mathcal{B}$; then, by utilizing the $\perp$-preservation of L, we build an OS $\{\sigma_n\}$, such that $\sigma_n = L(\sigma_{n-1}) = L^n(\sigma_0)$ and $\sigma_{n-1} \perp \sigma_n$ for every $n \in \mathbb{N}$. Note that, if $\sigma_n = L(\sigma_n)$, then σ_n is FP of L for each $n \geq 0$. Let $\sigma_n \neq \sigma_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Let $y_n = \vartheta(\sigma_n, \sigma_{n+1}, \varsigma)$ for all $n \geq 0$. By the first part of (ii) and (25), we obtain

$$\begin{aligned}
\mathcal{J}(y_n) &\geq \mathcal{J}(\vartheta(L\sigma_{n-1}, L\sigma_n, \varsigma)) \\
&\geq \mathcal{S}\left(\begin{array}{c} (\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{v} * (\vartheta(\sigma_{n-1}, L\sigma_{n-1}, \varsigma))^{\eta} * (\vartheta(\sigma_n, L\sigma_n, \varsigma))^{\gamma} \\ *(\vartheta(\sigma_{n-1}, L\sigma_n, \varsigma))^{\delta} * (\vartheta(\sigma_n, L\sigma_{n-1}, \varsigma))^{1-v-\eta-\gamma-\delta} \end{array} \right) \\
&\geq \mathcal{S}\left(\begin{array}{c} (\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{v} * (\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{\eta} * (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{\gamma} \\ *(\vartheta(\sigma_{n-1}, \sigma_{n+1}, \varsigma))^{\delta} * (\vartheta(\sigma_n, \sigma_n, \varsigma))^{1-v-\eta-\gamma-\delta} \end{array} \right) \\
&\geq \mathcal{S}\left(\begin{array}{c} (\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{v} * (\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{\eta} * (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{\gamma} \\ *(\vartheta(\sigma_{n-1}, \sigma_{n+1}, \varsigma))^{\delta} * (1)^{1-v-\eta-\gamma-\delta} \end{array} \right) \\
&\geq \mathcal{S}\left(\begin{array}{c} (\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{v} * (\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{\eta} * (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{\gamma} \\ *(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{\delta} * (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{\delta} \end{array} \right) \\
&\geq \mathcal{S}\left((y_{n-1})^{v+\eta+\delta} (y_n)^{\gamma+\delta} \right) \\
&> \mathcal{J}\left((y_{n-1})^{v+\eta+\delta} (y_n)^{\gamma+\delta} \right).
\end{aligned} \tag{26}$$

Suppose that $y_n > y_{n-1}$ for some $n \geq 1$. By the monotonicity of $\mathcal{J}$ and (26), we have $(y_n)^{\gamma+\delta} > (y_n)^{\gamma+\delta}$. This is not possible. Consequently, we obtain $y_n > y_{n-1}$ for each $n \geq 1$. Next, by following the steps as taken in Theorem 4, we deduce $\sigma_n \to u$ as $n \to \infty$,, with the support of the $\perp$-regularity of $(\mathcal{B}, \vartheta, *, \perp)$. Then, we have $\sigma_n \perp u$ or $u \perp \sigma_n$. We must prove that $\vartheta(u, Lu, \varsigma) = 1$. Letting $\vartheta(\sigma_{n+1}, Lu, \varsigma) < 1$ and using (25), we obtain

$$\begin{aligned}
\mathcal{J}(\vartheta(\sigma_{n+1}, Lu, \varsigma)) &\geq \mathcal{J}(\vartheta(L\sigma_n, Lu, \varsigma)) \\
&\geq \mathcal{S}\left(\begin{array}{c} (\vartheta(\sigma_n, u, \varsigma))^{v} * (\vartheta(\sigma_n, L\sigma_n, \varsigma))^{\eta} * (\vartheta(u, Lu, \varsigma))^{\gamma} \\ *(\vartheta(\sigma_n, Lu, \varsigma))^{\delta} * (\vartheta(u, L\sigma_n, \varsigma))^{1-v-\eta-\gamma-\delta} \end{array} \right) \\
&\geq \mathcal{S}\left(\begin{array}{c} (\vartheta(\sigma_n, u, \varsigma))^{v} * (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{\eta} * (\vartheta(u, Lu, \varsigma))^{\gamma} \\ *(\vartheta(\sigma_n, Lu, \varsigma))^{\delta} * (\vartheta(u, \sigma_{n+1}, \varsigma))^{1-v-\eta-\gamma-\delta} \end{array} \right) \\
&> \mathcal{J}\left(\begin{array}{c} (\vartheta(\sigma_n, u, \varsigma))^{v} * (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{\eta} * (\vartheta(u, Lu, \varsigma))^{\gamma} \\ *(\vartheta(\sigma_n, Lu, \varsigma))^{\delta} * (\vartheta(u, \sigma_{n+1}, \varsigma))^{1-v-\eta-\gamma-\delta} \end{array} \right).
\end{aligned}$$

Using (ii), we obtain

$$\vartheta(\sigma_{n+1}, Lu, \varsigma) > \left(\begin{array}{c} (\vartheta(\sigma_n, u, \varsigma))^{v} * (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{\eta} * (\vartheta(u, Lu, \varsigma))^{\gamma} \\ *(\vartheta(\sigma_n, Lu, \varsigma))^{\delta} * (\vartheta(u, \sigma_{n+1}, \varsigma))^{1-v-\eta-\gamma-\delta} \end{array} \right).$$

Then, for large n, the last inequality implies that $\vartheta(u, Lu, \varsigma) \geq 1$. Hence, $\vartheta(u, Lu, \varsigma) = 1$, or $u = Lu$. $\square$

Theorem 11. *Let $\perp$ be a TOR. Then, every $\perp$-PSM defined on a $\perp$-regular OCFMS $(\mathcal{B}, \vartheta, *, \perp)$ verifying (25) and (i), (iii), (v)–(viii) has a fixed point in $\mathcal{B}$.*

Proof. By following the steps as taken in Theorems 5 and 10, the proof is obvious. $\square$

4. Applications

In this section, we discuss the applications of fractional differential equations and Volterra-type Fredholm integral equations.

4.1. An Application to Fractional Differential Equation

A variety of useful fractional differential features is postulated and searched by Lacroix (1819). Caputo and Fabrizio announced [19] a new fractional technique in 2015. The need to characterize a class of non-local systems that cannot be properly represented by traditional local theories or fractional models with a singular kernel [19] sparked interest in this description. The different kernels that can be selected to satisfy the requirements of different applications are the fundamental difference among fractional derivatives. The Caputo fractional derivative [20], the Cauto–Fabrizio derivative [19], and the Atangana–Baleanu fractional derivative [16], for example, are determined by power laws, the Caputo–Fabrizio derivative by an exponential decay law, and the Atangana–Baleanu derivative by the Mittag–Leffler law. A variety of new Caputo–Fabrizio (CFD) models were lately investigated in [15,17,18].

In OFMSs, we will look at one of these models (represent $C_{(I,\mathbb{R})}$ by $\Bbbk$).

Let $\vartheta : \Bbbk^2 \to [1, \infty)$ be defined by

$$\vartheta(u, v, \varsigma) = e^{-\frac{\|u-v\|}{\varsigma}} = e^{-\sup_{l \in I} \frac{|u(l)-v(l)|}{\varsigma}}, \text{ for all } u, v \in C_{(I,\mathbb{R})}.$$

Then, $(\Bbbk, \vartheta, \varsigma)$ is a complete fuzzy metric space, where $I = [0, 1]$ and

$$\Bbbk = \{u | u : I \to \mathbb{R} \text{ and } u \text{ is continuous}\}.$$

The relation $\perp$ on $\Bbbk$ is as follows:

$$u \perp v \text{ iff } u(l)v(l) \geq u(l) \vee v(l), \text{ for all } u, v \in C_{(I,\mathbb{R})},$$

is an orthogonal relation and $(\Bbbk, \vartheta, *, \perp)$ is an OCFMS. Let the function $K_1 : I \times \mathbb{R} \to \mathbb{R}$ be taken as $K_1(s, r) \geq 0$ for all $s \in I$ and $\tau \geq 0$. We shall apply Theorem 2 to resolve the following CFDE:

$$^{C}D^{v}w(s) = K_1(s, w(s)); w \in C_{(I,\mathbb{R})} : \tag{27}$$

$$W(0) = 0, Iw(1) = w'(0).$$

We denote CFD of order v by $^{C}D^{v}$ and for $v \in (m-1, m)$; $m = [v] + 1$, we have

$$^{C}D^{v}w(s) = \frac{1}{\Gamma(m-v)} \int_0^s (s-z)^{m-v-1} w(z) \sigma_d z.$$

The notation $I^v w$ is interpreted, as follows:

$$w(s) = \frac{1}{\Gamma(m-v)}\int_0^s (s-z)^{v-1} K_1(z, w(z))\sigma_d z + \frac{s}{\Gamma(m-v)}\int_0^1\int_0^z (z-p)^{v-1} K_1(p, w(p))\sigma_d p\sigma_d z.$$

For the mapping $K_1 : I \times \mathbb{R} \to \mathbb{R}$ and $u_0 \in \Bbbk$, we state the following conditions:

(A) For $\tau \geq 0$, let

$$| K_1(s, w(s)) - K_1(s, u(s)) | \leq \frac{\Gamma(v+1)}{\Gamma(v)} | w(s) - u(s) |,$$

for each $w, u \in \Bbbk$ following the order $w \perp u$.

(B) There exists $u_0 \in \Bbbk$, such that

$$\begin{aligned} u_0(s) &\leq \frac{1}{\Gamma(v)}\int_0^s (s-z)^{v-1} K_1(z, w_0(z))\sigma_d z \\ &+ \frac{l}{\Gamma(v)}\int_0^1\int_0^z (z-p)^{v-1} K_1(p, u_0(p))\sigma_d p\sigma_d z. \end{aligned}$$

We noticed that $K_1 : I \times \mathbb{R} \to \mathbb{R}$ is not necessarily Lipschitz continuous.
For instance, K_1 is given by

$$K_1(s, w(s)) = sw(s) \text{ if } w(s) \leq \frac{1}{2}, 0 \text{ if } w(s) > \frac{1}{2}.$$

Following (A), K_1 is not continuous and monotone. Moreover, for $s = e^{-\tau}\Gamma(v+1)$,

$$\vartheta(K_1(s, w(s)), K_1(t, u(t)), \varsigma) = e^{-\frac{|K_1(l,\frac{1}{2}) - K_1(l,\frac{2}{3})|}{\varsigma}} = e^{\frac{s}{2\varsigma}} \geq e^{\frac{s}{6\varsigma}} = e^{-s\frac{|\frac{1}{2}-\frac{1}{3}|}{\varsigma}} = \vartheta(w, u, \varsigma).$$

Theorem 12. *Let the mappings $K_1 : I \times \mathbb{R} \to \mathbb{R}$ and $u_0 \in C_{(I,\mathbb{R})}$ satisfy the conditions (A)–(B). Then, the Equation (23) admits a solution in $\Bbbk$.*

Proof. Let $X = \left\{ J \in C_{(I,\mathbb{R})} : J(s) \geq 0 \text{ for all } s \in I \right\}$ and define $\Psi : X \to X$ by

$$(\Psi J)(s) = \frac{1}{\Gamma(v)}\int_0^s (s-z)^{v-1} K_1(z, J(z))\sigma_d z + \frac{s}{\Gamma(v)}\int_0^1\int_0^z (z-p)^{v-1} K_1(p, J(p))\sigma_d p\sigma_d z.$$

We define an orthogonal relation $\perp$ on X by

$$u \perp v \text{ iff } u(s)v(s) \geq u(s)v(s), \text{ for each } u, v \in X.$$

According to above conditions ,Ψ is preserving and there is $u_0 \in \Bbbk$ verifying (B) such that $u_n = \mathbb{R}^n(u_0)$ with $u_n \perp u_{n+1}$ or $u_{n+1} \perp u_n$ for each $n \geq 0$. we work on the validation of (3) in the next lines.

$$\vartheta((\Psi J)(s), \Psi(U)(s), \varsigma) = \exp\left(\begin{array}{c} \sup | \frac{1}{\Gamma(v)}\int_0^s (s-z)^{v-1} K_1(z, J(z))\sigma_d z \\ -\frac{1}{\Gamma(v)}\int_0^s (s-z)^{v-1} K_1(z, U(z))\sigma_d z \\ +\frac{s}{\Gamma(v)}\int_0^1\int_0^z (p-z)^{v-1} K_1(p, J(p))\sigma_d p\sigma_d z \\ -\frac{s}{\Gamma(v)}\int_0^1\int_0^z (p-z)^{v-1} K_1(p, U(p))\sigma_d p\sigma_d z \end{array} \right)$$

$$\geq \exp\left(\sup_{s,z\in I} \left\{ \begin{array}{c} \frac{1}{\Gamma(v)}\Gamma(v+1)\int_0^s (s-z)^{v-1}\frac{|J(z)-U(z)|}{\varsigma}\sigma_d z \\ -\frac{s}{\Gamma(v)}\Gamma(v+1)\int_0^1\int_0^z (p-z)^{v-1}\frac{|J(z)-U(z)|}{\varsigma}\sigma_d z\sigma_d p \end{array} \right\} \right)$$

$$\geq \exp\left(\begin{array}{c}\frac{1}{\Gamma(v)}\Gamma(v+1)\sup_{z\in I}\frac{|J(z)-U(z)|}{\varsigma}\\ \sup_{s\in I}\left\{\int_0^s(s-z)^{v-1}\mid\sigma_d z - s\int_0^1\int_0^z(p-z)^{v-1}\sigma_d z\sigma_d p\right\}\end{array}\right)$$

$$\geq \exp\left(\begin{array}{c}\frac{\Gamma(v)\Gamma(v+1)}{\Gamma(v)\Gamma(v+1)}\sup_{z\in I}\frac{|J(z)-U(z)|}{\varsigma}\\ -sB(v+1,1)\frac{\Gamma(v)\Gamma(v+1)}{\Gamma(v)\Gamma(v+1)}\sup_{s,z\in I}\frac{|J(z)-U(z)|}{\varsigma}\end{array}\right)$$

$$\geq \exp(1-sB(v+1,1))\sup_{s,z\in I}\frac{\mid J(z)-U(z)\mid}{\varsigma}$$

$$\geq \exp\left((1-sB(v+1))\sup_{s,z\in I}\frac{\mid J(z)-U(z)\mid}{\varsigma}\right)$$

$$= \left(\exp\left(\sup_{s,z\in I}\frac{\mid J(z)-U(z)\mid}{\varsigma}\right)\right)^{1-sB(v+1,1)}$$

$$= (\vartheta(J(z,U(z),\varsigma))^{1-sB(v+1,1)};\ \text{where } B \text{ is a beta function.}$$

By defining $\mathcal{J}(w) = \ln(w)$ and $\mathcal{S}(w) = D\mathcal{J}(w)$; $w > 0, \tau > 0$, and putting $1 - sB(v+1,1) = D < 1$, the last inequality has the form:

$$\mathcal{J}(\vartheta(\Psi(J)(s),\Psi(U)(s)),\tau) \geq \mathcal{S}(\vartheta(J,U,\tau)).$$

□

4.2. Application to Volterra-Type Integral Equation

There are several types of integral equations, but they are only used in the "model scientific process", in which the value, or the rate of change of the change of value, of some quantity (or quantities), depends on the past history. This opposes the present value, in which we can obtain the rate at which a quantity evolves. Just as for differential equations, the integral equation need to be "solved" to describe and predict how a physical quantity is going to behave as time passes. For solving integral equations, there are things such as Fredholm theorems, fixed point methods, boundary element methods, and Nystrom methods. In this paper, we apply Theorem 2 to demonstrate the existence of the multiplicative Volterra-type integral equation given below;

$$f(k) = \int_0^k L(k,h,f,\varsigma)\sigma_d h \tag{28}$$

for each $k \in H$ and $L : H \times H \times \Bbbk \to \mathbb{R}$. We demonstrate the existence of the solution to (27).

Let $\vartheta : \Bbbk \times \Bbbk \times (0,\infty) \to \mathbb{R}$ be defined as

$$\vartheta(u,v,\varsigma) = e^{-\frac{|u(l)-v(l)|}{\varsigma}}, \text{ for all } u,v \in C_{(I,\mathbb{R})}.$$

Then, $(\Bbbk,\vartheta,*)$ is a CFMS, where $I = [0,1]$ and

$$\Bbbk = \{u|u : I \to \mathbb{R} \text{ and } u \text{ is a continuous}\}.$$

The relation $\perp$ on $\Bbbk$ is as follows

$$u \perp v \text{ iff } u(l)v(l) \geq u(l) \vee v(l), \text{ for each } u,v \in C_{(I,\mathbb{R})},$$

is an orthogonal relation and $(\Bbbk,\vartheta,*,\perp)$ is an OCFMS.

The following is the existence theorem for the integral Equation (28).

Theorem 13. *Assume that the following conditions are satisfied.*

(a) *Assume that* $L : H \times H \times \Bbbk \to \mathbb{R}$ *is continuous.*

(b) *Suppose there exists* $\tau > 0$*, such that*

$$e^{-\frac{|L(k,h,f)-L(k,h,q)|m}{\varsigma}} \geq e^{-\frac{|\vartheta(f,q)-\left(\tau\left(\sqrt{\vartheta(f,q)}\right)+1\right)^2|}{\varsigma}} \tag{29}$$

for all $k, h \in [0,1]$ *and* $f, q \in C_{(I,\mathbb{R}^+)}$. *Then, the integral Equation (28) admits a solution in* $C_{(I,\mathbb{R}^+)}$.

Proof. Let $\mathbb{R} = \Bbbk$ and endow it with the relation $\perp$ and fuzzy metric space ϑ. Define the mapping $\Psi : \mathbb{R} \to \mathbb{R}$ by

$$(\Psi f)(k) = \int_0^k L(k,h,f,\varsigma)\sigma_d h \tag{30}$$

so that the fixed point of Ψ is a solution of the integral Equation (28). According to the above definitions, ψ is $\perp$-preserving; there is $u_0 \in \Bbbk$ verifying $u_n = \mathbb{R}^n(u_0)$ with $u_n \perp u_{n+1}$ or $u_{n+1} \perp u_n$ for each $n \geq 0$. We work on the validation of (3) in the next few lines. By assumption (b), we have

$$\begin{aligned}
\vartheta(\Psi(f),\Psi(q),\varsigma) &= e^{-\frac{|(\Psi f)(k)-(\Psi q)(k)|}{\varsigma}} \\
&\geq \int_0^k e^{-\frac{|(\Psi f)(k)-(\Psi q)(k)|}{\varsigma}} \sigma_d h \\
&\geq \int_0^k e^{-\frac{|\vartheta(f,q)-\left(\tau\left(\sqrt{\vartheta(f,q)}\right)+1\right)^2|}{\varsigma}} \sigma_d h \\
&= e^{-\frac{|\vartheta(f,q)-\left(\tau\left(\sqrt{\vartheta(f,q)}\right)+1\right)^2|}{\varsigma}} \int_0^k \sigma_d h \\
&= k e^{-\frac{|\vartheta(f,q)-\left(\tau\left(\sqrt{\vartheta(f,q)}\right)+1\right)^2|}{\varsigma}} \\
&= \vartheta(f,q,\varsigma)
\end{aligned}$$

Hence, by defining $\mathcal{J}(w) = \ln(w)$ and $\mathcal{S}(w) = D\mathcal{J}(w)$;

$$\mathcal{J}(\vartheta(\Psi(f),\Psi(q),\varsigma)) \geq \mathcal{S}(\vartheta(f,q,\varsigma)).$$

Thus, all the conditions of Theorem 2 are satisfied and $v = k$. Therefore, the integral Equation (28) admits, at most, one solution. □

5. Discussion and Conclusions

The study of interpolative contractions is an important research subject, with applications in optimization, functional analysis, dynamic systems, and other domains where the existence of fixed points is critical. The study of interpolative contractions, like any other mathematical idea, is evolving, and new conclusions and applications may emerge in the future. Interpolative contractions are more relaxed than strict contractions, allowing for a broader class of mappings while still ensuring the existence of fixed points. Several fixed-point theorems for interpolative contractions have been established in diverse scenarios, including metric spaces, partial metric spaces, and probabilistic metric spaces. In this paper, we studied the $(\mathcal{J},\mathcal{S})$-orthogonal fuzzy interpolative contraction proved to be a source of generalization of many well-known contractions, i.e., the Banach-type $(\mathcal{J},\mathcal{S})$-orthogonal fuzzy interpolative contraction, Kannan-type $(\mathcal{J},\mathcal{S})$-orthogonal fuzzy interpolative contraction, Chatarjea-type $(\mathcal{J},\mathcal{S})$-orthogonal fuzzy interpolative contraction, and Hardy–Rogers-type $(\mathcal{J},\mathcal{S})$-orthogonal fuzzy interpolative contraction. The methodology applied for the investigation of the fixed point of $(\mathcal{J},\mathcal{S})$-orthogonal fuzzy interpolative contraction encapsulated existing corresponding methodologies. Further, we provided several non-trivial examples with applications to integral equations and fractional differential equations to support the theory. The results extend the earlier results of [8,11–14]. This work can be extended in the framework of controlled fuzzy metric spaces, intuitionistic fuzzy metric spaces, and neutrosophic metric spaces, by increasing the number of mappings and many other contexts.

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