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Matrix Approaches for Gould–Hopper–Laguerre–Sheffer Matrix Polynomial Identities

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Abstract: The Gould–Hopper–Laguerre–Sheffer matrix polynomials were initially studied using operational methods, but in this paper, we investigate them using matrix techniques. By leveraging properties of Pascal functionals and Wronskian matrices, we derive several identities for these polynomials, including recurrence relations. It is highlighted that these identities, acquired via matrix techniques, are distinct from the ones obtained when using operational methods.

Keywords: Gould–Hopper–Laguerre–Sheffer matrix polynomials; generalized Pascal functional matrix; Wronskian matrix; recurrence relations; geometric function theory; Bell polynomials

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1. Introduction and Preliminaries

Ongoing research in the field of special functions, including special polynomials, is driven primarily by their wide-ranging applications in mathematics, statistics, physics, and engineering. Numerous special polynomials have been introduced, studied, and utilized in various applications. Special matrix functions and polynomials represent a notable generalization of special functions, with their study being crucial due to numerous applications in fields such as statistics, physics, and engineering. The special matrix polynomials are found in a variety of mathematical and engineering disciplines. For instance, the matrix polynomials are used to form matrix polynomial equations, which encode a variety of problems ranging from elementary world problems to complicated problems in the sciences, in settings ranging from basic chemistry and physics to economics and social science, as well as in calculus and numerical analysis to approximate other functions. Thus, many researchers have investigated the special matrix polynomials (see, e.g., [1–5] and references therein).

The focus of this research is on the methodologies employed in the study of these polynomials. Conveniently, we classify the methodologies about the cited articles in this work into three rough categories: conventional methods, operational methods, and matrix methods. For example, conventional techniques are used in [1,6]; operational techniques are used in [2,3]; matrix techniques are used in [4,5]; conventional and operational techniques are used in [7].

This study concentrates on the matrix-based approaches by selecting the matrix polynomials given in [2]. It highlights that the identities obtained in this article using matrix techniques differ from those in [2] obtained using operational methods. This article utilizes matrix techniques, specifically certain identities of the generalized Pascal functional matrix of an analytic function and the Wronskian matrix of several analytic functions, to present several identities and recurrence relations for the Gould–Hopper–Laguerre–Sheffer matrix polynomials in (12). This article is structured as follows: Section 2 establishes recurrence relations for the Gould–Hopper–Laguerre–Sheffer matrix polynomials, while Section 3

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provides certain identities for these polynomials. Additionally, particular cases are demonstrated to showcase the applications of the results presented in Sections 2 and 3. Section 4 presents several particular cases of the results in Section 2. In conclusion, Section 5 of this article provides final remarks, emphasizing that the identities obtained in this study using matrix techniques differ from those derived in [2] through operational methods.

Exploring the properties of multivariable special polynomials is facilitated by a powerful method provided by linear algebra. The features of the generalized Pascal functional matrix and the Wronskian matrix [4] play a crucial role in establishing the theoretical basis for this method. Let us recall certain definitions and properties pertaining to the Pascal and Wronskian matrices: Let $\mathcal{F}$ be a class of functions which are analytic at the origin. The generalized Pascal functional matrix $\mathbf{P}_\ell[\mathsf{a}(u)]$ $(\mathsf{a}(u) \in \mathcal{F})$ is a lower triangular matrix of order $(\ell+1)$ (see, e.g., [4,5]):

$$\mathbf{P}_\ell[\mathsf{a}(u)]_{\eta,\xi} = \begin{cases} \binom{\eta}{\xi}\mathsf{a}^{(\eta-\xi)}(u), & \text{if } \eta \geqslant \xi, \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

for all η, $\xi = 0, 1, 2, \ldots, \ell$. In the present and subsequent instances, for any nonnegative integer τ, $\mathsf{a}^{(\tau)}$ denotes the τ^{th} order derivative of a, while a^τ stands for the τ^{th} power of a. In particular, $\mathsf{a}^{(0)} = \mathsf{a}$ and $\mathsf{a}^0 = 1$.

The ℓ^{th} order Wronskian matrix of analytic functions $\mathsf{a}_1(u)$, $\mathsf{a}_2(u)$, $\mathsf{a}_3(u)$, $\cdots$, $\mathsf{a}_m(u) \in \mathcal{F}$ is an $(\ell+1) \times m$ matrix which is defined as

$$\mathbf{W}_\ell[\mathsf{a}_1(u), \mathsf{a}_2(u), \mathsf{a}_3(u), \cdots, \mathsf{a}_m(u)] = \begin{bmatrix} \mathsf{a}_1(u) & \mathsf{a}_2(u) & \mathsf{a}_3(u) & \cdots & \mathsf{a}_m(u) \\ \mathsf{a}_1'(u) & \mathsf{a}_2'(u) & \mathsf{a}_3'(u) & \cdots & \mathsf{a}_m'(u) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathsf{a}_1^{(\ell)}(u) & \mathsf{a}_2^{(\ell)}(u) & \mathsf{a}_3^{(\ell)}(u) & \cdots & \mathsf{a}_m^{(\ell)}(u) \end{bmatrix}. \tag{2}$$

For any constants α, β and any analytic functions $\mathsf{a}(u)$ and $\mathsf{b}(u)$, the following properties hold true (see, e.g., [5]):

$$\mathbf{P}_\ell[\alpha\mathsf{a}(u) + \beta\mathsf{b}(u)] = \alpha\mathbf{P}_\ell[\mathsf{a}(u)] + \beta\mathbf{P}_\ell[\mathsf{b}(u)]; \tag{3}$$

$$\mathbf{W}_\ell[\alpha\mathsf{a}(u) + \beta\mathsf{b}(u)] = \alpha\mathbf{W}_\ell[\mathsf{a}(u)] + \beta\mathbf{W}_\ell[\mathsf{b}(u)]; \tag{4}$$

$$\mathbf{P}_\ell[\mathsf{a}(u)]\mathbf{P}_\ell[\mathsf{b}(u)] = \mathbf{P}_\ell[\mathsf{b}(u)]\mathbf{P}_\ell[\mathsf{a}(u)] = \mathbf{P}_\ell[\mathsf{a}(u)\mathsf{b}(u)]; \tag{5}$$

$$\mathbf{P}_\ell[\mathsf{a}(u)]\mathbf{W}_\ell[\mathsf{b}(u)] = \mathbf{P}_\ell[\mathsf{b}(u)]\mathbf{W}_\ell[\mathsf{a}(u)] = \mathbf{W}_\ell[\mathsf{a}(u)\mathsf{b}(u)]; \tag{6}$$

$$\mathbf{W}_\ell[\mathsf{b}(\mathsf{a}(u))]_{u=0} = \mathbf{W}_\ell[1, \mathsf{a}(u), \mathsf{a}^2(u), \mathsf{a}^3(u), \cdots, \mathsf{a}^\ell(u)]_{u=0}\Lambda_\ell^{-1}\mathbf{W}_\ell[\mathsf{b}(u)]_{u=0}, \tag{7}$$

where $\Lambda_\ell := \mathrm{diag}[0!, 1!, 2!, \cdots, \ell!]$ (and elsewhere), $\mathsf{a}(0) = 0$ and $\mathsf{a}'(0) \neq 0$. Equations (3)–(5) reveal that $\mathbf{W}_\ell$ exhibits linearity, whereas $\mathbf{P}_\ell$ satisfies both linearity and the multiplicative law. Furthermore, the multiplication of $\mathbf{P}_\ell$ is commutative. Furthermore, for any analytic functions $\mathsf{b}(u)$ and $\mathsf{a}_1(u), \mathsf{a}_2(u), \cdots, \mathsf{a}_m(u)$, the following property holds true:

$$\mathbf{P}_\ell[\mathsf{b}(u)]\mathbf{W}_\ell[\mathsf{a}_1(u), \mathsf{a}_2(u), \cdots, \mathsf{a}_m(u)] = \mathbf{W}_\ell[(\mathsf{b}\mathsf{a}_1)(u), (\mathsf{b}\mathsf{a}_2)(u), \cdots, (\mathsf{b}\mathsf{a}_m)(u)]. \tag{8}$$

Here and elsewhere, $\mathsf{b}(u)$ is an invertible analytic function, that is, $\mathsf{b}(0) \neq 0$, while $\mathsf{a}(u)$ is analytic function with $\mathsf{a}(0) = 0$ and $\mathsf{a}'(0) \neq 0$ that admits compositional inverse a^{-1} (see Remark 1).

It is also noted that the involved functions are analytic at $u = 0$ and have power series representations centered at the origin and its suitable neighborhood. It is recalled (see Theorem 2.3.4 in [8]) that a sequence $s_\ell(x)$ is Sheffer for $(\mathsf{b}(u), \mathsf{a}(u))$ if and only if it is generated by

$$\frac{1}{\mathsf{b}(\mathsf{a}^{-1}(u))}e^{x\mathsf{a}^{-1}(u)}=\sum_{\ell=0}^{\infty}\frac{s_\ell(x)}{\ell!}\,u^\ell\quad(x\in\mathbb{C}).\tag{9}$$

Furthermore, the Gould–Hopper–Laguerre–Sheffer matrix polynomials (GHLSMaP) ${}_{\mathsf{g}L}s_\ell^m(x,y,z;A,B)$ are defined by the following generating function (see [2]):

$$\frac{1}{\mathsf{b}(\mathsf{a}^{-1}(u))}e^{\left(x\sqrt{2A}\mathsf{a}^{-1}(u)+By\left(\mathsf{a}^{-1}(u)\right)^m\right)}C_0(z\mathsf{a}^{-1}(u))=\sum_{\ell=0}^{\infty}{}_{\mathsf{g}L}s_\ell^m(x,y,z;A,B)\frac{u^\ell}{\ell!},\tag{10}$$

where $C_0(xu)$ is the 0th-order Bessel–Tricomi function and given by (see Equations (5)–(7) in [2])

$$C_0(\alpha x)=\exp(-\alpha D_x^{-1})\{1\},\tag{11}$$

and A, B are matrices in $\mathbb{C}^{k\times k}$ $(k\in\mathbb{N})$, which denotes the set of k by k square matrices with entries of complex numbers, such that A is positive stable, $\mathbb{C}$ and $\mathbb{N}$ being, respectively, the sets of complex numbers and positive integers, and elsewhere. In addition, the other notations and restrictions about (10) would follow from those in [2]. In view of (11), the generating relation (10) takes the following form:

$$\begin{aligned}F(x,y,z;A,B;u)&:=\frac{1}{\mathsf{b}(\mathsf{a}^{-1}(u))}e^{\left(x\sqrt{2A}-D_z^{-1}\right)\mathsf{a}^{-1}(u)+By\left(\mathsf{a}^{-1}(u)\right)^m}\\&=\sum_{\ell=0}^{\infty}{}_{\mathsf{g}L}s_\ell^m(x,y,z;A,B)\frac{u^\ell}{\ell!}.\end{aligned}\tag{12}$$

For $\mathsf{a}(u)=u$, the GHLSMaP ${}_{\mathsf{g}L}s_\ell^m(x,y,z;A,B)$ gives the Gould–Hopper–Laguerre–Appell matrix polynomials (GHLAMaPs), ${}_{\mathsf{g}L}\mathcal{A}_\ell^m(x,y,z;A,B)$, which are generated by the following function (see [3]):

$$G(x,y,z;A,B;u):=\frac{1}{\mathsf{b}(u)}e^{\left(x\sqrt{2A}-D_z^{-1}\right)u+By\,u^m}=\sum_{\ell=0}^{\infty}{}_{\mathsf{g}L}\mathcal{A}_\ell^m(x,y,z;A,B)\frac{u^\ell}{\ell!}.\tag{13}$$

Remark 1. In (9), the functions a and b are called *delta* series and *invertible* series, respectively, because they are treated as series centered at 0 (see [8]). The followings are well-known in the geometric function theory (see [9]). Let $\mathscr{S}$ denote the class of functions $\mathsf{a}(u)$ normalized by the following Maclaurin series:

$$\mathsf{a}(u)=u+\sum_{\ell=2}^{\infty}a_\ell u^\ell\quad(u\in\mathbb{U}),\tag{14}$$

which are analytic in the open unit disk $\mathbb{U}:=\{u\in\mathbb{C}:|u|<1\}$ and univalent in $\mathbb{U}$. Then, it is well known that every function $\mathsf{a}\in\mathscr{S}$ has compositional inverse a^{-1}, given by

$$\mathsf{a}^{-1}(\mathsf{a}(u))=u\quad(u\in\mathbb{U})$$

and

$$\mathsf{a}\left(\mathsf{a}^{-1}(v)\right)=v\quad\left(|v|<r_0(\mathsf{a});\,r_0(\mathsf{a})\geqslant\frac{1}{4}\right),$$

the latter one of which means that v belongs to the range of a and $r_0(\mathsf{a})$ is the radius centered at 0 including the range of a. Indeed, the inverse function a^{-1} is given by

$$\mathsf{a}^{-1}(v)=v-a_2v^2+\left(2a_2^2-a_3\right)v^3-\left(5a_2^3-5a_2a_3+a_4\right)v^4+\cdots.$$

It is found that the analytic function $\mathsf{a}\in\mathscr{S}$ and the delta series a are the same when their domains are adjusted and the delta series $\mathsf{a}(u)$ is normalized. Take an example $\mathsf{a}(u):=\frac{u}{1-u}$. Then, $\mathsf{a}^{-1}(u)=\frac{u}{1+u}$. Since both $\mathsf{a}(u)$ and $\mathsf{a}^{-1}(u)$ are univalent in $\mathbb{U}$, $\mathsf{a}(u)$ is called bi-

univalent in $\mathbb{U}$ (see [9]). The sequence $s_\ell(x)$ is Sheffer for $(1, \mathsf{a}(u))$, which is also called associated with $\mathsf{a}(u)$, and generated by

$$e^{x\frac{u}{1+u}} = \sum_{\ell=0}^{\infty} \frac{s_\ell(x)}{\ell!}\, u^\ell \quad (x \in \mathbb{C}). \tag{15}$$

It is observed from the Taylor–Maclaurin expansion that

$$s_0(x) = 1 \quad \text{and} \quad s_k(x) = \frac{\partial^k}{\partial u^k} e^{x\frac{u}{1+u}}\Big|_{u=0} \quad (k \in \mathbb{N}). \tag{16}$$

By using Faá di Bruno's formula (see [10]), which is a recursive formula for finding higher-order derivatives of a composite function, we have

$$s_\ell(x) = B_\ell(x_1, \ldots, x_\ell), \quad x_k := (-1)^{k-1}\, k!\, x \quad (k \in \mathbb{N}), \tag{17}$$

where $B_\ell(x_1, \ldots, x_\ell)$ are the ℓth complete exponential Bell polynomials which can be expressed as determinants (see [11]):

$$\begin{aligned} &B_\ell(x_1, \ldots, x_\ell) \\ &= \det \begin{bmatrix} \frac{x_1}{0!} & \frac{x_2}{1!} & \frac{x_3}{2!} & \frac{x_4}{3!} & \cdots & \cdots & \frac{x_\ell}{(\ell-1)!} \\ -1 & \frac{x_1}{0!} & \frac{x_2}{1!} & \frac{x_3}{2!} & \cdots & \cdots & \frac{x_{\ell-1}}{(\ell-2)!} \\ 0 & -2 & \frac{x_1}{0!} & \frac{x_2}{1!} & \cdots & \cdots & \frac{x_{\ell-2}}{(\ell-3)!} \\ 0 & 0 & -3 & \frac{x_1}{0!} & \cdots & \cdots & \frac{x_{\ell-3}}{(\ell-4)!} \\ 0 & 0 & 0 & -4 & \cdots & \cdots & \frac{x_{\ell-4}}{(\ell-5)!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -(\ell-1) & \frac{x_1}{0!} \end{bmatrix} \end{aligned} \tag{18}$$

The first few associated polynomials $s_\ell(x)$ for $\mathsf{a}(u) = \frac{u}{1-u}$ are

$$\begin{aligned} &s_0(x) = 1, \quad s_1(x) = x, \quad s_2(x) = x^2 - 2x, \quad s_3(x) = x^3 - 6x^2 + 6x, \\ &s_4(x) = x^4 - 12x^3 + 36x^2 - 24x. \end{aligned}$$

2. Recurrence Relations

Since $F(x, y, z; A, B; u)$ is analytic at $u = 0$, it follows from (12) and Taylor–Maclaurin series expansion that, for $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ (and elsewhere),

$${}_{\mathsf{g}L}s_k^m(x, y, z; A, B) = \left(\frac{d}{du}\right)^k F(x, y, z; A, B; u)\Big|_{u=0}. \tag{19}$$

Thus, it is found from (2) and (19) that

$$\begin{aligned} &\mathbf{W}_\ell\big[F(x, y, z; A, B; u)\big]_{u=0} \\ &\quad = [{}_{\mathsf{g}L}s_0^m(x, y, z; A, B)\ {}_{\mathsf{g}L}s_1^m(x, y, z; A, B)\ \ldots\ {}_{\mathsf{g}L}s_\ell^m(x, y, z; A, B)]^T, \end{aligned} \tag{20}$$

where T denotes the transpose of a matrix.

The following lemma is a modified version of Lemma 3.3 in [5].

Lemma 1. *The following identity holds true:*

$$\begin{aligned}
&\Big(\mathbf{W}_\ell[{}_{gL}s_0^m(x,y,z;A,B)\ {}_{gL}s_1^m(x,y,z;A,B)\cdots {}_{gL}s_\ell^m(x,y,z;A,B)]\big|_x\Big)^T\Lambda_\ell^{-1}\Xi_\ell^{-1}\\
&=\mathbf{W}_\ell\Big[1,\mathsf{a}^{-1}(u),(\mathsf{a}^{-1}(u))^2,\cdots,(\mathsf{a}^{-1}(u))^\ell\Big]_{u=0}\Lambda_\ell^{-1}\mathbf{P}_\ell\left[\frac{e^{\left(-D_z^{-1}u+Byu^m\right)}}{\mathsf{b}(u)}\right]_{u=0}\mathbf{P}_\ell[e^{x\sqrt{2A}u}]_{u=0}\\
&=\mathbf{W}_\ell\Big[1,\mathsf{a}^{-1}(u),(\mathsf{a}^{-1}(u))^2,\cdots,(\mathsf{a}^{-1}(u))^\ell\Big]_{u=0}\Lambda_\ell^{-1}\mathbf{P}_\ell[G(x,y,z;A,B;u)]_{u=0}\\
&=\mathbf{W}_\ell\Big[1,\mathsf{a}^{-1}(u),(\mathsf{a}^{-1}(u))^2,\cdots,(\mathsf{a}^{-1}(u))^\ell\Big]_{u=0}\Lambda_\ell^{-1}\mathbf{P}_\ell\Big[\sum_{\ell=0}^{\infty}{}_{gL}\mathcal{A}_\ell^m(x,y,z;A,B)\frac{u^\ell}{\ell!}\Big]_{u=0},
\end{aligned}\tag{21}$$

where $\Lambda_\ell^{-1}=\mathrm{diag}[1,1/1!,1/2!,\cdots,1/\ell!]$, $\Xi_\ell^{-1}:=\mathrm{diag}\Big[1,1/\sqrt{2A},\cdots,1/(\sqrt{2A})^\ell\Big]$, *and* $G(x,y,z;A,B;u)$ *is the same as in* (13).

Proof. Using (7) in the right member of (20) provides

$$\begin{aligned}
&[{}_{gL}s_0^m(x,y,z;A,B)\ {}_{gL}s_1^m(x,y,z;A,B)\ \ldots\ {}_{gL}s_\ell^m(x,y,z;A,B)]^T\\
&\quad=\mathbf{W}_\ell[1,\mathsf{a}^{-1}(u),(\mathsf{a}^{-1}(u))^2,\cdots,(\mathsf{a}^{-1}(u))^\ell]_{u=0}\\
&\qquad\times\Lambda_\ell^{-1}\,\mathbf{W}_\ell\left[\frac{e^{x\sqrt{2A}u}e^{\left(-D_z^{-1}u+Byu^m\right)}}{\mathsf{b}(u)}\right]_{u=0}.
\end{aligned}\tag{22}$$

Employing (6) in (22) and noting that

$$\mathbf{W}_\ell[e^{x\sqrt{2A}u}]_{u=0}=[1\ x\sqrt{2A}\ (x\sqrt{2A})^2\ \cdots\ (x\sqrt{2A})^\ell]^T,$$

one obtains

$$\begin{aligned}
&[{}_{gL}s_0^m(x,y,z;A,B)\ {}_{gL}s_1^m(x,y,z;A,B)\ \ldots\ {}_{gL}s_\ell^m(x,y,z;A,B)]^T\\
&\quad=\mathbf{W}_\ell[1,\mathsf{a}^{-1}(u),\cdots,(\mathsf{a}^{-1}(u))^\ell]_{u=0}\,\Lambda_\ell^{-1}\\
&\qquad\times\mathbf{P}_\ell\left[\frac{e^{\left(-D_z^{-1}u+Byu^m\right)}}{\mathsf{b}(u)}\right]_{u=0}[1\ x\sqrt{2A}\ (x\sqrt{2A})^2\ \cdots\ (x\sqrt{2A})^\ell]^T.
\end{aligned}\tag{23}$$

Taking k^{th} order partial derivatives with respect to x on both sides of (23) and dividing each side of the resultant equation by $(\sqrt{2A})^k\,k!$ yields the following identity:

$$\begin{aligned}
&\frac{1}{k!}\frac{1}{(\sqrt{2A})^k}\left[\frac{\partial^k}{\partial x^k}{}_{gL}s_0^m(x,y,z;A,B)\ \frac{\partial^k}{\partial x^k}{}_{gL}s_1^m(x,y,z;A,B)\ \ldots\ \frac{\partial^k}{\partial x^k}{}_{gL}s_\ell^m(x,y,z;A,B)\right]^T\\
&\quad=\mathbf{W}_\ell\Big[1,\mathsf{a}^{-1}(u),(\mathsf{a}^{-1}(u))^2,\cdots,(\mathsf{a}^{-1}(u))^\ell\Big]_{u=0}\Lambda_\ell^{-1}\mathbf{P}_\ell\left[\frac{e^{\left(-D_z^{-1}u+Byu^m\right)}}{\mathsf{b}(u)}\right]_{u=0}\\
&\qquad\times\Big[\underbrace{0\cdots0}_{k\text{ terms}}\ \ 1\ \ \binom{k+1}{k}x\sqrt{2A}\ \ \binom{k+2}{k}(x\sqrt{2A})^2\cdots\ \ \binom{\ell}{k}(x\sqrt{2A})^{\ell-k}\Big]^T.
\end{aligned}\tag{24}$$

The last observation is that the right and left members of (24) are equal to the $(k+1)^{\text{th}}$ columns of the corresponding members of the first equality of (21), respectively. The second and third equalities of (21) follow from (5) and (13). This completes the proof. □

Theorem 1. *For finite order* $\ell \in \mathbb{N}_0$, *the* $\mathrm{GHLSMaP}_{gL}s_{\ell}^{m}(x,y,z;A,B)$ *satisfies the following differential matrix recurrence relation:*

$$\begin{aligned} &{}_{gL}s_{\ell+1}^{m}(x,y,z;A,B) \\ &\quad = \sum_{k=0}^{\ell} \frac{((x\sqrt{2A} - D_z^{-1})P_k + mByQ_k + R_k)}{(\sqrt{2A})^k\, k!} \frac{\partial^k}{\partial x^k}\, {}_{gL}s_{\ell}^{m}(x,y,z;A,B), \end{aligned} \tag{25}$$

where

$$\begin{aligned} &{}_{gL}s_{0}^{m}(x,y,z;A,B) = F(x,y,z;A,B;0), \quad P_k := \left(\frac{1}{\mathsf{a}'(u)}\right)^{(k)}\Big|_{u=0}, \\ &Q_k := \left(\frac{u^{m-1}}{\mathsf{a}'(u)}\right)^{(k)}\Big|_{u=0}, \quad R_k := \left(-\frac{\mathsf{b}'(u)}{\mathsf{b}(u)\mathsf{a}'(u)}\right)^{(k)}\Big|_{u=0}. \end{aligned}$$

Proof. In view of Definition 2 and Equation (19), we have

$$\begin{aligned} &\mathbf{W}_{\ell}\left[\frac{d}{du}F(x,y,z;A,B;u)\right]_{u=0} \\ &= \left[{}_{gL}s_{1}^{m}(x,y,z;A,B)\ {}_{gL}s_{2}^{m}(x,y,z;A,B)\ {}_{gL}s_{3}^{m}(x,y,z;A,B)\ \ldots\ {}_{gL}s_{\ell+1}^{m}(x,y,z;A,B)\right]^{T}. \end{aligned} \tag{26}$$

On the other hand, performing the differentiation in the expression

$$\mathbf{W}_{\ell}\left[\frac{d}{du}F(x,y,z;A,B;u)\right]_{u=0}$$

and then using properties (5)–(7) in a suitable manner, we find that

$$\begin{aligned} &\mathbf{W}_{\ell}\left[\frac{d}{du}F(x,y,z;A,B;u)\right]_{u=0} \\ &= \mathbf{W}_{\ell}[1,\mathsf{a}^{-1}(u),(\mathsf{a}^{-1}(u))^2,\cdots,(\mathsf{a}^{-1}(u))^{\ell}]_{u=0}\Lambda_{\ell}^{-1}\mathbf{P}_{\ell}\left[\frac{e^{\left(-D_z^{-1}u+Byu^m\right)}}{\mathsf{b}(u)}\right]_{u=0} \\ &\quad \times \mathbf{P}_{\ell}[e^{\left(x\sqrt{2A}u\right)}]_{u=0}\mathbf{W}_{\ell}\left[\left(x\sqrt{2A} - D_z^{-1} + mByu^{m-1} - \frac{\mathsf{b}'(u)}{\mathsf{b}(u)}\right)\frac{1}{\mathsf{a}'(u)}\right]_{u=0}. \end{aligned} \tag{27}$$

Using Lemma 1, Equation (27) yields

$$\begin{aligned} &\mathbf{W}_{\ell}\left[\frac{d}{du}F(x,y,z;A,B;u))\right]_{u=0} \\ &\quad = (\mathbf{W}_{\ell}[{}_{gL}s_{0}^{m}(x,y,z;A,B)\ {}_{gL}s_{1}^{m}(x,y,z;A,B)\ \ldots\ {}_{gL}s_{\ell}^{m}(x,y,z;A,B)])^{T}\Lambda_{\ell}^{-1}\Xi_{\ell}^{-1} \\ &\qquad \times \mathbf{W}_{\ell}\left[\frac{(x\sqrt{2A} - D_z^{-1})}{\mathsf{a}'(u)} + \frac{mByu^{m-1}}{\mathsf{a}'(u)} - \frac{\mathsf{b}'(u)}{\mathsf{b}(u)\mathsf{a}'(u)}\right]_{u=0}, \end{aligned}$$

which, upon using Definition 2, leads to

$$\begin{aligned}
&\mathbf{W}_{\ell}\left[\frac{d}{du}F(x,y,z;A,B;u)\right]_{u=0}\\
&=\begin{bmatrix}
{}_{gL}s_0^m(x,y,z;A,B) & \frac{\partial}{\partial x}\frac{{}_{gL}s_0^m(x,y,z;A,B)}{(\sqrt{2A})\ 1!} & \frac{\partial^2}{\partial x^2}\frac{{}_{gL}s_0^m(x,y,z;A,B)}{(\sqrt{2A})^2\ 2!} & \cdots & \frac{\partial^{\ell}}{\partial x^{\ell}}\frac{{}_{gL}s_0^m(x,y,z;A,B)}{(\sqrt{2A})^{\ell}\ \ell!}\\
{}_{gL}s_1^m(x,y,z;A,B) & \frac{\partial}{\partial x}\frac{{}_{gL}s_1^m(x,y,z;A,B)}{(\sqrt{2A})\ 1!} & \frac{\partial^2}{\partial x^2}\frac{{}_{gL}s_1^m(x,y,z;A,B)}{(\sqrt{2A})^2\ 2!} & \cdots & \frac{\partial^{\ell}}{\partial x^{\ell}}\frac{{}_{gL}s_1^m(x,y,z;A,B)}{(\sqrt{2A})^{\ell}\ \ell!}\\
{}_{gL}s_2^m(x,y,z;A,B) & \frac{\partial}{\partial x}\frac{{}_{gL}s_2^m(x,y,z;A,B)}{(\sqrt{2A})\ 1!} & \frac{\partial^2}{\partial x^2}\frac{{}_{gL}s_2^m(x,y,z;A,B)}{(\sqrt{2A})^2\ 2!} & \cdots & \frac{\partial^{\ell}}{\partial x^{\ell}}\frac{{}_{gL}s_2^m(x,y,z;A,B)}{(\sqrt{2A})^{\ell}\ \ell!}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
{}_{gL}s_{\ell}^m(x,y,z;A,B) & \frac{\partial}{\partial x}\frac{{}_{gL}s_{\ell}^m(x,y,z;A,B)}{(\sqrt{2A})\ 1!} & \frac{\partial^2}{\partial x^2}\frac{{}_{gL}s_{\ell}^m(x,y,z;A,B)}{(\sqrt{2A})^2\ 2!} & \cdots & \frac{\partial^{\ell}}{\partial x^{\ell}}\frac{{}_{gL}s_{\ell}^m(x,y,z;A,B)}{(\sqrt{2A})^{\ell}\ \ell!}
\end{bmatrix}\\
&\times\begin{bmatrix}
(x\sqrt{2A}-D_z^{-1})P_0+mByQ_0+R_0\\
(x\sqrt{2A}-D_z^{-1})P_1+mByQ_1+R_1\\
(x\sqrt{2A}-D_z^{-1})P_2+mByQ_2+R_2\\
\vdots\\
(x\sqrt{2A}-D_z^{-1})P_{\ell}+mByQ_{\ell}+R_{\ell}
\end{bmatrix}.
\end{aligned}\tag{28}$$

Equating the last rows of Equations (26) and (28), our assertion (25) is proved. □

Set $\mathsf{a}(u)=u$ and find that $P_0=1$, $P_k=0$ $(k\in\mathbb{N})$, $Q_{m-1}=(m-1)!$, $Q_k=0$ $(k\in\mathbb{N}_0\setminus\{m-1\})$. Therefore, setting $\mathsf{a}(u)=u$ in Theorem 1 provides the following corollary.

Corollary 1. *The* GHLAMaP ${}_{gL}\mathcal{A}_{\ell}^m(x,y,z;A,B)$ *satisfies the following differential matrix recurrence relation of finite order* $\ell\in\mathbb{N}_0$*:*

$$\begin{aligned}
{}_{gL}\mathcal{A}_{\ell+1}^m(x,y,z;A,B)=&(x\sqrt{2A}-D_z^{-1})\,{}_{gL}\mathcal{A}_{\ell}^m(x,y,z;A,B)\\
&+\frac{mBy}{(\sqrt{2A})^{m-1}}\frac{\partial^{m-1}}{\partial x^{m-1}}\,{}_{gL}\mathcal{A}_{\ell}^m(x,y,z;A,B)\\
&+\sum_{k=0}^{\ell}\frac{D_k}{(\sqrt{2A})^k\,k!}\frac{\partial^k}{\partial x^k}\,{}_{gL}\mathcal{A}_{\ell}^m(x,y,z;A,B),
\end{aligned}\tag{29}$$

where

$$D_k=\left(-\frac{b'(u)}{b(u)}\right)^{(k)}\Big|_{u=0}.$$

Theorem 2. *For finite order* $\ell\in\mathbb{N}_0$*, the* GHLSMaP ${}_{gL}s_{\ell}^m(x,y,z;A,B)$ *satisfies the following recurrence relation:*

$$\begin{aligned}
T_0\,{}_{gL}s_{\ell+1}^m(x,y,z;A,B)=&\sum_{k=0}^{\ell}\frac{1}{(\sqrt{2A})^k\,k!}\left(U_k\left(x\sqrt{2A}-D_z^{-1}\right)-U_{k+1}+mByV_k\right)\\
&\times\frac{\partial^k}{\partial x^k}\,{}_{gL}s_{\ell}^m(x,y,z;A,B)-\sum_{k=1}^{\ell}\binom{\ell}{k-1}T_{\ell-k+1}\,{}_{gL}s_k^m(x,y,z;A,B),
\end{aligned}\tag{30}$$

where

$${}_{gL}s_0^m(x,y,z;A,B)=F(x,y,z;A,B;0),\quad T_k:=\left(\mathsf{a}'(\mathsf{a}^{-1}(u))g(\mathsf{a}^{-1}(u))\right)^{(k)}\Big|_{u=0},$$
$$U_k:=(b(u))^{(k)}\Big|_{u=0},\quad V_k:=\left(u^{m-1}b(u)\right)^{(k)}\Big|_{u=0}.$$

Proof. Consider the following:

$$\mathbf{W}_{\ell}\left[\mathsf{a}'(\mathsf{a}^{-1}(u))\mathsf{b}(\mathsf{a}^{-1}(u))\frac{d}{du}F(x,y,z;A,B;u)\right]_{u=0}. \tag{31}$$

By using the relation (6) in Equation (31), we obtain

$$\mathbf{P}_{\ell}\left[\mathsf{a}'(\mathsf{a}^{-1}(u))\mathsf{b}(\mathsf{a}^{-1}(u))\right]_{u=0}\mathbf{W}_{\ell}\left[\frac{d}{du}F(x,y,z;A,B;u)\right]_{u=0}, \tag{32}$$

which, in view of (2) and Equation (19), becomes

$$\begin{aligned}&\mathbf{P}_{\ell}\left[\mathsf{a}'(\mathsf{a}^{-1}(u))\mathsf{b}(\mathsf{a}^{-1}(u))\right]_{u=0}\\&\quad\times\left[{}_{g}Ls_{1}^{m}(x,y,z;A,B)\ {}_{g}Ls_{2}^{m}(x,y,z;A,B)\ \ldots\ {}_{g}Ls_{\ell+1}^{m}(x,y,z;A,B)\right]^{T}.\end{aligned} \tag{33}$$

Using (1) in Equation (33), we find

$$\begin{bmatrix}T_0 & 0 & 0 & \cdots & 0\\ T_1 & T_0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ T_{\ell} & \binom{\ell}{1}T_{\ell-1} & \binom{\ell}{2}T_{\ell-2} & \cdots & T_0\end{bmatrix}\begin{bmatrix}{}_{g}Ls_{1}^{m}(x,y,z;A,B)\\ {}_{g}Ls_{2}^{m}(x,y,z;A,B)\\ \vdots\\ {}_{g}Ls_{\ell+1}^{m}(x,y,z;A,B)\end{bmatrix}. \tag{34}$$

On the other hand, we can write (31) as

$$\mathbf{W}_{\ell}\left[\left(x\sqrt{2A}-D_z^{-1}+mBy\left(\mathsf{a}^{-1}(u)\right)^{m-1}-\frac{\mathsf{b}'(\mathsf{a}^{-1}(u))}{\mathsf{b}(\mathsf{a}^{-1}(u))}\right)e^{(x\sqrt{2A}-D_z^{-1})\mathsf{a}^{-1}(u)+By(\mathsf{a}^{-1}(u))^{m}}\right]_{u=0},$$

which, upon using (7), gives

$$\begin{aligned}&\mathbf{W}_{\ell}[1,\mathsf{a}^{-1}(u),(\mathsf{a}^{-1}(u))^{2},\cdots,(\mathsf{a}^{-1}(u))^{\ell}]_{u=0}\Lambda_{\ell}^{-1}\\&\times\mathbf{W}_{\ell}\left[\left\{\left(x\sqrt{2A}-D_z^{-1}+mByu^{m-1}\right)\mathsf{b}(u)-\mathsf{b}'(u)\right\}\frac{e^{(x\sqrt{2A}-D_z^{-1})u+Byu^{m}}}{\mathsf{b}(u)}\right]_{u=0}.\end{aligned} \tag{35}$$

Using (4)–(6) and Lemma 1 in Equation (35), we have

$$\begin{aligned}&\left(\mathbf{W}_{\ell}\left[{}_{g}Ls_{0}^{m}(x,y,z;A,B)\ {}_{g}Ls_{1}^{m}(x,y,z;A,B)\ \ldots\ {}_{g}Ls_{\ell}^{m}(x,y,z;A,B)\right]\big|_{x}\right)^{T}\Lambda_{\ell}^{-1}\Xi_{\ell}^{-1}\\&\times\left(\left(x\sqrt{2A}-D_z^{-1}\right)\mathbf{W}_{\ell}[\mathsf{b}(u)]_{u=0}+mBy\mathbf{W}_{\ell}\left[u^{m-1}\mathsf{b}(u)\right]_{u=0}-\mathbf{W}_{\ell}\left[\mathsf{b}'(u)\right]_{u=0}\right),\end{aligned} \tag{36}$$

which, upon using (2), yields

$$\begin{aligned}&\begin{bmatrix}{}_{g}Ls_{0}^{m}(x,y,z;A,B) & \frac{\partial}{\partial x}\frac{{}_{g}Ls_{0}^{m}(x,y,z;A,B)}{(\sqrt{2A})\ 1!} & \frac{\partial^2}{\partial x^2}\frac{{}_{g}Ls_{0}^{m}(x,y,z;A,B)}{(\sqrt{2A})^2\ 2!} & \cdots & \frac{\partial^{\ell}}{\partial x^{\ell}}\frac{{}_{g}Ls_{0}^{m}(x,y,z;A,B)}{(\sqrt{2A})^{\ell}\ \ell!}\\ {}_{g}Ls_{1}^{m}(x,y,z;A,B) & \frac{\partial}{\partial x}\frac{{}_{g}Ls_{1}^{m}(x,y,z;A,B)}{(\sqrt{2A})\ 1!} & \frac{\partial^2}{\partial x^2}\frac{{}_{g}Ls_{1}^{m}(x,y,z;A,B)}{(\sqrt{2A})^2\ 2!} & \cdots & \frac{\partial^{\ell}}{\partial x^{\ell}}\frac{{}_{g}Ls_{1}^{m}(x,y,z;A,B)}{(\sqrt{2A})^{\ell}\ \ell!}\\ {}_{g}Ls_{2}^{m}(x,y,z;A,B) & \frac{\partial}{\partial x}\frac{{}_{g}Ls_{2}^{m}(x,y,z;A,B)}{(\sqrt{2A})\ 1!} & \frac{\partial^2}{\partial x^2}\frac{{}_{g}Ls_{2}^{m}(x,y,z;A,B)}{(\sqrt{2A})^2\ 2!} & \cdots & \frac{\partial^{\ell}}{\partial x^{\ell}}\frac{{}_{g}Ls_{2}^{m}(x,y,z;A,B)}{(\sqrt{2A})^{\ell}\ \ell!}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ {}_{g}Ls_{\ell}^{m}(x,y,z;A,B) & \frac{\partial}{\partial x}\frac{{}_{g}Ls_{\ell}^{m}(x,y,z;A,B)}{(\sqrt{2A})\ 1!} & \frac{\partial^2}{\partial x^2}\frac{{}_{g}Ls_{\ell}^{m}(x,y,z;A,B)}{(\sqrt{2A})^2\ 2!} & \cdots & \frac{\partial^{\ell}}{\partial x^{\ell}}\frac{{}_{g}Ls_{\ell}^{m}(x,y,z;A,B)}{(\sqrt{2A})^{\ell}\ \ell!}\end{bmatrix}\\&\times\begin{bmatrix}(x\sqrt{2A}-D_z^{-1})U_0+mByV_0-U_1\\ (x\sqrt{2A}-D_z^{-1})U_1+mByV_1-U_2\\ (x\sqrt{2A}-D_z^{-1})U_2+mByV_2-U_3\\ \vdots\\ (x\sqrt{2A}-D_z^{-1})U_{\ell}+mByV_{\ell}-U_{\ell+1}\end{bmatrix}.\end{aligned} \tag{37}$$

Equating the last rows of Equations (34) and (37), our assertion (30) is proved. □

3. Identities

This section provides certain identities for the GHLSMaP ${}_{gL}s_{\ell}^{m}(x,y,z;A,B)$.

Theorem 3. *The following relation between* GHLSMaP ${}_{gL}s_{\ell}^{m}(x,y,z;A,B)$ *and* GHLAMaP ${}_{gL}\mathcal{A}_{\ell}^{m}(x,y,z;A,B)$ *holds true: For* $\ell \in \mathbb{N}_0$,

$$\sum_{k=0}^{\ell} \frac{{}_{gL}\mathcal{A}_{k}^{m}(-x,-y,-z;A,B)}{(\sqrt{2A})^{k}\,k!}\,\frac{\partial^{k}}{\partial x^{k}}\,{}_{gL}s_{\ell}^{m}(x,y,z;A,B) = \alpha_{\ell}, \tag{38}$$

where

$$\alpha_{\ell} := \left(\frac{1}{(b(a^{-1}(u)))^{2}}\right)^{(\ell)}\Big|_{u=0}.$$

Proof. We find from (2) that

$$\mathbf{W}_{\ell}\left[\frac{1}{(\mathsf{b}(\mathsf{a}^{-1}(u)))^{2}}\right]_{u=0} = [\alpha_0\,\alpha_1\,\cdots\,\alpha_{\ell}]^{T}. \tag{39}$$

We can write

$$\begin{aligned}&\mathbf{W}_{\ell}\left[\frac{1}{(\mathsf{b}(\mathsf{a}^{-1}(u)))^{2}}\right]_{u=0}\\ &= \mathbf{W}_{\ell}\left[\left(\frac{e^{(x\sqrt{2A}-D_z^{-1})\mathsf{a}^{-1}(u)+By(\mathsf{a}^{-1}(u))^{m}}}{\mathsf{b}(\mathsf{a}^{-1}(u))\mathsf{b}(\mathsf{a}^{-1}(u))}\right)e^{-\left\{(x\sqrt{2A}-D_z^{-1})\mathsf{a}^{-1}(u)+By(\mathsf{a}^{-1}(u))^{m}\right\}}\right]_{u=0}.\end{aligned} \tag{40}$$

Utilizing the identities (5), (6), and (7) in the right member of Equation (40) suitably, we obtain

$$\begin{aligned}\mathbf{W}_{\ell}\left[\frac{1}{(\mathsf{b}(\mathsf{a}^{-1}(u)))^{2}}\right]_{u=0} &= \mathbf{W}_{\ell}[1,\mathsf{a}^{-1}(u),(\mathsf{a}^{-1}(u))^{2},\cdots,(\mathsf{a}^{-1}(u))^{\ell}]_{u=0}\Lambda_{\ell}^{-1}\mathbf{P}_{\ell}\left[\frac{1}{\mathsf{b}(u)}\right]_{u=0}\\ &\quad\times\mathbf{P}_{\ell}\left[e^{(x\sqrt{2A}-D_z^{-1})u+Byu^{m}}\right]_{u=0}\mathbf{W}_{\ell}\left[\frac{e^{(-x\sqrt{2A}+D_z^{-1})u+B(-y)u^{m}}}{\mathsf{b}(u)}\right]_{u=0},\end{aligned}$$

which, upon noting that $D_{-z}^{-1}\{1\} = -D_z^{-1}\{1\}$ (see, e.g., Equation (6) in [2]) and employing Equation (13) and Lemma 1, yields

$$\begin{aligned}&\mathbf{W}_{\ell}\left[\frac{1}{(\mathsf{b}(\mathsf{a}^{-1}(u)))^{2}}\right]_{u=0}\\ &= \left(\mathbf{W}_{\ell}[{}_{gL}s_{0}^{m}(x,y,z;A,B)\;{}_{gL}s_{1}^{m}(x,y,z;A,B)\cdots{}_{gL}s_{\ell}^{m}(x,y,z;A,B)]\big|_{x}\right)^{T}\\ &\quad\times\Lambda_{\ell}^{-1}\Xi_{\ell}^{-1}\mathbf{W}_{\ell}\left[\sum_{j=0}^{\infty}{}_{gL}\mathcal{A}_{j}^{m}(-x,-y,-z;A,B)\frac{u^{j}}{j!}\right]_{u=0}.\end{aligned} \tag{41}$$

Then, the use of (2) in Equation (41) gives

$$\begin{aligned}
&\mathbf{W}_\ell\left[\frac{1}{(g(\mathsf{a}^{-1}(u)))^2}\right]_{u=0}\\
&=\begin{bmatrix}
{}_{gL}s_0^m(x,y,z;A,B) & \frac{\partial}{\partial x}\frac{{}_{gL}s_0^m(x,y,z;A,B)}{(\sqrt{2A})\ 1!} & \frac{\partial^2}{\partial x^2}\frac{{}_{gL}s_0^m(x,y,z;A,B)}{(\sqrt{2A})^2\ 2!} & \cdots & \frac{\partial^\ell}{\partial x^\ell}\frac{{}_{gL}s_0^m(x,y,z;A,B)}{(\sqrt{2A})^\ell\ \ell!}\\
{}_{gL}s_1^m(x,y,z;A,B) & \frac{\partial}{\partial x}\frac{{}_{gL}s_1^m(x,y,z;A,B)}{(\sqrt{2A})\ 1!} & \frac{\partial^2}{\partial x^2}\frac{{}_{gL}s_1^m(x,y,z;A,B)}{(\sqrt{2A})^2\ 2!} & \cdots & \frac{\partial^\ell}{\partial x^\ell}\frac{{}_{gL}s_1^m(x,y,z;A,B)}{(\sqrt{2A})^\ell\ \ell!}\\
{}_{gL}s_2^m(x,y,z;A,B) & \frac{\partial}{\partial x}\frac{{}_{gL}s_2^m(x,y,z;A,B)}{(\sqrt{2A})\ 1!} & \frac{\partial^2}{\partial x^2}\frac{{}_{gL}s_2^m(x,y,z;A,B)}{(\sqrt{2A})^2\ 2!} & \cdots & \frac{\partial^\ell}{\partial x^\ell}\frac{{}_{gL}s_2^m(x,y,z;A,B)}{(\sqrt{2A})^\ell\ \ell!}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
{}_{gL}s_\ell^m(x,y,z;A,B) & \frac{\partial}{\partial x}\frac{{}_{gL}s_\ell^m(x,y,z;A,B)}{(\sqrt{2A})\ 1!} & \frac{\partial^2}{\partial x^2}\frac{{}_{gL}s_\ell^m(x,y,z;A,B)}{(\sqrt{2A})^2\ 2!} & \cdots & \frac{\partial^\ell}{\partial x^\ell}\frac{{}_{gL}s_\ell^m(x,y,z;A,B)}{(\sqrt{2A})^\ell\ \ell!}
\end{bmatrix}\\
&\times\begin{bmatrix}
{}_{gL}\mathcal{A}_0^m(-x,-y,-z;A,B)\\
{}_{gL}\mathcal{A}_1^m(-x,-y,-z;A,B)\\
{}_{gL}\mathcal{A}_2^m(-x,-y,-z;A,B)\\
\vdots\\
{}_{gL}\mathcal{A}_\ell^m(-x,-y,-z;A,B)
\end{bmatrix}.
\end{aligned}\tag{42}$$

Equating the last row of Equations (39) and (42), the desired identity (38) is proved. □

Theorem 4. GHLSMaP ${}_{gL}s_\ell^m(x,y,z;A,B)$ *satisfies the following identity:*

$$\begin{aligned}
&\sum_{k=0}^{\ell}\binom{\ell}{k}\beta_k\ {}_{gL}s_{\ell+1-k}^m(x,y,z;A,B)\\
&\quad=\sum_{k=0}^{\ell}\binom{\ell}{k}\left\{(x\sqrt{2A}-D_z^{-1})\gamma_k+mBy\delta_k+\epsilon_k\right\}{}_{gL}s_{\ell-k}^m(x,y,z;A,B).
\end{aligned}\tag{43}$$

where

$$\begin{aligned}
\beta_k&:=\Big(b(a^{-1}(u))\Big)^{(k)}\Big|_{u=0},\quad \gamma_k:=\left(\frac{b(a^{-1}(u))}{a'(a^{-1}(u))}\right)^{(k)}\Big|_{u=0},\\
\delta_k&:=\left(\frac{(a^{-1}(u))^{m-1}b(a^{-1}(u))}{a'(a^{-1}(u))}\right)^{(k)}\Big|_{u=0},\quad \epsilon_k:=\left(-\frac{b'(a^{-1}(u))}{a'(a^{-1}(u))}\right)^{(k)}\Big|_{u=0}.
\end{aligned}$$

Proof. Using (6), with the aid of Definitions (1) and (2), yields

$$\begin{aligned}
&\mathbf{W}_\ell\left[\mathsf{b}(\mathsf{a}^{-1}(u))\frac{d}{du}F(x,y,z;A,B;u)\right]_{u=0}\\
&=\mathbf{P}_\ell\left[\frac{d}{du}\left(\frac{e^{(x\sqrt{2A}-D_z^{-1})\mathsf{a}^{-1}(u)+By(\mathsf{a}^{-1}(u))^m}}{\mathsf{b}(\mathsf{a}^{-1}(u))}\right)\right]_{u=0}\mathbf{W}_\ell\Big[\mathsf{b}(\mathsf{a}^{-1}(u))\Big]_{u=0}\\
&=\begin{bmatrix}
{}_{gL}s_1^m(x,y,z;A,B) & 0 & \cdots & 0\\
{}_{gL}s_2^m(x,y,z;A,B) & {}_{gL}s_1^m(x,y,z;A,B) & \cdots & 0\\
{}_{gL}s_3^m(x,y,z;A,B) & \binom{2}{1}{}_{gL}s_2^m(x,y,z;A,B) & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots\\
{}_{gL}s_{\ell+1}^m(x,y,z;A,B) & \binom{\ell}{1}{}_{gL}s_\ell^m(x,y,z;A,B) & \cdots & {}_{gL}s_1^m(x,y,z;A,B)
\end{bmatrix}\begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\\ \vdots\\ \beta_\ell\end{bmatrix}.
\end{aligned}\tag{44}$$

From the left-most term of Equation (44) and $\frac{d}{du}\mathsf{a}^{-1}(u) = 1/\mathsf{a}'(\mathsf{a}^{-1}(u))$, we find

$$\begin{aligned}\mathbf{W}_\ell\left[\mathsf{b}(\mathsf{a}^{-1}(u))\frac{d}{du}F(x,y,z;A,B;u)\right]_{u=0}\\ = \mathbf{W}_\ell\left[\left(\frac{(x\sqrt{2A}-D_z^{-1})\mathsf{b}(\mathsf{a}^{-1}(u))}{\mathsf{a}'(\mathsf{a}^{-1}(u))} - \frac{\mathsf{b}'(\mathsf{a}^{-1}(u))}{\mathsf{a}'(\mathsf{a}^{-1}(u))}\right.\right.\\ \left.\left. + \frac{mBy(\mathsf{a}^{-1}(u))^{m-1}\mathsf{b}(\mathsf{a}^{-1}(u))}{\mathsf{a}'(\mathsf{a}^{-1}(u))}\right)F(x,y,z;A,B;u)\right]_{u=0}.\end{aligned} \tag{45}$$

Using (6) and (4) in the right member of (45), we obtain

$$\begin{aligned}\mathbf{W}_\ell\left[\mathsf{b}(\mathsf{a}^{-1}(u))\frac{d}{du}F(x,y,z;A,B;u)\right]_{u=0} = \mathbf{P}_\ell[F(x,y,z;A,B;u)]_{u=0}\\ \times\left((x\sqrt{2A}-D_z^{-1})\mathbf{W}_\ell\left[\frac{\mathsf{b}(\mathsf{a}^{-1}(u))}{\mathsf{a}'(\mathsf{a}^{-1}(u))}\right]_{u=0} + mBy\mathbf{W}_\ell\left[\frac{(\mathsf{a}^{-1}(u))^{m-1}\mathsf{b}(\mathsf{a}^{-1}(u))}{\mathsf{a}'(\mathsf{a}^{-1}(u))}\right]_{u=0}\right.\\ \left. + \mathbf{W}_\ell\left[-\frac{\mathsf{b}'(\mathsf{a}^{-1}(u))}{\mathsf{a}'(\mathsf{a}^{-1}(u))}\right]_{u=0}\right).\end{aligned} \tag{46}$$

Employing (1) in the first factor and γ_k, δ_k, and ϵ_k in the second factor in the right member of (46), we obtain a column $(\ell+1)\times 1$ matrix, say M. Finally, identifying the $(\ell+1)$-th row of the resulting matrix in the right-hand side of (44) with $(\ell+1)$-th row of M, we prove the desired identity (43). □

Theorem 5. *Let $\ell \in \mathbb{N}$. Then,* GHLSMaP ${}_{gL}s_\ell^m(x,y,z;A,B)$ *satisfies the following identity:*

$$\ell\ {}_{gL}s_{\ell-1}^m(x,y,z;A,B) = \sum_{k=0}^{\ell}\frac{\zeta_k}{(\sqrt{2A})^k\,k!}\frac{\partial^k}{\partial x^k}{}_{gL}s_\ell^m(x,y,z;A,B), \tag{47}$$

where

$$\zeta_k := (\mathsf{a}(u))^{(k)}\big|_{u=0} \quad (k\in\mathbb{N}_0).$$

Proof. Using (6), in view of Definitions (1) and (2), and (12), we have

$$\begin{aligned}\mathbf{W}_\ell[uF(x,y,z;A,B;u)]_{u=0} = \mathbf{P}_\ell[u]_{u=0}\mathbf{W}_\ell[F(x,y,z;A,B;u)]_{u=0}\\ = \begin{bmatrix}0&0&0&0&\cdots&0&0&0\\1&0&0&0&\cdots&0&0&0\\0&2&0&0&\cdots&0&0&0\\0&0&3&0&\cdots&0&0&0\\\vdots&\vdots&\vdots&\ddots&\ddots&\vdots&\vdots&\vdots\\&&&&\ddots&\ddots&&\\0&0&0&0&\cdots&\ell-1&0&0\\0&0&0&0&\cdots&0&\ell&0\end{bmatrix}\begin{bmatrix}{}_{gL}s_0^m(x,y,z;A,B)\\{}_{gL}s_1^m(x,y,z;A,B)\\{}_{gL}s_2^m(x,y,z;A,B)\\\vdots\\{}_{gL}s_{\ell-1}^m(x,y,z;A,B)\\{}_{gL}s_\ell^m(x,y,z;A,B)\end{bmatrix}.\end{aligned} \tag{48}$$

Furthermore, since $\mathsf{a}(\mathsf{a}^{-1}(u)) = u$, in terms of (12), the left-most term of (48) can be written as

$$\begin{aligned}\mathbf{W}_\ell[uF(x,y,z;A,B;u)]_{u=0}\\ = \mathbf{W}_\ell\left[\mathsf{a}(\mathsf{a}^{-1}(u))\,\frac{1}{\mathsf{b}(\mathsf{a}^{-1}(u))}e^{(x\sqrt{2A}-D_z^{-1})\mathsf{a}^{-1}(u)+By\left(\mathsf{a}^{-1}(u)\right)^m}\right]_{u=0},\end{aligned}$$

the right member of which, in view of (7), becomes

$$\begin{aligned}&\mathbf{W}_{\ell}[uF(x,y,z;A,B;u)]_{u=0}\\&\quad=\mathbf{W}_{\ell}[1,\mathsf{a}^{-1}(u),(\mathsf{a}^{-1}(u))^{2},\cdots,(\mathsf{a}^{-1}(u))^{\ell}]_{u=0}\Lambda_{\ell}^{-1}\,\mathbf{W}_{\ell}\left[\mathsf{a}(u)\,G(x,y,z;A,B;u)\right]_{u=0}.\end{aligned}\tag{49}$$

Using (6) at the right-most factor in (49), we find

$$\begin{aligned}&\mathbf{W}_{\ell}[uF(x,y,z;A,B;u)]_{u=0}\\&\quad=\mathbf{W}_{\ell}[1,\mathsf{a}^{-1}(u),(\mathsf{a}^{-1}(u))^{2},\cdots,(\mathsf{a}^{-1}(u))^{\ell}]_{u=0}\Lambda_{\ell}^{-1}\mathbf{P}_{\ell}[G(x,y,z;A,B;u)]_{u=0}\,\mathbf{W}_{\ell}[\mathsf{a}(u)]_{u=0}.\end{aligned}\tag{50}$$

Applying Lemma 1 to the right member of (50), we obtain

$$\begin{aligned}&\mathbf{W}_{\ell}[uF(x,y,z;A,B;u)]_{u=0}\\&=\left(\mathbf{W}_{\ell}[{}_{gL}s_{0}^{m}(x,y,z;A,B)\;{}_{gL}s_{1}^{m}(x,y,z;A,B)\cdots{}_{gL}s_{\ell}^{m}(x,y,z;A,B)]\big|_{x}\right)^{T}\Lambda_{\ell}^{-1}\Xi_{\ell}^{-1}\\&\quad\times\mathbf{W}_{\ell}[\mathsf{a}(u)]_{u=0}.\end{aligned}\tag{51}$$

Using Definition 2 in Equation (51), and, then, equating the last row of the resultant equation with the last row of Equation (48), our desired identity (47) is proved. □

Theorem 6. *For the GHLSMaP* ${}_{gL}s_{\ell}^{m}(x,y,z;A,B)$*, the following identity holds true:*

$$\begin{aligned}&\ell\sum_{k=1}^{\ell}\binom{\ell-1}{k-1}\beta_{\ell-k}\;{}_{gL}s_{k}^{m}(x,y,z;A,B)\\&\qquad=\sum_{k=0}^{\ell}\frac{1}{(\sqrt{2A})^{k}\,k!}\Big((x\sqrt{2A}-D_{z}^{-1})\theta_{k}+mBy\phi_{k}+\psi_{k}\Big)\frac{\partial^{k}}{\partial x^{k}}\,{}_{gL}s_{\ell}^{m}(x,y,z;A,B),\end{aligned}\tag{52}$$

where

$$\begin{aligned}&\theta_{k}:=\left(\frac{b(u)a(u)}{a'(u)}\right)^{(k)}\Big|_{u=0},\quad\phi_{k}:=\left(\frac{u^{m-1}b(u)a(u)}{a'(u)}\right)^{(k)}\Big|_{u=0},\\&\psi_{k}:=\left(-\frac{b'(u)a(u)}{a'(u)}\right)^{(k)}\Big|_{u=0}.\end{aligned}$$

Proof. Using (5) and (6), Definitions 1 and 2, and Equation (12), we obtain

$$\begin{aligned}&\mathbf{W}_{\ell}\left[u\;\mathsf{b}(\mathsf{a}^{-1}(u))\frac{d}{du}F(x,y,z;A,B;u)\right]_{u=0}\\&=\mathbf{P}_{\ell}[u]_{u=0}\mathbf{P}_{\ell}\left[\mathsf{b}(\mathsf{a}^{-1}(u))\right]_{u=0}\mathbf{W}_{\ell}\left[\frac{d}{du}F(x,y,z;A,B;u)\right]_{u=0}\\&=\begin{bmatrix}0&0&0&\cdots&0&0\\\beta_{0}&0&0&\cdots&0&0\\2\binom{1}{0}\beta_{1}&2\binom{1}{1}\beta_{0}&0&\cdots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\\ell\binom{\ell-1}{0}\beta_{\ell-1}&\ell\binom{\ell-1}{1}\beta_{\ell-2}&\ell\binom{\ell-1}{2}\beta_{\ell-3}&\cdots&\ell\binom{\ell-1}{\ell-1}\beta_{0}&0\end{bmatrix}\begin{bmatrix}{}_{gL}s_{1}^{m}(x,y,z;A,B)\\{}_{gL}s_{2}^{m}(x,y,z;A,B)\\{}_{gL}s_{3}^{m}(x,y,z;A,B)\\\vdots\\{}_{gL}s_{\ell+1}^{m}(x,y,z;A,B)\end{bmatrix}.\end{aligned}\tag{53}$$

On the other hand,

$$\begin{aligned}&\frac{d}{du}F(x,y,z;A,B;u)=\frac{1}{\{\mathsf{b}(\mathsf{a}^{-1}(u))\}^{2}}\,e^{(x\sqrt{2A}-D_{z}^{-1})\mathsf{a}^{-1}(u)+By\left(\mathsf{a}^{-1}(u)\right)^{m}}\\&\times\left[\left\{(x\sqrt{2A}-D_{z}^{-1})+mBy\left(\mathsf{a}^{-1}(u)\right)^{m-1}\right\}\mathsf{b}(\mathsf{a}^{-1}(u))\;-\mathsf{b}'(\mathsf{a}^{-1}(u))\right]\frac{1}{\mathsf{a}'(\mathsf{a}^{-1}(u))},\end{aligned}$$

which, upon using $u = \mathsf{a}(\mathsf{a}^{-1}(u))$, gives

$$u\,\mathsf{b}(\mathsf{a}^{-1}(u))\frac{d}{du}F(x,y,z;A,B;u) = \frac{\mathsf{a}(\mathsf{a}^{-1}(u))}{\mathsf{b}(\mathsf{a}^{-1}(u))}\,e^{(x\sqrt{2A}-D_z^{-1})\mathsf{a}^{-1}(u)+By\left(\mathsf{a}^{-1}(u)\right)^m} \times \left[\left\{(x\sqrt{2A}-D_z^{-1})+mBy\left(\mathsf{a}^{-1}(u)\right)^{m-1}\right\}\mathsf{b}(\mathsf{a}^{-1}(u)) - \mathsf{b}'(\mathsf{a}^{-1}(u))\right]\frac{1}{\mathsf{a}'(\mathsf{a}^{-1}(u))}. \tag{54}$$

Employing (7) in the right member of (54), with the aid of (6), we have

$$\begin{aligned}&\mathbf{W}_\ell[u\,\mathsf{b}(\mathsf{a}^{-1}(u))\frac{d}{du}F(x,y,z;A,B;u)]_{u=0}\\ &= \mathbf{W}_\ell[1,\mathsf{a}^{-1}(u),(\mathsf{a}^{-1}(u))^2,\cdots,(\mathsf{a}^{-1}(u))^\ell]_{u=0}\Lambda_\ell^{-1}\mathbf{P}_\ell[G(x,y,z;A,B;u)]_{u=0}\\ &\quad\times\mathbf{W}_\ell\Big[(x\sqrt{2A}-D_z^{-1})\frac{\mathsf{b}(u)\mathsf{a}(u)}{\mathsf{a}'(u)}+mBy\,\frac{u^{m-1}\mathsf{b}(u)\mathsf{a}(u)}{\mathsf{a}'(u)}-\frac{\mathsf{b}'(u)\mathsf{a}(u)}{\mathsf{a}'(u)}\Big]_{u=0}\end{aligned} \tag{55}$$

Using (4) and Lemma 1 in the right member of (55), we obtain the resultant $(\ell+1)\times 1$ matrix, say M_1. Finally, equating the $(\ell+1)$-th row of the resulting matrix in (53) and the $(\ell+1)$-th row of M_1 yields the desired identity (52). This completes the proof. □

4. Particular Cases

This section considers several particular cases of the results in Section 2.

Example 1. *The case of* $\mathsf{b}(u)=1$ *and* $\mathsf{a}(u)=\frac{e^u-1}{e^u+1}$. *Then, the Sheffer polynomials* $s_\ell(x)$ *in* (9) *become the Mittag–Leffler polynomials* $M_\ell(x)$ *(see [8] (pp. 75–78) and the references therein). In this regard, the GHLSMaPs* ${}_{gL}s_\ell^m(x,y,z;A,B)$ *are named as Gould–Hopper–Laguerre–Mittag—Leffler matrix polynomials and denoted by (GHLMLMaP)* ${}_{gL}M_\ell^m(x,y,z;A,B)$.

In Theorems 2, 3 and 5, we find

$$U_0=1,\ U_k=0\ (k\in\mathbb{N});\quad V_{m-1}=(m-1)!,\ V_k=0\ (k\neq m-1);$$

$$\alpha_0=1,\ \alpha_k=0\ (k\in\mathbb{N});\quad \zeta_k=\frac{1}{2}\left(E_k(1)-E_k^{(1)}(0)\right);\quad T_k=\begin{cases}\frac{1}{2}, & \text{if } k=0,\\ -1, & \text{if } k=2,\\ 0, & \text{if } k\neq 0,\,2.\end{cases}$$

Here, $E_k(x)$ are the Euler polynomials and $E_k^{(\alpha)}(x)$ are the generalized Euler polynomials which are generated by (see, e.g., [12] (pp. 86 and 88))

$$\frac{2e^{xu}}{e^u+1}=\sum_{k=0}^{\infty}E_k(x)\,\frac{u^k}{k!}\quad(|u|<\pi), \tag{56}$$

and

$$\left(\frac{2}{e^u+1}\right)^\alpha e^{xu}=\sum_{k=0}^{\infty}E_k^{(\alpha)}(x)\,\frac{u^k}{k!}\quad(|u|<\pi;\,1^\alpha=1). \tag{57}$$

Therefore Equations (30), (38), and (47) provide the following identities for GHLMLMaP ${}_{gL}M_\ell^m(x,y,z;A,B)$:

$$\begin{aligned}\frac{1}{2}{}_{gL}M_{\ell+1}^m(x,y,z;A,B) &= \frac{(x\sqrt{2A}-D_z^{-1})}{(\sqrt{2A})}\,{}_{gL}M_\ell^m(x,y,z;A,B)\\ &\quad+\binom{\ell}{\ell-2}{}_{gL}M_{\ell-1}^m(x,y,z;A,B)+\frac{mBy}{(\sqrt{2A})^{m-1}}\frac{\partial^{m-1}}{\partial x^{m-1}}{}_{gL}M_\ell^m(x,y,z;A,B).\end{aligned} \tag{58}$$

$${}_{gL}A_0(-x,-y,-z;A,B)\ {}_{gL}M_\ell^m(x,y,z;A,B)=1. \tag{59}$$

$$ {}_{gL}M_{\ell-1}^{m}(x,y,z;A,B)=\frac{1}{\ell}\sum_{k=0}^{\ell}\frac{(E_k(1)-E_k)}{(\sqrt{2A})^k\,k!}\frac{\partial^k}{\partial x^k}{}_{gL}M_{\ell}^{m}(x,y,z;A,B). \tag{60} $$

Example 2. *The case of* $b(u)=1$ *and* $a(u)=\ln(1+u)$. *Then, the Sheffer polynomials in* (9) *become the exponential polynomials* $e_\ell(x)$ *(see, e.g., [7] and [8] (pp. 63–67)). In this connection, the GHLSMaPs* ${}_{gL}s_{\ell}^{m}(x,y,z;A,B)$ *are named as the Gould–Hopper–Laguerre exponential matrix polynomials and denoted by (GHLEMaP)* ${}_{gL}e_{\ell}^{m}(x,y,z;A,B)$.

In Theorem 2, Theorem 3 and Theorem 5, we find

$$ U_0=1,\ U_k=0\ (k\in\mathbb{N});\quad V_{m-1}=(m-1)!,\ V_k=0\ (k\neq m-1);\quad T_k=(-1)^k; $$

$$ \alpha_0=1,\ \alpha_k=0\ (k\in\mathbb{N});\quad \beta_0=1,\ \beta_k=0\ (k\in\mathbb{N});\quad \gamma_0=\gamma_1=1,\ \gamma_k=0\ (k\in\mathbb{N}\setminus\{1\}); $$

$$ \epsilon_k=0;\quad \delta_{m-1}=(m-1)!,\ \delta_m=m!,\ \delta_k=0\ (k\neq m,\ m-1);\quad \zeta_k=(-1)^{k-1}(k-1)!. $$

Therefore Equations (30), (38), (43) and (47) give the following identities for GHLEMaP ${}_{gL}e_{\ell}^{m}(x,y,z;A,B)$:

$$ \begin{aligned} &\sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^{\ell-k}\,{}_{gL}e_{k+1}^{m}(x,y,z;A,B)\\ &\quad=\frac{(x\sqrt{2A}-D_z^{-1})}{(\sqrt{2A})}\,{}_{gL}e_{\ell}^{m}(x,y,z;A,B)+\frac{mBy}{(\sqrt{2A})^{m-1}}\frac{\partial^{m-1}}{\partial x^{m-1}}\,{}_{gL}e_{\ell}^{m}(x,y,z;A,B). \end{aligned} \tag{61} $$

$$ {}_{gL}A_0(-x,-y,-z;A,B)\,{}_{gL}e_{\ell}^{m}(x,y,z;A,B)=1. \tag{62} $$

$$ \begin{aligned} {}_{gL}e_{\ell+1}^{m}(x,y,z;A,B)&=(x\sqrt{2A}-D_z^{-1})\,{}_{gL}e_{\ell}^{m}(x,y,z;A,B)\\ &+\frac{(x\sqrt{2A}-D_z^{-1})}{(\sqrt{2A})}\frac{\partial}{\partial x}\,{}_{gL}e_{\ell}^{m}(x,y,z;A,B)+\frac{mBy}{(\sqrt{2A})^{m-1}}\frac{\partial^{m-1}}{\partial x^{m-1}}\,{}_{gL}e_{\ell}^{m}(x,y,z;A,B)\\ &+\frac{mBy}{(\sqrt{2A})^{m}}\frac{\partial^{m}}{\partial x^{m}}\,{}_{gL}e_{\ell}^{m}(x,y,z;A,B). \end{aligned} \tag{63} $$

$$ {}_{gL}e_{\ell-1}^{m}(x,y,z;A,B)=\frac{1}{\ell}\sum_{k=0}^{\ell}\frac{(-1)^{k-1}}{k\,(\sqrt{2A})^k}\frac{\partial^k}{\partial x^k}\,{}_{gL}e_{\ell}^{m}(x,y,z;A,B). \tag{64} $$

Example 3. *The case of* $b(u)=(1-u)^{-\alpha-1}$ *and* $a(u)=\frac{u}{u-1}$. *Then the Sheffer polynomials in* (9) *become the generalized Laguerre polynomials* $L_{\ell}^{(\alpha)}(x)$ *(see, e.g., [8,13] (pp. 108–113)). In this regard, GHLSMaP* ${}_{gL}s_{\ell}^{m}(x,y,z;A,B)$ *are named as the Gould-Hopper-Laguerre-generalized Laguerre Matrix polynomials and denoted by (GHLgLMaP)* ${}_{gL}L_{\ell}^{(\alpha,m)}(x,y,z;A,B)$.

In Theorem 2, Theorem 3 and Theorem 5, we find

$$ \begin{aligned} &U_k=(\alpha+k)_k;\quad V_k=(u^{m-1}g(u))^{(k)}\big|_{u=0};\quad T_k=(-1)^k(\alpha+1)_k;\\ &\alpha_\ell=(2\alpha+\ell+1)_n;\quad \zeta_k=-k!. \end{aligned} $$

Therefore Equations (30), (38) and (47) offer the following identities for GHLgLMaP ${}_{gL}L_{\ell}^{(\alpha,m)}(x,y,z;A,B)$:

$$\begin{aligned}&\sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^{\ell-k}(\alpha+1)_{\ell-k}\,{}_{gL}L_{k+1}^{(\alpha,m)}(x,y,z;A,B)\\&=\sum_{k=0}^{\ell}\frac{1}{(\sqrt{2A})^k\,k!}\big((x\sqrt{2A}-D_z^{-1})\\&\quad\times(\alpha+k)_k-(\alpha+k+1)_{k+1}+mByV_k\big)\frac{\partial^k}{\partial x^k}\,{}_{gL}L_{\ell}^{(\alpha,m)}(x,y,z;A,B).\end{aligned}\tag{65}$$

$$\sum_{\ell=0}^{\infty}\frac{{}_{gL}A_k(-x,-y,-z;A,B)}{(\sqrt{2A})^k\,k!}\frac{\partial^k}{\partial x^k}\,{}_{gL}L_{\ell}^{(\alpha,m)}(x,y,z;A,B)=(2\alpha+\ell+1)_{\ell}.\tag{66}$$

$${}_{gL}L_{\ell-1}^{(\alpha,m)}(x,y,z;A,B)=\frac{-1}{\ell}\sum_{k=0}^{\ell}\frac{1}{(\sqrt{2A})^k}\frac{\partial^k}{\partial x^k}\,{}_{gL}L_{\ell}^{(\alpha,m)}(x,y,z;A,B).\tag{67}$$

Example 4. *The case of* $b(u)=\frac{e^u-1}{u}$ *and* $a(u)=u$. *The Sheffer polynomials in* (9) *become the Bernoulli polynomials* $B_{\ell}(x)$ *(see, e.g., [8] (pp. 93–100) and [6]). In this connection, the GHLSMaPs* ${}_{gL}S_{\ell}^{m}(x,y,z;A,B)$ *are named as the Gould-Hopper-Laguerre-Bernoulli matrix polynomials and denoted by (GHLBMaP)* ${}_{gL}B_{\ell}^{m}(x,y,z;A,B)$.

In Theorems 2–6, we find

$$U_0=1,\ U_k=\frac{1}{k+1}\ (k\neq 0);\quad V_0=0,\ V_k=\frac{k!}{(k-m+2)!}\ (k\neq 0);$$

$$T_0=1,\ T_k=\frac{1}{k+1}\ (k\neq 0);$$

$$\alpha_{\ell}=\sum_{k=0}^{\ell}\binom{\ell}{k}B_{\ell-k}B_k;\quad \beta_k=\frac{1}{k+1};\quad \gamma_k=\frac{1}{k+1};\quad \epsilon_k=\frac{-1}{k+2};\quad \delta_k=\frac{k!}{(k-m+2)!};$$

$$\zeta_1=1,\ \zeta_k=0\ (k\neq 1);\quad \theta_0=0,\ \theta_k=1\ (k\neq 0);\quad \phi_k=\frac{k!}{(k-m+1)!};\quad \psi_k=\frac{-k}{k+1}.$$

Therefore Equations (30), (38), (43), (47), and (53) give the following identities for GHLBMaP ${}_{gL}B_{\ell}^{m}(x,y,z;A,B)$:

$$\begin{aligned}&\sum_{k=0}^{\ell}\binom{\ell}{k}\frac{1}{\ell-k+1}\,{}_{gL}B_{k+1}^{m}(x,y,z;A,B)\\&=\sum_{k=0}^{\ell}\frac{1}{(\sqrt{2A})^k\,k!}\left(\frac{(x\sqrt{2A}-D_z^{-1})}{k+1}-\frac{1}{k+2}+\frac{mBy\,k!}{(k-m+2)!}\right)\\&\quad\times\frac{\partial^k}{\partial x^k}\,{}_{gL}B_{\ell}^{m}(x,y,z;A,B).\end{aligned}\tag{68}$$

$$\sum_{k=0}^{\ell}\frac{{}_{gL}B_k(-x,-y,-z;A,B)}{(\sqrt{2A})^k\,k!}\frac{\partial^k}{\partial x^k}\,{}_{gL}B_{\ell}^{m}(x,y,z;A,B)=\sum_{k=0}^{\ell}\binom{\ell}{k}B_{\ell-k}B_k.\tag{69}$$

$$\sum_{k=0}^{\ell}\binom{\ell}{k}\frac{1}{k+1}\,{}_{gL}B_{\ell+1-k}^{m}(x,y,z;A,B)$$
$$=\sum_{k=0}^{\ell}\frac{1}{(\sqrt{2A})^{k}\,k!}\left(\frac{(x\sqrt{2A}-D_{z}^{-1})}{k+1}-\frac{1}{k+2}+\frac{mBy\,k!}{(k-m+2)!}\right)\tag{70}$$
$$\times\frac{\partial^{k}}{\partial x^{k}}\,{}_{gL}B_{\ell}^{m}(x,y,z;A,B).$$

$$(\sqrt{2A})\,{}_{gL}B_{\ell-1}^{m}(x,y,z;A,B)=\frac{1}{\ell}\frac{\partial}{\partial x}\,{}_{gL}B_{\ell}^{m}(x,y,z;A,B).\tag{71}$$

$$\ell\sum_{k=1}^{\ell}\binom{\ell-1}{k-1}\frac{{}_{gL}B_{k}^{m}(x,y,z;A,B)}{\ell-k+1}$$
$$=\sum_{k=1}^{\ell}\frac{1}{(\sqrt{2A})^{k}\,k!}\left((x\sqrt{2A}-D_{z}^{-1})+\frac{mBy\,k!}{(k-m+1)!}-\frac{k}{k+1}\right)\tag{72}$$
$$\times\frac{\partial^{k}}{\partial x^{k}}\,{}_{gL}B_{\ell}^{m}(x,y,z;A,B).$$

5. Concluding Remarks

In [2], a novel type of polynomials, named as the Gould–Hopper–Laguerre–Sheffer matrix polynomials, was introduced, and various identities of these polynomials were demonstrated through the utilization of operational methods. In this article, the Gould–Hopper–Laguerre–Sheffer matrix polynomials in [2] were also investigated using matrix techniques, and several of their identities, such as recurrence relations, were derived. Some particular cases of the main findings in this article were further demonstrated. It is pointed out that the identities derived in this article via matrix techniques are different from the ones in [2] derived via operational methods.

The Sheffer polynomials and their hybrid forms have been extensively explored (see, e.g., [8] and references therein). The Sheffer matrix polynomials and their hybrid matrix forms have recently been investigated (see, e.g., [1–5] and references therein). In this article, some properties of the generalized Pascal functional matrix and Wronskian matrix were mainly utilized.

The operational methods (referenced as [2]) and the matrix methods employed in this article each exhibit distinct advantages. To conclude, we raise an essential question: Is it possible to attain the identities derived using the matrix methods presented in this article by employing the operational methods described in [2]?

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