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Solvability of a Boundary Value Problem Involving Fractional Difference Equations

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Abstract: In this current work, we apply the topological degree and fixed point theorems to investigate the existence, uniqueness, and multiplicity of solutions for a boundary value problem associated with a fractional-order difference equation. Moreover, we provide some appropriate examples to verify our main conclusions.

Keywords: fractional difference equations; boundary value problems; solvability; fixed-point theory

MSC: 34B10; 34B18; 34A34; 45G15; 45M20

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1. Introduction

Let $[\kappa,\delta]_{\mathbb{N}_\kappa} := \{\kappa,\kappa+1,\kappa+2,\cdots,\delta\}$ $(\delta-\kappa\in\mathbb{N}_1)$, where $\mathbb{N}_\kappa := \{\kappa,\kappa+1,\kappa+2,\cdots\}$. In the current work, we shall discuss the solvability of the fractional difference boundary value problem

$$\begin{cases} -\Delta_{v-3}^{v}\psi(t) = g(t+v-1,\psi(t+v-1)), \quad t\in[0,\mathfrak{b}+2]_{\mathbb{N}_0}, \\ \psi(v-3) = \left[\Delta_{v-3}^{\alpha}\psi(t)\right]\big|_{t=v-\alpha-2} = \left[\Delta_{v-3}^{\beta}\psi(t)\right]\Big|_{t=v+\mathfrak{b}+2-\beta} = 0, \end{cases} \tag{1}$$

where $v\in(2,3], \beta\in(1,2), v-\beta\in(1,+\infty), \alpha\in(0,1), \mathfrak{b}\in(3,+\infty)(\mathfrak{b}\in\mathbb{N})$, and Δ_{v-3}^{v} is a discrete fractional-order operator defined by

$$\Delta_a^\nu\psi(t) := \begin{cases} \frac{1}{\Gamma(-\nu)}\sum\limits_{s=a}^{t+\nu}(t-s-1)^{\underline{-\nu-1}}\psi(s), & N-1<\nu<N, \\ \Delta^N\psi(t), & \nu=N, \end{cases}$$

where $N\in\mathbb{N}$ with $0\leqslant N-1<v\leqslant N$. As in [1], this definition is equivalent to (2) in Section 2.

The theory of fractional calculus has been widely used in modern mathematics for more than 300 years, and the study of solutions of fractional differential (difference) equations arises in real-world problems in the field of physics, mechanics, chemistry, and engineering. For example, in [2], the authors extended the variational approach to the fractional discrete case and introduced the Gompertz fractional difference equation

$$\Delta_0^\alpha \ln\mathcal{G}(t-\alpha+1) = (b-1)\ln\mathcal{G}(t) + a,$$

which can be used to describe tumor growth, a special relationship between tumor size and time, and is of special interest since growth estimation is very critical in clinical practice. Here, a, b are parameters and $\alpha\in(0,1]$. One can also find some other applications for the

Gompertz fractional difference equation in [1]. In [3], the authors introduced the following discrete logistic map and investigated the chaotic behavior:

$$\begin{cases} {}^{C}\Delta_a^v \psi(t) = \mu\psi(t+v-1)(1-\psi(t+v-1)), t \in \mathbb{N}_{a+1-v}, 0 < v \leq 1, \\ \psi(a) = c, \end{cases}$$

where ${}^{C}\Delta_a^v$ is the left Caputo-like delta difference defined by

$${}^{C}\Delta_a^v \psi(t) = \frac{1}{\Gamma(m-v)} \sum_{s=a}^{t-(m-v)} (t-s-1)^{\underline{m-v-1}} \Delta_s^m \psi(s),$$

where $t \in \mathbb{N}_{a+m-v}, m = [v] + 1$.

We note that in [4], the author mentioned that discretization is inevitable for fractional differential equations. To date, they are only used as the starting point for approximate solution calculations, and there is no special research on fractional difference equations. Therefore, from the perspective of theory and application, this is a big gap. Many developments in the theory are now taking place, and two books [5,6] are sources for mathematicians who are interested in this area. However, we still note that most works focus on fractional-order differential equations, while the research on fractional-order difference equations is quite small (we refer the reader to [5–26]). In [7], the authors investigated positive solutions for the discrete fractional boundary value problems

$$\begin{cases} -\Delta_{v-2}^v \chi(t) = \mathcal{F}(t+v-1, \chi(t+v-1)), 1 < v \leq 2, \\ \chi(v-2) = \Delta_{v-1}^{v-1}\chi(v+N) = 0, \end{cases}$$

where $t \in [0, N+1]_{\mathbb{N}_0}$ and $\mathcal{F} : [v-1, v+N]_{\mathbb{N}_{v-1}} \times \mathbb{R} \to \mathbb{R}^+$ satisfies some superlinear or sublinear conditions. In [8], the authors utilized fixed-point methods to investigate the solvability of a fractional difference equation with a p-Laplacian operator

$$\begin{cases} \Delta^\beta[\phi_p(\Delta^\alpha \chi)](t) + \mathcal{F}(\alpha+\beta+t-1, \chi(\alpha+\beta+t-1)) = 0, t \in [0, \mathfrak{b}]_{\mathbb{N}_0}, \\ \Delta^\alpha \chi(\beta-2) = \Delta^\alpha \chi(\beta+\mathfrak{b}) = 0, \\ \chi(\alpha+\beta-4) = \chi(\alpha+\beta+\mathfrak{b}) = 0, \end{cases}$$

where $\mathcal{F} : [\alpha+\beta-4, \alpha+\beta+\mathfrak{b}]_{\mathbb{N}_{\alpha+\beta-4}} \times \mathbb{R} \to \mathbb{R}$ satisfies a Lipschitz condition, and $\phi_p(z) = |z|^{p-2}z, p > 1, z \in \mathbb{R}$. In [9], the authors utilized the fixed point index to consider the solvability of the system of fractional-order difference boundary value problems

$$\begin{cases} \Delta^v \chi_1(t) = \mathcal{F}_1(t+v-1, \chi_1(t+v-1), \chi_2(t+v-1)), t \in [0, T]_{\mathbb{Z}}, \\ \Delta^v \chi_2(t) = \mathcal{F}_2(t+v-1, \chi_1(t+v-1), \chi_2(t+v-1)), t \in [0, T]_{\mathbb{Z}}, \\ \chi_1(v-1) = \chi_1(v+T), \chi_2(v-1) = \chi_2(v+T), \end{cases}$$

where $\mathcal{F}_i (i = 1, 2)$ are semipositone nonlinearities.

We note that usually one expresses the solutions of fractional-order equations by a Green's function. However, not all fractional-order difference equations can be obtained in this way, for example, in [10], the authors studied the problem

$$\begin{cases} \Delta^\alpha \chi(t) = \mathcal{F}(t+\alpha-1, \chi(t+\alpha-1)),\ t \in [0, T]_{\mathbb{N}_0}, \alpha \in (1, 2], \\ \chi(\alpha-2) = 0, \chi(\alpha+T) = \Delta^{-\beta}\chi(\zeta+\beta),\ \zeta \in \mathbb{N}_{\alpha-2, \alpha+T-1}, \beta > 0, \end{cases}$$

and showed it is equivalent to

$$\chi(t) = -\frac{t^{\underline{\alpha-1}}}{\Theta\Gamma(\alpha)}\left[\frac{1}{\Gamma(\beta)}\sum_{s=\alpha}^{\zeta}\sum_{\xi=0}^{s-\alpha}(\zeta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}\mathcal{F}(\xi+\alpha-1,\chi(\xi+\alpha-1))\right.$$
$$\left.-\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}\mathcal{F}(s+\alpha-1,\chi(s+\alpha-1))\right]+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}\mathcal{F}(s+\alpha-1,\chi(s+\alpha-1)),$$

where
$$\Theta = \sum_{s=1}^{\zeta-\alpha+2}\frac{(\zeta+\beta-s-\alpha+1)^{\underline{\beta-1}}\Gamma(s+\alpha-1)}{\Gamma(\beta)\Gamma(s)} - \frac{\Gamma(\alpha+T+1)}{\Gamma(T+2)}.$$

Clearly, the integral form is very complicated and cannot be formulated via some suitable Green's function.

Inspired by the aforementioned works, in this paper, via a Green's function, we use the topological degree and fixed point theorems to consider the existence, uniqueness, and multiplicity of solutions to (1). Furthermore, we present some examples to illustrate our main results.

2. Preliminary

In this section, we first offer some basic materials for discrete fractional calculus; see [5–26] and the references therein.

Definition 1. *Let* $t^{\underline{v}} := \frac{\Gamma(t+1)}{\Gamma(t+1-v)}$, $\forall t, v \in \mathbb{R}$. *If* $t+1-v$ *is a pole of* $\Gamma(\cdot)$ *and* $t+1$ *is not a pole, then* $t^{\underline{v}} = 0$.

Definition 2. *For* $v > 0$, *a function* $\mathcal{F}$*'s* v*-th fractional sum is defined by*

$$\Delta_a^{-v}\mathcal{F}(t) = \frac{1}{\Gamma(v)}\sum_{s=a}^{t-v}(t-s-1)^{\underline{v-1}}\mathcal{F}(s), \ \ t \in \mathbb{N}_{a+v}.$$

$\mathcal{F}$*'s* v*-th fractional difference is defined by*

$$\Delta_a^{v}\mathcal{F}(t) = \Delta^N\Delta_a^{v-N}\mathcal{F}(t), \ \ t \in \mathbb{N}_{a+N-v}, \tag{2}$$

where $N \in \mathbb{N}$ *with* $0 \leqslant N-1 < v \leqslant N$.

Let $\chi : [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}} \to \mathbb{R}$ *be a given function. Then, we consider the problem*

$$\begin{cases} -\Delta_{v-3}^{v}\psi(t) = \chi(t+v-1), \quad t \in [0, \mathfrak{b}+2]_{\mathbb{N}_0}, \\ \psi(v-3) = \left[\Delta_{v-3}^{\alpha}\psi(t)\right]\Big|_{t=v-\alpha-2} = \left[\Delta_{v-3}^{\beta}\psi(t)\right]\Big|_{t=v+\mathfrak{b}+2-\beta} = 0, \end{cases} \tag{3}$$

where $v, \alpha, \beta, \mathfrak{b}$ *can be founded in* (1).

Lemma 1 (see [11])**.** *Problem* (3) *has a unique solution*

$$\psi(t) = \sum_{s=0}^{\mathfrak{b}+2}\mathbb{G}(t,s)\chi(s+v-1), \quad t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}},$$

where $\mathbb{G}$ *is the Green's function given by*

$$\mathbb{G}(t,s) = \frac{1}{\Gamma(v)}\begin{cases} \frac{t^{\underline{v-1}}(v+\mathfrak{b}-\beta-s+1)^{\underline{v-\beta-1}}}{(v+\mathfrak{b}-\beta+2)^{\underline{v-\beta-1}}} - (t-s-1)^{\underline{v-1}}, 0 \le s < t-v+1 \le \mathfrak{b}+2, \\ \frac{t^{\underline{v-1}}(v+\mathfrak{b}-\beta-s+1)^{\underline{v-\beta-1}}}{(v+\mathfrak{b}-\beta+2)^{\underline{v-\beta-1}}}, 0 \le t-v+1 \le s \le \mathfrak{b}+2. \end{cases} \tag{4}$$

Lemma 2 (see [11])**.** *The Green's function* (4) *has the properties*

(G1) $\mathbb{G}(t,s) > 0, (t,s) \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}} \times [0, \mathfrak{b}+2]_{\mathbb{N}_0}$,

(G2) $\frac{t^{\underline{v-1}}\mathbb{G}(\mathfrak{b}+v+1,s)}{(\mathfrak{b}+v+1)^{\underline{v-1}}} \leqslant \mathbb{G}(t,s) \leqslant \frac{t^{\underline{v-1}}(v+\mathfrak{b}-\beta-s+1)^{\underline{v-\beta-1}}}{\Gamma(v)(v+\mathfrak{b}-\beta+2)^{\underline{v-\beta-1}}}, (t,s) \in [v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}} \times [0,\mathfrak{b}+2]_{\mathbb{N}_0}$,

(G3) $q(t)\mathbb{G}(\mathfrak{b}+v+1,s) \leqslant \mathbb{G}(t,s) \leqslant \mathbb{G}(\mathfrak{b}+v+1,s), q(t) = \frac{t^{\underline{v-1}}}{(\mathfrak{b}+v+1)^{\underline{v-1}}}, (t,s) \in [v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}} \times [0,\mathfrak{b}+2]_{\mathbb{N}_0}$.

Lemma 3 (see [11]). *Let $\varphi(s+v-1) = \mathbb{G}(\mathfrak{b}+v+1,s)$, $s \in [0,\mathfrak{b}+2]_{\mathbb{N}_0}$. Then, the following inequalities hold:*

$$\sum_{t=v-1}^{\mathfrak{b}+v+1} \mathbb{G}(t,s)\varphi(t) \leqslant \kappa_2\varphi(s+v-1), \kappa_2 = \sum_{t=v-1}^{\mathfrak{b}+v+1} \varphi(t), s \in [0,\mathfrak{b}+2]_{\mathbb{N}_0}, \tag{5}$$

and

$$\sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)q(s+v-1) \geq \kappa_1 q(t), \kappa_1 = \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(\mathfrak{b}+v+1,s)q(s+v-1), t \in [v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}. \tag{6}$$

Let E be a set of all maps from $[v-3,\mathfrak{b}+v+1]_{\mathbb{N}_{v-3}}$ to $\mathbb{R}$, and $\|\psi\| = \max_{t\in[v-3,\mathfrak{b}+v+1]_{\mathbb{N}_{v-3}}} |\psi(t)|$. Then, E is a Banach space. Moreover, define a set $P = \{\psi \in E : \psi(t) \geqslant 0, t \in [v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}\}$. Then, P is a cone on E. Lemma 3 enables us to obtain that (1) is equivalent to the sum equation

$$\psi(t) = \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)g(s+v-1,\psi(s+v-1)) := (\mathcal{B}\psi)(t), \quad t \in [v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}},$$

where G is defined in Lemma 3. Obviously, $\psi \in E\backslash\{0\}$ is a solution for (1) when $\psi \in E\backslash\{0\}$ is a fixed point of $\mathcal{B}$.

Lemma 4. *Let $P_0 = \{\psi \in P : \psi(t) \geqslant q(t)\|\psi\|, \forall t \in [v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}\}$. Then, $\mathcal{L}(P) \subset P_0$, where*

$$(\mathcal{L}\psi)(t) = \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)\psi(s+v-1), \quad t \in [v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}.$$

Lemma 5 (see [27] Theorem A.3.3). *Let E be a Banach space, $\Omega \subset E$ a bounded open set, and $\mathcal{T} : \Omega \to E$ be a continuous compact operator. If there is an $\mathfrak{x}_0 \in E\backslash\{0\}$ such that*

$$\mathfrak{x} - \mathcal{T}\mathfrak{x} \neq \mu\mathfrak{x}_0, \forall \mathfrak{x} \in \partial\Omega, \mu \geq 0,$$

then $\deg(I-\mathcal{T},\Omega,0) = 0$*, where deg denotes the topological degree.*

Lemma 6 (see [27] Lemma 2.5.1). *Let E be a Banach space, $\Omega \subset E$ a bounded open set with $0 \in \Omega$, and $\mathcal{T} : \Omega \to E$ be a continuous compact operator. If*

$$\mathcal{T}\mathfrak{x} \neq \mu\mathfrak{x}, \forall \mathfrak{x} \in \partial\Omega, \mu \geq 1,$$

then $deg(I-\mathcal{T},\Omega,0) = 1$.

Lemma 7 (see [28,29]). *Let X be a Banach space and P be a cone on X. Define functionals as follows: $\alpha, \gamma : P \to \mathbb{R}^+$ are continuous increasing and $\beta : P \to \mathbb{R}^+$ is continuous. Moreover, there exists $M > 0, 0 < \widetilde{a} < \widetilde{c}$ such that*

$$\alpha(0) < \widetilde{a}, \gamma(\mathfrak{x}) \leq \beta(\mathfrak{x}) \leq \alpha(\mathfrak{x}) \text{ and } \|\mathfrak{x}\| \leq M\gamma(\mathfrak{x}), \ \forall \mathfrak{x} \in \overline{P(\gamma,\widetilde{c})} := \overline{\{\mathfrak{x} \in P : \gamma(\mathfrak{x}) < \widetilde{c}\}}.$$

Furthermore, there is a completely continuous operator $\mathcal{T} : \overline{P(\gamma,\widetilde{c})} \to P$ and a constant $\widetilde{b} > 0$ with $0 < \widetilde{a} < \widetilde{b} < \widetilde{c}$ such that $\beta(\lambda\mathfrak{x}) \leq \lambda\beta(\mathfrak{x})$ for $\lambda \in (0,1]$, $\mathfrak{x} \in \partial P(\beta,\widetilde{b})$, and

(E1) $\gamma(\mathcal{T}\mathfrak{x}) < \widetilde{c}, \forall \mathfrak{x} \in \partial P(\gamma, \widetilde{c})$;
(E2) $\beta(\mathcal{T}\mathfrak{x}) > \widetilde{b}, \forall \mathfrak{x} \in \partial P(\beta, \widetilde{b})$;
(E3) $\alpha(\mathcal{T}\mathfrak{x}) < \widetilde{a}, \forall \mathfrak{x} \in \partial P(\alpha, \widetilde{a})$.
Then, $\mathcal{T}$ has at least three fixed points $\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3 \in \overline{P(\gamma, \widetilde{c})}$ such that

$$0 \leq \alpha(\mathfrak{x}_1) < \widetilde{a} < \alpha(\mathfrak{x}_2), \beta(\mathfrak{x}_2) < \widetilde{b} < \beta(\mathfrak{x}_3), \gamma(\mathfrak{x}_3) < \widetilde{c}.$$

In the following, we present some lemmas involving the theory of mixed monotone operators. Let $(E, \|\cdot\|)$ be a real Banach space which is partially ordered by a cone $P \subset E$, i.e., $x \leq y \Leftrightarrow y - x \in P$. If $x \leq y$ and $x \neq y$, then we mean that $x < y$ or $y > x$. Moreover, for a fixed $h > 0$, we define $P_h = \{x \in E \mid x \sim h\}$, in which $\sim$ is an equivalence relation, i.e., $x \sim y$ implies that there are $\lambda, \mu > 0$ such that $\lambda x \geq y \geq \mu x, \forall x, y \in E$.

Definition 3 (see [30,31])**.** *If $\mathfrak{u}_i$, $\mathfrak{v}_i (i = 1, 2) \in P, \mathfrak{u}_1 < \mathfrak{u}_2, \mathfrak{v}_1 > \mathfrak{v}_2$ imply $A(\mathfrak{u}_1, \mathfrak{v}_1) \leq A(\mathfrak{u}_2, \mathfrak{v}_2)$, then $A : P \times P \to P$ is called a mixed monotone operator.*

Definition 4 (see [30,31])**.** *If $A(t\mathfrak{x}) \geq tA\mathfrak{x}$, $\forall t \in (0, 1), \mathfrak{x} \in P$, then $A : P \to P$ is said to be sub-homogeneous.*

Lemma 8 (see [30,31])**.** *Let $\mathcal{B} : P \to P$ be an increasing sub-homogeneous operator, $\mathcal{A} : P \times P \to P$ a mixed monotone operator and satisfy*

$$\mathcal{A}\Big(t\mathfrak{x}, t^{-1}\mathfrak{y}\Big) \geq t^{\alpha}\mathcal{A}(\mathfrak{x}, \mathfrak{y}),\ t, \alpha \in (0, 1), \mathfrak{x}, \mathfrak{y} \in P. \tag{7}$$

If
(C1) There is a $\mathfrak{h}_0 \in P_{\mathfrak{h}}$ such that $\mathcal{A}(\mathfrak{h}_0, \mathfrak{h}_0) \in P_{\mathfrak{h}}$ and $\mathcal{B}\mathfrak{h}_0 \in P_{\mathfrak{h}}$;
(C2) There is a constant $\delta_0 > 0$ such that $\mathcal{A}(\mathfrak{x}, \mathfrak{y}) \geq \delta_0 \mathcal{B}\mathfrak{x}, \forall \mathfrak{x}, \mathfrak{y} \in P$.
Then,
(D1) $\mathcal{A} : P_{\mathfrak{h}} \times P_{\mathfrak{h}} \to P_{\mathfrak{h}}, \mathcal{B} : P_{\mathfrak{h}} \to P_{\mathfrak{h}}$;
(D2) There are $\mathfrak{u}_0, \mathfrak{v}_0 \in P_{\mathfrak{h}}$ and $r \in (0, 1)$ such that $r\mathfrak{v}_0 \leq \mathfrak{u}_0 < \mathfrak{v}_0$, $\mathfrak{u}_0 \leq \mathcal{A}(\mathfrak{u}_0, \mathfrak{v}_0) + \mathcal{B}\mathfrak{u}_0 \leq \mathcal{A}(\mathfrak{v}_0, \mathfrak{u}_0) + \mathcal{B}(\mathfrak{v}_0) \leq \mathfrak{v}_0$;
(D3) $\mathcal{A}(\mathfrak{x}, \mathfrak{x}) + \mathcal{B}\mathfrak{x} = \mathfrak{x}$ has a unique solution $\mathfrak{x}^*$ in $P_{\mathfrak{h}}$;
(D4) For any initial values $\mathfrak{x}_0, \mathfrak{y}_0 \in P_{\mathfrak{h}}$, the sequences $\mathfrak{x}_n = \mathcal{A}(\mathfrak{x}_{n-1}, \mathfrak{y}_{n-1}) + \mathcal{B}\mathfrak{x}_{n-1}$, $\mathfrak{y}_n = \mathcal{A}(\mathfrak{y}_{n-1}, \mathfrak{x}_{n-1}) + \mathcal{B}\mathfrak{y}_{n-1}$ converge to $\mathfrak{x}^*$ as $n \to \infty$.

3. Main Results

In the section, we will state our main theorems and give their proof. In the first theorem, we obtain an existence result on nontrivial solutions for (1) when the nonlinearity can change sign.

Theorem 1. *Suppose that the following assumptions hold:*
(H1) $g(t, \psi) : [v - 1, \mathfrak{b} + v + 1]_{\mathbb{N}_{v-1}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function;
(H2) There are nonnegative continuous functions $\gamma_1(t), \gamma_2(t)$ and $\mathcal{M}(\psi)$ with $\gamma_2(t) \not\equiv 0$, $t \in [v - 1, \mathfrak{b} + v + 1]_{\mathbb{N}_{v-1}}$ such that

$$g(t, \psi) \geq -\gamma_1(t) - \gamma_2(t)\mathcal{M}(\psi), (t, \psi) \in [v - 1, \mathfrak{b} + v + 1]_{\mathbb{N}_{v-1}} \times \mathbb{R};$$

(H3) $\lim_{|\psi| \to +\infty} \frac{\mathcal{M}(\psi)}{|\psi|} = 0$;
(H4) $\liminf_{|\psi| \to +\infty} \frac{g(t,\psi)}{|\psi|} > \kappa_1^{-1}$, *uniformly in* $t \in [v - 1, \mathfrak{b} + v + 1]_{\mathbb{N}_{v-1}}$;
(H5) $\liminf_{|\psi| \to 0^+} \frac{|g(t,\psi)|}{|\psi|} < \kappa_2^{-1}$, *uniformly in* $t \in [v - 1, \mathfrak{b} + v + 1]_{\mathbb{N}_{v-1}}$.
Then, (1) has one nontrivial solution.

Proof. From (H3), for any given $\varepsilon > 0$, there exists $Y_0 > 0$ such that $\mathcal{M}(\psi) \leq \varepsilon|\psi|$ for $|\psi| > Y_0$. Let $\mathcal{M}^* = \max_{|\psi|\in[0,Y_0]} \mathcal{M}(\psi)$. Then, we have

$$\mathcal{M}(\psi) \leq \varepsilon|\psi| + \mathcal{M}^*, \ \psi \in \mathbb{R}. \tag{8}$$

By (H4), there exist $\delta_1 > 0$ and $Y_1 \geq Y_0$ such that $g(t,\psi) \geq (\kappa_1^{-1} + \delta_1)|\psi|$ for $|\psi| > Y_1$ and $t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}$. Furthermore, let $\widetilde{C}_g = \max_{(t,\psi)\in[v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}\times[0,Y_1]} |g(t,\psi)|$. Then, we obtain

$$g(t,\psi) \geq (\kappa_1^{-1} + \delta_1)|\psi| - \widetilde{C}_g, \ t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}, \psi \in \mathbb{R}.$$

Note that δ_1 can be greater than $\varepsilon\|\gamma_2\|$; then, from (H2) and (H3) and (8), we have

$$g(t,\psi) \geq (\kappa_1^{-1} + \delta_1 - \varepsilon\|\gamma_2\|)|\psi| - \gamma_1(t) - C_g, \ t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}, \psi \in \mathbb{R}, \tag{9}$$

where $C_g = \widetilde{C}_g + \|\gamma_2\|\mathcal{M}^*$. Let

$$\mathcal{R} > \max\left\{ \frac{\kappa_2(\|\gamma_1\| + \|\gamma_2\|\mathcal{M}^* + C_g)}{1 - \varepsilon\kappa_2\|\gamma_2\|}, \frac{(\|\gamma_1\| + \|\gamma_2\|\mathcal{M}^* + C_g)\left[\frac{\kappa_2(\delta_1 - \varepsilon\|\gamma_2\|)}{(\mathfrak{b}+v+1)^{\underline{v-1}}} + (\kappa_1^{-1} + \delta_1 - \varepsilon\|\gamma_2\|)\mathcal{N}_1\right]}{\frac{\delta_1 - \varepsilon\|\gamma_2\|}{(\mathfrak{b}+v+1)^{\underline{v-1}}}(1 - \varepsilon\kappa_2\|\gamma_2\|) - \varepsilon\|\gamma_2\|\mathcal{N}_1(\kappa_1^{-1} + \delta_1 - \varepsilon\|\gamma_2\|)} \right\}, \tag{10}$$

where

$$\mathcal{N}_1 = \sum_{\tau=0}^{\mathfrak{b}+2} \frac{(v+\mathfrak{b}-\beta-\tau+1)^{\underline{v-\beta-1}}}{(v+\mathfrak{b}-\beta+2)^{\underline{v-\beta-1}}\Gamma(v)}.$$

We prove that

$$\psi - \mathcal{B}\psi \neq \mu q, \ \psi \in \partial B_{\mathcal{R}}, \mu \geq 0, \tag{11}$$

where q is given in Lemma 4, and

$$B_{\mathcal{R}} = \{\psi \in E : \|\psi\| < \mathcal{R}\}, \ \partial B_{\mathcal{R}} = \{\psi \in E : \|\psi\| = \mathcal{R}\}.$$

Proof by contradiction. Then, there are $\psi \in \partial B_{\mathcal{R}}, \mu \geq 0$ such that

$$\psi - \mathcal{B}\psi = \mu q. \tag{12}$$

Note that if $\mu = 0$ and $\psi \in \partial B_{\mathcal{R}}$ is a nontrivial solution to (1), the theorem has been obtained. So, we only consider the case $\mu > 0$. Moreover, we also find that

$$q \in P_0.$$

In order to prove our theorem, we need to define a function $\widetilde{\psi}$ as follows:

$$\widetilde{\psi}(t) = \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)\left[\gamma_1(s+v-1) + \gamma_2(s+v-1)\mathcal{M}(\psi(s+v-1)) + C_g\right], t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}, \psi \in \partial B_{\mathcal{R}}.$$

Then, we get the following claims:
Claim i. Note that $\gamma_1 + \gamma_2\mathcal{M}(\psi) + C_g \in P$, and Lemma 6 implies that

$$\widetilde{\psi} \in P_0. \tag{13}$$

Claim ii. From (12), we find

$$\begin{aligned}\psi(t) + \widetilde{\psi}(t) &= (\mathcal{B}\psi)(t) + \widetilde{\psi}(t) + \mu q(t) \\ &= \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)\left[g(s+v-1, \psi(s+v-1)) + \gamma_1(s+v-1) + \gamma_2(s+v-1)\mathcal{M}(\psi(s+v-1)) + C_g\right] + \mu q(t),\end{aligned}$$

for all $t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}$. Note that $g + \gamma_1 + \gamma_2\mathcal{M} + C_g \in P$ and $q \in P_0$, and we have

$$\psi + \widetilde{\psi} \in P_0. \tag{14}$$

Claim iii. From (8) and (10), we have

$$\begin{aligned}
\|\widetilde{\psi}\| &\leq \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(\mathfrak{b}+v+1, s)\left[\|\gamma_1\| + \|\gamma_2\|(\varepsilon\|\psi\| + \mathcal{M}^*) + C_g\right] \\
&= \kappa_2\left[\|\gamma_1\| + \|\gamma_2\|(\varepsilon\|\psi\| + \mathcal{M}^*) + C_g\right] \\
&< \mathcal{R}.
\end{aligned}$$

From Claim ii and (9), we have

$$\begin{aligned}
(\mathcal{B}\psi)(t) + \widetilde{\psi}(t) &= \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)\left[g(s+v-1, \psi(s+v-1)) + \gamma_1(s+v-1) + \gamma_2(s+v-1)\mathcal{M}(\psi(s+v-1)) + C_g\right] \\
&\geq \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)\left[g(s+v-1, \psi(s+v-1)) + \gamma_1(s+v-1) + C_g\right] \\
&\geq \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)\left[(\kappa_1^{-1} + \delta_1 - \varepsilon\|\gamma_2\|)|\psi(s+v-1)| - \gamma_1(s+v-1) - C_g + \gamma_1(s+v-1) + C_g\right] \\
&\geq (\kappa_1^{-1} + \delta_1 - \varepsilon\|\gamma_2\|) \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)\psi(s+v-1) \\
&= (\kappa_1^{-1} + \delta_1 - \varepsilon\|\gamma_2\|)\left[\sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)\left[\psi(s+v-1) + \widetilde{\psi}(s+v-1)\right] - \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)\widetilde{\psi}(s+v-1)\right] \\
&\geq \kappa_1^{-1} \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)\left[\psi(s+v-1) + \widetilde{\psi}(s+v-1)\right].
\end{aligned} \tag{15}$$

The last inequality in (15) holds if

$$(\delta_1 - \varepsilon\|\gamma_2\|) \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)\left[\psi(s+v-1) + \widetilde{\psi}(s+v-1)\right] - (\kappa_1^{-1} + \delta_1 - \varepsilon\|\gamma_2\|) \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)\widetilde{\psi}(s+v-1) \geq 0, \tag{16}$$

for $t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}$. In what follows, we prove (16). Indeed, from Claim ii we have $\psi(t) + \widetilde{\psi}(t) \geq q(t)\|\psi + \widetilde{\psi}\| \geq q(t)(\|\psi\| - \|\widetilde{\psi}\|), t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}$. Therefore, from (4) and (10), we obtain

$$\begin{aligned}
&(\delta_1 - \varepsilon\|\gamma_2\|) \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)\left[\psi(s+v-1) + \widetilde{\psi}(s+v-1)\right] - (\kappa_1^{-1} + \delta_1 - \varepsilon\|\gamma_2\|) \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)\widetilde{\psi}(s+v-1) \\
&\geq (\delta_1 - \varepsilon\|\gamma_2\|) \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)q(s+v-1)(\|\psi\| - \|\widetilde{\psi}\|) \\
&\quad - (\kappa_1^{-1} + \delta_1 - \varepsilon\|\gamma_2\|) \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s) \sum_{\tau=0}^{\mathfrak{b}+2} \mathbb{G}(s+v-1, \tau)\left[\gamma_1(\tau+v-1) + \gamma_2(\tau+v-1)\mathcal{M}(\psi(\tau+v-1)) + C_g\right] \\
&\geq (\delta_1 - \varepsilon\|\gamma_2\|) \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)\frac{(s+v-1)^{\underline{v-1}}}{(\mathfrak{b}+v+1)^{\underline{v-1}}}(\|\psi\| - \|\widetilde{\psi}\|) \\
&\quad - (\kappa_1^{-1} + \delta_1 - \varepsilon\|\gamma_2\|) \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s) \sum_{\tau=0}^{\mathfrak{b}+2} \frac{(s+v-1)^{\underline{v-1}}(v+\mathfrak{b}-\beta-\tau+1)^{\underline{v-\beta-1}}}{(v+\mathfrak{b}-\beta+2)^{\underline{v-\beta-1}}\Gamma(v)}\left[\|\gamma_1\| + \|\gamma_2\|(\varepsilon\|\psi\| + \mathcal{M}^*) + C_g\right] \\
&\geq \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)(s+v-1)^{\underline{v-1}}\left[\frac{\delta_1 - \varepsilon\|\gamma_2\|}{(\mathfrak{b}+v+1)^{\underline{v-1}}}(\mathcal{R} - \kappa_2[\|\gamma_1\| + \|\gamma_2\|(\varepsilon\mathcal{R} + \mathcal{M}^*) + C_g])\right. \\
&\quad \left. - (\kappa_1^{-1} + \delta_1 - \varepsilon\|\gamma_2\|)[\|\gamma_1\| + \|\gamma_2\|(\varepsilon\mathcal{R} + \mathcal{M}^*) + C_g] \sum_{\tau=0}^{\mathfrak{b}+2} \frac{(v+\mathfrak{b}-\beta-\tau+1)^{\underline{v-\beta-1}}}{(v+\mathfrak{b}-\beta+2)^{\underline{v-\beta-1}}\Gamma(v)}\right] \\
&\geq 0.
\end{aligned}$$

This implies that (15) holds, as required. Consequently, we have

$$(\mathcal{B}\psi)(t)+\widetilde{\psi}(t)\geq \kappa_1^{-1}\mathcal{L}(\psi+\widetilde{\psi})(t), t\in[v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}.$$

Using (12), we obtain

$$\psi+\widetilde{\psi}=\mathcal{B}\psi+\widetilde{\psi}+\mu q\geq \kappa_1^{-1}\mathcal{L}(\psi+\widetilde{\psi})+\mu q\geq \mu q,\ \psi\in\partial B_{\mathcal{R}},\mu>0.$$

Define

$$\mu^*=\sup\{\mu>0:\psi+\widetilde{\psi}\geq \mu q\}.$$

Note that $\mu^*\geq\mu$ and $\psi+\widetilde{\psi}\geq\mu^* q$, and from (6), we have

$$\psi+\widetilde{\psi}\geq \kappa_1^{-1}\mathcal{L}(\mu^* q)+\mu q=\kappa_1^{-1}\mu^*\mathcal{L}q+\mu q\geq(\mu^*+\mu)q,$$

which contradicts the definition of μ^*. Hence, (11) holds, and Lemma 7 enables us to find

$$\deg(I-\mathcal{B},B_{\mathcal{R}},0)=0. \tag{17}$$

From (H5), there exist $\delta_2\in(0,\kappa_2^{-1})$ and $r>0$ such that

$$|g(t,\psi)|\leq(\kappa_2^{-1}-\delta_2)|\psi|,\ |\psi|\in[0,r], t\in[v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}. \tag{18}$$

For this r, we prove that

$$\mathcal{B}\psi\neq\mu\psi,\psi\in\partial B_r,\mu\geq 1. \tag{19}$$

Proof by contradiction. Then, there are $\psi\in\partial B_r,\mu\geq 1$ such that

$$\mathcal{B}\psi=\mu\psi\Rightarrow|\psi|=\frac{1}{\mu}|\mathcal{B}\psi|\leq|\mathcal{B}\psi|.$$

This, together with (18), implies that

$$|\psi(t)|\leq\sum_{s=0}^{\mathfrak{b}+2}\mathbb{G}(t,s)|g(s+v-1,\psi(s+v-1))|\leq(\kappa_2^{-1}-\delta_2)\sum_{s=0}^{\mathfrak{b}+2}\mathbb{G}(t,s)|\psi(s+v-1)|. \tag{20}$$

Multiplying by $\varphi(t)$ on the both sides of (20) and summing over $[v-1,\mathfrak{b}+v+1]$, then (5) implies that

$$\begin{aligned}\sum_{t=v-1}^{\mathfrak{b}+v+1}|\psi(t)|\varphi(t)&\leq(\kappa_2^{-1}-\delta_2)\sum_{t=v-1}^{\mathfrak{b}+v+1}\sum_{s=0}^{\mathfrak{b}+2}\mathbb{G}(t,s)\varphi(t)|\psi(s+v-1)|\\&\leq(\kappa_2^{-1}-\delta_2)\kappa_2\sum_{s=0}^{\mathfrak{b}+2}|\psi(s+v-1)|\varphi(s+v-1)\\&=(\kappa_2^{-1}-\delta_2)\kappa_2\sum_{t=v-1}^{\mathfrak{b}+v+1}|\psi(t)|\varphi(t).\end{aligned}$$

This implies that $\sum_{t=v-1}^{\mathfrak{b}+v+1}|\psi(t)|\varphi(t)=0$, and thus $\psi(t)\equiv 0, t\in[v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}$. Clearly, this is contradictory to $\psi\in\partial B_r$. Hence, Lemma 8 shows that

$$\deg(I-\mathcal{B},B_r,0)=1. \tag{21}$$

Equations (17) and (21) enable us to obtain

$$\deg\big(I-\mathcal{B},B_{\mathcal{R}}\backslash\overline{B}_r,0\big)=\deg(I-\mathcal{B},B_{\mathcal{R}},0)-\deg(I-\mathcal{B},B_r,0)=-1.$$

This implies that $\mathcal{B}$ has a fixed point in $B_{\mathcal{R}} \backslash \overline{B}_r$, and (1) has a nontrivial solution. □

In the following theorem, using the generalized Avery–Henderson fixed point theorem (Lemma 9), we obtain triple positive solutions for (1) when the nonlinearity satisfies some bounded conditions.

Theorem 2. *Suppose that there exist positive constants* $\widetilde{a}, \widetilde{b}, \widetilde{c}$ *with* $\widetilde{a} < \widetilde{b} < \widetilde{c}$, $\frac{\widetilde{c}}{\widetilde{b}} > \frac{\sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(\mathfrak{b}+v+1,s)}{\sum_{t=r}^{\mathfrak{b}+v+1} \mathbb{G}(\mathfrak{b}+v+1,t-v+1)}$ *(r is a fixed point in* $(v-1, \mathfrak{b}+v+1)_{\mathbb{N}_{v-1}}$*) such that*

(H6) $g(t,\psi) : [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}} \times \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function, and* $g(t,0) \not\equiv 0$, $t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}$*;*

(H7) $g(t,\psi) < \frac{\widetilde{c}}{q_0 \sum_{s=0}^{\mathfrak{b}+2} G(\mathfrak{b}+v+1,s)}$*, for* $t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}, \psi \in [0, \widetilde{c}q_0^{-2}]$*;*

(H8) $g(t,\psi) > \frac{\widetilde{b}}{q_0 \sum_{t=r}^{\mathfrak{b}+v+1} G(\mathfrak{b}+v+1,t-v+1)}$ *for* $t \in [r, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}, \psi \in [\widetilde{b}, \widetilde{b}q_0^{-2}]$*;*

(H9) $g(t,\psi) < \frac{\widetilde{a}}{\sum_{s=0}^{\mathfrak{b}+2} G(\mathfrak{b}+v+1,s)}$ *for* $t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}, \psi \in [0, \widetilde{a}]$.

Then, (1) *has at least three positive solutions* $\mathfrak{y}_1, \mathfrak{y}_2$ *and* $\mathfrak{y}_3$ *satisfying*

$$0 < \alpha(\mathfrak{y}_1) < \widetilde{a} < \alpha(\mathfrak{y}_2), \beta(\mathfrak{y}_2) < \widetilde{b} < \beta(\mathfrak{y}_3), \gamma(\mathfrak{y}_3) < \widetilde{c}.$$

Proof. Note that if $q_0 = \min_{t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}} q(t) > 0$, then from Lemma 6 and (H6) we have

$$\mathcal{B}(P) \subset P_0.$$

Let $\alpha(\psi) = \max_{t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}} \psi(t)$, $\beta(\psi) = \min_{t \in [r, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}} \psi(t)$ and $\gamma(\psi) = q_0 \max_{t \in [v-1, r]_{\mathbb{N}_{v-1}}} \psi(t)$. We easily know that $\alpha, \gamma : P \to \mathbb{R}^+$ are continuous, increasing functionals with $\alpha(0) = 0$, $\forall t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}$, $\psi \in P$ and $\beta(\lambda\psi) = \lambda\beta(\psi)$. Moreover, for $\psi \in P_0$, we have

$$\gamma(\psi) = q_0 \max_{t \in [v-1, r]_{\mathbb{N}_{v-1}}} \psi(t) \le q_0 \|\psi\| \le \min_{t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}} \psi(t) \le \min_{t \in [r, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}} \psi(t) = \beta(\psi) \le \alpha(\psi),$$

and

$$\gamma(\psi) \ge q_0 \min_{t \in [v-1, r]_{\mathbb{N}_{v-1}}} \psi(t) \ge q_0 \min_{t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}} \psi(t) \ge q_0^2 \|\psi\|,$$

i.e.,

$$\|\psi\| \le \frac{1}{q_0^2} \gamma(\psi).$$

(i) For $\psi \in \partial P(\gamma, \widetilde{c})$, we have

$$\widetilde{c} = \gamma(\psi) \ge q_0^2 \|\psi\|,$$

which implies that

$$0 \le \psi(t) \le \widetilde{c}q_0^{-2}, \ t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}.$$

By (H7), we find

$$\begin{aligned}
\gamma(\mathcal{B}\psi) &= q_0 \max_{t\in[v-1,r]_{\mathbb{N}_{v-1}}} \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)g(s+v-1,\psi(s+v-1)) \\
&\leq q_0 \max_{t\in[v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}} \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)g(s+v-1,\psi(s+v-1)) \\
&< \frac{\widetilde{c}}{q_0 \sum\limits_{s=0}^{\mathfrak{b}+2} G(\mathfrak{b}+v+1,s)} q_0 \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(\mathfrak{b}+v+1,s) \\
&= \widetilde{c}.
\end{aligned}$$

(ii) For $\psi \in \partial P(\beta, \widetilde{b})$, we have

$$\widetilde{b} = \beta(\psi) = \min_{t\in[r,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}} \psi(t) \leq \|\psi\| \leq \frac{1}{q_0^2}\gamma(\psi) \leq \frac{1}{q_0^2}\beta(\psi) = \frac{\widetilde{b}}{q_0^2}.$$

This implies that

$$\widetilde{b} \leq \psi(t) \leq \frac{\widetilde{b}}{q_0^2}, \psi \in \partial P(\beta,\widetilde{b}), t \in [r, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}.$$

This, combined with (H8), enables us to obtain

$$\begin{aligned}
\beta(\mathcal{B}\psi) &= \min_{t\in[r,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}} \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)g(s+v-1,\psi(s+v-1)) \\
&\geq \sum_{s=0}^{\mathfrak{b}+2} \min_{t\in[r,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}} q(t)\mathbb{G}(\mathfrak{b}+v+1,s)g(s+v-1,\psi(s+v-1)) \\
&\geq \sum_{s=0}^{\mathfrak{b}+2} \min_{t\in[v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}} q(t)\mathbb{G}(\mathfrak{b}+v+1,s)g(s+v-1,\psi(s+v-1)) \\
&= q_0 \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(\mathfrak{b}+v+1,s)g(s+v-1,\psi(s+v-1)) \\
&= q_0 \sum_{t=v-1}^{\mathfrak{b}+v+1} \mathbb{G}(\mathfrak{b}+v+1,t-v+1)g(t,\psi(t)) \\
&\geq q_0 \sum_{t=r}^{\mathfrak{b}+v+1} \mathbb{G}(\mathfrak{b}+v+1,t-v+1)g(t,\psi(t)) \\
&> \frac{\widetilde{b}}{q_0 \sum\limits_{t=r}^{\mathfrak{b}+v+1} \mathbb{G}(\mathfrak{b}+v+1,t-v+1)} q_0 \sum_{t=r}^{\mathfrak{b}+v+1} \mathbb{G}(\mathfrak{b}+v+1,t-v+1) \\
&= \widetilde{b}.
\end{aligned}$$

(iii) For $\psi \in \partial P(\alpha, \widetilde{a})$, we have

$$0 \leq \alpha(\psi) = \max_{t\in[v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}} \psi(t) = \widetilde{a},$$

and

$$0 \leq \psi(t) \leq \widetilde{a},\ \psi \in \partial P(\alpha,\widetilde{a}), t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}.$$

This, together with (H9), implies that

$$\begin{aligned}\alpha(\mathcal{B}\psi) &= \max_{t\in[v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}} \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)g(s+v-1,\psi(s+v-1)) \\ &< \frac{\widetilde{a}}{\sum\limits_{s=0}^{\mathfrak{b}+2} \mathbb{G}(\mathfrak{b}+v+1,s)} \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(\mathfrak{b}+v+1,s) \\ &= \widetilde{a}.\end{aligned}$$

Now, we have established that all the conditions in Lemma 9 hold, and note that $\mathcal{B}0 \neq 0$, so we conclude that (1) has at least three positive solutions $\mathfrak{y}_i \in P\backslash\{0\}$ such that $0 < \alpha(\mathfrak{y}_1) < \widetilde{a} < \alpha(\mathfrak{y}_2), \beta(\mathfrak{y}_2) < \widetilde{b} < \beta(\mathfrak{y}_3), \gamma(\mathfrak{y}_3) < \widetilde{c}$. $\square$

In what follows, we study the problem

$$\begin{cases} -\Delta_{v-3}^{v}\psi(t) = f(t+v-1,\psi(t+v-1),\psi(t+v-1)) + g(t+v-1,\psi(t+v-1)), & t \in [0,\mathfrak{b}+2]_{\mathbb{N}_0}, \\ \psi(v-3) = \left[\Delta_{v-3}^{\alpha}\psi(t)\right]\big|_{t=v-\alpha-2} = \left[\Delta_{v-3}^{\beta}\psi(t)\right]\Big|_{t=v+\mathfrak{b}+2-\beta} = 0, \end{cases} \tag{22}$$

where $v,\alpha,\beta,\mathfrak{b}$ are founded in (1). By Lemma 3, (22) is equivalent to the following equation

$$\psi(t) = \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)[f(s+v-1,\psi(s+v-1),\psi(s+v-1)) + g(s+v-1,\psi(s+v-1))],\ t \in [v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}},$$

and let $\mathcal{A} : P\times P \to P$ and $\mathcal{B} : P\times P \to P$ be defined by

$$\mathcal{A}(\mathfrak{y},\mathfrak{x})(t) = \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)f(s+v-1,\mathfrak{y}(s+v-1),\mathfrak{x}(s+v-1)),\ (\mathcal{B}\mathfrak{y})(t) = \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)g(s+v-1,\mathfrak{y}(s+v-1)).$$

Obviously, $\mathfrak{y}^*$ is a solution of (1) when $\mathfrak{y}^* = \mathcal{A}(\mathfrak{y}^*,\mathfrak{y}^*) + \mathcal{B}\mathfrak{y}^*$. In the following theorem, we study the operators $\mathcal{A},\mathcal{B}$ to help us to obtain the existence of solutions to (22). Moreover, the positive solution is unique, and it can be uniformly approximated by two appropriate iterative sequences.

Now, we list some assumptions for our nonlinearities f,g as follows:

(H10) $f(t,\mathfrak{u},\mathfrak{v}) : [v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+, g(t,\mathfrak{u}) : [v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}} \times \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions;

(H11) $f(t,\mathfrak{u},\mathfrak{v})$ is increasing about $\mathfrak{u} \in \mathbb{R}^+$ for fixed $t \in [v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}$ and $\mathfrak{v} \in \mathbb{R}^+$ and decreasing about $\mathfrak{v} \in \mathbb{R}^+$ for fixed $t \in [v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}$ and $\mathfrak{u} \in \mathbb{R}^+$, and $g(t,\mathfrak{u})$ is increasing about $\mathfrak{u} \in \mathbb{R}^+$ for fixed $t \in [v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}$;

(H12) For every $t \in [v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}, \gamma \in (0,1), \mathfrak{u},\mathfrak{v} \in \mathbb{R}^+$, there is a constant $\xi \in (0,1)$ such that $f(t,\gamma\mathfrak{u},\gamma^{-1}\mathfrak{v}) \geq \gamma^{\xi} f(t,\mathfrak{u},\mathfrak{v})$ and $g(t,\gamma\mathfrak{u}) \geq \gamma g(t,\mathfrak{u})$;

(H13) For every $t \in [v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}$ and $\mathfrak{u},\mathfrak{v} \in \mathbb{R}^+$, there is a constant $\delta_0 > 0$ such that $f(t,\mathfrak{u},\mathfrak{v}) \geq \delta_0 g(t,\mathfrak{u})$.

Theorem 3. *Suppose that (H10)–(H13) hold. Then, we get*

(T1) There are $\mathfrak{y}_0,\mathfrak{x}_0 \in P_{\mathfrak{h}}$ and $r \in (0,1)$ such that $r\mathfrak{x}_0 \leq \mathfrak{y}_0 < \mathfrak{x}_0$,

$$\mathfrak{y}_0(t) \leq \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)[f(s+v-1,\mathfrak{y}_0(s+v-1),\mathfrak{x}_0(s+v-1)) + g(s+v-1,\mathfrak{y}_0(s+v-1))],$$

and

$$\mathfrak{x}_0(t) \geq \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)[f(s+v-1,\mathfrak{x}_0(s+v-1),\mathfrak{y}_0(s+v-1)) + g(s+v-1,\mathfrak{x}_0(s+v-1))],$$

where $\mathfrak{h}(t) = t^{\underline{v-1}}$;

(T2) (22) *has a unique positive solution* $\mathfrak{y}^* \in P_{\mathfrak{h}}$;

(T3) For each initial value $\mathfrak{x}_0, \mathfrak{y}_0 \in P_{\mathfrak{h}}$, *the sequences*

$$\mathfrak{x}_n = \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)[f(s+v-1, \mathfrak{x}_{n-1}(s+v-1), \mathfrak{y}_{n-1}(s+v-1)) + g(s+v-1, \mathfrak{x}_{n-1}(s+v-1))],$$

$$\mathfrak{y}_n = \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s)[f(s+v-1, \mathfrak{y}_{n-1}(s+v-1), \mathfrak{x}_{n-1}(s+v-1)) + g(s+v-1, \mathfrak{y}_{n-1}(s+v-1))],$$

converge to $\mathfrak{y}^*$ *as* $n \to \infty$.

Proof. From (H10) and (H11), we know that $\mathcal{A} : P \times P \to P$ is a mixed monotone operator and $\mathcal{B} : P \to P$ is an increasing operator. Using (H12), for all $\gamma \in (0,1)$ and $\mathfrak{x}, \mathfrak{y} \in P$, we obtain

$$\begin{aligned}
\mathcal{A}(\gamma\mathfrak{y}, \gamma^{-1}\mathfrak{x})(t) &= \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s) f(s+v-1, \gamma\mathfrak{y}(s+v-1), \gamma^{-1}\mathfrak{x}(s+v-1)) \\
&\geq \gamma^{\xi} \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s) f(s+v-1, \mathfrak{y}(s+v-1), \mathfrak{x}(s+v-1)) \\
&= \gamma^{\xi} \mathcal{A}(\mathfrak{y}, \mathfrak{x})(t),
\end{aligned}$$

and hence $\mathcal{A}$ satisfies (7) in Lemma 12. In addition, for any $\mathfrak{y} \in P$ and $\gamma \in (0,1)$ we find

$$\begin{aligned}
(\mathcal{B}\gamma\mathfrak{y})(t) &= \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s) g(s+v-1, \gamma\mathfrak{y}(s+v-1)) \\
&\geq \gamma \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s) g(s+v-1, \mathfrak{y}(s+v-1)) \\
&= \gamma(\mathcal{B}\mathfrak{y})(t).
\end{aligned}$$

Thus, $\mathcal{B}$ is a sub-homogeneous operator.

Let $\mathfrak{h}_0 = \mathfrak{h} = t^{\underline{v-1}}$, and $\mathfrak{h}_0 \in P_h$. From Lemma 4, we have

$$\begin{aligned}
\mathcal{A}(\mathfrak{h}_0, \mathfrak{h}_0)(t) &= \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s) f(s+v-1, (s+v-1)^{\underline{v-1}}, (s+v-1)^{\underline{v-1}}) \\
&\leq \sum_{s=0}^{\mathfrak{b}+2} \frac{t^{\underline{v-1}}(v+\mathfrak{b}-\beta-s+1)^{\underline{v-\beta-1}}}{\Gamma(v)(v+\mathfrak{b}-\beta+2)^{\underline{v-\beta-1}}} f(s+v-1, (s+v-1)^{\underline{v-1}}, (s+v-1)^{\underline{v-1}}) \\
&\leq \sum_{s=0}^{\mathfrak{b}+2} \frac{t^{\underline{v-1}}(v+\mathfrak{b}-\beta-s+1)^{\underline{v-\beta-1}}}{\Gamma(v)(v+\mathfrak{b}-\beta+2)^{\underline{v-\beta-1}}} f(s+v-1, (\mathfrak{b}+v+1)^{\underline{v-1}}, 0) \\
&= \sum_{s=0}^{\mathfrak{b}+2} \frac{(v+\mathfrak{b}-\beta-s+1)^{\underline{v-\beta-1}}}{\Gamma(v)(v+\mathfrak{b}-\beta+2)^{\underline{v-\beta-1}}} f(s+v-1, (\mathfrak{b}+v+1)^{\underline{v-1}}, 0) \cdot \mathfrak{h}_0,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{A}(\mathfrak{h}_0, \mathfrak{h}_0)(t) &= \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s) f(s+v-1, (s+v-1)^{\underline{v-1}}, (s+v-1)^{\underline{v-1}}) \\
&\geq \sum_{s=0}^{\mathfrak{b}+2} \frac{t^{\underline{v-1}}\mathbb{G}(\mathfrak{b}+v+1, s)}{(\mathfrak{b}+v+1)^{\underline{v-1}}} f(s+v-1, (s+v-1)^{\underline{v-1}}, (s+v-1)^{\underline{v-1}}) \\
&\geq \sum_{s=0}^{\mathfrak{b}+2} \frac{t^{\underline{v-1}}\mathbb{G}(\mathfrak{b}+v+1, s)}{(\mathfrak{b}+v+1)^{\underline{v-1}}} f(s+v-1, 0, (\mathfrak{b}+v+1)^{\underline{v-1}}) \\
&= \sum_{s=0}^{\mathfrak{b}+2} \frac{\mathbb{G}(\mathfrak{b}+v+1, s)}{(\mathfrak{b}+v+1)^{\underline{v-1}}} f(s+v-1, 0, (\mathfrak{b}+v+1)^{\underline{v-1}}) \cdot \mathfrak{h}_0.
\end{aligned}$$

Let $l = \sum_{s=0}^{\mathfrak{b}+2} \frac{\mathbb{G}(\mathfrak{b}+v+1,s)}{(\mathfrak{b}+v+1)^{\underline{v-1}}} f(s+v-1, 0, (\mathfrak{b}+v+1)^{\underline{v-1}}), L = \sum_{s=0}^{\mathfrak{b}+2} \frac{(v+\mathfrak{b}-\beta-s+1)^{\underline{v-\beta-1}}}{\Gamma(v)(v+\mathfrak{b}-\beta+2)^{\underline{v-\beta-1}}} f(s+v-1, (\mathfrak{b}+v+1)^{\underline{v-1}}, 0)$. Then, we have $l\mathfrak{h}_0 \le \mathcal{A}(\mathfrak{h}_0, \mathfrak{h}_0) \le L\mathfrak{h}_0$, i.e., $\mathcal{A}(\mathfrak{h}_0, \mathfrak{h}_0) \in P_{\mathfrak{h}_0}$. Similarly, from (H11), we have

$$\begin{aligned}
&t^{\underline{v-1}} \sum_{s=0}^{\mathfrak{b}+2} \frac{\mathbb{G}(\mathfrak{b}+v+1,s)}{(\mathfrak{b}+v+1)^{\underline{v-1}}} g(s+v-1, 0) \\
&\le (\mathcal{B}\mathfrak{h}_0)(t) = \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s) g(s+v-1, (s+v-1)^{\underline{v-1}}) \\
&\le t^{\underline{v-1}} \sum_{s=0}^{\mathfrak{b}+2} \frac{(v+\mathfrak{b}-\beta-s+1)^{\underline{v-\beta-1}}}{\Gamma(v)(v+\mathfrak{b}-\beta+2)^{\underline{v-\beta-1}}} g(s+v-1, (\mathfrak{b}+v+1)^{\underline{v-1}}).
\end{aligned}$$

Thus, we obtain $\mathcal{B}\mathfrak{h}_0 \in P_{\mathfrak{h}_0}$. Therefore, (C1) in Lemma 12 holds.

Finally, for every $\mathfrak{x}, \mathfrak{y} \in P$, from (H13) we have

$$\begin{aligned}
\mathcal{A}(\mathfrak{y}, \mathfrak{x})(t) &= \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s) f(s+v-1, \mathfrak{y}(s+v-1), \mathfrak{x}(s+v-1)) \\
&\ge \delta_0 \sum_{s=0}^{\mathfrak{b}+2} \mathbb{G}(t,s) g(s+v-1, \mathfrak{y}(s+v-1)) \\
&= \delta_0 (\mathcal{B}\mathfrak{y})(t).
\end{aligned}$$

Thus, (C2) in Lemma 12 holds. Then, our conclusions are true from Lemma 12. □

4. Examples

In this section, we will provide some examples to verify our main results.

Example 1. *Let* $g(t,\psi) = a|\psi| - b\mathcal{M}(\psi), \mathcal{M}(\psi) = \ln(|\psi|+1), \psi \in \mathbb{R}, t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}$, *where* $a \in \left(\kappa_1^{-1}, +\infty\right)$ *and* $b \in \left(a, a+\kappa_2^{-1}\right)$. *Then,* $\lim_{|\psi|\to+\infty} \frac{\mathcal{M}(\psi)}{|\psi|} = 0$, *and* $\lim_{|\psi|\to+\infty} \frac{a|\psi|-b\mathcal{M}(\psi)}{|\psi|} = a > \kappa_1^{-1}$, $\lim_{|\psi|\to0^+} \frac{|a|\psi|-b\mathcal{M}(\psi)|}{|\psi|} = |a-b| < \kappa_2^{-1}$. *Therefore, (H1)–(H5) hold.*

Example 2. *Let* $\mathfrak{b} = 4, v = 2.5, \alpha = 0.5, \beta = 1.4$. *Then,* $[v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}} = \{1.5, 2.5, 3.5, 4.5, 5.5, 6.5, 7.5\}$, $[0, \mathfrak{b}+2]_{\mathbb{N}_0} = \{0,1,2,3,4,5,6\}$, $q_0 = \min_{t\in[v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}} \frac{t^{\underline{1.5}}}{7.5^{\underline{1.5}}} = 0.068$, *and if* $r = 6.5$ *we also obtain*

$$\sum_{s=0}^{6} \mathbb{G}(7.5, s) = \sum_{s=0}^{6} \frac{7.5^{\underline{1.5}}(6.1-s)^{\underline{0.1}}}{7.1^{\underline{0.1}}} - 5.5^{\underline{1.5}} = 112.26,$$

$$\sum_{t=r}^{\mathfrak{b}+v+1} \mathbb{G}(\mathfrak{b}+v+1, t-v+1) = \sum_{t=r+1-v}^{\mathfrak{b}+2} \mathbb{G}(\mathfrak{b}+v+1, t) = \sum_{t=5}^{6} \mathbb{G}(7.5, t) = 19.86.$$

Let $\widetilde{a} = 1, \widetilde{b} = 4, \widetilde{c} = 24$, *and*

$$g(t,\psi) = \begin{cases} 0.008, & \psi \in [0,1], \\ \psi - 0.992, & \psi \in [1,4], \\ 3.008, & \psi \in [4, +\infty). \end{cases}$$

Then, g satisfies

(I) $g(t,\psi) < \frac{\widetilde{c}}{q_0 \sum_{s=0}^{\mathfrak{b}+2} G(\mathfrak{b}+v+1,s)} = 3.144$, *for* $t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}, \psi \in [0, 5190.3]$;

(II) $g(t,\psi) > \frac{\widetilde{b}}{q_0 \sum\limits_{t=r}^{\mathfrak{b}+v+1} G(\mathfrak{b}+v+1,t-v+1)} = 2.96$ *for* $t \in [r, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}, \psi \in [4, 865.1]$;

(III) $g(t,\psi) < \frac{\widetilde{a}}{\sum\limits_{s=0}^{\mathfrak{b}+2} G(\mathfrak{b}+v+1,s)} = 0.009$ *for* $t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}, \psi \in [0,1]$.

Therefore, (H6)–(H9) hold.

Example 3. *Let* $f(t,\mathfrak{u},\mathfrak{v}) = (\mathfrak{b}+v+1-t)^{-\frac{1}{3}} t^{-\frac{2}{3}} \mathfrak{u}^{\frac{1}{3}} + \mathfrak{v}^{-\frac{1}{5}}, g(t,\mathfrak{u}) = (\mathfrak{b}+v+1-t)^{-\frac{1}{3}} t^{-\frac{2}{3}} \mathfrak{u}^{\frac{1}{3}}$, $(t,\mathfrak{u},\mathfrak{v}) \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}} \times \mathbb{R}^+ \times \mathbb{R}^+$. *Then, f is increasing about $\mathfrak{u}$ and decreasing about $\mathfrak{v}$, and g is increasing about $\mathfrak{u}$. For any $\gamma \in (0,1), \mathfrak{u}, \mathfrak{v} \in \mathbb{R}^+$, taking $\xi = \frac{1}{2}$, then $\gamma^{\xi} \in (\gamma, 1)$ and we obtain*

$$\begin{aligned} f\left(t, \gamma\mathfrak{u}, \gamma^{-1}\mathfrak{v}\right) &= (\mathfrak{b}+v+1-t)^{-\frac{1}{3}} t^{-\frac{2}{3}} (\gamma\mathfrak{u})^{\frac{1}{3}} + \left(\gamma^{-1}\mathfrak{v}\right)^{-\frac{1}{5}} \\ &= \gamma^{\frac{1}{3}} (\mathfrak{b}+v+1-t)^{-\frac{1}{3}} t^{-\frac{2}{3}} \mathfrak{u}^{\frac{1}{3}} + \gamma^{\frac{1}{5}} \mathfrak{v}^{-\frac{1}{5}} \\ &\geq \gamma^{\frac{1}{2}} \left[(\mathfrak{b}+v+1-t)^{-\frac{1}{3}} t^{-\frac{2}{3}} \mathfrak{u}^{\frac{1}{3}} + \mathfrak{v}^{-\frac{1}{5}} \right] \\ &= \gamma^{\xi} f(t,\mathfrak{u},\mathfrak{v}), \end{aligned}$$

and

$$\begin{aligned} g(t,\gamma\mathfrak{u}) &= (\mathfrak{b}+v+1-t)^{-\frac{1}{3}} t^{-\frac{2}{3}} (\gamma\mathfrak{u})^{\frac{1}{3}} \\ &= \gamma^{\frac{1}{3}} (\mathfrak{b}+v+1-t)^{-\frac{1}{3}} t^{-\frac{2}{3}} \mathfrak{u}^{\frac{1}{3}} \\ &\geq \gamma \left[(\mathfrak{b}+v+1-t)^{-\frac{1}{3}} t^{-\frac{2}{3}} \mathfrak{u}^{\frac{1}{3}} \right] \\ &= \gamma g(t,\mathfrak{u}). \end{aligned}$$

Moreover, it is easy to see that $f(t,\mathfrak{u},\mathfrak{v}) \geq g(t,\mathfrak{u})$ for $(t,\mathfrak{u},\mathfrak{v}) \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}} \times \mathbb{R}^+ \times \mathbb{R}^+$. Therefore, (H10)–(H13) hold.

Example 4. *In [31], the authors consider nonlinearities like:*

$$f(t,\mathfrak{u},\mathfrak{v}) = \mathfrak{u}^{\frac{1}{4}} + [\mathfrak{v}+2]^{-\frac{1}{3}} + b(t) + d,\ g(t,\mathfrak{u}) = \frac{\mathfrak{u}}{1+\mathfrak{u}} a(t) + c - d,\ (t,\mathfrak{u},\mathfrak{v}) \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}} \times \mathbb{R}^+ \times \mathbb{R}^+,$$

where $a, b : [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}} \to \mathbb{R}^+$ with $a \not\equiv 0$, and c, d are positive constants with $c > d > 0$. Note that f is increasing about $\mathfrak{u}$ and decreasing about $\mathfrak{v}$, and g is increasing about $\mathfrak{u}$. Moreover, for $\gamma \in (0,1), t \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}, \mathfrak{u}, \mathfrak{v}s. \in \mathbb{R}^+$, we have

$$g(t,\gamma\mathfrak{u}) = \frac{\gamma\mathfrak{u}}{1+\gamma\mathfrak{u}} a(t) + c - d \geq \frac{\gamma\mathfrak{u}}{1+\mathfrak{u}} a(t) + \gamma(c-d) = \gamma g(t,\mathfrak{u}),$$

and

$$f\left(t, \gamma\mathfrak{u}, \gamma^{-1}\mathfrak{v}\right) = \gamma^{\frac{1}{4}} \mathfrak{u}^{\frac{1}{4}} + \gamma^{\frac{1}{3}} [\mathfrak{v}+2\gamma]^{-\frac{1}{3}} + b(t) + d \ \geq \gamma^{\frac{1}{3}} \left\{ \mathfrak{u}^{\frac{1}{4}} + [\mathfrak{v}+2]^{-\frac{1}{3}} + b(t) + d \right\} = \gamma^{\xi} f(t,\mathfrak{u},\mathfrak{v}).$$

Furthermore, we note that

$$\begin{aligned} f(t,\mathfrak{u},\mathfrak{v}) \geq d &= \frac{d}{\max_{t \in [v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}} a(t) + c - d} \times \left(\max_{t \in [v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}} a(t) + c - d \right) \\ &\geq \frac{d}{\max_{t \in [v-1,\mathfrak{b}+v+1]_{\mathbb{N}_{v-1}}} a(t) + c - d} \times \left(\frac{\mathfrak{u}}{1+\mathfrak{u}} a(t) + c - d \right) \\ &:= \delta_0 g(t,\mathfrak{u}),\ (t,\mathfrak{u},\mathfrak{v}) \in [v-1, \mathfrak{b}+v+1]_{\mathbb{N}_{v-1}} \times \mathbb{R}^+ \times \mathbb{R}^+. \end{aligned}$$

Therefore, (H10)–(H13) hold.

5. Conclusions

Fractional-order difference equations are a new form of differential equation that have wider applications compared to traditional integer-order differential equations. They are generalized differential equations whose derivative index can be a decimal or a fraction, rather than an integer. This form of differential equation has wide applications in fields such as physics, engineering, and finance. Therefore, the importance of studying fractional difference equations is now becoming apparent. In this paper, we consider a boundary value problem with a fractional-order difference equation and use Green's function to express its solution. Moreover, we obtain some existence theorems for the considered problem, i.e., when the nonlinearities satisfy some appropriate conditions, we study the existence, uniqueness, and multiplicity of solutions via the topological degree and fixed point theorems. Finally, we provide some examples to verify our main results.

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