

Article

A Fixed Point Theorem for Generalized Ćirić-Type Contraction in Kaleva–Seikkala’s Type Fuzzy b -Metric Spaces

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Abstract: In this paper, we state and establish a new fixed point theorem for generalized Ćirić-type contraction in Kaleva–Seikkala’s type fuzzy b -metric space. Our results improve and extend some well-known results in the literature. Some examples are given to support our result. Finally, as an application, we show the existence and uniqueness of solution to Volterra integral equation formulated in Kaleva–Seikkala’s type fuzzy b -metric space.

Keywords: fixed-point theorem; generalized Ćirić-type contraction; Kaleva–Seikkala’s type fuzzy b -metric space

MSC: 47H10; 54H25



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1. Introduction

In 1984, the Kaleva–Seikkala’s type fuzzy metric space (briefly, $\mathcal{KS}$ -FMS) was initiated by Kaleva and Seikkala [1]. As we all know, the $\mathcal{KS}$ -FMS possesses a rich structure with proper choices of binary operations. Much work has been performed in regard to $\mathcal{KS}$ -FMS, see, e.g., [2–5]. In 1989, the notion of b -metric space (briefly, b -MS) was introduced by Bakhtin [6] (see also Czerwik [7]). A mass of fixed-point results in b -MS were studied by many authors over the past few years, see, e.g., [8–11].

Recently, the concept of Kaleva–Seikkala’s type fuzzy b -metric space (briefly, $\mathcal{KS}$ -FbMS) was given by Li et al. [12], which generalizes the notions of $\mathcal{KS}$ -FMS and b -MS and Banach type, Chatterjea type and Reich type fixed-point theorems were obtained. Notice that the Ćirić type fixed-point theorem was not involved in [12]. It is widely known that the Ćirić-type fixed-point theorem [13] extends other well-known fundamental metrical fixed-point theorems in the research literature (Banach [14], Kannan [15], Chatterjea [16,17], etc.). Based on the importance and application potential of quantitative science, many authors investigated heavily the generalizations of Ćirić fixed-point theorem in different directions in the last 20 years; see, e.g., [8,9,18]. In particular, Kumam et al. [19] in 2015 obtained the generalized Ćirić-type fixed-point theorems in metric spaces.

In this paper, we establish a generalized Ćirić-type fixed-point theorem in $\mathcal{KS}$ -FbMS. This result improves and extends some well-known results in the literature. Roughly speaking, the geometric interpretation of the generalized Ćirić-type contraction is that the metric between $\mathcal{T}x$ and $\mathcal{T}y$ can be controlled by the other nine metrics. In fact, there are 10 metrics between the 5 points $x, y, \mathcal{T}x, \mathcal{T}y$, and $\mathcal{T}^2x$. In Section 2, we recall some related definitions, basic properties and lemmas on $\mathcal{KS}$ -FbMS. In Section 3, we state a definition of generalized Ćirić-type contraction in $\mathcal{KS}$ -FbMS. Moreover, we construct a new example to illustrate that a generalized Ćirić-type contraction map is obviously not a Ćirić-type contraction. In Section 4, we first give two sufficient conditions to show that a generalized Ćirić-type contraction has a unique fixed point in the complete $\mathcal{KS}$ -FbMS. Second, we

give two examples to illustrate our main result and show that two sufficient conditions are complete independence. Third, we give another example to show that the two conditions in our result are not necessary for the existence of unique fixed point. Finally, we give some corollaries on Ćirić-type fixed-point theorems in $\mathcal{KS}$ -FbMS. In Section 5, as an application, we show the existence of solution to Volterra equation formulated in $\mathcal{KS}$ -FbMS.

2. Preliminaries

Throughout this paper, let $\mathbb{N}$, $\mathbb{Z}^+$ and $\mathbb{R}$ denote the sets of natural numbers, positive integer numbers, real numbers, respectively.

Now, we recall some definitions about $\mathcal{KS}$ -FbMS as follows.

Definition 1 ([20]). Let $\eta : \mathbb{R} \rightarrow [0, 1]$ be a mapping, whose α -level set is denoted by $[\eta]_\alpha = \{\delta \in \mathbb{R} : \eta(\delta) \geq \alpha\}$, η is called a fuzzy real number or fuzzy interval, if the following two conditions are satisfied:

- (1) There exists $\delta_0 \in \mathbb{R}$ such that $\eta(\delta_0) = 1$.
- (2) $[\eta]_\alpha = [\hat{\lambda}_\alpha, \hat{\rho}_\alpha]$ is a closed interval of $\mathbb{R}$ for each $\alpha \in (0, 1]$, where $-\infty < \hat{\lambda}_\alpha \leq \hat{\rho}_\alpha < +\infty$.

Let $\mathcal{F}$ denote the set of all such fuzzy real numbers. If $\eta \in \mathcal{F}$ and $\eta(\delta) = 0$ whenever $\delta < 0$, then η is called a non-negative fuzzy real number, and $\mathcal{F}^+$ denotes the set of all non-negative fuzzy real numbers.

Definition 2 ([12]). Assume that $\mathcal{M}$ a non-empty set, $b \geq 1$ and that $\mathfrak{D}$ is a mapping from $\mathcal{M} \times \mathcal{M}$ into $\mathcal{F}^+$. Let $\mathcal{L}, \mathfrak{R} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be two non-decreasing and symmetric functions, such that $\mathcal{L}(0, 0) = 0$ and $\mathfrak{R}(1, 1) = 1$. For $\alpha \in (0, 1]$ and $x, y \in \mathcal{M}$, define

$$[\mathfrak{D}(x, y)]_\alpha = [\hat{\lambda}_\alpha(x, y), \hat{\rho}_\alpha(x, y)].$$

Then, $\mathfrak{D}$ is called a fuzzy b -metric, and the quintuple $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ is called a fuzzy b -metric space (briefly, $\mathcal{KS}$ -FbMS) with the coefficient b , if

- (BM1) $\mathfrak{D}(x, y) = \bar{0}$ if, and only if, $x = y$;
- (BM2) $\mathfrak{D}(x, y) = \mathfrak{D}(y, x)$ for all $x, y \in \mathcal{M}$;
- (BM3) for all $x, y, z \in \mathcal{M}$:
 - (BM3L) $\mathfrak{D}(x, y)(b(\theta + \delta)) \geq \mathcal{L}(\mathfrak{D}(x, z)(\theta), \mathfrak{D}(z, y)(\delta))$, whenever $\theta \leq \hat{\lambda}_1(x, z)$, $\delta \leq \hat{\lambda}_1(z, y)$ and $(b(\theta + \delta)) \leq \hat{\lambda}_1(x, y)$;
 - (BM3R) $\mathfrak{D}(x, y)(b(\theta + \delta)) \leq \mathfrak{R}(\mathfrak{D}(x, z)(\theta), \mathfrak{D}(z, y)(\delta))$, whenever $\theta \geq \hat{\lambda}_1(x, z)$, $\delta \geq \hat{\lambda}_1(z, y)$ and $(b(\theta + \delta)) \geq \hat{\lambda}_1(x, y)$.

In the following propositions we state some properties. (For details see [12]).

Proposition 1. Let $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ be a $\mathcal{KS}$ -FbMS, $[\mathfrak{D}(x, y)]_t = [\hat{\lambda}_t(x, y), \hat{\rho}_t(x, y)]$ for $t \in (0, 1]$, where $x, y \in \mathcal{M}$. Then

- (1) $\lim_{\delta \rightarrow -\infty} \mathfrak{D}(x, y)(\delta) = 0 = \lim_{\delta \rightarrow +\infty} \mathfrak{D}(x, y)(\delta)$.
- (2) $\mathfrak{D}(x, y)(\delta)$ is a left continuous and non-increasing function for $\delta \in (\hat{\lambda}_1(x, y), +\infty)$.
- (3) $\hat{\rho}_t(x, y)$ is a left continuous and non-increasing function for $t \in (0, 1]$.

Proposition 2. Let $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ be a $\mathcal{KS}$ -FbMS, and assume that

- (R1) $\max\{x, y\} \geq \mathfrak{R}(x, y)$;
- (R2) for each $t \in (0, 1]$ there exists $s \in (0, t]$ such that $\mathfrak{R}(s, r) < t$ for all $r \in (0, t)$;
- (R3) $\lim_{t \rightarrow 0^+} \mathfrak{R}(t, t) = 0$.

Then $(R1) \Rightarrow (R2) \Rightarrow (R3)$.

Proposition 3. Let $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ be a $\mathcal{KS}$ -FbMS. Then $(R2) \Rightarrow$ for each $t \in (0, 1]$, there exists $s = s(t) \in (0, t]$ such that $\hat{\rho}_t(x, y) \leq b[\hat{\rho}_s(x, z) + \hat{\rho}_t(z, y)]$ for all $x, y, z \in \mathcal{M}$.

The following definition and lemma were introduced by Li et al. [12].

Definition 3. Let $(\mathcal{M}, \mathcal{D}, \mathcal{L}, \mathfrak{R}, b)$ be a $\mathcal{KS}$ -FbMS and $\{x_n\}$ be a sequence in $\mathcal{M}$.

- (1) $\{x_n\}$ is said to be converge to $x \in \mathcal{M}$, if $\lim_{n \rightarrow \infty} \mathcal{D}(x_n, x) = \bar{0}$, equivalently, $\lim_{n \rightarrow \infty} \hat{\rho}_t(x_n, x) = 0$ for each $t \in (0, 1]$;
- (2) $\{x_n\}$ is called a Cauchy sequence, if $\lim_{n, m \rightarrow \infty} \mathcal{D}(x_n, x_m) = \bar{0}$, i.e., for any given $\epsilon > 0$ and $t \in (0, 1]$, there exists $N = N(\epsilon, t) \in \mathbb{Z}^+$, such that $\hat{\rho}_t(x_n, x_m) < \epsilon$, whenever $n, m \geq N$;
- (3) If every Cauchy sequence in $\mathcal{M}$ converges, $(\mathcal{M}, \mathcal{D}, \mathcal{L}, \mathfrak{R}, b)$ is called complete.

Under $(\mathfrak{R}2)$, the limit of the sequence in $\mathcal{KS}$ -FbMS is unique.

Lemma 1. Let $(\mathcal{M}, \mathcal{D}, \mathcal{L}, \mathfrak{R}, b)$ be a $\mathcal{KS}$ -FbMS with $(\mathfrak{R}2)$ and $\{x_n\} \subseteq \mathcal{M}$ be a sequence. If there exist $x, y \in \mathcal{M}$ such that

$$\lim_{n \rightarrow \infty} \hat{\rho}_t(x_n, x) = \lim_{n \rightarrow \infty} \hat{\rho}_t(x_n, y) = 0,$$

then $x = y$.

3. Generalized Ćirić-Type Contraction

The definition of generalized Ćirić-type contraction on an ordinary metric space was introduced by Kumam et al. [19]. Analogously, we shall give the notion of generalized Ćirić-type contraction in $\mathcal{KS}$ -FbMS.

Definition 4. Assume that $(\mathcal{M}, \mathcal{D}, \mathcal{L}, \mathfrak{R}, b)$ be a $\mathcal{KS}$ -FbMS with $(\mathfrak{R}2)$ and the coefficient $b \geq 1$, $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{M}$ be a selfmap. Then $\mathcal{T}$ is called a generalized Ćirić-type contraction, if there exists $\hat{\lambda} \in [0, 1)$ such that

$$\begin{aligned} \hat{\rho}_t(\mathcal{T}x, \mathcal{T}y) \leq \hat{\lambda} \max \{ & \hat{\rho}_t(x, y), \hat{\rho}_t(x, \mathcal{T}x), \hat{\rho}_t(y, \mathcal{T}y), \hat{\rho}_t(x, \mathcal{T}y), \hat{\rho}_t(\mathcal{T}x, y), \hat{\rho}_t(\mathcal{T}^2x, x), \\ & \hat{\rho}_t(\mathcal{T}^2x, \mathcal{T}x), \hat{\rho}_t(\mathcal{T}^2x, y), \hat{\rho}_t(\mathcal{T}^2x, \mathcal{T}y) \}, \end{aligned} \quad (1)$$

for all $t \in (0, 1]$ and $x, y \in \mathcal{M}$.

Remark 1. The definition of usual Ćirić-type contraction was obtained by reducing four values $\hat{\rho}_t(\mathcal{T}^2x, x), \hat{\rho}_t(\mathcal{T}^2x, \mathcal{T}x), \hat{\rho}_t(\mathcal{T}^2x, y), \hat{\rho}_t(\mathcal{T}^2x, \mathcal{T}y)$ to a generalized Ćirić-type contraction.

Notice that the Ćirić-type contraction is the generalized Ćirić-type contraction in $\mathcal{KS}$ -FbMS, but, in general, the converse is not true. Next, we give a new example to show that there exists a generalized Ćirić-type contraction and it is not a Ćirić-type contraction.

Example 1. Let $\mathcal{M} = X_a \cup X_c \cup X_e \cup X_f \cup X_g$, where a, c, e, f and g are five distinct indexes, $X_i := \{i\} \times [0, 1], i \in \{a, c, e, f, g\}$, and define $d : \mathcal{M} \times \mathcal{M} \rightarrow [0, 2]$, as follows

$$d(x, y) = \begin{cases} |\tilde{x} - \tilde{y}|^2, & x, y \in X_i, \\ 2, & (x, y) \in \{(X_a, X_c), (X_c, X_a), (X_a, X_e), (X_e, X_a)\}, \\ 1, & \text{otherwise,} \end{cases}$$

where $x := (i, \tilde{x}), y := (j, \tilde{y}); \tilde{x}, \tilde{y} \in [0, 1]$.

Let $T : \mathcal{M} \rightarrow \mathcal{M}$ be a mapping defined by

$$Tx = \begin{cases} (a, \frac{1}{2}\tilde{x}), & x \in X_i, i \in \{a, f, g\}, \\ (f, \tilde{x}), & x \in X_c, \\ (g, \tilde{x}), & x \in X_e, \end{cases}$$

for any $x = (i, \tilde{x}), \tilde{x} \in [0, 1]$.

Let $\mathfrak{D}(x, y) : \mathbb{R} \rightarrow \mathbb{R}$ be a mapping. If $x = y \in \mathcal{M}$, we define $\mathfrak{D}(x, y)(\xi) = \bar{0}(\xi)$ for any $\xi \in \mathbb{R}$. If $x, y \in \mathcal{M}$ with $x \neq y$, $\mathfrak{D}(x, y)$ is defined by

$$\mathfrak{D}(x, y)(\xi) = \begin{cases} 0, & \xi < 0, \\ e^{-\frac{\xi}{d(x, y)}}, & \xi \geq 0, \end{cases}$$

and $\mathcal{L}(\alpha, \beta) = \min\{\alpha, \beta\}$, $\mathfrak{R}(\alpha, \beta) = \max\{\alpha, \beta\}$,

then the following assertions hold:

- (1) $(\mathcal{M}, d)$ is a complete b -MS with the coefficient $b = 2$;
- (2) $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ is a complete $\mathcal{KS}$ -FbMS with the coefficient $b = 2$;
- (3) $T : \mathcal{M} \rightarrow \mathcal{M}$ is a generalized Ćirić-type contraction in $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ with contraction constant $\hat{\lambda} = \frac{1}{2}$. However, T is not a Ćirić-type contraction.

Proof. (1) For any $x, y, z \in \mathcal{M}$, the following five cases are considered:

Case b1. If $x, y, z \in X_i, i \in \{a, c, e, f, g\}$;

Case b2. If $x, y \in X_i$ and $z \in X_j$, where $i \neq j$;

Case b3. If $x, z \in X_i$ and $y \in X_j$, where $i \neq j$;

Case b4. If $y, z \in X_i$ and $x \in X_j$, where $i \neq j$;

Case b5. If $x \in X_i, y \in X_j$ and $z \in X_k$, where $i \neq j \neq k$.

It is easy to see that $d(x, y) \leq 2d(x, z) + 2d(y, z)$.

In addition, for any Cauchy sequence $\{x_n\}$ in $(\mathcal{M}, d)$, then there exists $\tilde{n} \in \mathbb{N}$ such that $x_n \in X_i$ for all $n \geq \tilde{n}$, where $i \in \{a, c, e, f, g\}$. Thus, we can easily prove that $(\mathcal{M}, d)$ is complete. So, $(\mathcal{M}, d)$ is a complete b -MS with the coefficient $b = 2$.

(2) Clearly, (BM1) and (BM2) of Definition 2 hold.

To see (BM3). By a simple calculation, we obtain $\hat{\lambda}_1(x, y) = \hat{\rho}_1(x, y) = 0$ for all $x, y \in \mathcal{M}$.

(i) We prove (BM3 $\mathcal{L}$) with $b = 2$, equivalently, if $\theta, \delta \in \mathbb{R}$ satisfy

$$\begin{cases} \theta \leq \hat{\lambda}_1(x, z), \\ \delta \leq \hat{\lambda}_1(z, y), \\ 2(\theta + \delta) \leq \hat{\lambda}_1(x, y), \end{cases}$$

then $\mathfrak{D}(x, y)(2(\theta + \delta)) \geq \mathcal{L}(\mathfrak{D}(x, z)(\theta), \mathfrak{D}(z, y)(\delta)) = \min\{\mathfrak{D}(x, z)(\theta), \mathfrak{D}(z, y)(\delta)\}$ holds.

Now, the following three cases are considered, since $\theta \leq 0, \delta \leq 0$ and $2(\theta + \delta) \leq 0$.

Case $\mathcal{L}1$. Assume that $\theta = 0$ and $\delta = 0$. We have

$$\mathfrak{D}(x, y)(2(\theta + \delta)) = 1 = \min\{\mathfrak{D}(x, z)(\theta), \mathfrak{D}(z, y)(\delta)\}.$$

Case $\mathcal{L}2$. Assume that $\theta < 0$ and $\delta < 0$. We have

$$\mathfrak{D}(x, y)(2(\theta + \delta)) = 0 = \min\{\mathfrak{D}(x, z)(\theta), \mathfrak{D}(z, y)(\delta)\}.$$

Case $\mathcal{L}3$. Assume that $\theta = 0, \delta < 0$ or $\delta = 0, \theta < 0$. We have

$$\mathfrak{D}(x, y)(2(\theta + \delta)) = 0 = \min\{\mathfrak{D}(x, z)(\theta), \mathfrak{D}(z, y)(\delta)\}.$$

(ii) We prove (BM3 $\mathfrak{R}$) with $b = 2$, equivalently, if $\theta, \delta \in \mathbb{R}$ satisfy

$$\begin{cases} \theta \geq \hat{\lambda}_1(x, z), \\ \delta \geq \hat{\lambda}_1(z, y), \\ 2(\theta + \delta) \geq \hat{\lambda}_1(x, y), \end{cases}$$

then $\mathfrak{D}(x, y)(2(\theta + \delta)) \leq \mathfrak{R}(\mathfrak{D}(x, z)(\theta), \mathfrak{D}(z, y)(\delta)) = \max(\mathfrak{D}(x, z)(\theta), \mathfrak{D}(z, y)(\delta))$ holds. Now, the following three cases are considered, since $\theta \geq 0, \delta \geq 0$ and $2(\theta + \delta) \geq 0$.

Case $\mathfrak{R}1$. Assume that $\theta = 0$ and $\delta = 0$. We have

$$\mathfrak{D}(x, y)(2(\theta + \delta)) = 1 = \max\{\mathfrak{D}(x, z)(\theta), \mathfrak{D}(z, y)(\delta)\}.$$

Case $\mathfrak{R}2$. Assume that $\theta = 0, \delta > 0$ or $\delta = 0, \theta > 0$. We have

$$\mathfrak{D}(x, y)(2(\theta + \delta)) < 1 = \max\{\mathfrak{D}(x, z)(\theta), \mathfrak{D}(z, y)(\delta)\}.$$

Case $\mathfrak{R}3$. Assume that $\theta > 0$ and $\delta > 0$. For $x, y, z \in \mathcal{M}$.

(i) If $x = y$ and $z \in \mathcal{M}$, then

$$\mathfrak{D}(x, y)(2(\theta + \delta)) = 0 \leq \max\{\mathfrak{D}(x, z)(\theta), \mathfrak{D}(z, y)(\delta)\}.$$

(ii) If $z = x \neq y$ or $x \neq y = z$, we may assume that $z = x \neq y$, then

$$\begin{aligned} \mathfrak{D}(x, y)(2(\theta + \delta)) &= e^{-\frac{2(\theta + \delta)}{d(x, y)}} \\ &= e^{-\frac{2(\theta + \delta)}{d(z, y)}} \\ &\leq e^{-\frac{\delta}{d(z, y)}} \\ &= \mathfrak{D}(z, y)(\delta) \\ &\leq \max\{\mathfrak{D}(x, z)(\theta), \mathfrak{D}(z, y)(\delta)\}. \end{aligned}$$

(iii) If $x \neq y \neq z$. Note that

$$\min\left\{\frac{\theta}{d(x, z)}, \frac{\delta}{d(z, y)}\right\} \leq \frac{\theta + \delta}{d(x, z) + d(z, y)}.$$

then

$$\begin{aligned} \mathfrak{D}(x, y)(2(\theta + \delta)) &= e^{-\frac{2(\theta + \delta)}{d(x, y)}} \\ &\leq e^{-\frac{2(\theta + \delta)}{2(d(x, z) + d(z, y))}} \\ &\leq e^{-\min\left\{\frac{\theta}{d(x, z)}, \frac{\delta}{d(z, y)}\right\}} \\ &\leq \max\left\{e^{-\frac{\theta}{d(x, z)}}, e^{-\frac{\delta}{d(z, y)}}\right\} \\ &= \max\{\mathfrak{D}(x, z)(\theta), \mathfrak{D}(z, y)(\delta)\}. \end{aligned}$$

We thus conclude that $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ is a $\mathcal{KS}$ -FbMS.

On the other hand, it follows from the definition of $\mathfrak{D}(x, y)$ that

$$\mathfrak{D}(x, y)(\hat{\rho}_t(x, y)) = t,$$

for all $t \in (0, 1]$. Hence, we have

$$\hat{\rho}_t(x, y) = -\ln t \cdot d(x, y).$$

Moreover, from (1), we obtain that $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ is a complete $\mathcal{KS}$ -FbMS with $b = 2$.

(3) For any $x, y \in \mathcal{M}$, the following four cases are considered:

Case 1. If $x \in X_c, y \in X_e$ or $x \in X_e, y \in X_c$; we may assume that $x = (c, \tilde{x}) \in X_c, y = (e, \tilde{y}) \in X_e$, where $\tilde{x}, \tilde{y} \in [0, 1]$, we have $\hat{\rho}_t(Tx, Ty) = \hat{\rho}_t(x, y) = \hat{\rho}_t(x, Tx) = \hat{\rho}_t(y, Ty) = \hat{\rho}_t(x, Ty) = \hat{\rho}_t(y, Tx) = \hat{\rho}_t(T^2x, Tx) = \hat{\rho}_t(T^2x, Ty) = -\ln t, \hat{\rho}_t(T^2x, x) = \hat{\rho}_t(T^2x, y) = -2\ln t$.

Case 2. If $x, y \in X_c$ or $x, y \in X_e$; we may assume that $x = (c, \tilde{x}), y = (c, \tilde{y}) \in X_c$, we have $\hat{\rho}_t(Tx, Ty) = -\ln t \cdot |\tilde{x} - \tilde{y}|^2 \leq -\ln t, \hat{\rho}_t(T^2x, x) = -2\ln t$, where $t \in (0, 1]$.

Case 3. If $x \in X_i, y \in X_j$, where $i, j \in \{a, f, g\}$; for $x = (i, \tilde{x}), y = (j, \tilde{y})$, we have $\hat{\rho}_t(Tx, Ty) = -\frac{\ln t}{4} |\tilde{x} - \tilde{y}|^2 \leq \frac{1}{2} \hat{\rho}_t(x, y)$, where $t \in (0, 1]$.

Case 4. If $x \in X_i$, where $i \in \{a, f, g\}, y \in X_c$ or X_e ; for $x = (i, \tilde{x}) \in X_i$, we may assume that $y = (c, \tilde{y}) \in X_c$, we have $\hat{\rho}_t(Tx, Ty) = -\ln t, \hat{\rho}_t(Tx, y) = -2\ln t$, where $t \in (0, 1]$.

The above calculations show that T is not Ćirić-type contraction for $x = (c, \tilde{x}) \in X_c, y = (e, \tilde{y}) \in X_e$, because there is no a non-negative number $\hat{\lambda} < 1$ satisfying the Ćirić-type contraction condition. However, it follows that T is a generalized Ćirić-type contraction in $(\mathcal{M}, \mathcal{D}, \mathcal{L}, \mathfrak{R}, b)$ with contraction constant $\hat{\lambda} = \frac{1}{2}$ and for all $x, y \in \mathcal{M}$. $\square$

Remark 2. In Example 1, the domain for a, c, e, f , and g is less important than one might expect. In fact, one may exploit the same kind of argument with real (complex) numbers instead of indexes.

4. Fixed-Point Theorem for Generalized Ćirić-Type Contraction

Now, we prove and state a fixed-point theorem for a generalized Ćirić-type contraction in $\mathcal{KS}\text{-FbMS}$.

The following lemmas play a crucial role in the proof of Cauchy sequence.

Lemma 2. Let $(\mathcal{M}, \mathcal{D}, \mathcal{L}, \mathfrak{R}, b)$ be a $\mathcal{KS}\text{-FbMS}$ with $(\mathfrak{R}2)$, $T : \mathcal{M} \rightarrow \mathcal{M}$ be a map and $x_0 \in \mathcal{M}$. Let $\{x_n\}_{n=0}^\infty$ be a sequence by $x_n = Tx_{n-1} = T^n x_0$, for each $n \in \mathbb{N}$, (where it is understood that $T^0 x_0 = x_0$). Define

$$D(\mu, \nu) := \max\{\hat{\rho}_t(x_i, x_j) : \mu \leq i < j \leq \nu\},$$

where $i, j, \mu, \nu \in \mathbb{N}$ and for any $t \in (0, 1]$. If T is a generalized Ćirić-type contraction with contraction constant $\hat{\lambda} \in [0, 1)$, Then $D(\mu + 1, \nu) \leq \hat{\lambda} D(\mu, \nu)$, where $\mu, \nu \in \mathbb{N}$.

Proof. For $x_0 \in \mathcal{M}, i, j, \mu, \nu \in \mathbb{N}$, with $\nu - \mu > 1$ and $\mu + 1 \leq i < j \leq \nu$. Suppose that $t \in (0, 1]$, Since T is a generalized Ćirić-type contraction with $\hat{\lambda} \in [0, 1)$, we have

$$\begin{aligned} \hat{\rho}_t(x_i, x_j) &= \hat{\rho}_t(Tx_{i-1}, Tx_{j-1}) \\ &\leq \hat{\lambda} \max\{\hat{\rho}_t(x_{i-1}, x_{j-1}), \hat{\rho}_t(x_{i-1}, x_i), \hat{\rho}_t(x_{j-1}, x_j), \hat{\rho}_t(x_{i-1}, x_j), \\ &\quad \hat{\rho}_t(x_i, x_{j-1}), \hat{\rho}_t(x_{i+1}, x_{i-1}), \hat{\rho}_t(x_{i+1}, x_i), \hat{\rho}_t(x_{i+1}, x_{j-1}), \hat{\rho}_t(x_{i+1}, x_j)\} \\ &\leq \hat{\lambda} D(\mu, \nu). \end{aligned}$$

thus we have $D(\mu + 1, \nu) \leq \hat{\lambda} D(\mu, \nu)$. $\square$

Remark 3. It follows from Lemma 2 that $D(\mu, \nu) = \max\{\hat{\rho}_t(x_\mu, x_k) : \mu < k \leq \nu\}$, where $\mu, \nu \in \mathbb{N}$ and for any $t \in (0, 1]$. Indeed, since $D(\mu + 1, \nu) \leq \hat{\lambda} D(\mu, \nu) < D(\mu, \nu)$, where $0 \leq \hat{\lambda} < 1$, we have

$$\begin{aligned} D(\mu, \nu) &= \max\{\hat{\rho}_t(x_i, x_j) : \mu \leq i < j \leq \nu\} \\ &= \max\{\max\{\hat{\rho}_t(x_\mu, x_k) : \mu < k \leq \nu\}, D(\mu + 1, \nu)\} \\ &= \max\{\hat{\rho}_t(x_\mu, x_k) : \mu < k \leq \nu\}. \end{aligned}$$

Lemma 3. Let $(\mathcal{M}, \mathcal{D}, \mathcal{L}, \mathfrak{R}, b)$ be a $\mathcal{KS}\text{-FbMS}$ with $(\mathfrak{R}2)$, $T : \mathcal{M} \rightarrow \mathcal{M}$ be a generalized Ćirić-type contraction with contraction constant $\hat{\lambda} \in [0, 1)$. For $x_0 \in \mathcal{M}$, let $\{x_n\}_{n=0}^\infty$ be a sequence by $x_n = Tx_{n-1} = T^n x_0$, for all $n \in \mathbb{N}$, (where it is understood that $T^0 x_0 = x_0$). Then $\{x_n\}$ is a Cauchy sequence in $\mathcal{M}$.

Proof. First we prove that the following assertion holds. There exists $M > 0$, such that $D(0, n) \leq M$, for any $n \in \mathbb{N}$.

Since for any $\hat{\lambda} \in [0, 1)$, there exists $p \in \mathbb{N}$, such that $\hat{\lambda}^p < \frac{1}{b}$. If $D(0, n) \leq D(0, p)$, for any

$n \in \mathbb{N}$, then assertion holds. Otherwise, if $D(0, n_p) > D(0, p)$, for some $n_p \in \mathbb{N}$, then there exists integer $r \leq n_p$, such that $D(0, n_p) = \hat{\rho}_t(x_0, x_r)$ for any $t \in (0, 1]$ and $p < r$. Applying a triangle inequality and Lemma 2. For any $t \in (0, 1]$, then there exists $s = s(t) \in (0, t]$, such that

$$\begin{aligned} D(0, n_p) = \hat{\rho}_t(x_0, x_r) &\leq b\hat{\rho}_s(x_0, x_p) + b\hat{\rho}_t(x_p, x_r) \\ &\leq b\hat{\rho}_s(x_0, x_p) + bD(p, n_p) \\ &\leq b\hat{\rho}_s(x_0, x_p) + b\hat{\lambda}D(p-1, n_p) \\ &\leq b\hat{\rho}_s(x_0, x_p) + b\hat{\lambda}^p D(0, n_p), \end{aligned}$$

Therefore,

$$D(0, n_p) \leq \frac{b}{1 - b\hat{\lambda}^p} \hat{\rho}_s(x_0, x_p).$$

Let

$$M := \max \left\{ D(0, p), \frac{b}{1 - b\hat{\lambda}^p} \hat{\rho}_s(x_0, x_p) \right\}.$$

Therefore, we have $D(0, n) \leq M$ for any $n \in \mathbb{N}$, where $t \in (0, 1]$, $s = s(t) \in (0, t]$.

Next, we shall show that $\{x_n\}$ is a Cauchy sequence in $\mathcal{M}$. For any $m, n \in \mathbb{N}$ with $m < n$ and $t \in (0, 1]$, T be a generalized Ćirić-type contraction, it follows from Lemma 2 that $\hat{\rho}_t(x_m, x_n) \leq D(m, n) \leq \hat{\lambda}^m D(0, n)$. Moreover, it follows from $\hat{\lambda} \in [0, 1)$ that $\lim_{n, m \rightarrow \infty} \hat{\rho}_t(x_m, x_n) = 0$. Hence, $\{x_n\}$ is a Cauchy sequence in $\mathcal{M}$. $\square$

As we have seen that the metric d in b -MS is discontinuous in general and d does not have Fatou property. Moreover, it follows that Fatou property is strictly weaker than continuity (see [21]). It is easily seen that the metric $\mathfrak{D}$ has the analogous property in the $\mathcal{KS}$ -FbMS $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$. In order to study the existence and uniqueness of the fixed-point for such mappings, we assume that $\mathfrak{D}$ has the Fatou property.

The following notion of the Fatou property is due to Li et al. [12].

Definition 5. Let $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ be a $\mathcal{KS}$ -FbMS with $(\mathfrak{R}2)$. $\mathfrak{D}$ is called to have the Fatou property if, for any $t \in (0, 1]$,

$$\hat{\rho}_t(x, y) \leq \liminf_{n \rightarrow \infty} \hat{\rho}_t(x_n, y), \quad (2)$$

whenever $\{x_n\} \subseteq \mathcal{M}$ with $\lim_{n \rightarrow \infty} \hat{\rho}_t(x_n, x) = 0$ and any $y \in \mathcal{M}$.

Now we can state our main result.

Theorem 1. Let $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ be a complete $\mathcal{KS}$ -FbMS with $(\mathfrak{R}2)$, $T : \mathcal{M} \rightarrow \mathcal{M}$ be a generalized Ćirić-type contraction with contraction constant $\hat{\lambda} \in [0, 1)$. If one of the following conditions is satisfied

(1) $\mathfrak{D}$ has the Fatou property,

(2) $\hat{\lambda} \in \left[0, \frac{1}{b}\right)$,

then T has a unique fixed-point in $\mathcal{M}$.

Proof. Given $x_0 \in \mathcal{M}$, set $\{x_n\}_{n=0}^\infty$ a sequence by $x_n = Tx_{n-1} = T^n x_0$, for each $n \in \mathbb{N}$, (where it is understood that $T^0 x_0 = x_0$). It follows from Lemma 3 that $\{x_n\}$ is a Cauchy sequence in $\mathcal{M}$. Notice that $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ be a complete $\mathcal{KS}$ -FbMS. Therefore, there exists $v \in \mathcal{M}$ such that $\lim_{n \rightarrow \infty} \hat{\rho}_t(x_n, v) = 0$, for all $t \in (0, 1]$.

If (1) holds, we show that v is a fixed-point of T . Assume that, on the contrary, $Tv \neq v$, that

is, $\hat{\rho}_{t_0}(Tv, v) > 0$ for some $t_0 \in (0, 1]$. Since T is a generalized Ćirić-type contraction with $\hat{\lambda} \in [0, 1)$, we have

$$\begin{aligned} \hat{\rho}_{t_0}(x_{n+1}, Tv) &= \hat{\rho}_{t_0}(Tx_n, Tv) \\ &\leq \hat{\lambda} \max\{\hat{\rho}_{t_0}(x_n, v), \hat{\rho}_{t_0}(x_n, x_{n+1}), \hat{\rho}_{t_0}(v, Tv), \hat{\rho}_{t_0}(x_n, Tv), \hat{\rho}_{t_0}(x_{n+1}, v), \\ &\quad \hat{\rho}_{t_0}(x_{n+2}, x_n), \hat{\rho}_{t_0}(x_{n+2}, x_{n+1}), \hat{\rho}_{t_0}(x_{n+2}, v), \hat{\rho}_{t_0}(x_{n+2}, Tv)\} \end{aligned} \quad (3)$$

Using the fact that $\lim_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_n, v) = 0$ and $\mathfrak{D}$ has the Fatou property, we obtain

$$\hat{\rho}_{t_0}(v, Tv) \leq \liminf_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_n, Tv) = \liminf_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_{n+1}, Tv). \quad (4)$$

On the other hand, since:

$$\begin{aligned} \lim_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_n, v) &= \lim_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_n, x_{n+1}) = \lim_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_{n+1}, v) = \lim_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_{n+2}, x_n) \\ &= \lim_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_{n+2}, x_{n+1}) = \lim_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_{n+2}, v) = 0. \end{aligned} \quad (5)$$

Combining Equations (3)–(5), we obtain that:

$$\begin{aligned} \hat{\rho}_{t_0}(v, Tv) &\leq \liminf_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_{n+1}, Tv) \\ &\leq \hat{\lambda} \max\left\{\lim_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_n, v), \lim_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_n, x_{n+1}), \hat{\rho}_{t_0}(v, Tv), \liminf_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_n, Tv), \right. \\ &\quad \lim_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_{n+1}, v), \lim_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_{n+2}, x_n), \lim_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_{n+2}, x_{n+1}), \lim_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_{n+2}, v), \\ &\quad \left. \liminf_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_{n+2}, Tv)\right\} \\ &= \hat{\lambda} \max\left\{\hat{\rho}_{t_0}(v, Tv), \liminf_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_n, Tv), \liminf_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_{n+2}, Tv)\right\} \\ &= \hat{\lambda} \liminf_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_n, Tv). \end{aligned}$$

Thus, we find that

$$\liminf_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_n, Tv) = \liminf_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_{n+1}, Tv) \leq \hat{\lambda} \liminf_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_n, Tv).$$

We obtain

$$\liminf_{n \rightarrow \infty} \hat{\rho}_{t_0}(x_n, Tv) = 0,$$

and hence

$$\hat{\rho}_{t_0}(v, Tv) = 0,$$

which is a contradiction. Thus, $Tv = v$ and T has a fixed-point in $\mathcal{M}$.

If T has another fixed-point $v^* \in \mathcal{M}$, that is $Tv^* = v^*$ and $v^* \neq v$, then $\hat{\rho}_{t_0}(v^*, v) > 0$ for some $t_0 \in (0, 1]$. Since

$$\begin{aligned} \hat{\rho}_{t_0}(v^*, v) &= \hat{\rho}_{t_0}(Tv^*, Tv) \\ &\leq \hat{\lambda} \max\left\{\hat{\rho}_{t_0}(v^*, v), \hat{\rho}_{t_0}(v^*, Tv^*), \hat{\rho}_{t_0}(v, Tv), \hat{\rho}_{t_0}(v^*, Tv), \hat{\rho}_{t_0}(Tv^*, v), \hat{\rho}_{t_0}(T^2v^*, v^*), \right. \\ &\quad \left. \hat{\rho}_{t_0}(T^2v^*, Tv^*), \hat{\rho}_{t_0}(T^2v^*, v), \hat{\rho}_{t_0}(T^2v^*, Tv)\right\} \\ &= \hat{\lambda} \hat{\rho}_{t_0}(v^*, v) \\ &< \hat{\rho}_{t_0}(v^*, v), \end{aligned}$$

which is a contradiction. Hence $v^* = v$, i.e., T has a unique fixed-point in $\mathcal{M}$.

Next, we verify (2). Since T is a generalized Ćirić-type contraction with $\hat{\lambda} \in [0, 1)$, we have

$$\begin{aligned}\hat{\rho}_t(x_{n+1}, Tv) &= \hat{\rho}_t(Tx_n, Tv) \\ &\leq \hat{\lambda} \max\{\hat{\rho}_t(x_n, v), \hat{\rho}_t(x_n, x_{n+1}), \hat{\rho}_t(v, Tv), \hat{\rho}_t(x_n, Tv), \hat{\rho}_t(x_{n+1}, v), \\ &\quad \hat{\rho}_t(x_{n+2}, x_n), \hat{\rho}_t(x_{n+2}, x_{n+1}), \hat{\rho}_t(x_{n+2}, v), \hat{\rho}_t(x_{n+2}, Tv)\},\end{aligned}$$

for any $n \in \mathbb{N}$ and $t \in (0, 1]$.

Note that for any $n \in \mathbb{N}$, the following three cases are discussed.

(i) If $\hat{\rho}_t(x_{n+1}, Tv) \leq \hat{\lambda} \hat{\rho}_t(v, Tv)$, since for any $t \in (0, 1]$, there exists $s = s(t) \in (0, t]$, such that

$$\hat{\rho}_t(x_{n+1}, Tv) \leq \hat{\lambda} b \hat{\rho}_s(x_{n+1}, v) + \hat{\lambda} b \hat{\rho}_t(x_{n+1}, Tv),$$

we obtain

$$\hat{\rho}_t(x_{n+1}, Tv) \leq \frac{\hat{\lambda} b}{1 - \hat{\lambda} b} \hat{\rho}_s(x_{n+1}, v).$$

(ii) If $\hat{\rho}_t(x_{n+1}, Tv) \leq \hat{\lambda} \hat{\rho}_t(x_n, Tv)$, then

$$\hat{\rho}_t(x_{n+1}, Tv) \leq \hat{\lambda} b \hat{\rho}_s(x_n, x_{n+1}) + \hat{\lambda} b \hat{\rho}_t(x_{n+1}, Tv),$$

which deduces

$$\hat{\rho}_t(x_{n+1}, Tv) \leq \frac{\hat{\lambda} b}{1 - \hat{\lambda} b} \hat{\rho}_s(x_n, x_{n+1}).$$

(iii) If $\hat{\rho}_t(x_{n+1}, Tv) \leq \hat{\lambda} \hat{\rho}_t(x_{n+2}, Tv)$, then

$$\hat{\rho}_t(x_{n+1}, Tv) \leq \hat{\lambda} b \hat{\rho}_s(x_{n+2}, x_{n+1}) + \hat{\lambda} b \hat{\rho}_t(x_{n+1}, Tv),$$

which deduces

$$\hat{\rho}_t(x_{n+1}, Tv) \leq \frac{\hat{\lambda} b}{1 - \hat{\lambda} b} \hat{\rho}_s(x_{n+2}, x_{n+1}).$$

Therefore, by the above, we obtain that

$$\begin{aligned}\hat{\rho}_t(x_{n+1}, Tv) &\leq \hat{\lambda} \max\left\{\hat{\rho}_t(x_n, v), \hat{\rho}_t(x_n, x_{n+1}), \frac{\hat{\lambda} b}{1 - \hat{\lambda} b} \hat{\rho}_s(x_{n+1}, v), \frac{\hat{\lambda} b}{1 - \hat{\lambda} b} \hat{\rho}_s(x_n, x_{n+1}), \right. \\ &\quad \left. \hat{\rho}_t(x_{n+1}, v), \hat{\rho}_t(x_{n+2}, x_n), \hat{\rho}_t(x_{n+2}, x_{n+1}), \hat{\rho}_t(x_{n+2}, v), \frac{\hat{\lambda} b}{1 - \hat{\lambda} b} \hat{\rho}_s(x_{n+2}, x_{n+1})\right\},\end{aligned}\quad (6)$$

for any $t \in (0, 1]$ and $n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} \hat{\rho}_t(x_n, v) = 0$, for any $t \in (0, 1]$, we have that $\lim_{n \rightarrow \infty} \hat{\rho}_s(x_n, v) = 0$, where $s = s(t) \in (0, t]$.

Hence, by Equation (6), we conclude that

$$\lim_{n \rightarrow \infty} \hat{\rho}_t(x_{n+1}, Tv) = 0.$$

By Lemma 1, we have $Tv = v$. It is immediate from the proof of (1) that T has the unique fixed-point v in $\mathcal{M}$. $\square$

Remark 4. It is immediate from Theorem 1 that we can conclude Theorem 3.1 in [19] and Theorem 1 in [13].

The next result readily follows from the above theorem.

Corollary 1. Let $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ be a complete $\mathcal{KS}$ -FbMS with $(\mathfrak{R}2)$, $T^k : \mathcal{M} \rightarrow \mathcal{M}$ be a generalized Ćirić-type contraction with contraction constant $\hat{\lambda} \in [0, 1)$, for some $k \in \mathbb{Z}^+$. If one of the following conditions is satisfied

(1) $\mathfrak{D}$ has the Fatou property,

(2) $\hat{\lambda} \in [0, \frac{1}{b})$,

then T has a unique fixed-point in $\mathcal{M}$.

Proof. By the conclusion of Theorem 1, T^k has a unique fixed-point u in $\mathcal{M}$ and $T^k(Tu) = T(T^k u) = Tu$, it implies that $Tu = u$, that is, T has a fixed-point u . Its uniqueness is obvious. $\square$

To illustrate Theorem 1 we construct two examples, and show that these two conditions in Theorem 1 are complete independence. Next, we present an example that satisfies (1) in Theorem 1, but nor is (2) satisfied.

Example 2. Let $\mathcal{M} = X_a \cup X_c \cup X_e \cup X_f \cup X_g$, where a, c, e, f , and g are five distinct indexes, $X_i := \{i\} \times [0, 1]$, $i \in \{a, c, e, f, g\}$, and define $d : \mathcal{M} \times \mathcal{M} \rightarrow [0, 2]$, as follows

$$d(x, y) = \begin{cases} |\tilde{x} - \tilde{y}|, & x, y \in X_i, \tilde{x}\tilde{y} \neq 0, \\ \frac{1}{3}|\tilde{x} - \tilde{y}|, & x, y \in X_i, \tilde{x}\tilde{y} = 0, \\ 2, & (x, y) \in \{(X_a, X_c), (X_c, X_a), (X_a, X_e), (X_e, X_a)\}, \\ 1, & \text{otherwise,} \end{cases}$$

where $x := (i, \tilde{x})$, $y := (j, \tilde{y})$; $\tilde{x}, \tilde{y} \in [0, 1]$.

Let $T : \mathcal{M} \rightarrow \mathcal{M}$ be a mapping defined by

$$Tx = \begin{cases} (a, \frac{1}{2}\tilde{x}), & x \in X_i, i \in \{a, f, g\}, \\ (f, \tilde{x}), & x \in X_c, \\ (g, \tilde{x}), & x \in X_e, \end{cases}$$

for any $x = (i, \tilde{x})$, $\tilde{x} \in [0, 1]$, and $\mathfrak{D}(x, y) : \mathbb{R} \rightarrow \mathbb{R}$ be a mapping. If $x = y \in \mathcal{M}$, we define $\mathfrak{D}(x, y)(\xi) = \bar{0}(\xi)$ for any $\xi \in \mathbb{R}$. If $x, y \in \mathcal{M}$ with $x \neq y$, $\mathfrak{D}(x, y)$ is defined by

$$\mathfrak{D}(x, y)(\xi) = \begin{cases} 0, & \xi < 0, \\ \frac{1}{1 + \frac{\xi}{d(x, y)}}, & \xi \geq 0, \end{cases}$$

and $\mathcal{L}(\alpha, \beta) = \min\{\alpha, \beta\}$, $\mathfrak{R}(\alpha, \beta) = \max\{\alpha, \beta\}$,

then the following assertions hold:

(1) $(\mathcal{M}, d)$ is a complete b -MS with the coefficient $b = 3$;

(2) $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ is a complete $\mathcal{KS}$ -FbMS with $(\mathfrak{R}2)$ and the coefficient $b = 3$;

(3) $T : \mathcal{M} \rightarrow \mathcal{M}$ is a generalized Ćirić-type contraction in $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ with contraction constant $\hat{\lambda} = \frac{1}{2} > \frac{1}{b}$. However, T is not a Ćirić-type contraction;

(4) $\mathfrak{D}$ has the Fatou property;

(5) T has a unique fixed-point in $\mathcal{M}$.

Proof. (1) The proof is analogous to the proof of Example 1.

(2) It is clear that $(\mathfrak{R}2)$ holds. Moreover, the inclusion can be proved in the same way as in the proof of Example 1, the only difference being that Case $\mathfrak{R}3$ with $b = 3$.

Case $\mathfrak{R}3$. Assume that $\theta > 0$ and $\delta > 0$. For $x, y, z \in \mathcal{M}$.

(i) If $x = y$ and $z \in \mathcal{M}$, then

$$\mathfrak{D}(x, y)(3(\theta + \delta)) = 0 \leq \max\{\mathfrak{D}(x, z)(\theta), \mathfrak{D}(z, y)(\delta)\}.$$

(ii) If $z = x \neq y$ or $x \neq y = z$, we may assume that $z = x \neq y$, then

$$\begin{aligned}\mathfrak{D}(x, y)(3(\theta + \delta)) &= \frac{1}{1 + \frac{3(\theta + \delta)}{d(x, y)}} \\ &= \frac{1}{1 + \frac{3(\theta + \delta)}{d(z, y)}} \\ &\leq \frac{1}{1 + \frac{\delta}{d(z, y)}} \\ &= \mathfrak{D}(z, y)(\delta) \\ &\leq \max\{\mathfrak{D}(x, z)(\theta), \mathfrak{D}(z, y)(\delta)\}.\end{aligned}$$

(iii) If $x \neq y \neq z$. Note that

$$\min\left\{\frac{\theta}{d(x, z)}, \frac{\delta}{d(z, y)}\right\} \leq \frac{\theta + \delta}{d(x, z) + d(z, y)}.$$

Then

$$\begin{aligned}\mathfrak{D}(x, y)(3(\theta + \delta)) &= \frac{1}{1 + \frac{3(\theta + \delta)}{d(x, y)}} \\ &\leq \frac{1}{1 + \frac{3(\theta + \delta)}{3(d(x, z) + d(z, y))}} \\ &\leq \frac{1}{1 + \min\left\{\frac{\theta}{d(x, z)}, \frac{\delta}{d(z, y)}\right\}} \\ &\leq \max\left\{\frac{1}{1 + \frac{\theta}{d(x, z)}}, \frac{1}{1 + \frac{\delta}{d(z, y)}}\right\} \\ &= \max\{\mathfrak{D}(x, z)(\theta), \mathfrak{D}(z, y)(\delta)\}.\end{aligned}$$

We thus conclude that $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ is a $\mathcal{KS}$ -FbMS.

On the other hand, it follows from the definition of $\mathfrak{D}(x, y)$ that

$$\mathfrak{D}(x, y)(\hat{\rho}_t(x, y)) = t,$$

for all $t \in (0, 1]$. Hence, we have

$$\hat{\rho}_t(x, y) = \frac{1-t}{t}d(x, y).$$

Moreover, by (1), we obtain that $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ is a complete $\mathcal{KS}$ -FbMS with $(\mathfrak{R}2)$ and the coefficient $b = 3$.

(3) For any $x, y \in \mathcal{M}$, the following four cases are considered:

Case 1. If $x \in X_c, y \in X_e$ or $x \in X_e, y \in X_c$; we may assume that $x = (c, \tilde{x}) \in X_c, y = (e, \tilde{y}) \in X_e$, where $\tilde{x}, \tilde{y} \in [0, 1]$, we have $\hat{\rho}_t(Tx, Ty) = \hat{\rho}_t(x, y) = \hat{\rho}_t(x, Tx) = \hat{\rho}_t(y, Ty) = \hat{\rho}_t(x, Ty) = \hat{\rho}_t(y, Tx) = \hat{\rho}_t(T^2x, Tx) = \hat{\rho}_t(T^2x, Ty) = \frac{1-t}{t}, \hat{\rho}_t(T^2x, x) = \hat{\rho}_t(T^2x, y) = 2 \cdot \frac{1-t}{t}$, where $t \in (0, 1]$.

Case 2. If $x, y \in X_c$ or $x, y \in X_e$; we may assume that $x = (c, \tilde{x}), y = (c, \tilde{y}) \in X_c$, we have $\hat{\rho}_t(Tx, Ty) = \hat{\rho}_t(x, y) \leq \frac{1-t}{t}, \hat{\rho}_t(T^2x, x) = 2 \cdot \frac{1-t}{t}$, where $t \in (0, 1]$.

Case 3. If $x \in X_i, y \in X_j$, where $i, j \in \{a, f, g\}$; for $x = (i, \tilde{x}), y = (j, \tilde{y})$, we have $\hat{\rho}_t(Tx, Ty) \leq \frac{1}{2}\hat{\rho}_t((i, \tilde{x}), (j, \tilde{y})) = \frac{1}{2}\hat{\rho}_t(x, y)$, where $t \in (0, 1]$.

Case 4. If $x \in X_i$, where $i \in \{a, f, g\}$, $y \in X_c$ or X_e ; for $x = (i, \tilde{x}) \in X_i$, we may assume that $y = (c, \tilde{y}) \in X_c$, we have $\hat{\rho}_t(Tx, Ty) = \frac{1-t}{t}$, $\hat{\rho}_t(Tx, y) = 2 \cdot \frac{1-t}{t}$, where $t \in (0, 1]$.

The above calculations present that T is not Ćirić-type contraction for $x = (c, \tilde{x}) \in X_c, y = (e, \tilde{y}) \in X_e$, because there is no a non-negative number $\hat{\lambda} < 1$ satisfying the Ćirić-type contraction condition. However, it follows that T is a generalized Ćirić-type contraction in $(\mathcal{M}, \mathcal{D}, \mathcal{L}, \mathfrak{R}, b)$ with $\hat{\lambda} = \frac{1}{2} > \frac{1}{b}$ and for all $x, y \in \mathcal{M}$.

(4) Now, we will show that $\mathcal{D}$ has the Fatou property.

Given $x, y \in \mathcal{M}$. Let $\{x_n\}$ be a sequence in $\mathcal{M}$ and x_n converges to x . If there exists $\hat{n} \in \mathbb{N}$, s.t. $x_n = x$ for any $n \geq \hat{n}$, then, for any $t \in (0, 1]$,

$$\hat{\rho}_t(x, y) = \liminf_{n \rightarrow \infty} \hat{\rho}_t(x_n, y).$$

Otherwise, we may assume that $x_n \neq x$ for all $n \in \mathbb{N}$. The following two cases are considered.

Case F1. If $x \in X_i, y \in X_j$, where $i \neq j$ and $i, j \in \{a, c, e, f, g\}$. It follows from $\hat{\rho}_t(x_n, x) \rightarrow 0$ that there exists $\bar{n} \in \mathbb{N}$, s.t. $x_n \in X_i$ for all $n \geq \bar{n}$. Hence, for each $t \in (0, 1]$, we have $\hat{\rho}_t(x, y) = \liminf_{n \rightarrow \infty} \hat{\rho}_t(x_n, y)$.

Case F2. If $x = (i, \tilde{x}), y = (i, \tilde{y}) \in X_i$, where $i \in \{a, c, e, f, g\}$. In this case, we discuss the following three subcases.

Case F2a. If $\tilde{x}\tilde{y} \neq 0$. Since x_n converges to $x \neq (i, 0)$, then there exists $n_0 \in \mathbb{N}$, s.t. $x_n \neq (i, 0)$ for any $n \geq n_0$. We thus conclude that

$$\hat{\rho}_t(x, y) = \frac{1-t}{t} |\tilde{x} - \tilde{y}| \leq \frac{1-t}{t} |\tilde{x} - \tilde{x}_n| + \frac{1-t}{t} |\tilde{x}_n - \tilde{y}| = \hat{\rho}_t(x, x_n) + \hat{\rho}_t(x_n, y),$$

for any $n \geq n_0$. Hence we have $\hat{\rho}_t(x, y) \leq \liminf_{n \rightarrow \infty} \hat{\rho}_t(x_n, y)$.

Case F2b. If $\tilde{x} \neq 0, \tilde{y} = 0$. Thus, there exists $n_0 \in \mathbb{N}$, s.t. $x_n \neq (i, 0)$ for any $n \geq n_0$. Hence, we find that

$$\begin{aligned} \hat{\rho}_t(x, y) &= \frac{1-t}{t} \cdot \frac{1}{3} |\tilde{x} - \tilde{y}| \leq \frac{1-t}{t} \cdot \frac{1}{3} |\tilde{x} - \tilde{x}_n| + \frac{1-t}{t} \cdot \frac{1}{3} |\tilde{x}_n - \tilde{y}| \\ &= \frac{1}{3} \hat{\rho}_t(x, x_n) + \hat{\rho}_t(x_n, y). \end{aligned}$$

Hence we find $\hat{\rho}_t(x, y) \leq \liminf_{n \rightarrow \infty} \hat{\rho}_t(x_n, y)$.

Case F2c. If $\tilde{x} = 0$. In this case, if $\tilde{y} = 0$, then $x = y$ and $\hat{\rho}_t(x, y) \leq \liminf_{n \rightarrow \infty} \hat{\rho}_t(x_n, y)$, where $t \in (0, 1]$. So we suppose $\tilde{y} \neq 0$. Since $x_n \neq x$ for all $n \in \mathbb{N}$, we obtain that

$$\begin{aligned} \hat{\rho}_t(x, y) &= \frac{1-t}{t} \cdot \frac{1}{3} |\tilde{x} - \tilde{y}| \leq \frac{1-t}{t} \cdot \frac{1}{3} |\tilde{x} - \tilde{x}_n| + \frac{1-t}{t} \cdot \frac{1}{3} |\tilde{x}_n - \tilde{y}| \\ &= \hat{\rho}_t(x, x_n) + \frac{1}{3} \hat{\rho}_t(x_n, y). \end{aligned}$$

Hence, we find

$$\hat{\rho}_t(x, y) \leq \frac{1}{3} \liminf_{n \rightarrow \infty} \hat{\rho}_t(x_n, y) \leq \liminf_{n \rightarrow \infty} \hat{\rho}_t(x_n, y).$$

In light of the above, it is clear that $\mathcal{D}$ has the Fatou property.

(5) It is obvious that there exist a unique $x = (a, 0) \in \mathcal{M}$, such that $Tx = x$. Hence, T has a unique fixed-point in $\mathcal{M}$. $\square$

In the following example, (2) in Theorem 1 holds, but nor (1) is satisfied.

Example 3. Let $\mathcal{M} = X_a \cup X_c \cup X_e \cup X_f \cup X_g$, where a, c, e, f and g are five distinct indexes, $X_i := \{i\} \times [0, 1], i \in \{a, c, e, f, g\}$, and define $d : \mathcal{M} \times \mathcal{M} \rightarrow [0, 6]$, as follows

$$d(x, y) = \begin{cases} |\tilde{x} - \tilde{y}|, & x, y \in X_i, \tilde{x}\tilde{y} \neq 0, \\ 2|\tilde{x} - \tilde{y}|, & x, y \in X_i, \tilde{x}\tilde{y} = 0, \\ 6, & (x, y) \in \{(X_a, X_c), (X_c, X_a), (X_a, X_e), (X_e, X_a)\}, \\ 2, & \text{otherwise,} \end{cases}$$

where $x := (i, \tilde{x}), y := (j, \tilde{y}); \tilde{x}, \tilde{y} \in [0, 1]$.

Let $T : \mathcal{M} \rightarrow \mathcal{M}$ be a mapping defined by

$$Tx = \begin{cases} (a, \frac{1}{3}\tilde{x}), & x \in X_i, i \in \{a, f, g\}, \\ (f, \tilde{x}), & x \in X_c, \\ (g, \tilde{x}), & x \in X_e, \end{cases}$$

for any $x = (i, \tilde{x}), \tilde{x} \in [0, 1]$, and $\mathfrak{D}(x, y) : \mathbb{R} \rightarrow \mathbb{R}$ be a mapping. If $x = y \in \mathcal{M}$, we define $\mathfrak{D}(x, y)(\xi) = \bar{0}(\xi)$ for any $\xi \in \mathbb{R}$. If $x, y \in \mathcal{M}$ with $x \neq y$, $\mathfrak{D}(x, y)$ is defined by

$$\mathfrak{D}(x, y)(\xi) = \begin{cases} 0, & \xi < 0, \\ \frac{1}{1 + \frac{\xi}{d(x, y)}}, & \xi \geq 0, \end{cases}$$

and $\mathcal{L}(\alpha, \beta) = \min\{\alpha, \beta\}, \mathfrak{R}(\alpha, \beta) = \max\{\alpha, \beta\}$,

then the following assertions hold:

- (1) $(\mathcal{M}, d)$ is a complete b -MS with the coefficient $b = 2$;
- (2) $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ is a complete $\mathcal{KS}$ -FbMS with $(\mathfrak{R}2)$ and the coefficient $b = 2$;
- (3) $T : \mathcal{M} \rightarrow \mathcal{M}$ is a generalized Ćirić-type contraction in $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ with $\hat{\lambda} = \frac{1}{3} < \frac{1}{b}$. However, T is not a Ćirić-type contraction;
- (4) $\mathfrak{D}$ fails to have the Fatou property;
- (5) T has a unique fixed-point in $\mathcal{M}$.

Proof. (1). The proof is analogous to the proof of Example 1.

(2). The inclusion can be proved in the same way as in the proof of Example 2.

(3). For any $x, y \in \mathcal{M}$, we discuss the following four cases:

Case 1. If $x \in X_c, y \in X_e$ or $x \in X_e, y \in X_c$; we may suppose that $x = (c, \tilde{x}) \in X_c, y = (e, \tilde{y}) \in X_e$, where $\tilde{x}, \tilde{y} \in [0, 1]$, we have $\hat{\rho}_t(Tx, Ty) = \hat{\rho}_t(x, y) = \hat{\rho}_t(x, Tx) = \hat{\rho}_t(y, Ty) = \hat{\rho}_t(x, Ty) = \hat{\rho}_t(y, Tx) = \hat{\rho}_t(T^2x, Tx) = \hat{\rho}_t(T^2x, Ty) = 2 \cdot \frac{1-t}{t}, \hat{\rho}_t(T^2x, x) = \hat{\rho}_t(T^2x, y) = 6 \cdot \frac{1-t}{t}$, where $t \in (0, 1]$.

Case 2. If $x, y \in X_c$ or $x, y \in X_e$; we may suppose that $x = (c, \tilde{x}), y = (c, \tilde{y}) \in X_c$, we find $\hat{\rho}_t(Tx, Ty) = \hat{\rho}_t(x, y) \leq 2 \cdot \frac{1-t}{t}, \hat{\rho}_t(T^2x, x) = 6 \cdot \frac{1-t}{t}$, where $t \in (0, 1]$.

Case 3. If $x \in X_i, y \in X_j$, where $i, j \in \{a, f, g\}$; for $x = (i, \tilde{x}), y = (j, \tilde{y})$, we have $\hat{\rho}_t(Tx, Ty) \leq \frac{1}{3}\hat{\rho}_t((i, \tilde{x}), (j, \tilde{y})) = \frac{1}{3}\hat{\rho}_t(x, y)$, where $t \in (0, 1]$.

Case 4. If $x \in X_i$, where $i \in \{a, f, g\}, y \in X_c$ or X_e ; for $x = (i, \tilde{x}) \in X_i$, we may suppose that $y = (c, \tilde{y}) \in X_c$, we have $\hat{\rho}_t(Tx, Ty) = 2 \cdot \frac{1-t}{t}, \hat{\rho}_t(Tx, y) = 6 \cdot \frac{1-t}{t}$, where $t \in (0, 1]$.

The above calculations show that T is not Ćirić-type contraction for $x = (c, \tilde{x}) \in X_c, y = (e, \tilde{y}) \in X_e$, because there is no a non-negative number $\hat{\lambda} < 1$ satisfying the Ćirić-type contraction condition. However, it follows that T is a generalized Ćirić-type contraction in $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ with contraction constant $\hat{\lambda} = \frac{1}{3} < \frac{1}{b}$ and for any $x, y \in \mathcal{M}$.

(3). Now, we show that $\mathfrak{D}$ fails to have the Fatou property.

Choose $x_n = (a, \frac{1}{n})$, $n \in \mathbb{N}$ and $y = (a, 1)$. Then, for $x_n \neq (a, 0)$, we have that $\hat{\rho}_t(x_n, (a, 0)) = \hat{\rho}_t((a, \frac{1}{n}), (a, 0)) = \frac{2(1-t)}{nt} \rightarrow 0$ as $n \rightarrow \infty$. Since

$$\liminf_{n \rightarrow \infty} \hat{\rho}_t(x_n, y) = \liminf_{n \rightarrow \infty} \frac{1-t}{t} \left(1 - \frac{1}{n}\right) = \frac{1-t}{t},$$

$$\hat{\rho}_t((a, 0), (a, 1)) = \frac{2(1-t)}{t} > \liminf_{n \rightarrow \infty} \hat{\rho}_t(x_n, y),$$

where $t \in (0, 1)$, we obtain that $\mathfrak{D}$ fails to have the Fatou property.

(4). It is obvious that there exist a unique $x = (a, 0) \in \mathcal{M}$, such that $Tx = x$. Hence, T has a unique fixed-point in $\mathcal{M}$. $\square$

Remark 5. It is noteworthy that in Example 2 and Example 3, if we took $d(x, y) = \alpha|\tilde{x} - \tilde{y}|$, when $x, y \in X_i$ and $\tilde{x}\tilde{y} = 0$, it is easy to obtain that the following assertions hold:

(1) If $\alpha > 1$, then $b = \alpha$ and $\mathfrak{D}$ fails to have the Fatou property.

(2) If $\alpha < 1$, then $b = \frac{1}{\alpha}$ and $\mathfrak{D}$ has Fatou property.

The following example illustrates that the conditions in Theorem 1 are sufficient but not necessary for the existence of unique fixed-points.

Example 4. Let $\mathcal{M} = X_a \cup X_c \cup X_e \cup X_f \cup X_g$, where a, c, e, f , and g are five distinct indexes, $X_i := \{i\} \times [0, 1]$, $i \in \{a, c, e, f, g\}$, and define $d : \mathcal{M} \times \mathcal{M} \rightarrow [0, 3]$, as follows

$$d(x, y) = \begin{cases} |\tilde{x} - \tilde{y}|, & x, y \in X_i, \tilde{x}\tilde{y} \neq 0, \\ 2|\tilde{x} - \tilde{y}|, & x, y \in X_i, \tilde{x}\tilde{y} = 0, \\ 3, & (x, y) \in \{(X_a, X_c), (X_c, X_a), (X_a, X_e), (X_e, X_a)\}, \\ 2, & \text{otherwise,} \end{cases}$$

where $x := (i, \tilde{x})$, $y := (j, \tilde{y})$; $\tilde{x}, \tilde{y} \in [0, 1]$.

Let $T : \mathcal{M} \rightarrow \mathcal{M}$ be a map defined by

$$Tx = \begin{cases} (a, \frac{2}{3}\tilde{x}), & x \in X_i, i \in \{a, f, g\}, \\ (f, \tilde{x}), & x \in X_c, \\ (g, \tilde{x}), & x \in X_e, \end{cases}$$

for any $x = (i, \tilde{x})$, $\tilde{x} \in [0, 1]$, and $\mathfrak{D}(x, y) : \mathbb{R} \rightarrow \mathbb{R}$ be a mapping. If $x = y \in \mathcal{M}$, we define $\mathfrak{D}(x, y)(\xi) = \bar{0}(\xi)$, for any $\xi \in \mathbb{R}$. If $x, y \in \mathcal{M}$ with $x \neq y$, $\mathfrak{D}(x, y)$ is defined by

$$\mathfrak{D}(x, y)(\xi) = \begin{cases} 0, & \xi < 0, \\ \frac{1}{1 + \frac{\xi}{d(x, y)}}, & \xi \geq 0, \end{cases}$$

and $\mathcal{L}(\alpha, \beta) = \min\{\alpha, \beta\}$, $\mathfrak{R}(\alpha, \beta) = \max\{\alpha, \beta\}$,

then the following assertions hold:

(1) $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ is a complete $\mathcal{KS}$ -FbMS with $(\mathfrak{R}2)$ and the coefficient $b = 2$;

(2) $T : \mathcal{M} \rightarrow \mathcal{M}$ is a generalized Ćirić-type contraction in $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ with $\hat{\lambda} = \frac{2}{3} > \frac{1}{b}$.

However, T is not a Ćirić-type contraction;

(3) $\mathfrak{D}$ fails to have the Fatou property;

(4) T has a unique fixed-point in $\mathcal{M}$.

Proof. (1). The inclusion can be proved in the same way as in the proof of Example 3.

(2). For any $x, y \in \mathcal{M}$, we discuss the following four cases:

Case 1. If $x \in X_c$, $y \in X_e$ or $x \in X_e$, $y \in X_c$; we may suppose that $x = (c, \tilde{x}) \in X_c$, $y =$

$(e, \tilde{y}) \in X_e$, where $\tilde{x}, \tilde{y} \in [0, 1]$, we have $\hat{\rho}_t(Tx, Ty) = \hat{\rho}_t(x, y) = \hat{\rho}_t(x, Tx) = \hat{\rho}_t(y, Ty) = \hat{\rho}_t(x, Ty) = \hat{\rho}_t(y, Tx) = \hat{\rho}_t(T^2x, Tx) = \hat{\rho}_t(T^2x, Ty) = 2 \cdot \frac{1-t}{t}, \hat{\rho}_t(T^2x, x) = \hat{\rho}_t(T^2x, y) = 3 \cdot \frac{1-t}{t}$, where $t \in (0, 1]$.

Case 2. If $x, y \in X_c$ or $x, y \in X_e$; we may suppose that $x = (c, \tilde{x}), y = (c, \tilde{y}) \in X_c$, we have $\hat{\rho}_t(Tx, Ty) = \hat{\rho}_t(x, y) \leq 2 \cdot \frac{1-t}{t}, \hat{\rho}_t(T^2x, x) = 3 \cdot \frac{1-t}{t}$, where $t \in (0, 1]$.

Case 3. If $x \in X_i, y \in X_j$, where $i, j \in \{a, f, g\}$; for $x = (i, \tilde{x}), y = (j, \tilde{y})$, we have $\hat{\rho}_t(Tx, Ty) \leq \frac{2}{3}\hat{\rho}_t((i, \tilde{x}), (j, \tilde{y})) = \frac{2}{3}\hat{\rho}_t(x, y)$, where $t \in (0, 1]$.

Case 4. If $x \in X_i$, where $i \in \{a, f, g\}$, $y \in X_c$ or X_e ; for $x = (i, \tilde{x}) \in X_i$, we may suppose that $y = (c, \tilde{y}) \in X_c$, we have $\hat{\rho}_t(Tx, Ty) = 2 \cdot \frac{1-t}{t}, \hat{\rho}_t(Tx, y) = 3 \cdot \frac{1-t}{t}$, where $t \in (0, 1]$.

The above calculations show that T is not a Ćirić-type contraction for $x = (c, \tilde{x}) \in X_c, y = (e, \tilde{y}) \in X_e$, because there is no a non-negative number $\hat{\lambda} < 1$ satisfying the Ćirić-type contraction condition. However, it follows that T is a generalized Ćirić-type contraction in $(\mathcal{M}, \mathcal{D}, \mathcal{L}, \mathfrak{R}, b)$ with contraction constant $\hat{\lambda} = \frac{2}{3} > \frac{1}{b}$ and for all $x, y \in \mathcal{M}$.

(3). It follows from Example 3 that $\mathcal{D}$ fails to have the Fatou property.

(4). It is obvious that there exist a unique $x = (a, 0) \in \mathcal{M}$, such that $Tx = x$. Hence, T has a unique fixed-point in $\mathcal{M}$. $\square$

Next, we give some corollaries on Ćirić-type fixed-point theorems in $\mathcal{KS}$ -FbMS.

Corollary 2. Let $(\mathcal{M}, \mathcal{D}, \mathcal{L}, \mathfrak{R}, b)$ be a complete $\mathcal{KS}$ -FbMS with $(\mathfrak{R}2)$, $T : \mathcal{M} \rightarrow \mathcal{M}$ be a Ćirić-type contraction with contraction constant $\hat{\lambda} \in [0, 1)$, which satisfies one of the following conditions,

- (1) $\hat{\lambda} \in [0, \frac{1}{b})$,
- (2) $\mathcal{D}$ has the Fatou property,

then T has a unique fixed-point in $\mathcal{M}$.

Remark 6. It is immediate from Corollary 2 that we can conclude Theorem 13 in [12].

Corollary 3. Let $(\mathcal{M}, \mathcal{D}, \mathcal{L}, \mathfrak{R}, b)$ be a complete $\mathcal{KS}$ -FbMS with $(\mathfrak{R}2)$, $T^k : \mathcal{M} \rightarrow \mathcal{M}$ be a Ćirić-type contraction with contraction constant $\hat{\lambda} \in [0, 1)$, for some $k \in \mathbb{Z}^+$. If one of the following conditions are satisfied

- (1) $\hat{\lambda} \in [0, \frac{1}{b})$,
- (2) $\mathcal{D}$ satisfies the Fatou property,

then T has a unique fixed-point in $\mathcal{M}$.

Remark 7. It is immediate from Corollary 3 that we can conclude one of the main results of Theorem 2 in [13].

5. Existence of Solution to Volterra Integral Equation

Integral equation theory plays an important role in applied mathematics. Volterra integral equations are used in many fields of physics, such as actuarial sciences, demography, potential theory and Dirichlet problems, reactor theory, electrostatics, among others [10,22,23].

Let us discuss the following Volterra integral equation:

$$x(s) = \phi(s) + \mu \int_0^s k(s, v)x(v)dv, s \in [0, 1]. \quad (7)$$

By applying Corollary 1, we show the existence of solution to the Equation (7).

Example 5. Suppose that the Equation (7) satisfies the following conditions; $k(s, v)$ and $\phi(s)$ are continuous for $0 \leq s \leq 1, 0 \leq v \leq 1$. Then, the Equation (7) has a unique solution in $C[0, 1]$.

Indeed, let $\mathcal{M} = C[0, 1]$, and define $d : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$, as follows

$$d(x, y) = \begin{cases} \frac{1}{3} \max_{0 \leq s \leq 1} |x(s) - y(s)|, & x(s)y(s) \equiv 0, \forall s \in [0, 1]; \\ \max_{0 \leq s \leq 1} |x(s) - y(s)|, & \text{other}; \end{cases}$$

where $x, y \in \mathcal{M}$.

Let $T : \mathcal{M} \rightarrow \mathcal{M}$ defined by

$$Tx(s) = \phi(s) + \mu \int_0^s k(s, v)x(v)dv. \quad (8)$$

$\mathfrak{D}(x, y) : \mathbb{R} \rightarrow \mathbb{R}$ be a mapping. If $x = y \in \mathcal{M}$, we define $\mathfrak{D}(x, y)(\xi) = \bar{0}(\xi)$, for any $\xi \in \mathbb{R}$. If $x, y \in \mathcal{M}$ with $x \neq y$, $\mathfrak{D}(x, y)$ is defined by

$$\mathfrak{D}(x, y)(\xi) = \begin{cases} 0, & \xi < 0, \\ e^{-\frac{\xi}{d(x, y)}}, & \xi \geq 0, \end{cases}$$

and $\mathcal{L}(\alpha, \beta) = \min\{\alpha, \beta\}$, $\mathfrak{R}(\alpha, \beta) = \max\{\alpha, \beta\}$,

Notice that $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$ is a complete $\mathcal{KS}$ -FbMS with $(\mathfrak{R}2)$ and the coefficient $b = 3$, and $\mathfrak{D}$ satisfies the Fatou property.

Next, we shall show that $T^k : \mathcal{M} \rightarrow \mathcal{M}$ is a generalized Ćirić-type contraction in $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$, for some $k \in \mathbb{Z}^+$.

$$\begin{aligned} |Tx - Ty| &= |\mu| \left| \int_0^s k(s, v)[x(v) - y(v)]dv \right| \\ &\leq |\mu| M s \max_{0 \leq s \leq 1} |x(s) - y(s)|, \end{aligned}$$

where $M = \max_{0 \leq s \leq 1, 0 \leq vs \leq 1} |k(s, v)|$. Moreover, we have:

$$\begin{aligned} |T^2x - T^2y| &= |T(Tx) - T(Ty)| \\ &\leq |\mu|^2 M^2 \int_0^s v \max_{0 \leq s \leq 1} |x(v) - y(v)| dv \\ &= |\mu|^2 M^2 \frac{s^2}{2!} \max_{0 \leq s \leq 1} |x(s) - y(s)|. \end{aligned}$$

By induction, for any $n \in \mathbb{Z}^+$, we conclude that

$$|T^n x - T^n y| \leq |\mu|^n M^n \frac{s^n}{n!} \max_{0 \leq s \leq 1} |x(s) - y(s)|.$$

Furthermore, we have that:

$$\begin{aligned} d(T^n x, T^n y) &\leq \max_{0 \leq s \leq 1} |T^n x - T^n y| \\ &\leq |\mu|^n M^n \frac{1}{n!} \max_{0 \leq s \leq 1} |x(s) - y(s)| \\ &\leq \frac{3}{n!} |\mu|^n M^n d(x, y). \end{aligned}$$

By the proof of Example 1, we have $\hat{\rho}_t(T^n x, T^n y) \leq \frac{3}{n!} |\mu|^n M^n \hat{\rho}_t(x, y)$, for all $t \in (0, 1]$.

Since $\frac{3}{n!} |\mu|^n M^n \rightarrow 0$ ($n \rightarrow \infty$), then there exists a finite positive integer k such that $0 < \frac{3}{k!} |\mu|^k M^k < 1$. Hence, we have that $T^k : \mathcal{M} \rightarrow \mathcal{M}$ is a generalized Ćirić-type contraction in $(\mathcal{M}, \mathfrak{D}, \mathcal{L}, \mathfrak{R}, b)$, for some $k \in \mathbb{Z}^+$.

So, by applying Corollary 1, T has a unique fixed-point in $\mathcal{M}$, i.e., the Equation (7) has a unique solution in $C[0, 1]$.

In particular, the following Volterra integral equation is considered:

$$x(s) = s + \int_0^s (v - s)x(v)dv, s \in [0, 1]. \quad (9)$$

Define $T : \mathcal{M} \rightarrow \mathcal{M}$ as follows:

$$Tx(s) = s + \int_0^s (v - s)x(v)dv. \quad (10)$$

Note that, since $\phi(s) = s, \mu = 1, k(s, v) = v - s$, we have $\frac{3}{n!}|\mu|^n M^n = \frac{3}{n!}$. Thus, $T^3 : \mathcal{M} \rightarrow \mathcal{M}$ is a generalized Ćirić-type contraction.

It is not difficult to verify that $x(s) = \sin s$ is the unique solution for Equation (9).

6. Conclusions

In this paper, we research a generalized Ćirić-type fixed-point theorem in $\mathcal{KS}$ -FbMS. Roughly speaking, the $\mathcal{KS}$ -FbMS is much broader and the generalized Ćirić-type contraction is much weaker. Our main result improves some fixed-point theorems in the literature, such as a fixed-point theorem of the Ćirić-type contraction in metric spaces ([13], Theorem 1), a fixed-point theorem of the generalized Ćirić-type contraction in metric spaces ([19], Theorem 3.1), a fixed-point theorem of the Ćirić-type contraction in b -MS ([21], Theorem 3) and a fixed-point theorem of the Reich type contraction in $\mathcal{KS}$ -FbMS ([12], Theorem 13). As an application, we show the existence of solution to Volterra equation in $\mathcal{KS}$ -FbMS.

In addition, from Example 4, we see that two sufficient conditions are not necessary for the existence of unique fixed-point. Naturally, the question arises, what is the only necessary condition for the existence of unique fixed-point theorems for generalized Ćirić-type contractions in $\mathcal{KS}$ -FbMS?

As future research direction, we point out the following:

1. To study necessary and sufficient conditions for the existence of unique fixed-point theorems for generalized Ćirić-type contractions in $\mathcal{KS}$ -FbMS;
2. To study fixed-point theorems for contractions of Boyd–Wong type in $\mathcal{KS}$ -FbMS.

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