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Solving Integral Equations via Fixed Point Results Involving Rational-Type Inequalities

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Abstract: In this study, we establish unique and common fixed point results in the context of a complete complex-valued b-metric space using rational-type inequalities. The presented work generalizes some well-known results from the existing literature. Furthermore, to ensure the validity of the findings, we have included some examples and a section on the existence of solutions for the systems of Volterra–Hammerstein integral equations and Urysohn integral equations, respectively.

Keywords: complex valued b-metric space; fixed point; compatible mappings; rational contractions; integral equations

MSC: 34A08; 54H25; 47H10

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1. Introduction

Integral equations have received considerable attention due to their wide range of applications in many branches of engineering, economics, and mathematics. The solution of integral equations has been studied by different researchers, and one of the most significant tools for solving them is the theory of fixed points. Over the last few decades, this area has drawn the attention of many researchers due to its substantial applications in many disciplines, notably nonlinear analysis, engineering problems, and topology.

The development in the research areas mentioned above began in 1922 with the first fixed point theorem known as Banach's contraction principle [1]. In 1989, Bakhtin introduced the concept of b-metric spaces, which is a generalization of metric spaces and was a new concept at that time [2]. Since then, many significant results in b-metric spaces, such as those in [3–6], have been proven. Similarly, many common fixed point results for mappings satisfying rational-type inequalities, which are not worthwhile in cone metric spaces, have also been proven [7–10]. Furthermore, Azam et al. [11] introduced the concept of metric spaces in the complex version in 2011 and Rao and their coauthors introduced the concept of b-metric spaces in the complex plane in 2013 [12]. Different fixed point results for mappings satisfying conditions of different types of contractions in complex and complex b-metric spaces have been justified and studied [13–18]. Although the concept of a complex-valued b-metric space is more general than that of a complex-valued metric space, both have been extensively studied in the literature.

Fixed point theory is one of the prominent ways to solve integral equations. Common fixed point results in complex-valued b-metric spaces are applied to find the unique common solution of systems of integral equations. Many researchers, notably Sintunavarat et al. [19] and Rashwan and Salch [20], have used fixed point methods to put forward solutions for integral equations of Urysohn. Similarly, Pathak et al. [21] and Rashwan and Salch [20] have studied system of Volterra–Hammerstein and nonlinear integral equations using fixed point methods.

Bahadur and Sarwar in [22] also used fixed point results with the help of the (CLR) property and common (E.A) in a complex-valued metric space to solve nonlinear integral equations and investigate the unique common solution. Similarly, Khaled and Abdelkrim in [16] investigated the existence of a unique solution for Urysohn's integral equations using fixed point results for four mappings in a b-metric space in the complex plane.

This manuscript presents results for the existence and uniqueness of a common fixed point for six self-maps holding a rational-type inequality in complex-valued b-metric spaces, subject to compatibility and continuity conditions. In addition, the existence of a unique common solution is provided for the following systems of Urysohn integral equations and Volterra–Hammerstein integral equations in the complex plane.

$$\mu(z) = \varphi(z) + \int_g^h K_i\big(z, t, \mu(t)\big)dt$$

where $z \in (g, h) \subseteq R; \mu, \varphi_i \in C\big((g, h), R^\eta\big)$, $K_i : (g, h) \times (g, h) \times R^\eta \to R^\eta$, $i = 1, 2, \ldots, 6$ and

$$\mu(z) = \rho_i(z) + \gamma \int_0^x w(z, \alpha) g_i\big(\alpha, \mu(\alpha)\big)d\alpha + \upsilon \int_0^\infty \eta(z, \alpha) h_i\big(\alpha, \mu(\alpha)\big)d\alpha$$

for all $x \in (0, \infty)$, where γ, υ are real numbers, $\varrho_i \in C$ is known and $w(z, \alpha), \eta(z, \alpha), g_i\big(\alpha, \mu(\alpha)\big)$ and $h_i\big(\alpha, \mu(\alpha)\big)$, $i = 1, 2, \ldots, 6$ are real-valued measurable functions in both z and α on $(0, \infty)$.

2. Preliminaries

In this sequel, we need the following definitions and notations.

Definition 1 ([16]). *For a set of complex numbers C, a partial order $\preceq$ on C is given below: $\zeta_1 \preceq \zeta_2 \iff Real(\zeta_1) \leqslant Real(\zeta_2)$ and $Img(\zeta_1) \leqslant Img(\zeta_2)$.*

Therefore, we can say that $\zeta_1 \precsim \zeta_2$ if one of the following is necessities fulfilled:

(1) $Real(\zeta_1) = Real(\zeta_2)$,$Img(\zeta_1) < Img(\zeta_2)$;
(2) $Real(\zeta_1) < Real(\zeta_2)$,$Img(\zeta_1) = Img(\zeta_2)$;
(3) $Real(\zeta_1) < Real(\zeta_2)$,$Img(\zeta_1) < Img(\zeta_2)$;
(4) $Real(\zeta_1) = Real(\zeta_2)$,$Img(\zeta_1) = Img(\zeta_2)$.

We can say that $\zeta_1 \precnsim \zeta_2$ if $\zeta_1 \neq \zeta_2$ and one of the above conditions is satisfied and similarly $\zeta_1 < \zeta_2$ if amd only if condition (3) is fulfilled.

Definition 2 ([16]). *Let $\gamma \neq \phi$ and let $d : \gamma \times \gamma \to C$ be a complex valued metric space on C, if it fulfils the following necessities;*

(1) $0 \preceq d(\aleph, \hbar)$ for all $\aleph, \hbar \in C$ and $d(\aleph, \hbar) = 0$ if and only if $\aleph = \hbar$;
(2) $d(\aleph, \hbar) = d(\hbar, \aleph)$ for all $\aleph, \hbar \in C$;
(3) $d(\aleph, \hbar) \preceq d(\aleph, c) + d(c, \hbar)$ for all $\aleph, \hbar, c \in C$.

Then, d is known as complex valued metric on γ and (γ, d) is known as a complex valued metric space.

Example 1 ([23]). *Let $Z \doteq C$ be a set of complex numbers defined $d; Z \times Z \to C$ by $d(\kappa_1, \kappa_2) =| a_1 - a_2 | +\imath | b_1 - b_2 |$ where $\kappa_1 = a_1 + \imath b_1$ and $\kappa_2 = a_2 + \imath b_2$. Then, (C, d) would be a complex valued metric space.*

Example 2 ([16]). *Let $Z \doteq C$ define mapping $d : Z \times Z \to C$ by $d(\kappa_1, \kappa_2) = e^{\imath x} | \kappa_1 - \kappa_2 |$ where $x \in (0, \pi/2)$. Then, (Z, d) is said to be a complex valued metric space.*

Definition 3 ([16]). *For a provided real number $b \geqslant 1$ and a nonempty set $\mathbb{Z}$, a function d_b: $\mathbb{Z} \times \mathbb{Z} \to C$ is termed as complex valued b-metric on $\mathbb{Z}$. If for all $\hbar, \ell, \wp \in \mathbb{Z}$ the following necessities are fulfilled.*

(1) $d_b(\hbar, \wp) = 0$ if and only if $\hbar = \wp$, $\forall\ \hbar, \wp \in \mathbb{Z}$;

(2) $d_b(\hbar,\wp) \geq 0$ *for all* $\hbar,\wp \in \mathbb{Z}$*;*
(3) $d_b(\hbar,\wp) = d_b(\wp,\hbar)$ *for all* $\hbar,\wp \in \mathbb{Z}$*;*
(4) $d_b(\hbar,\wp) \preceq b[d_b(\hbar,\ell) + d_b(\ell,\wp)]$ *for all* $\hbar,\wp,\ell \in \mathbb{Z}$*.*

Example 3 ([16]). *Let* $Z \doteq [0,1]$ *define* $d_b : Z \times Z \to C$ *by* $d_b(\nu_1,\nu_2) = \mid \nu_1 - \nu_2 \mid^2 + \iota \mid \nu_1 - \nu_2 \mid^2$ *for all* $\nu_1,\nu_2,\nu_3 \in Z$*.*

Then, (Z,d_b) *is a complex valued b-metric space with* $b \doteq 2$*.*

Definition 4 ([16]). *Suppose* (Z,d_b) *is a complex valued b-metric space; then, a sequence* s_η *is a Cauchy sequence if for every* $0 \prec \varepsilon \in Z$*, there exists a positive number* δ *such that* $\eta,\mu \geqslant \delta$ *implies* $d_b(s_\eta - s_\mu) \prec \varepsilon$*.*

Definition 5 ([16]). *A sequence* (s_η) *in a complex valued b-metric space* (Z,d_b) *will converge to* $\alpha \in Z$ *if for a given* $\varepsilon \succ 0$*, there exists a positive integer* δ *depending on* ε *such that* $d_b(s_\eta,\alpha) \prec \varepsilon$ *whenever* $\eta \geqslant \delta$*.*

Definition 6 ([16]). *If every cauchy sequence in Z converges, then the space* (Z,d_b) *will be declared as a complete complex valued b-metric space.*

Example 4 ([24]). *Let* $Z = \mathbb{C}$*. Define a function* $d_b : Z \times Z \to \mathbb{C}$ *such that* $d_b(\kappa_1,\kappa_2) = |\varsigma_1 - \varsigma_2|^2 + i|\varrho_1 - \varrho_2|^2$*, where* $\kappa_1 = \varsigma_1 + i\varrho_1$ *and* $\kappa_2 = \varsigma_2 + i\varrho_2$*.*

Then, (Z,d_b) *is a complete complex valued b-metric space with* $b = 2$*.*

Definition 7 ([16]). *Two self mappings H and T of a complex valued b-metric space* (Z,d) *would be declared compatible if these mandatory requirements are fulfilled.*

$\lim_{n\to\infty} d(HTs_\eta, THs_\eta) = 0$*;*
Whenever for a sequence s_η *in Z;*
$\lim_{n\to\infty} Ts_\eta = \lim_{n\to\infty} Hs_\eta = k$ *for some* $k \in Z$*.*

Definition 8 ([25]). *A positive term series* Σs_η *such that* $\lim_{\eta\to\infty}(s_\eta)^{\frac{1}{\eta}} = K$*,*

(a) For $K < 1$*; the series converges;*
(b) For $K > 1$*; the series diverges;*
(c) The test fails and does not provide any proper information if $K = 1$*.*

Theorem 1 ([26]). *(1) If* $\Delta(z)$ *is a complex function and it is analytic on a simple closed curve, then* $\int_C \Delta(z)dz = 0$*.*

(2) If $\Delta(z)$ *is an analytic function in a closed curve C and if 'k' is any point contained in C, then* $\Delta(k) = \frac{1}{2\pi\iota}\int_C \frac{\Delta(z)}{z-k}dz$*.*

Example 5. $\int_C \frac{e^{k^2}}{k-2}dk$ *, where C is a curve.*

Let $\Delta(k) = e^{k^2}$*. Thus, C is a simple closed curve and* $k = 2$ *is inside C. Thus, the solution is* $2\pi\iota\Delta(2) = 2\pi\iota e^4$*.*

Lemma 1 ([27]). *Let* (s_η) *be a sequence of real numbers and let* $(s_\eta) \to K$*. If* $\mho$ *is a continuous function at K and it is defined for all* s_η*, then* $\mho(s_\eta) \to \mho(K)$*.*

3. Main Results

In this section, we present the proof of a common fixed point theorem for six mappings in a complex-valued b-metric space. Additionally, we provide examples and applications based on the theorem. The first new result is presented below:

Theorem 2. *Let (Z,d) be a complex valued b-metric space and A,W,C,T,N,Q: Z* $\to$ *Z be six self mappings fulfilling the following necessities;*

(CM1) $A(Z) \subseteq T(Z)$, $A(Z) \subseteq N(Z)$, $W(Z) \subseteq C(Z)$ and $W(Z) \subseteq Q(Z)$.
(CM2) $d(A_s, W_m) \precsim \frac{\rho}{b^2} R(s,m)$, if $b \geqslant 1$ and $\rho \in (0,1)$ for all $s, m \in Z$.
where

$$R(s,m) = \max\left\{d(Ns,Tm), d(As,Cm), d(Am,Nm), d(Wm,Qm), \frac{d(As,Ts)d(Ws,Ns)}{1+d(Cs,Qs)}\right\}.$$

(CM3) The pairs (N,A), (C,A), (T,A), (A,Q) and (W,A) are compatible.
(CM4) N, C, T, Q and W are continuous.
Then, A, W, C, N, T and Q have a unique common fixed point.

Proof. Let $s_0 \in Z$ be an arbitrary point in Z, then from condition (CM1) there exist s_1, s_2, s_3 and s_4 such that

$$\begin{aligned} m_0 &= Ts_1 = As_0, & m_1 &= Ns_2 = As_1, \\ m_2 &= Cs_3 = Ws_2\ , & m_3 &= Qs_4 = Ws_3. \end{aligned}$$

We can construct sequences m_η and s_η in Z. Therefore,

$$\left.\begin{aligned} m_{2\eta} &= Ts_{2\eta+1} = As_{2\eta}, \\ m_{2\eta+1} &= Qs_{2\eta+2} = Ws_{2\eta+1}, \\ m_{2\eta+2} &= Cs_{2\eta+3} = Ws_{2\eta+2}, \\ m_{2\eta+3} &= Ns_{2\eta+4} = As_{2\eta+3}. \end{aligned}\right\} \tag{1}$$

$$m_{2\eta} = Ts_{2\eta+1} = As_{2\eta},$$

$$m_{2\eta+1} = Qs_{2\eta+2} = Bs_{2\eta+1},$$

$$m_{2\eta+2} = Cs_{2\eta+3} = Bs_{2\eta+2},$$

$$m_{2\eta+3} = Ps_{2\eta+4} = As_{2\eta+3}.$$

Using (1) in (CM2), we get

$$d(m_{2\eta}, m_{2\eta+1}) = d(As_{2\eta}, Ws_{2\eta+1}) \preceq \tfrac{\rho}{b^2} R(s_{2\eta}, s_{2\eta+1}),$$

where

$$\begin{aligned} R(s_{2\eta}, s_{2\eta+1}) = \\ \max\Big\{& d(Nm_{2\eta}, Tm_{2\eta+1}), d(As_{2\eta}, Cs_{2\eta+1}), d(As_{2\eta+1}, Ns_{2\eta+1}), d(Ws_{2\eta+1}, Qs_{2\eta+1}), \\ & \frac{d(As_{2\eta}, Ts_{2\eta})d(Ws_{2\eta+1}, Ns_{2\eta+1})}{1 + d(Cs_{2\eta}, Qs_{2\eta+1})}\Big\}. \end{aligned}$$

Therefore,

$$\begin{aligned} R(s_{2\eta}, s_{2\eta+1}) = \\ \max\Big\{& d(m_{2\eta-1}, m_{2\eta}), d(m_{2\eta}, m_{2\eta}), d(m_{2\eta+1}, m_{2\eta}), d(m_{2\eta+1}, m_{2\eta}), \\ & \frac{d(m_{2\eta}, m_{2\eta+1})d(m_{2\eta+1}, m_{2\eta+1})}{1 + d(m_{2\eta-1}, m_{2\eta})}\Big\}. \end{aligned}$$

Thus,

$$R(s_{2\eta}, s_{2\eta+1}) = \max[d(m_{2\eta-1}, m_{2\eta}), d(m_{2\eta+1}, m_{2\eta})].$$

Let $\max = d(m_{2\eta+1}, m_{2\eta})$. Then,

$$R(s_{2\eta}, s_{2\eta+1}) = d(m_{2\eta+1}, m_{2\eta}).$$

So,

$$d(m_{2\eta+1}, m_{2\eta}) \precsim \frac{\rho}{b^2} d(m_{2\eta+1}, m_{2\eta}).$$

Thus,

$$d(m_{2\eta+1}, m_{2\eta}) - \frac{\rho}{b^2} d(m_{2\eta+1}, m_{2\eta}) \precsim 0.$$

Therefore,

$$(1 - \frac{\rho}{b^2})(d(m_{2\eta+1}, m_{2\eta})) \precsim 0.$$

$(1 - \frac{\rho}{b^2}) \not\precsim 0.$
ρ and b are positive and also $d(m_{2\eta+1}, m_{2\eta}) \succ 0$.
Hence, there arises a contradiction.
Thus, $d(m_{2\eta+1}, m_{2\eta})$ is not a maximum.

Thus, we deduce that $max = d(m_{2\eta-1}, m_{2\eta})$.
So,

$$d(m_{2\eta+1}, m_{2\eta}) \precsim \frac{\rho}{b^2} d(m_{2\eta-1}, m_{2\eta}).$$

Similarly, we get

$$d(m_{2\eta+2}, m_{2\eta+1}) \precsim \frac{\rho}{b^2} d(m_{2\eta}, m_{2\eta+1}).$$

It follows that

$$d(m_{2\eta+1}, m_{2\eta}) \precsim \frac{\rho}{b^2} d(m_{2\eta-1}, m_{2\eta}) \ldots \precsim (\frac{\rho}{b^2})^\eta d(m_0, m_1).$$

Which implies that:

$$\mid d(m_{\eta+1}, m_\eta) \mid \leqslant \frac{\rho}{b^2} \mid d(m_{2\eta-1}, m_{2\eta}) \mid \ldots \leqslant (\frac{\rho}{b^2})^\eta \mid d(m_0, m_1 \mid.$$

For $\nu < \eta$,

$$\mid d(m_\eta, m_\nu) \mid \leqslant b(\frac{\rho}{b^2})^\eta \mid d(m_0, m_1) \mid + b^2(\frac{\rho}{b^2})^{\eta+1} \mid d(m_0, m_1) \mid + b^3(\frac{\rho}{b^2})^{\eta+2} \mid d(m_0, m_1) \mid$$

$$+ \ldots + b^{\nu-\eta-1}(\frac{\rho}{b^2})^{\nu-1} \mid d(m_0, m_1) \mid.$$

$$= \sum_{i=1}^{\nu-\eta} b^i (\frac{\rho}{b^2})^{i+\eta-1} \mid d(m_0, m_1) \mid .$$

Therefore,

$$\mid d(m_\eta, m_\nu) \mid \leqslant \sum_{i=1}^{\nu-\eta} b(\frac{\rho}{b^2})^{i+\eta-1} \mid d(m_0, m_1) \mid = \sum_{j=\eta}^{\nu-1} b^j (\frac{\rho}{b^2})^j \mid d(m_0, m_1) \mid$$
$$\leqslant \sum_{i=\eta}^{\infty} (\frac{\rho}{b})^j \mid d(m_0, m_1) \mid = \frac{(\frac{\rho}{b})^\eta}{(1 - \frac{\rho}{b})} \mid d(m_0, m_1) \mid.$$

This is a geometric sequence.

Hence, by the Cauchy root test,

let

$$s_\eta = \frac{(\frac{\rho}{b})^\eta}{(1 - \frac{\rho}{b})}.$$

Then,

$$(s_\eta)^{\frac{1}{\eta}} = \frac{(\frac{\rho}{b})}{(1 - \frac{\rho}{b})^{\frac{1}{\eta}}}.$$

Which implies

$$\lim_{\eta\to\infty}(s_\eta)^{\frac{1}{\eta}} = \lim_{\eta\to\infty}\frac{(\frac{\rho}{b})}{(1-\frac{\rho}{b})^{\frac{1}{\eta}}} = \frac{\rho}{b}.$$

Since, $\frac{\rho}{b} < 1$, and because $b > 1$ and $\rho \in (0,1)$.

Thus, the series s_η converges and from a real analysis we know that the necessary condition for the convergence of the series is that when $\eta \to \infty$ then $s_\eta \to 0$.

Therefore,

$$\mid d(m_\eta, m_v) \mid = \frac{(\frac{\rho}{b})^\eta}{(1-\frac{\rho}{b})} \mid d(m_0, m_1) \mid \to 0 \text{ as } \eta \to \infty$$

Thus, m_η is a cauchy sequence in Z which proves that Z is complete, so there exists $\mu \in Z$ such that $m_\eta \to \mu$ as $\eta \to \infty$.

So, we have

$$\begin{array}{ll} Ts_{2\eta+1} \to \mu, & As_{2\eta} \to \mu\ , \\ Ns_{2\eta+1} \to \mu, & Ws_{2\eta} \to \mu\ , \\ Cs_{2\eta+1} \to \mu, & Qs_{2\eta+1} \to \mu\ . \end{array}$$

for sub-sequences.

Now, from (CM4), the mapping N is continuous.

So, by Lemma 1

$NNs_{2\eta} \to N\mu$ and $NAs_{2\eta} \to N\mu$ as $\eta \to \infty$.

In addition, (N,A) is compatible; thus, it implies that $ANs_{2\eta} \to N\mu$.

Indeed,

$$d(ANs_{2\eta}, N\mu) \precsim b[d(ANs_{2\eta}, NAs_{2\eta}) + d(NAs_{2\eta}, N\mu)].$$

So,

$$\mid d(ANs_{2\eta}, N\mu) \mid \leqslant b \mid d(ANs_{2\eta}, NAs_{2\eta}) \mid +b \mid d(NAs_{2\eta}, N\mu) \mid \to 0 \text{ as } \eta \to \infty.$$

Now, we prove that

(1) $N\mu = \mu$

On contrary let

$N\mu \neq \mu$.

$$d(N\mu, \mu) \precsim bd(N\mu, ANs_{2\eta}) + b^2 d(ANs_{2\eta}, Ws_{2\eta+1}) + b^2 d(Ws_{2\eta+1}, \mu).$$

By using (CM2) with $s = Ns_{2\eta}$ and $m = s_{2\eta+1}$, we get:

$$d(ANs_{2\eta}, Ws_{2\eta+1}) \precsim \frac{\rho}{b^2} R(Ns_{2\eta}, s_{2\eta+1})$$

where

$$R(Ns_{2\eta}, s_{2\eta+1}) = \max\left\{ d(NNs_{2\eta}, Ts_{2\eta+1}), d(ANs_{2\eta}, Cs_{2\eta+1}), d(As_{2\eta+1}, Ns_{2\eta+1}), d(Ws_{2\eta+1}, Qs_{2\eta+1}), \frac{d(ANs_{2\eta}, TNs_{2\eta}) d(Ws_{2\eta+1}, Ns_{2\eta+1})}{1 + d(CNs_{2\eta}, Qs_{2\eta+1})} \right\}.$$

Let $\eta \to \infty$, then we get:

$$R(Ns_{2\eta}, s_{2\eta+1}) = \max\left\{ d(N\mu, \mu), d(N\mu, \mu), d(\mu, \mu), d(\mu, \mu), \frac{d(ANs_{2\eta}, TNs_{2\eta}) d(\mu, \mu)}{1 + d(CNs_{2\eta}, Qs_{2\eta+1})} \right\}.$$

$R(Ns_{2\eta}, s_{2\eta+1}) = d(N\mu, \mu)$.

Further,

$$| d(N\mu, \mu) | \leqslant \frac{\rho}{b^2} | d(N\mu, \mu) | \quad \Rightarrow \quad (1 - \frac{\rho}{b^2}) | d(N\mu, \mu) | \leqslant 0,$$

which is a contradiction.

So $| d(N\mu, \mu) | = 0$, which is possible only if $N\mu = \mu$.

Next,

(2) $A\mu = \mu$.

On contrary, let $A\mu \neq \mu$

$$d(A\mu, \mu) \precsim bd(A\mu, Ws_{2\eta}) + bd(Ws_{2\eta+1}, \mu).$$

By using (CM2) with $s = \mu$ and $m = s_{2\eta+1}$, we get:

$$d(A\mu, Ws_{2\eta+1}) \precsim \frac{\rho}{b^2} R(\mu, s_{2\eta+1})$$

where

$$R(\mu, s_{2\eta+1}) = \max\left\{ d(N\mu, Ts_{2\eta+1}), d(A\mu, Cs_{2\eta+1}), d(As_{2\eta+1}, Ns_{2\eta+1}), d(Ws_{2\eta+1}, Qs_{2\eta+1}), \frac{d(A\mu, TNs_{2\eta}) d(Ws_{2\eta+1}, Ns_{2\eta+1})}{1 + d(C\mu, Qs_{2\eta+1})} \right\}.$$

Let $\eta \to \infty$, then we get:

$$R(\mu, s_{2\eta+1}) = \max\left\{ d(\mu, \mu), d(A\mu, \mu), d(\mu, \mu), d(\mu, \mu), \frac{d(ANs_{2\eta}, TNs_{2\eta}) d(\mu, \mu)}{1 + d(C\mu, \mu)} \right\}.$$

$R(\mu, s_{2\eta+1}) = d(A\mu, \mu)$.

Further,

$$| d(A\mu, \mu) | \leqslant \frac{\rho}{b^2} | d(A\mu, \mu) | \quad \Rightarrow \quad (1 - \frac{\rho}{b^2}) | d(A\mu, \mu) | \leqslant 0,$$

which is a contradiction.

So, $| d(A\mu, \mu) | = 0$, which is possible only if $A\mu = \mu$.

Further, we show

(3) $C\mu = \mu$.

On the contrary, let $C\mu \neq \mu$.

From (CM4), C is continuous.

Then, by Lemma 1

$CAs_{2\eta} \to C\mu$ and $CCs_{2\eta} \to C\mu$ as $\eta \to \infty$.

In addition, the pair (C,A) is compatible, which implies that:

$ACs_{2\eta} \to C\mu$.

Indeed,

$$d(ACs_{2\eta}, C\mu) \precsim b[d(ACs_{2\eta}, CAs_{2\eta}) + d(CAs_{2\eta}, C\mu)].$$

So,

$$| d(ACs_{2\eta}, C\mu) | \leqslant b \, | d(ACs_{2\eta}, CAs_{2\eta}) | + b \, | d(CAs_{2\eta}, C\mu) | \to 0, \text{ as } \eta \to \infty.$$

$$d(C\mu, \mu) \precsim bd(C\mu, ACs_{2\eta}) + b^2 d(ACs_{2\eta}, Ws_{2\eta+1}) + b^2 d(Ws_{2\eta+1}, \mu).$$

By using (CM2) with $s = Cs_{2\eta}$ and $m = s_{2\eta+1}$,

we get:

$$d(ACs_{2\eta}, Ws_{2\eta+1}) \precsim \frac{\rho}{b^2} R(Cs_{2\eta}, s_{2\eta+1})$$

where

$$R(Cs_{2\eta}, s_{2\eta+1}) = \max\left\{d(NCs_{2\eta}, Ts_{2\eta+1}), d(ACs_{2\eta}, Cs_{2\eta+1}), d(As_{2\eta+1}, Ns_{2\eta+1}), d(Ws_{2\eta+1}, Qs_{2\eta+1}), \frac{d(ACs_{2\eta}, TCs_{2\eta})d(Ws_{2\eta+1}, Ns_{2\eta+1})}{1 + d(CCs_{2\eta}, Qs_{2\eta+1})}\right\}.$$

Let $\eta \to \infty$, then we get:

$$R(Cs_{2\eta}, s_{2\eta+1}) = \max\left\{d(N\mu, \mu), d(C\mu, \mu), d(\mu, \mu), d(\mu, \mu), \frac{d(C\mu, TNs_{2\eta})d(\mu, \mu)}{1+d(C\mu, \mu)}\right\}.$$

$R(Cs_{2\eta}, s_{2\eta+1}) = d(C\mu, \mu)$.
Further,

$$| d(C\mu, \mu) | \leqslant \frac{\rho}{b^2} | d(C\mu, \mu) | \quad \Rightarrow \quad (1 - \frac{\rho}{b^2}) | d(C\mu, \mu) | \leqslant 0,$$

which is a contradiction.

So, $| d(C\mu, \mu) | = 0$, which is possible only if $C\mu = \mu$.
Next, we prove,
(4) $T\mu = \mu$.
On the contrary, let $T\mu \neq \mu$,
again from (CM4), C is continuous, then by Lemma 1

$TAs_{2\eta} \to T\mu$ and $TTs_{2\eta} \to T\mu$ as $\eta \to \infty$.
Furthermore, the pair (T,A) is compatible, which implies that $ATs_{2\eta} \to T\mu$.
Indeed,

$$d(ATs_{2\eta}, T\mu) \precsim b[d(ATs_{2\eta}, TAs_{2\eta}) + d(TAs_{2\eta}, T\mu)]$$

So,

$$| d(ATs_{2\eta}, T\mu) | \leqslant b | d(ATs_{2\eta}, TAs_{2\eta}) | + b | d(TAs_{2\eta}, T\mu) | \to 0 \text{ as } \eta \to \infty$$

$$d(T\mu, \mu) \precsim bd(T\mu, ACs_{2\eta}) + b^2 d(ATs_{2\eta}, Ws_{2\eta+1}) + b^2 d(Ws_{2\eta+1}, \mu)$$

By using (CM2) with $s = Cs_{2\eta}$ and $m = s_{2\eta+1}$, we get:

$$d(ATs_{2\eta}, Ws_{2\eta+1}) \precsim \frac{\rho}{b^2} R(Ts_{2\eta}, s_{2\eta+1})$$

where

$$R(Ts_{2\eta}, s_{2\eta+1}) = \max\left\{(NTs_{2\eta}, Ts_{2\eta+1}), d(ATs_{2\eta}, Cs_{2\eta+1}), d(As_{2\eta+1}, Ns_{2\eta+1}), d(Ws_{2\eta+1}, Qs_{2\eta+1}), \frac{d(ATs_{2\eta}, TTs_{2\eta})d(Ws_{2\eta+1}, Ns_{2\eta+1})}{1 + d(CTs_{2\eta}, Qs_{2\eta+1})}\right\}.$$

Let $\eta \to \infty$, then
we get:

$$R(Ts_{2\eta}, s_{2\eta+1}) = \max\left\{(N\mu, \mu), d(T\mu, \mu), d(\mu, \mu), d(\mu, \mu), \frac{d(Au, TNs_{2\eta})d(\mu, \mu)}{1+d(C\mu, \mu)}\right\}.$$

$R(Ts_{2\eta}, s_{2\eta+1}) = d(T\mu, \mu)$.
Further,
$$| d(T\mu,\mu) | \leqslant \frac{\rho}{b^2} | d(T\mu,\mu) | \quad \Rightarrow \quad (1-\frac{\rho}{b^2}) | d(T\mu,\mu) | \leqslant 0,$$
which is a contradiction.
So, $| d(T\mu,\mu) | = 0$, which is possible only if $T\mu = \mu$.
Furthermore, we prove,
(5) $Q\mu = \mu$.
On the contrary, let $Q\mu \neq \mu$

Again from (CM4), Q is continuous, then by Lemma 1

$QAs_{2\eta} \to Q\mu$ and $QQs_{2\eta} \to Q\mu$ as $\eta \to \infty$.
Furthermore, the pair (Q,A) is compatible, which implies that $QAs_{2\eta} \to Q\mu$.
Indeed,
$$d(AQs_{2\eta}, Q\mu) \precsim b[d(AQs_{2\eta}, QAs_{2\eta}) + d(QAs_{2\eta}, Q\mu)].$$
So,
$$| d(AQs_{2\eta}, Q\mu) | \leqslant b | d(AQs_{2\eta}, QAs_{2\eta}) | + b | d(QAs_{2\eta}, Q\mu) | \to 0 \text{ as } \eta \to \infty$$
$$d(Q\mu,\mu) \precsim bd(Q\mu, AQs_{2\eta}) + b^2 d(AQs_{2\eta}, Ws_{2\eta+1}) + b^2 d(Ws_{2\eta+1}, \mu).$$
By using (CM2) with $s = Qs_{2\eta}$ and $m = s_{2\eta+1}$, we get:

$$d(AQs_{2\eta}, Ws_{2\eta+1}) \precsim \frac{\rho}{b^2} R(Qs_{2\eta}, s_{2\eta+1})$$
where
$$R(Qs_{2\eta}, s_{2\eta+1}) =$$
$$\max\Big\{d(NQs_{2\eta}, Ts_{2\eta+1}), d(AQs_{2\eta}, Cs_{2\eta+1}), d(As_{2\eta+1}, Ns_{2\eta+1}), d(Ws_{2\eta+1}, Qs_{2\eta+1}),$$
$$\frac{d(AQs_{2\eta}, TQs_{2\eta}) d(Ws_{2\eta+1}, Qs_{2\eta+1})}{1 + d(CQs_{2\eta}, Qs_{2\eta+1})}\Big\}.$$
Let $\eta \to \infty$, then we get:
$$R(Qs_{2\eta}, s_{2\eta+1}) = \max\Big\{d(N\mu,\mu), d(Q\mu,\mu), d(\mu,\mu), d(\mu,\mu), \frac{d(Q\mu, TQs_{2\eta}) d(\mu,\mu)}{1+d(C\mu,\mu)}\Big\}.$$
$R(Qs_{2\eta}, s_{2\eta+1}) = d(Q\mu, \mu)$.
Further,
$$| d(Q\mu,\mu) | \leqslant \frac{\rho}{b^2} | d(Q\mu,\mu) | \quad \Rightarrow \quad (1-\frac{\rho}{b^2}) | d(Q\mu,\mu) | \leqslant 0,$$
which is a contradiction.
So, $| d(Q\mu,\mu) | = 0$, which is possible only if $Q\mu = \mu$.
Next, we need to prove,
(6) $W\mu = \mu$.
On the contrary, let $W\mu \neq \mu$
Again from (CM4), W is continuous, then by Lemma 1

$WAs_{2\eta} \to W\mu$ and $WWs_{2\eta} \to Q\mu$ as $\eta \to \infty$.
Furthermore, the pair (W,A) is compatible, which implies that $WAs_{2\eta} \to W\mu$.
Indeed,
$$d(AWs_{2\eta}, W\mu) \precsim b[d(AWs_{2\eta}, WAs_{2\eta}) + d(WAs_{2\eta}, Q\mu)].$$
So,

$$| d(AWs_{2\eta}, W\mu) | \leqslant b \mid d(AWs_{2\eta}, WAs_{2\eta}) \mid + b \mid d(WAs_{2\eta}, W\mu) | \to 0 \text{ as } \eta \to \infty.$$

$$d(W\mu, \mu) \precsim bd(W\mu, AWs_{2\eta}) + b^2 d(AWs_{2\eta}, Ws_{2\eta+1}) + b^2 d(Ws_{2\eta+1}, \mu).$$

By using (CM2) with $s = Qs_{2\eta}$ and $m = s_{2\eta+1}$, we get:

$$d(AWs_{2\eta}, Ws_{2\eta+1}) \precsim \frac{\rho}{b^2} R(Ws_{2\eta}, s_{2\eta+1})$$

where

$$R(Ws_{2\eta}, s_{2\eta+1}) = \max\left\{ d(NWs_{2\eta}, Ts_{2\eta+1}), d(AWs_{2\eta}, Cs_{2\eta+1}), d(As_{2\eta+1}, Ns_{2\eta+1}), d(Ws_{2\eta+1}, Qs_{2\eta+1}), \frac{d(AWs_{2\eta}, TWs_{2\eta}) d(Ws_{2\eta+1}, Qs_{2\eta+1})}{1 + d(CWs_{2\eta}, Qs_{2\eta+1})} \right\}.$$

Let $\eta \to \infty$, then we get:

$$R(Ws_{2\eta}, s_{2\eta+1}) = \max\left\{ d(P\mu, \mu), d(W\mu, \mu), d(\mu, \mu), d(\mu, \mu), \frac{d(W\mu, TWs_{2\eta}) d(\mu, \mu)}{1 + d(C\mu, \mu)} \right\}.$$

$R(Ws_{2\eta}, s_{2\eta+1}) = d(W\mu, \mu)$.
Further,

$$| d(W\mu, \mu) | \leqslant \frac{\rho}{b^2} \mid d(W\mu, \mu) \mid \quad \Rightarrow \quad (1 - \frac{\rho}{b^2}) \mid d(W\mu, \mu) | \leqslant 0$$

which is a contradiction. So, $| d(W\mu, \mu) | = 0$, which is possible only if $W\mu = \mu$.
Thus, we conclude that:

$$N\mu = Q\mu = T\mu = C\mu = W\mu = A\mu = \mu$$

Uniqueness:
Now, we have to look forward for uniqueness.
Let us consider ϑ as another common fixed point of A, W, C, Q, N and T. Then,

$$N\vartheta = Q\vartheta = T\vartheta = C\vartheta = W\vartheta = A\vartheta = \vartheta$$

By putting $s = \mu$ and $m = \vartheta$ in (CM2), we get

$$d(\mu, \vartheta) = d(A\mu, T\vartheta) \precsim \frac{\rho}{b^2} (\mu, \vartheta)$$

where

$$R(\mu, \vartheta) = \max\left\{ d(N\mu, T\vartheta), d(A\mu, C\vartheta), d(W\vartheta, Q\vartheta), d(A\mu, N\vartheta), \frac{d(A\mu, T\mu) d(W\vartheta, N\vartheta)}{1 + d(C\mu, Q\vartheta)} \right\}.$$

Thus,

$$R(\mu, \vartheta) = \max\left\{ d(\mu, \vartheta), d(\mu, \vartheta), d(\vartheta, \vartheta), d(\mu, \vartheta), \frac{d(\mu, \mu) d(\vartheta, \vartheta)}{1 + d(\mu, \vartheta)} \right\}.$$

So,

$$R(\mu, \vartheta) = d(\mu, \vartheta).$$

Further,

$$| d(\mu, \vartheta) | \leqslant \frac{\rho}{b^2} \mid d(\mu, \vartheta) \mid \quad \Rightarrow \quad (1 - \frac{\rho}{b^2}) \mid d(\mu, \vartheta) | \leqslant 0,$$

which is a contradiction. So, $| d(W\mu, \mu) | = 0$, which implies that $\mu = \vartheta$.
Thus, the common fixed point for A, W, C, T, N and Q is unique. □

The above theorem yields the following corollaries.

Corollary 1. *Let (Z,d) be a complex valued b-metric space and A,C,T,N,Q: Z → Z be five self mappings fulfilling the following necessities;*

(CM1) $A(z) \subseteq T(z)$, $A(z) \subseteq N(z)$, $A(z) \subseteq C(z)$ *and* $A(z) \subseteq Q(z)$.
(CM2) $d(A_s, A_m) \precsim \frac{\rho}{b^2} R(s,m)$, *if* $b \geqslant 1$ *and* $\rho \in (0,1)$ *for all* $s, m \in Z$ *where*

$$R(s,m) = \max\left\{d(Ns,Tm), d(As,Cm), d(Am,Nm), d(Am,Qm), \frac{d(As,Ts)d(Am,Nm)}{1+d(Cs,Qm)}\right\}.$$

(CM3) The pairs (N,A), (C,A), (T,A) and (A,Q) are compatible.
(CM4) N, C, T and Q are continuous.
Then, A, C, N, T and Q have a unique common fixed point.

Proof. For A = W in Theorem 2, this result can easily be obtained. □

Corollary 2. *Let (Z,d) be a complex valued b-metric space and Q,A,W,T: Z → Z be four self mappings fulfilling the following necessities;*
(CM1) $A(z) \subseteq T(z)$, $W(z) \subseteq Q(z)$.
(CM2) $d(A_s, A_m) \precsim \frac{\rho}{b^2} R(s,m)$, *if* $b \geqslant 1$ *and* $\rho \in (0,1)$ *for all* $s, w \in Z$, *where*

$$R(z,w) = \max\left\{d(Ts,Tm), d(As,Qw), d(Am,Tm), d(Am,Qm), \frac{d(As,Ts)d(Tm,Tm)}{1+d(Ws,Qm)}\right\}.$$

(CM3) The pairs (W,A), (T,A) and (A,Q) are compatible.
(CM4) T, W and Q are continuous.
Then, W, A, T and C have a unique common fixed point.

Proof. For T = N and C = Q in Theorem 2, this result can easily be obtained. □

Corollary 3. *Let (Z,d) be a complex valued b-metric space and Q,A,T,: Z → Z be three self mappings fulfilling the following necessities;*
(CM1) $A(z) \subseteq T(z)$, $A(z) \subseteq Q(z)$.
(CM2) $d(A_s, A_m) \precsim \frac{\rho}{b^2} R(s,m)$, *if* $b \geqslant 1$ *and* $\rho \in (0,1)$ *for all* $s, m \in Z$ *where*

$$R(s,m) = \max\left\{d(Qs,Tm), d(As,Tm), d(Am,Tm), \frac{d(As,Ts)d(Am,Qm)}{1+d(Ts,Qm)}\right\}.$$

(CM3) The pairs (T,A) and (A,Q) are compatible.
(CM4) Q and T are continuous.
Then, A, T and Q have a unique common fixed point

Proof. For A=B, C=T and N=C in Theorem 2, this result can easily be achieved. □

Corollary 4. *Let (Z,d) be a complex valued b-metric space and A,Q: Z → Z be two self mappings fulfilling the following necessities;*
(CM1) $A(z) \subseteq Q(z)$.
(CM2) $d(A_s, A_m) \precsim \frac{\rho}{b^2} R(s,m)$, *if* $b \geqslant 1$ *and* $\rho \in (0,1)$ *for all* $s, m \in Z$, *where*

$$R(s,m) = \max\left\{d(Qs,Qm), d(As,Qm), d(Am,Qm), \frac{d(As,Qs)d(Am,Qm)}{1+d(Qs,Qm)}\right\}.$$

(CM3) The pair (A,Q) is compatible.
(CM4) Q is continuous.
Then, A and Q have a unique common fixed point.

Proof. For A = W, C = T = N = Q in Theorem 2, this result can easily be obtained. □

Remark 1. *Corollary 2 is the result of [16].*

Example 6. *Let Z = [0,1],* $\forall\, s, m \in Z$. *Define* $d : Z \times Z \to C$ *as a complex valued b-metric space with b = 2 by;*

$$d(s,m) = | s - m |^2 + \iota | s - m |^2$$

Now, define the mappings A, W, C, N, T and Q such that

$$\begin{array}{lll} As = \frac{s}{32}, & Ws = \frac{s^2}{48}, & Ns = \frac{s}{2}, \\ Qs = \frac{s}{8}, & Ts = \frac{s^2}{3}, & Cs = \frac{s}{4}. \end{array}$$

Clearly,
(1) $A(s) \subseteq T(s)$,$A(s) \subseteq P(s)$,$B(s) \subseteq C(s)$,$B(s) \subseteq Q(s)$.
(2) The pairs (N,A), (C,A), (T,A), (A,Q) and (W,A) are compatible for $s_\eta = \frac{1}{\eta}$.
(3) A, W, C, N, T and Q are continuous.
(4)

$$d(As, Wm) = | \tfrac{s}{32} - \tfrac{m^2}{48} |^2 + \iota | \tfrac{s}{32} - \tfrac{m^2}{48} |^2.$$

$$d(As, Wm) = \frac{1}{256}\left\{ | \tfrac{s}{2} - \tfrac{m^2}{3} |^2 + \iota | \tfrac{s}{2} - \tfrac{m^2}{3} |^2 \right\}$$

$$d(Ns, Tm) = | \tfrac{s}{2} - \tfrac{m^2}{3} |^2 + \iota | \tfrac{s}{2} - \tfrac{m^2}{3} |^2.$$

Thus, $d(As, Wm) = \frac{1}{256} d(Ns, Tm)$.
This means that $d(As, Wm) \precsim \frac{\rho}{b^2} R(s,m)$, *where* $\rho = \frac{1}{64}$ *and b=2.*

Thus, all the conditions of Theorem 2 are satisfied; therefore, A, W, C, T, N and Q have a unique common fixed point.

Example 7. *Let* $\mathcal{X} = \beta(0,\kappa)$, $\kappa > 1$, $\forall\, s, m \in \mathcal{X}$ *and* $\Upsilon : \mathcal{X} \times \mathcal{X} \to C$ *be defined by*

$$\Upsilon(s(w), m(w)) = \frac{\iota}{2\pi} \left| \int_\daleth \frac{s(w)}{w} - \int_\daleth \frac{m(w)}{w} \right|^2,$$

which is a complete CVM_b, *and let* $\daleth$ *be a closed path in* Υ *containing a zero.*

We first prove that Υ *is a complex valued b-metric space with b = 2*

$$\begin{aligned}
\Upsilon(s(w), m(w)) &= \frac{\iota}{2\pi} \left| \int_\daleth \frac{s(w)}{w} - \int_\daleth \frac{m(w)}{w} \right|^2 \\
&= \frac{\iota}{2\pi} \left| \int_\daleth \frac{s(w)}{w} - \int_\daleth \frac{y(w)}{w} + \int_\daleth \frac{y(w)}{w} - \int_\daleth \frac{m(w)}{w} \right|^2 \\
&\precsim \frac{\iota}{2\pi} \left| \int_\daleth \frac{s(w)}{w} - \int_\daleth \frac{y(w)}{w} \right|^2 + \frac{\iota}{2\pi} \left| \int_\daleth \frac{y(w)}{w} - \int_\daleth \frac{m(w)}{w} \right|^2 \\
&\quad + 2\left[\frac{\iota}{2\pi} \left| \int_\daleth \frac{s(w)}{w} - \int_\daleth \frac{y(w)}{w} \right|^2 \left| \int_\daleth \frac{y(w)}{w} - \int_\daleth \frac{m(w)}{w} \right|^2 \right] \\
&\precsim \frac{\iota}{2\pi} \left| \int_\daleth \frac{s(w)}{w} - \int_\daleth \frac{y(w)}{w} \right|^2 + \frac{\iota}{2\pi} \left| \int_\daleth \frac{y(w)}{w} - \int_\daleth \frac{m(w)}{w} \right|^2 \\
&\quad + \frac{\iota}{2\pi} \left| \int_\daleth \frac{s(w)}{w} - \int_\daleth \frac{y(w)}{w} \right|^2 + \frac{\iota}{2\pi} \left| \int_\daleth \frac{y(w)}{w} - \int_\daleth \frac{m(w)}{w} \right|^2 \\
&\precsim 2\left\{ \frac{\iota}{2\pi} \left| \int_\daleth \frac{s(w)}{w} - \int_\daleth \frac{y(w)}{w} \right|^2 + \frac{\iota}{2\pi} \left| \int_\daleth \frac{y(w)}{w} - \int_\daleth \frac{m(w)}{w} \right|^2 \right\}
\end{aligned}$$

$\Upsilon(s(w), m(w)) \precsim 2\Upsilon(s(w), y(w)) + \Upsilon(y(w), m(w))$.
Hence, it is proven that Υ *is a complex valued b-metric space with b = 2.*
Now, we define the mappings A, W, C, T, N and Q $:\mathcal{X} \times \mathcal{X}$ *by:*

$$As(w) = 1 - e^{w}, \quad Ws(w) = w, \quad Cs(w) = \tfrac{1}{2}w^2 + w$$
$$Ts(w) = 1 - e^{\frac{w}{2}}, \quad Ns(w) = 1 - e^{\frac{w}{4}}, \quad Qs(w) = \tfrac{1}{4}w^2 + w\,.$$

Clearly,
(1) $A(x) \subseteq T(x)$, $A(x) \subseteq N(x)$, $W(x) \subseteq C(x)$ and $W(x) \subseteq Q(x)$.
(2) The pairs (N,A), (C,A), (T,A), (A,Q) and (W,A) are compatible.
(3) N, C, T, Q and W are continuous.
(4) By using the Cauchy integral formula when the mappings A, W, C, N, T and Q are analytic, we get:

$$d(As(w), Wm(w)) = \tfrac{\iota}{2\pi}\left|\int_{\daleth}\tfrac{1-e^{w}}{w} - \int_{\daleth}\tfrac{w}{w}\right|^2 = 0$$
$$d(Ns(w), Tm(w)) = \tfrac{\iota}{2\pi}\left|\int_{\daleth}\tfrac{1-e^{\frac{w}{4}}}{w} - \int_{\daleth}\tfrac{1-e^{\frac{w}{2}}}{w}\right|^2 = 0$$
$$d(Am(w), Nm(w)) = \tfrac{\iota}{2\pi}\left|\int_{\daleth}\tfrac{1-e^{w}}{w} - \int_{\daleth}\tfrac{1-e^{\frac{w}{4}}}{w}\right|^2 = 0$$
$$d(As(w), Ts(w)) = \tfrac{\iota}{2\pi}\left|\int_{\daleth}\tfrac{1-e^{w}}{w} - \int_{\daleth}\tfrac{1-e^{\frac{w}{2}}}{w}\right|^2 = 0$$
$$d(Wm(w), Qm(w)) = \tfrac{\iota}{2\pi}\left|\int_{\daleth}\tfrac{w}{w} - \int_{\daleth}\tfrac{\frac{1}{4}w^2+w}{w}\right|^2 = \tfrac{\iota(2\pi)^4}{128\pi}$$
$$d(As(w), Cm(w)) = \tfrac{\iota}{2\pi}\left|\int_{\daleth}\tfrac{1-e^{w}}{w} - \int_{\daleth}\tfrac{\frac{1}{2}w^2+w}{w}\right|^2 = \tfrac{\iota}{8}\pi^3.$$
$$d(Wm(w), Nm(w)) = \tfrac{\iota}{2\pi}\left|\int_{\daleth}\tfrac{w}{w} - \int_{\daleth}\tfrac{1-e^{\frac{w}{4}}}{w}\right|^2 = 0.$$
$$d(Cs(w), Qm(w)) = \tfrac{\iota}{2\pi}\left|\int_{\daleth}\tfrac{\frac{1}{2}w^2+w}{w} - \int_{\daleth}\tfrac{\frac{1}{4}w^2+w}{w}\right|^2 = \tfrac{\iota(2\pi)^2}{128\pi}.$$

So,
$R(s(w), m(w)) = \max\{\tfrac{\iota}{8}\pi^3, 0\} = \tfrac{\iota}{8}\pi^3.$
Further, $0 = R(As(w), Wm(w)) \precsim \tfrac{\iota}{8}\pi^3.$

Thus, all the conditions of Theorem 2 are satisfied, which shows that the mappings A, W, C, T, N and Q have a unique common fixed point.

4. Applications

In this section, our aim is to provide some applications based on our results.

4.1. Existence of a Unique Common Solution to the System of Urysohn Integral Equations

Now, in this section, we apply Theorem 2 for the existence of a unique common solution to the following system:

$$\mu(z) = \varphi_i(z) + \int_x^y K_i\big(z, t, \mu(t)\big)dt, \tag{2}$$

where $z \in (x, y) \subseteq R$; $\mu, \varphi_i \in C\big((x, y), R^{\eta}\big)$ and $K_i : (x, y) \times (x, y) \times R^{\eta} \to R^{\eta}$, $i = 1, 2, ..., 6$.

Let us denote

$$\Psi_i\big(\mu(z)\big) = \int_x^y K_i\big(z, t, \varphi(z)\big)dt$$

where $i = 1, 2, 3, 4, 5, 6$.

Suppose these conditions are true:

(U_1): For $i = 3, 4$,

$$\Psi_1\mu(z) + \varphi_1(z) + \varphi_i(z) - \Psi_i\big(\Psi_1(\mu(z) + \varphi_1(z) + \varphi_i(z)\big) = 0$$

(U_2): For $j = 5, 6$,

$$\Psi_2\mu(z) + \varphi_2(z) + \varphi_j(z) - \Psi_j\big(\Psi_2(\mu(z) + \varphi_2(z) + \varphi_j(z)\big) = 0$$

(U_3): For $i = 3, 4, 5, 6$,

$$\Psi_i \mu(z) + \varphi_i(z) + \Psi_5 \mu(z) + \varphi_5(z) = 2\mu(z)$$

(U_4): For $j = 2$,

$$\Psi_1 \mu(z) + \varphi_1(z) - \Psi_j \mu(z) - \varphi_j(z) = 0$$

(U_5): For $i = 3, 4, 5, 6$,

$$-\Psi_1 \Psi_i \mu(z) - \Psi_1 \varphi_i(z) - \varphi_1(z) + \Psi_i \Psi_1 \mu(z) + \Psi_i \varphi_1(z) + \varphi_i(z) - 2\varphi_1(z) = 0$$

(U_6) For $j = 2$,

$$\Psi_1 \Psi_j \mu(z) + \Psi_1 \varphi_j \mu(z) - \Psi_j \Psi_1 \mu(z) - \Psi_j \varphi_1(z) = 0$$

Let $Y = C\big((x, y), R^\eta\big)$, $x > 0$ be a complete complex valued b-metric space with metric

$$d(s, m) = \max_{a \in (x,y)} \parallel s(a) - m(a) \parallel_\infty \sqrt{1 + x^2} e^{\iota tan^{-1} x}$$

for all $s, m \in Y$.

Define six operators Γ_1, Γ_2, Γ_3, Γ_4, Γ_5 and Γ_6: $Y \to Y$ by

$$\left.\begin{array}{l} \Gamma_1 \mu(z) = \Psi_1 \mu(z) + \varphi_1(z) \\ \Gamma_2 \mu(z) = \Psi_2 \mu(z) + \varphi_2(z) \\ \Gamma_3 \mu(z) = 2\mu(z) - \Psi_3 \mu(z) - \varphi_3(z) \\ \Gamma_4 \mu(z) = 2\mu(z) - \Psi_4 \mu(z) - \varphi_4(z) \\ \Gamma_5 \mu(z) = 2\mu(z) - \Psi_5 \mu(z) - \varphi_5(z) \\ \Gamma_6 \mu(z) = 2\mu(z) - \Psi_6 \mu(z) - \varphi_6(z). \end{array}\right\} \tag{3}$$

Now, we have to formulate the existence results.

Theorem 3. *(1): Based on these assumptions $(U_1$–$U_6)$, if for each $s, m \in Y$ and $b \geqslant 1$, $\rho \in (0, 1)$*

$$\chi_1 \sqrt{1 + x^2} e^{\iota tan^{-1} x} \precsim \frac{\rho}{b^2} \left\{ \chi_2, \chi_3, \chi_4, \chi_5, \frac{\chi_6 \times \chi_7}{1 + \chi_8} \right\}$$

where

$$\begin{aligned}
\chi_1 &= \parallel \Psi_1 s(z) + \varphi_1(z) - \Psi_2 m(z) - \varphi_2(z) \parallel_\infty \sqrt{1 + x^2} e^{\iota tan^{-1} x}, \\
\chi_2 &= \parallel 2s(z) - \Psi_3 s(z) - \varphi_3(z) - 2m(z) + \Psi_4 m(z) + \varphi_4(z) \parallel_\infty \sqrt{1 + x^2} e^{\iota tan^{-1} x}, \\
\chi_3 &= \parallel \Psi_1 s(z) + \varphi_1(z) - 2m(z) + \Psi_6 m(z) + \varphi_6(z) \parallel_\infty \sqrt{1 + x^2} e^{\iota tan^{-1} x}, \\
\chi_4 &= \parallel \Psi_1 m(z) + \varphi_1(z) - 2m(z) + \Psi_3 m(z) + \varphi_3(z) \parallel_\infty \sqrt{1 + x^2} e^{\iota tan^{-1} x}, \\
\chi_5 &= \parallel \Psi_2 m(z) + \varphi_2(z) - 2m(z) + \Psi_5 m(z) + \varphi_5(z) \parallel_\infty \sqrt{1 + x^2} e^{\iota tan^{-1} x}, \\
\chi_6 &= \parallel \Psi_1 s(z) + \varphi_1(z) - 2s(z) + \Psi_4 s(z) + \varphi_4(z) \parallel_\infty \sqrt{1 + x^2} e^{\iota tan^{-1} x}, \\
\chi_7 &= \parallel \Psi_2 m(z) + \varphi_2(z) - 2m(z) + \Psi_3 m(z) + \varphi_3(z) \parallel_\infty \sqrt{1 + x^2} e^{\iota tan^{-1} x}, \\
\chi_8 &= \parallel 2s(z) - \Psi_6 s(z) - \varphi_6(z) - 2m(z) + \Psi_5 m(z) + \varphi_5(z) \parallel_\infty \sqrt{1 + x^2} e^{\iota tan^{-1} x}
\end{aligned}$$

(2): $\Gamma_1(Y) \subseteq \Gamma_4(Y)$, $\Gamma_1(Y) \subseteq \Gamma_3(Y)$, $\Gamma_2(Y) \subseteq \Gamma_5(Y)$ and $\Gamma_2(Y) \subseteq \Gamma_6(Y)$.
(3): (Γ_1, Γ_2), (Γ_1, Γ_3), (Γ_1, Γ_4), (Γ_1, Γ_5) and (Γ_1, Γ_6) are compatible.
Then, the system of Urysohn integral equations (2) has a unique common solution.

Proof. Note that System 2 of integral equations has a unique common solution if and only if System 3 of operators has a unique common fixed point.
Now,

$$\left.\begin{array}{l} d(\Gamma_1 s,\Gamma_2 m)=\max_{z\in(x,y)}\parallel \Psi_1 s(z)+\varphi_1(z)-\Psi_2 m(z)-\varphi_2(z)\parallel_\infty \sqrt{1+x^2}e^{\iota tan^{-1}x},\\ d(\Gamma_3 s,\Gamma_4 m)=\max_{z\in(x,y)}\parallel 2s(z)-\Psi_3 s(z)-\varphi_3(z)-2m(z)+\Psi_4 m(z)+\varphi_4(z)\parallel_\infty \sqrt{1+x^2}e^{\iota tan^{-1}x},\\ d(\Gamma_1 s,\Gamma_6 m)=\max_{z\in(x,y)}\parallel \Psi_1 s(z)+\varphi_1(z)-2m(z)+\Psi_6 m(z)+\varphi_6(z)\parallel_\infty \sqrt{1+x^2}e^{\iota tan^{-1}x},\\ d(\Gamma_1 m,\Gamma_3 m)=\max_{z\in(x,y)}\parallel \Psi_1 m(z)+\varphi_1(z)-2m(z)+\Psi_3 m(z)+\varphi_3(z)\parallel_\infty \sqrt{1+x^2}e^{\iota tan^{-1}x},\\ d(\Gamma_2 m,\Gamma_5 m)=\max_{z\in(x,y)}\parallel \Psi_2 m(z)+\varphi_2(z)-2m(z)+\Psi_5 m(z)+\varphi_5(z)\parallel_\infty \sqrt{1+x^2}e^{\iota tan^{-1}x},\\ d(\Gamma_1 s,\Gamma_2 s)=\max_{z\in(x,y)}\parallel \Psi_1 s(z)+\varphi_1(z)-2s(z)+\Psi_4 s(z)+\varphi_4(z)\parallel_\infty \sqrt{1+x^2}e^{\iota tan^{-1}x},\\ d(\Gamma_2 m,\Gamma_3 m)=\max_{z\in(x,y)}\parallel \Psi_2 m(z)+\varphi_2(z)-2m(z)+\Psi_3 m(z)+\varphi_3(z)\parallel_\infty \sqrt{1+x^2}e^{\iota tan^{-1}x},\\ d(\Gamma_6 s,\Gamma_5 m)=\max_{z\in(x,y)}\parallel 2s(z)-\Psi_6 s(z)-\varphi_6(z)-2m(z)+\Psi_5 m(z)+\varphi_5(z)\parallel_\infty \sqrt{1+x^2}e^{\iota tan^{-1}x} \end{array}\right\} \quad (4)$$

From condition (CM2) of Theorem 2, we have

$$\chi_1\sqrt{1+x^2}e^{\iota tan^{-1}x} \precsim \frac{\rho}{b^2}\left\{\chi_2,\chi_3,\chi_4,\chi_5,\frac{\chi_6\times\chi_7}{1+\chi_8}\right\},$$

which implies that

$$\max_{a\in(x,y)}\chi_1\sqrt{1+x^2}e^{\iota tan^{-1}x} \precsim \frac{\rho}{b^2}\left\{\max_{a\in(x,y)}\chi_2,\max_{a\in(x,y)}\chi_3,\max_{a\in(x,y)}\chi_4,\max_{a\in(x,y)}\chi_5,\frac{\max_{a\in(x,y)}\chi_6\times\max_{a\in(x,y)}\chi_7}{1+\max_{a\in(x,y)}\chi_8}\right\}.$$

Using the above distances in Equation 4, we obtain

$$d(\Gamma_1 s,\Gamma_2 m) \precsim \frac{\rho}{b^2}\left\{d(\Gamma_3 s,\Gamma_4 m),d(\Gamma_1 s,\Gamma_6 m),d(\Gamma_1 m,\Gamma_3 m),d(\Gamma_2 m,\Gamma_5 m),\frac{d(\Gamma_1 s,\Gamma_4 s)d(\Gamma_2 m,\Gamma_3 m)}{1+d(\Gamma_6 s,\Gamma_5 m)}\right\}.$$

Now, to show that $\Gamma_1(Y)\subseteq\Gamma_4(Y)$, we have

$$\begin{aligned}\Gamma_4\big(\Gamma_1\mu(z)+\varphi_4(z)\big) &= 2[\Gamma_1\mu(z)+\varphi_4(z)]-\Psi_4\big(\Gamma_1\mu(z)+\varphi_4(z)\big)-\varphi_4(z)\\ &= \Gamma_1\mu(z)+\Gamma_1\mu(z)+\varphi_4(z)-\Psi_4\big(\Gamma_1\mu(z)+\varphi_4(z)\big)\\ &= \Gamma_1\mu(z)+\Psi_1\mu(z)+\varphi_1(z)+\varphi_4(z)-\Psi_4\big(\Psi_1\mu(z)+\varphi_1(z)+\varphi_4(z)\big).\end{aligned}$$

Using (U_1);
we get $\Gamma_4\big(\Gamma_1\mu(z)+\varphi_4(z)\big)=\Gamma_1\mu(z)$,
which implies that $\Gamma_1(Y)\subseteq\Gamma_4(Y)$.
Now, $\Gamma_2(Y)\subseteq\Gamma_5(Y)$, and thus we have

$$\begin{aligned}\Gamma_5\big(\Gamma_2\mu(z)+\varphi_5(z)\big) &= 2[\Gamma_2\mu(z)+\varphi_5(z)]-\Psi_5\big(\Gamma_2\mu(z)+\varphi_5(z)\big)-\varphi_5(z)\\ &= \Gamma_2\mu(z)+\Gamma_2\mu(z)+\varphi_5(z)-\Psi_5\big(\Gamma_2\mu(z)+\varphi_5(z)\big)\\ &= \Gamma_2\mu(z)+\Psi_2\mu(z)+\varphi_2(z)+\varphi_5(z)-\Psi_5\big(\Psi_2\mu(z)+\varphi_2(z)+\varphi_5(z)\big).\end{aligned}$$

Using (U_2),
we get $\Gamma_5\big(\Gamma_2\mu(z)+\varphi_5(z)\big)=\Gamma_2\mu(z)$,
which implies that $\Gamma_2(Y)\subseteq\Gamma_5(Y)$.
Similarly, one can prove that $\Gamma_1(Y)\subseteq\Gamma_3(Y)$ and $\Gamma_2(Y)\subseteq\Gamma_6(Y)$.
Next, we need to show that the pair (Γ_1,Γ_5) is compatible.
For this, let us have a sequence x_η such that $\lim_{\eta\to\infty}\Gamma_1 x_\eta=\lim_{\eta\to\infty}\Gamma_5 x_\eta=x$.
To prove that (Γ_1,Γ_5) is compatible,
it is enough to prove that $d(\Gamma_1\Gamma_5 x,\Gamma_5\Gamma_1 x)=0$ when $d(\Gamma_1 x,\Gamma_5 x)=0$ for some $x\in Y$.
With the help of (U3),

$$\begin{aligned}\parallel \Gamma_1(x)-\Gamma_5(x)\parallel &= \parallel \Psi_1 x(z)+\varphi_1(z)-2x(z)+\Psi_5 x(z)+\varphi_5(z)\parallel\\ &= \parallel -2x(z)+\Psi_1 x(z)+\varphi_1(z)+\Psi_5 x(z)+\varphi_5(z)\parallel\\ &= \parallel -2x(z)+2x(z)\parallel=0.\end{aligned}$$

So, $d(\Gamma_1 x, \Gamma_5 x) = 0$.

Now,

$$\begin{aligned} \| \Gamma_1\Gamma_5(x) - \Gamma_5\Gamma_1(x) \| &= \| \Gamma_1\big(2x(z) - \Psi_5 x(z) - \varphi_5(z)\big) + \varphi_1(z) - \Gamma_5\big(\Psi_1 x(z) + \varphi_1(z)\big) \| \\ &= \| 2\Psi_1 x(z) - \Psi_1\Psi_5 x(z) - \Psi_1\varphi_5(z) + \varphi_1(z) - 2\Psi_1 x(z) - 2\varphi_1(z) \\ &\quad + \Psi_5\Psi_1 x(z) + \Psi_5\varphi_1(z) + \varphi_5(z) \| \\ &= \| -\Psi_1\varphi_5(z) - \Psi_1\Psi_5 x(z) + \varphi_1(z) + \Psi_5\Psi_1 x(z) \\ &\quad + \Psi_5\varphi_1(z) + \varphi_5(z) - 2\varphi_1(z) \| = 0 \end{aligned}$$

Thus, $d(\Gamma_1\Gamma_5 x, \Gamma_5\Gamma_1 x) = 0$,
which implies that (Γ_1, Γ_5) is compatible.

Similarly, by using (U_3) and (U_5), we can show that the pairs (Γ_1, Γ_3), (Γ_1, Γ_4) and (Γ_1, Γ_6) are also compatible and by using (U_4) and (U_6) one can prove the compatibility of (Γ_1, Γ_2).

Thus, by Theorem 2, we can find a unique common fixed point of Γ_1, Γ_2, Γ_3, Γ_4, Γ_5, and Γ_6 in Y, that is, System (2) of Urysohn integral equations has a unique common solution in Y. □

4.2. Existence of a Unique Common Solution to the System of Volterra–Hammerstein Integral Equations:

Here, we discuss the existence of a solution for the following System (5) of non-linear Volterra–Hammerstein integral equations.

Let $C = \big(L(0,\infty), R\big)$ be the space of real-valued measurable functions on $(0,\infty)$:

$$\mu(z) = \varrho_i(z) + \gamma \int_0^x w(z,\alpha) g_i\big(\alpha, \mu(\alpha)\big) d\alpha + v \int_0^\infty \eta(z,\alpha) h_i\big(\alpha, \mu(\alpha)\big) d\alpha \tag{5}$$

for all $x \in (0,\infty)$, where γ, v are real numbers, $\varrho_i \in C$ is known and $w(z,\alpha)$, $\eta(z,\alpha)$, $g_i\big(\alpha, \mu(\alpha)\big)$ and $h_i\big(\alpha, \mu(\alpha)\big)$, $i = 1, 2, \ldots, 6$ are real-valued measurable functions in both z and α on $(0,\infty)$.

Let us denote

$$\mho_i \mu(z) = \int_0^x w(z,\alpha) g_i\big(\alpha, \mu(\alpha)\big) d\alpha$$

and

$$\Lambda_i \mu(z) = \int_0^\infty \eta(z,\alpha) h_i\big(\alpha, \mu(\alpha)\big) d\alpha$$

where $i = 1, 2, \ldots, 6$.

Assume that

(V_1): For $i = 4, 3$,

$$\begin{aligned} 0 &= \mho_1\mu(z) + \Lambda_1\mu(z) + \varrho_1(z) + \varrho_i(z) - \mho_i\big(\mho_1\mu(z) + \Lambda_1\mu(z) + \varrho_1(z) + \varrho_i(z)\big) \\ &\quad - \Lambda_i\big(\mho_1\mu(z) + \Lambda_1\mu(z) + \varrho_1(z) + \varrho_i(z) \end{aligned}$$

(V_2): For $j = 5, 6$,

$$\begin{aligned} 0 &= \mho_2\mu(z) + \Lambda_2\mu(z) + \varrho_2(z) + \varrho_j(z) - \mho_j\big(\mho_2\mu(z) + \Lambda_2\mu(z) + \varrho_2(z) + \varrho_j(z)\big) \\ &\quad - \Lambda_j\big(\mho_2\mu(z) + \Lambda_2\mu(z) + \varrho_2(z) + \varrho_j(z)\big). \end{aligned}$$

(V_3): For $i = 3, 4, 5, 6$,

$$\mho_1\mu(z) + \Lambda_1\mu(z) + \varrho_1(z) + \varrho_i(z) + \mho_i\mu(z) + \Lambda_i\mu(z) = 2\mu(z)$$

(V_4): For $j = 2$,

$$\begin{aligned} 0 &= \mho_1\big(\mho_j\mu(z) + \Lambda_j\mu(z) + \varrho_j(z)\big) + \Lambda_1\big(\mho_j\mu(z) + \Lambda_j\mu(z) + \varrho_j(z)\big) + \varrho_2(z) \\ &\quad -\big(\mho_2\big(\mho_j\mu(z) + \Lambda_j\mu(z) + \varrho_j(z)\big) + \Lambda_2\big(\mho_j\mu(z) + \Lambda_j\mu(z) + \varrho_j(z)\big) + \varrho_2(z)\big). \end{aligned}$$

(V_5): For $i = 3, 4, 5, 6$,

$$\begin{aligned} 0 &= \mho_1\big(2\mu(z) - \mho_i\mu(z) - \Lambda_i\mu(z) - \varrho_i(z)\big) + \Lambda_1\big(2\mu(z) - \mho_i\mu(z) - \Lambda_i\mu(z) - \varrho_i(z)\big) \\ &\quad +\varrho_1(z) - \big(2\big(\mho_1\mu(z) + \Lambda_1\mu(z) + \varrho_1(z)\big) - \mho_i\big(\mho_1\mu(z) + \Lambda_1\mu(z) + \varrho_1(z)\big) \\ &\quad -\Lambda_i\big(\mho_1\mu(z) + \Lambda_1\mu(z) + \varrho_1(z)\big) - \varrho_i(z)\big) \end{aligned}$$

Let $C = \big(L(0,\infty), R\big)$ be a complex valued b-metric space with metric:

$$d(s, m) = \max_{a \in (0,\infty)} \parallel s(a) - m(a) \parallel_\infty \sqrt{1 + x^2}e^{\iota tan^{-1}x}$$

for all $s, m \in C$.

Define six operators $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4, \Upsilon_5$ and Υ_6: $C \to C$ by

$$\left.\begin{aligned} \Upsilon_1\mu(z) &= \mho_1\mu(z) + \Lambda_1\mu(z) + \varrho_1(z), \\ \Upsilon_2\mu(z) &= \mho_2\mu(z) + \Lambda_2\mu(z) + \varrho_2(z), \\ \Upsilon_3\mu(z) &= 2\mu(z) - \mho_3\mu(z) - \Lambda_3(z) - \varrho_3(z), \\ \Upsilon_4\mu(z) &= 2\mu(z) - \mho_4\mu(z) - \Lambda_4(z) - \varrho_4(z), \\ \Upsilon_5\mu(z) &= 2\mu(z) - \mho_5\mu(z) - \Lambda_5(z) - \varrho_5(z), \\ \Upsilon_6\mu(z) &= 2\mu(z) - \mho_6\mu(z) - \Lambda_6(z) - \varrho_6(z). \end{aligned}\right\} \tag{6}$$

Now, we have to prove our existence results.

Theorem 4. *(1): Based on these suppositions* $(V_1$–$V_5)$*, if for each* $s, m \in C$ *and* $b \geqslant 1$, $\gamma \in (0, 1)$

$$\gimel_1\sqrt{1 + x^2}e^{\iota tan^{-1}x} \precsim \frac{\gamma}{b^2}\left\{\gimel_2, \gimel_3, \gimel_4, \gimel_5, \frac{\gimel_6 \times \gimel_7}{1 + \gimel_8}\right\}$$

where

$$\left.\begin{aligned} \gimel_1 &= \parallel \mho_1\mu(z) + \Lambda_1\mu(z) + \varrho_1(z) - \mho_2\mu(z) - \Lambda_2\mu(z) - \varrho_2(z) \parallel_\infty \sqrt{1 + x^2}e^{\iota tan^{-1}x}, \\ \gimel_2 &= \parallel 2\mu(z) - \mho_3\mu(z) - \Lambda_3(z) - \varrho_3(z) - 2\mu(z) + \mho_4\mu(z) + \Lambda_4\mu(z) + \varrho_4(z) \parallel_\infty \sqrt{1 + x^2}e^{\iota tan^{-1}x}, \\ \gimel_3 &= \parallel \mho_1\mu(z) + \Lambda_1\mu(z) + \varrho_1(z) - 2\mu(z) + \mho_6\mu(z) + \Lambda_6\mu(z) + \varrho_6(z) \parallel_\infty \sqrt{1 + x^2}e^{\iota tan^{-1}x}, \\ \gimel_4 &= \parallel \mho_1\mu(z) + \Lambda_1\mu(z) + \varrho_1(z) - 2\mu(z) + \mho_3\mu(z) + \Lambda_3\mu(z) + \varrho_3(z) \parallel_\infty \sqrt{1 + x^2}e^{\iota tan^{-1}x}, \\ \gimel_5 &= \parallel \mho_2\mu(z) + \Lambda_2\mu(z) + \varrho_2(z) - 2\mu(z) + \mho_5\mu(z) + \Lambda_5\mu(z) + \varrho_5 z) \parallel_\infty \sqrt{1 + x^2}e^{\iota tan^{-1}x}, \\ \gimel_6 &= \parallel \mho_1\mu(z) + \Lambda_1\mu(z) + \varrho_1(z) - 2\mu(z) + \mho_4\mu(z) + \Lambda_4\mu(z) + \varrho_4(z) \parallel_\infty \sqrt{1 + x^2}e^{\iota tan^{-1}x}, \\ \gimel_7 &= \parallel \mho_2\mu(z) + \Lambda_2\mu(z) + \varrho_2(z) - 2\mu(z) + \mho_3\mu(z) + \Lambda_3\mu(z) + \varrho_3(z) \parallel_\infty \sqrt{1 + x^2}e^{\iota tan^{-1}x}, \\ \gimel_8 &= \parallel 2\mu(z) - \mho_6\mu(z) - \Lambda_6\mu(z) - \varrho_6(z) - 2\mu(z) + \mho_5\mu(z) + \Lambda_5\mu(z) + \varrho_5(z) \parallel_\infty \sqrt{1 + x^2}e^{\iota tan^{-1}x} \end{aligned}\right\} \tag{7}$$

(2): $\Upsilon_1(Z) \subseteq \Upsilon_4(Z)$,$\Upsilon_1(Z) \subseteq \Upsilon_3(Z)$,$\Upsilon_2(Z) \subseteq \Upsilon_5(Z)$ *and* $\Upsilon_2(Z) \subseteq \Upsilon_6(Z)$.
(3): (Υ_1, Υ_2), (Υ_1, Υ_3), (Υ_1, Υ_4), (Υ_1, Υ_5) *and* (Υ_1, Υ_6) *are compatible.*
Then, the system of Volterra–Hammertion equations (5) has a unique common solution.

Proof. Note that System (5) has a unique common solution if and only if System (6) of operators has a unique common fixed point.
Now,

$$\left.\begin{array}{l}
d(\Upsilon_1 s, \Upsilon_2 m) = \max_{z\in(0,\infty)} \parallel \mho_1 s(z) + \Lambda_1 s(z) + \varrho_1(z) - \mho_2 m(z) \\
-\Lambda_2 m(z) - \varrho_2(z) \parallel_\infty \sqrt{1+x^2}e^{\iota tan^{-1}x}, \\
d(\Upsilon_3 s, \Upsilon_4 m) = \max_{z\in(0,\infty)} \parallel 2s(z) - \mho_3 s(z) - \Lambda_3 s(z) - \varrho_3(z) - 2m(z) + \mho_4 m(z) \\
+\Lambda_4 m(z) + \varrho_4(z) \parallel_\infty \sqrt{1+x^2}e^{\iota tan^{-1}x}, \\
d(\Upsilon_1 s, \Upsilon_6 m) = \max_{z\in(0,\infty)} \parallel \mho_1 s(z) + \Lambda_1 s(z) + \varrho_1(z) - 2m(z) + \mho_6 m(z) \\
+\Lambda_6 m(z) + \varrho_6(z) \parallel_\infty \sqrt{1+x^2}e^{\iota tan^{-1}x}, \\
d(\Upsilon_1 m, \Upsilon_3 m) = \max_{z\in(0,\infty)} \parallel \mho_1 m(z) + \Lambda_1 m(z) + \varrho_1(z) - 2m(z) + \mho_3 m(z) \\
+\Lambda_3 m(z) + \varrho_3(z) \parallel_\infty \sqrt{1+x^2}e^{\iota tan^{-1}x}, \\
d(\Upsilon_2 m, \Upsilon_5 m) = \max_{z\in(0,\infty)} \parallel \mho_2 m(z) + \Lambda_2 m(z) + \varrho_2(z) - 2m(z) + \mho_5 m(z) \\
+\Lambda_5 m(z) + \varrho_5 z) \parallel_\infty \sqrt{1+x^2}e^{\iota tan^{-1}x}, \\
d(\Upsilon_1 s, \Upsilon_2 s) = \max_{z\in(0,\infty)} \parallel \mho_1 s(z) + \Lambda_1 s(z) + \varrho_1(z) - 2s(z) + \mho_4 s(z) \\
+\Lambda_4 s(z) + \varrho_4(z) \parallel_\infty \sqrt{1+x^2}e^{\iota tan^{-1}x}, \\
d(\Upsilon_2 m, \Upsilon_3 m) = \max_{z\in(0,\infty)} \parallel \mho_2 m(z) + \Lambda_2 m(z) + \varrho_2(z) - 2m(z) + \mho_3 m(z) \\
+\Lambda_3 m(z) + \varrho_3(z) \parallel_\infty \sqrt{1+x^2}e^{\iota tan^{-1}x}, \\
d(\Upsilon_6 s, \Upsilon_5 m) = \max_{z\in(0,\infty)} \parallel 2s(z) - \mho_6 s(z) - \Lambda_6 s(z) - \varrho_6(z) - 2m(z) + \mho_5 m(z) \\
+\Lambda_5 m(z) + \varrho_5(z) \parallel_\infty \sqrt{1+x^2}e^{\iota tan^{-1}x}
\end{array}\right\} \quad (8)$$

From condition (CM2) of Theorem 2, we have

$$\beth_1\sqrt{1+x^2}e^{\iota tan^{-1}x} \precsim \frac{\varrho}{b^2}\left\{\beth_2, \beth_3, \beth_4, \beth_5, \frac{\beth_6 \times \beth_7}{1+\beth_8}\right\},$$

which implies that

$$\max_{a\in(x,y)} \beth_1\sqrt{1+x^2}e^{\iota tan^{-1}x} \precsim \frac{\varrho}{b^2}\left\{\max_{a\in(0,\infty)} \beth_2, \max_{a\in(0,\infty)} \beth_3, \max_{a\in(0,\infty)} \beth_4, \max_{a\in(0,\infty)} \beth_5, \frac{\max_{a\in(0,\infty)} \beth_6 \times \max_{a\in(0,\infty)} \beth_7}{1+\max_{a\in(0,\infty)} \beth_8}\right\}.$$

Using the above distances in Equation (8), we obtain

$$d(\Upsilon_1 s, \Upsilon_2 m) \precsim \frac{\varrho}{b^2}\left\{d(\Upsilon_3 s, \Upsilon_4 m), d(\Upsilon_1 s, \Upsilon_6 m), d(\Upsilon_1 m, \Upsilon_3 m), d(\Upsilon_2 m, \Upsilon_5 m), \frac{d(\Upsilon_1 s, \Upsilon_4 s)d(\Upsilon_2 m, \Upsilon_3 m)}{1+d(\Upsilon_6 s, \Upsilon_5 m)}\right\}.$$

Now, to show that $\Upsilon_1(Z) \subseteq \Upsilon_3(Z)$, we have

$$\begin{aligned}
\Upsilon_1(\Upsilon_1\mu(z) + \varrho_3(z)) &= 2[\Upsilon_1\mu(z) + \varrho_3(z)] - \mho_3[\Upsilon_1\mu(z) + \varrho_3(z)] - \Lambda_3[\Upsilon_1\mu(z) + \varrho_3(z)] - \varrho_3(z) \\
&= \Upsilon_1\mu(z) + \Upsilon_1\mu(z) + \varrho_3(z) - \mho_3[\Upsilon_1\mu(z) + \Upsilon_1\mu(z) + \varrho_3(z)] \\
&\quad -\Lambda_3[\Upsilon_1\mu(z) + \Upsilon_1\mu(z) + \varrho_1(z) + \varrho_3(z)]
\end{aligned}$$

From condition (V_1),

$$\Upsilon_1(\Upsilon_1\mu(z) + \varrho_3(z)) = \Upsilon_1\mu(z).$$

This implies that $\Upsilon_1(Z) \subseteq \Upsilon_3(Z)$.

Similarly, we can show that $\Upsilon_1(Z) \subseteq \Upsilon_4(Z)$, $\Upsilon_2(Z) \subseteq \Upsilon_5(Z)$ and $\Upsilon_2(Z) \subseteq \Upsilon_6(Z)$ by using (V_1) and (V_2).

Next, we have to show the compatibility of the pairs (Υ_1, Υ_2), (Υ_1, Υ_3), (Υ_1, Υ_4), (Υ_1, Υ_5) and (Υ_1, Υ_6).

For this, let us have a sequence x_η such that $\lim_{\eta\to\infty} \Upsilon_1 x_\eta = \lim_{\eta\to\infty} \Upsilon_5 x_\eta = x$.

To prove that (Υ_1, Υ_5) is compatible, it is enough to prove that $d(\Upsilon_1\Upsilon_5 x, \Upsilon_5\Upsilon_1 x) = 0$ when $d(\Upsilon_1 x, \Upsilon_5 x) = 0$ for some $x \in C$.

With the help of (V_3),

$$\begin{aligned}
\| \Upsilon_1(x) - \Upsilon_5(x) \| &= \| \mho_1 x(z) + \Lambda_1 x(z) + \varrho_1(z) - 2x(z) + \mho_5 x(z) + \Lambda_5 x(z) + \varrho_5(z) \| \\
&= \| -2x(z) + \mho_1 x(z) + \Lambda_1 x(z) + \varrho_1(z) + \Lambda_5 x(z) + \varrho_5(z) + \varrho_5(z) \| \\
&= \| 2x(z) + 2x(z) \| = 0.
\end{aligned}$$

So, $d(\Upsilon_1 x, \Upsilon_5 x) = 0$.

Now,

$$\begin{aligned}
\| \Upsilon_1\Upsilon_5(x) - \Upsilon_5\Upsilon_1(x) \| &= \| \Upsilon_1\big(2x(z) - \mho_5 x(z) - \Lambda_5 x(z) - \varrho_5(z)\big) - \Upsilon_5\big(\mho_1 x(z) \\
&\quad + \Lambda_1 x(z) + \varrho_1(z)\big) \| \\
&= \| \mho_1\big(2x(z) - \mho_5 x(z) - \Lambda_5 x(z) - \varrho_5(z)\big) + \Lambda_1\big(2x(z) - \mho_5 x(z) \\
&\quad - \Lambda_5 x(z) - \varrho_5(z)\big) + \varrho_1(z) - [2\big(\mho_1 x(z) + \Lambda_1 x(z) + \varrho_1(z)\big) \\
&\quad - \mho_5\big(\mho_1 x(z) + \Lambda_1 x(z) + \varrho_1(z)\big) - \Lambda_5\big(\mho_1 x(z) \\
&\quad + \Lambda_1 x(z) + \varrho_1(z)\big) - \varrho_5(z)] \| = 0.
\end{aligned}$$

Thus, $d(\Upsilon_1\Upsilon_5 x, \Upsilon_5\Upsilon_1 x) = 0$, which implies that (Υ_1, Υ_5) is compatible.

Similarly, by using (V_3), (V_4) and (V_5), we can show that the pairs (Υ_1, Υ_2), (Υ_1, Υ_4) and (Υ_1, Υ_6) are also compatible.

Thus, by Theorem 2, we can find a unique common fixed point of Υ_1, Υ_2, Υ_3, Υ_4, Υ_5 and Υ_6 in C, that is, System (5) of Volterra–Hammerstein equations has a unique common solution in C. □

5. Conclusions

Many real-world problems can be described by integral equations, and there are various techniques for investigating the solution of a system of integral equations. One of the significant tools is the theory of fixed points.

In the current study, we establish new fixed point results for six self mappings satisfying rational-type inequalities that serve as a useful tool for investigating unique solutions to systems of integral equations. This approach offers new ways to examine complicated mathematical systems and has the potential to significantly advance the study of integral equations.

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