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Article

Kropina Metrics with Isotropic Scalar Curvature

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Abstract: In this paper, we study Kropina metrics with isotropic scalar curvature. First, we obtain the expressions of Ricci curvature tensor and scalar curvature. Then, we characterize the Kropina metrics with isotropic scalar curvature on by tensor analysis.

Keywords: Kropina metrics; Ricci curvature tensor; scalar curvature

MSC: 53C30; 53C60

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1. Introduction

The curvature properties of metrics play very important roles in Riemannian and Finsler geometry. Riemannian curvature and Ricci curvature are the most important Riemannian geometric quantities in Finsler geometry. In 1988, the concept of Ricci curvature was first proposed by Akbar-Zadeh, and its tensor form can be naturally obtained [1]. In recent years, many scholars have conducted a great deal of research on them. Cheng-Shen-Tian proved that the polynomial (α, β)-metric is an Einstein metric if and only if it is Ricci-flat [2]. Zhang-Shen gave the expression of Ricci curvature of Kropina metric. Furthermore, they proved that a non-Riemannian Kropina metric with a constant Killing form β is an Einstein metric if and only if α is also an Einstein metric [3]. By using navigation date (h, W), they proved that n (≥ 2)-dimensional Kropina metric is an Einstein metric if and only if Riemann metric h is an Einstein metric and W is a Killing vector field with respect to h. Xia gave the expression for the Riemannian curvature of Kropina metrics and proved that a Kropina metric is an Einstein metric if and only if it has non-negative constant flag curvature [4]. Cheng-Ma-Shen studied and characterized projective Ricci-flat Kropina metrics and obtained its equivalent characterization Equation [5].

Unlike the notion of Riemannian curvature, there is no unified definition of scalar curvature in Finsler geometry, although several geometers have offered several versions of the definition of the Ricci curvature tensor [1,6–8]. In 2015, Li–Shen introduced a new definition of the Ricci curvature tensor [6]. This tensor is symmetric. They proved that a Finsler metric F has isotropic Ricci curvature tensor if and only if it has isotropic Ricci curvature and χ-curvature tensor satisfies $\chi_i = f_{ij}(x)y^j$, where $f_{ij} + f_{ji} = 0$. It was further proven that for Randers metrics, they are isotropic Ricci curvature tensors if and only if they are of isotropic Ricci curvature.

In Finsler geometry, there are several versions of the definition of scalar curvature. We used Akbar-Zadeh's definition [1] of the scalar curvature, based on Li–Shen's definition of the (symmetric) Ricci curvature tensor [6]. For a Finsler metric F on an n-dimensional manifold M, the scalar curvature R of F is defined as $R := g^{ij} Ric_{ij}$. Tayebi studied general fourth-root metrics [9]. They characterized general fourth-root metrics with isotropic scalar curvature and also for general fourth-root metrics with isotropic scalar curvature under conformal variation. Finally, they characterized Bryant metric with isotropic scalar curvature. Chen–Xia studied a conformally flat (α, β)-metric with weakly isotropic scalar curvature [10]. They proved that if conformally flat polynomial (α, β)-metrics have weakly isotropic scalar curvature R, then R vanishes.

In this paper, we obtain a characterization of Kropina metrics with isotropic scalar curvature and have the following results.

Theorem 1. *Let F be a Kropina metric on an n (≥ 3)-dimensional manifold M. Then, F is of isotropic scalar curvature if and only if*

$$\begin{cases} {}^{\alpha}Ric = \dfrac{n-2}{b^4}(-g\alpha^2 + c^2\beta^2 + c\beta s_0 - b^2 c_0 \beta), \\ r_{00} = c(x)\alpha^2, \\ (2s^m s_m - b^2 s^m{}_{|m})\alpha^2 = (n-1)(c\beta s_0 + s_0^2 - b^2 s_{0|0}), \\ h\alpha^2 = \left[\dfrac{3(n+1)}{n-2}\,{}^{\alpha}R - \dfrac{6(n+1)}{(n-2)b^2} b^k b^l\,{}^{\alpha}Ric_{kl} + \dfrac{3(n-1)(n+1)}{b^2}c^2 + \dfrac{n(3n+5)}{b^4}s^m s_m\right]\beta^2 \\ \quad + \left[\dfrac{(3n^2+4n-4)}{b^2}s^m_0 s_m - 3(n+1)s^m_{0|m} + \dfrac{3(n-1)(n+1)}{b^2}cs_0\right]\beta + 2(n-1)s^m_0 s_{m0}, \end{cases} \tag{1}$$

where $b = ||\beta||_\alpha$, g, h, c^2, c_0 are expressed by (15), (23), (26), (27), respectively. In this case, scalar curvature is

$$R = -\frac{1}{12}\left[\frac{n(1-b^2)}{b^2\beta}s^m_0 s_m + \frac{2(3n+2)}{b^2}s^m s_m + (3n+4)s^m_t s^t{}_m\right].$$

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. A Finsler structure of M is a function

$$F : TM \to [0, \infty)$$

with the following properties:

(1) Regularity: F is C^∞ on the slit tangent bundle $TM\backslash\{0\}$;
(2) Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y), \forall \lambda > 0$;
(3) Strong convexity:

$$g_{ij}(x, y) := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y) = \frac{1}{2}(F^2)_{y^i y^j}$$

is positive-definite at every point of $TM\backslash\{0\}$.

Let (M, F) be an n-dimensional Finsler manifold. Suppose that $x \in M$. The geodesics of a Finsler metric $F = F(x, y)$ on M are classified by the following ODEs:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where

$$G^i := \frac{1}{4}g^{ik}\left[\left(F^2\right)_{x^j y^k} y^j - \left(F^2\right)_{x^k}\right],$$

$\left(g^{ij}\right) := \left(g_{ij}\right)^{-1}$. The local functions $G^i = G^i(x, y)$ are called geodesic coefficients (or spray coefficients). Then, the S curvature with respect to a volume form $dV = \sigma(x)dx$ is defined by

$$S = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial \ln \sigma}{\partial x^m}.$$

For $x \in M$ and $y \in T_xM\backslash\{0\}$, Riemann curvature $R_y := R^i{}_k(x,y)\frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i{}_k := 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The trace of Riemann curvature is called Ricci curvature of F, i.e., $Ric := R^k{}_k$.
Riemann curvature tensor is defined by

$$R_j{}^i{}_{kl} = \frac{1}{3}\left\{ \frac{\partial^2 R^i{}_k}{\partial y^l \partial y^j} - \frac{\partial^2 R^i{}_l}{\partial y^k \partial y^j} \right\}.$$

Let $\overline{Ric}_{ij} := R_j{}^k{}_{kl}$, and

$$Ric_{ij} := \frac{1}{2}\{\overline{Ric}_{ij} + \overline{Ric}_{ji}\}$$

is called a Ricci curvature tensor. The scalar curvature R of F is defined by

$$R := g^{ij} Ric_{ij}.$$

Let $\kappa(x)$ be a scalar function on M, $\theta := \theta_i(x)y^i$ be a 1-form on M. If

$$R = n(n-1)\left[\frac{\theta}{F} + \kappa(x)\right],$$

then it is said that F is of weak isotropic scalar curvature. Especially when $\theta = 0$, i.e., $R = n(n-1)\kappa(x)$, it is said that F is of isotropic scalar curvature.

Let F be a Finsler metric on M. If $F = \frac{\alpha^2}{\beta}$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form; then, F is a Kropina metric. Its fundamental tensor $g = g_{ij}dx^i \otimes dx^j$ is given by [4]

$$g_{ij} = \frac{F}{\beta}\left\{2a_{ij} + \frac{3F}{\beta}b_i b_j - \frac{4}{\beta}(b_i y_j + b_j y_i) + \frac{4y_i y_j}{\alpha^2}\right\},$$

where $y_i := a_{ij}y^j$. Moreover,

$$g^{ij} = \frac{\beta}{2F}\left\{a^{ij} - \frac{b^i b^j}{b^2} + \frac{2}{b^2 F}(b^i y^j + b^j y^i) + 2\left(1 - \frac{2\beta}{b^2 F}\right)\frac{y^i y^j}{\alpha^2}\right\},$$

where $(a^{ij}) := (a_{ij})^{-1}$, $b^i := a^{ij}b_j$.

Let $\nabla\beta = b_{i|j}y^i dx^j$ denote the covariant derivative of β with respect to α. Set

$$r_{ij} = \frac{1}{2}\left(b_{i|j} + b_{j|i}\right),\ s_{ij} = \frac{1}{2}\left(b_{i|j} - b_{j|i}\right),\ r_{00} = r_{ij}y^i y^j,\ r_i = b^j r_{ij},\ r_0 = r_i y^i,$$
$$r^i = a^{ij}r_j,\ r = b^i r_i,\ s^i{}_0 = a^{ij}s_{jk}y^k,\ s_i = b^j s_{ji},\ s_0 = s_i y^i,\ s^i = a^{ij}s_j.$$

The Ricci curvature of Kropina metrics is given by the following.

Lemma 1 ([3])**.** *Let F be a Kropina metric on M. Then, the Ricci curvature of F is given by*

$$Ric = {}^{\alpha}Ric + T,$$

where ${}^{\alpha}Ric$ is the Ricci curvature of α, and

$$
\begin{aligned}
T =& \frac{3(n-1)}{b^4\alpha^4} r_{00}^2\beta^2 + \frac{n-1}{b^2\alpha^2} r_{00|0}\beta - \frac{4(n-1)}{b^4\alpha^2} r_{00}r_0\beta + \frac{2(n-1)}{b^4\alpha^2} r_{00}s_0\beta - \frac{r}{b^4} r_{00} \\
&+ \frac{r^k{}_k}{b^2} r_{00} + \frac{2n}{b^2} r_{k0}s^k{}_0 + \frac{1}{b^2} b^k r_{00|k} + \frac{1}{b^4} r_0^2 - \frac{1}{b^2} r_{0|0} - \frac{2(2n-3)}{b^4} r_0 s_0 + \frac{n-2}{b^2} s_{0|0} \\
&- \frac{n-2}{b^4} s_0^2 - \frac{1}{b^2\beta} r_{k0}s^k\alpha^2 - \frac{1}{b^2\beta} r_k s^k{}_0\alpha^2 - \frac{1}{b^4\beta} r s_0\alpha^2 + \frac{1}{b^2\beta} r^k{}_k s_0\alpha^2 + \frac{n-1}{b^2\beta} s^k{}_0 s_k\alpha^2 \\
&- \frac{1}{\beta} s^k{}_{0|k}\alpha^2 + \frac{1}{b^2\beta} b^k s_{0|k}\alpha^2 - \frac{1}{4\beta^2} s^j{}_k s^k{}_j\alpha^4 - \frac{1}{2b^2\beta^2} s^k s_k\alpha^4.
\end{aligned}
$$

Lemma 2 ([4]). *Let F be a Kropina metric on n-dimensional M. Then, the followings are equivalent:*

(i) *F has an isotropic S curvature, i.e., $S = (n+1)cF$;*
(ii) $r_{00} = \sigma\alpha^2$;
(iii) $S = 0$;
(iv) *β is a conformal form with respect to α,*

where $c = c(x)$ and $\sigma = \sigma(x)$ are functions on M.

3. Ricci Curvature Tensor and Scalar Curvature Tensor of Kropina Metrics

By the definition of Ricci curvature tensor and Lemma 1, we obtain the Ricci curvature tensor of Kropina metrics.

Proposition 1. *Let F be a Kropina metric on an n-dimensional manifold M. Then, the Ricci curvature tensor of F is given by*

$$
\begin{aligned}
Ric_{kl} =& {}^{\alpha}Ric_{kl} + F_{.k.l}\Big\{ -\frac{(n-5)}{b^4F^3} r_{00}^2 + \frac{2}{b^4F^2} r_{00}(s_0 - 2r_0) + \frac{1}{b^2F^2} r_{00|0} + \frac{(3n+2)}{3b^2F} s^m_0 r_{m0} \\
&- \frac{(3n+4)}{3b^4F} r_0 s_0 - \frac{s_0^2}{3b^4F} + \frac{s_{0|0}}{3b^2F} + \frac{s^m_0 s_{m0}}{3\beta} + \frac{1}{2b^2}(r^m_m s_0 - s^m r_{m0} - s^m_0 r_m - \frac{r s_0}{b^2} \\
&- b^2 s^m_{0|m} + b^m s_{0|m} - \frac{b^2}{2} F s^m_t s^t_m - F s^m s_m) + \frac{1}{6b^2}\left[3n - 2 - (1-b^2)\frac{F}{2\beta}\right] s^m_0 s_m \Big\} \\
&+ F_{.k}F_{.l}\Big\{ \frac{3(n-5)}{b^4F^4} r_{00}^2 - \frac{4}{b^4F^3} r_{00}(s_0 - 2r_0) - \frac{2}{b^2F^3} r_{00|0} - \frac{(3n+2)}{3b^2F^2} s^m_0 r_{m0} \\
&+ \frac{(3n+4)}{3b^4F^2} r_0 s_0 + \frac{s_0^2}{3b^4F^2} - \frac{s_{0|0}}{3b^2F^2} - \frac{1}{4} s^m_t s^t_m - \frac{(1-b^2)}{12b^2\beta} s^m_0 s_m - \frac{1}{2b^2} s^m s_m \Big\} \\
&- \frac{1}{b^4}\left[\frac{6(n-3)}{F^3} r_{00} - \frac{(n-7)}{F^2} r_0 + \frac{(n-3)}{F^2} s_0\right](F_{.k}r_{l0} + F_{.l}r_{k0}) + \frac{2}{b^2F^2}(F_{.k}r_{l0|0} \\
&+ F_{.l}r_{k0|0}) - \frac{(n-1)}{2b^2F^2}\Big(F_{.k}r_{00|l} + F_{.l}r_{00|k}\Big) + \frac{8(n-2)}{b^4F^2} r_{k0}r_{l0} - \frac{1}{2b^2}(F_{.k}r_{ml} + F_{.l}r_{mk})s^m \\
&+ \frac{(3n+2)}{6b^2F}(F_{.k}r_{lm} + F_{.l}r_{km})s^m_0 + \frac{(3n+2)}{6b^2F}(F_{.k}s^m_l + F_{.l}s^m_k)r_{m0} - \frac{(3n-5)}{b^4F}(r_{k0}r_l \\
&+ r_{l0}r_k) + \frac{(n-3)}{b^4F}(r_{k0}s_l + r_{l0}s_k) - \frac{2}{b^2F} r_{kl|0} + \frac{(n-1)}{b^2F}\Big(r_{k0|l} + r_{l0|k}\Big) + \frac{1}{6b^2F}(F_{.k}s_{l|0} \\
&+ F_{.l}s_{k|0}) + \frac{1}{6b^2F}\Big(F_{.k}s_{0|l} + F_{.l}s_{0|k}\Big) + \frac{1}{b^2} b^m r_{kl|m} + \frac{1}{b^4} r_k r_l - \frac{1}{2b^2}(F_{.k}s^m_l + F_{.l}s^m_k) r_m \\
&+ \frac{r_{kl}}{b^4}\left[\frac{4(n-2)r_{00}}{F^2} - \frac{2(n-3)}{F} r_0 + \frac{2(n-1)}{F} s_0 + (b^2 r^m_m - r)\right] + \frac{(n-1)}{2b^2}(s^m_l r_{mk} \\
&+ s^m_k r_{ml}) - \frac{(3n-7)}{2b^4}(s_k r_l + s_l r_k) - \frac{1}{2b^2}\Big(r_{k|l} + r_{l|k}\Big) + \frac{1}{3\beta}(F_{.k}s^m_l + F_{.l}s^m_k)s_{m0} \\
&+ \frac{(1-b^2)F}{24b^2\beta^2}(F_{.k}b_l + F_{.l}b_k)s^m_0 s_m - \frac{1}{2}\Big(F_{.k}s^m_{l|m} + F_{.l}s^m_{k|m}\Big) + \frac{F}{3\beta^2}(b_k s^m_l + b_l s^m_k)s_{m0}
\end{aligned}
$$

$$
-\frac{2Fb_k b_l}{3\beta^3} s^m_{\ 0} s_{m0} + \frac{1}{12b^2}\left[6n-5-(1-b^2)\frac{F}{2\beta}\right](F_{.k}s^m_{\ l} + F_{.l}s^m_{\ k})s_m - \frac{(n-2)}{b^4}s_k s_l
+ \frac{(n-2)}{2b^2}\left(s_{k|l} + s_{l|k}\right) + \frac{b^m}{2b^2}(F_{.k}s_{l|m} + F_{.l}s_{k|m}),
$$

where ${}^{\alpha}Ric_{kl}$ *denotes the Ricci curvature tensor of* α.

Contracting the Ricci curvature tensor with g^{kl}, we can obtain the expression of the scalar curvature R of Kropina metrics as following.

Proposition 2. *Let F be a Kropina metric on an n-dimensional manifold M. Then, the scalar curvature of F is given by*

$$
\begin{aligned}
R \ =& -\tfrac{24(n-2)}{b^6F^5}r_{00}^2\beta + \tfrac{1}{F^4}\Big\{-\tfrac{(n-1)(n-8)}{b^4}r_{00}^2 + \tfrac{8(n-2)}{b^6}r_{00}(5r_0 - s_0)\beta \\
& -\tfrac{4(n-2)}{b^4}r_{00|0}\beta\Big\} + \tfrac{2}{F^3}\Big\{-\tfrac{1}{b^2}\,{}^{\alpha}Ric\beta - \tfrac{2(n-1)}{b^4}r_{00}(2r_0 - s_0) + \tfrac{(n-1)}{b^2}r_{00|0} \\
& -\tfrac{4(n-2)}{b^6}r_{00}r\beta + \tfrac{(n-3)}{b^4}r^m_{\ m}r_{00}\beta + \tfrac{(n-1)}{b^4}r^m_{\ 0}r_{m0}\beta + \tfrac{4}{b^4}s^m_{\ 0}r_{m0}\beta + \tfrac{(n-2)}{b^4}b^m r_{00|m}\beta \\
& -\tfrac{7(n-2)}{b^6}r_0^2\beta + \tfrac{6(n-2)}{b^6}r_0s_0\beta + \tfrac{(n-2)}{b^4}r_{0|0}\beta + \tfrac{(n-2)}{b^6}s_0^2\beta - \tfrac{(n-2)}{b^4}s_{0|0}\beta\Big\} \\
& +\tfrac{1}{F^2}\Big\{{}^{\alpha}Ric + \tfrac{2}{b^2}b^k y^l\,{}^{\alpha}Ric_{kl}\beta + \tfrac{1}{b^4}(r^m_{\ m}b^2 - r)r_{00} + \tfrac{(3n^2+5n-3)}{3b^2}s^m_{\ 0}r_{m0} + \tfrac{1}{b^2}b^m r_{00|m} \\
& +\tfrac{r_0^2}{b^4} - \tfrac{(3n^2+13n-21)}{3b^4}r_0s_0 - \tfrac{1}{b^2}r_{0|0} - \tfrac{2(2n-3)}{3b^4}(s_0^2 - b^2 s_{0|0}) - \tfrac{(n-5)}{b^4}r^m_{\ m}r_0\beta \\
& +\tfrac{(n-1)}{b^4}r^m_{\ m}s_0\beta - \tfrac{4(n-2)}{b^4}r^m_{\ 0}r_m\beta - \tfrac{3}{b^4}r^m_{\ 0}s_m\beta + \tfrac{(n-1)}{b^2}r^m_{\ 0|m}\beta - \tfrac{1}{b^2}r^m_{\ m|0}\beta \\
& +\tfrac{(n-1)}{b^4}s^m_{\ 0}r_m\beta + \tfrac{4(n-2)}{b^6}r(r_0 - s_0)\beta + \tfrac{(n-2)}{b^4}s^m_{\ 0}s_m\beta - \tfrac{(n-2)}{b^4}b^m(r_{0|m} - s_{0|m})\beta\Big\} \\
& +\tfrac{1}{F}\Big\{\tfrac{1}{2}\Big({}^{\alpha}R - \tfrac{b^kb^l}{b^2}\,{}^{\alpha}Ric_{kl}\Big)\beta + \tfrac{(n-1)}{3\beta}s^m_{\ 0}s_{m0} + \tfrac{(n+1)}{2b^2}(s_0r^m_{\ m} - s^m r_{m0} - s^m_{\ 0}r_m \\
& -\tfrac{rs_0}{b^2} + b^m s_{0|m} - b^2 s^m_{\ 0|m}) + \tfrac{(3n^2+n-7)}{6b^2}s^m_{\ 0}s_m + \tfrac{1}{2b^6}\left(r^m_{\ m}b^2 - r\right)^2\beta + \tfrac{(n-1)}{2b^2}r^k_{\ m}s^m_{\ k}\beta \\
& +\tfrac{b^m}{2b^2}r^k_{\ k|m}\beta + \tfrac{1}{2b^4}r^m r_m\beta - \tfrac{(3n-16)}{2b^4}s^m r_m\beta - \tfrac{1}{2b^2}r^m_{\ |m}\beta - \tfrac{1}{2b^6}r^2\beta + \tfrac{1}{2b^4}b^m r_{|m}\beta \\
& -\tfrac{\beta}{2b^4}r^m r_{|m} - \tfrac{(n-2)}{b^4}s^m s_m\beta + \tfrac{(n-2)}{2b^2}s^m_{\ |m}\beta\Big\} - \tfrac{n}{2b^2}s^m s_m - \tfrac{n}{4}s^m_{\ t}s^t_{\ m} - \tfrac{n(1-b^2)}{12b^2\beta}s^m_{\ 0}s_m,
\end{aligned} \tag{2}
$$

where ${}^{\alpha}R$ *denotes the scalar curvature of* α.

4. The Proof of Main Theorem

In this section, we will prove Theorem 1.

Proof. "Necessity". Assume Kropina metric F is of isotropic scalar curvature, i.e., $R = n(n-1)\kappa(x)$. Substituting (2) into $R = n(n-1)\kappa(x)$ yields

$$
\alpha^{10}\Gamma_0 + \alpha^8\Gamma_1 + \alpha^6\Gamma_2 + \alpha^4\Gamma_3 + \alpha^2\Gamma_4 + \Gamma_5 = 0, \tag{3}
$$

where

$$
\begin{aligned}
\Gamma_0 =& -n\left[(n-1)\kappa(x) + \frac{1}{2b^2}s^m s_m + \frac{1}{4}s^m_{\ t}s^t_{\ m}\right]\beta - \frac{n(1-b^2)}{12b^2}s^m_{\ 0}s_m, \\
\Gamma_1 =& \left[\frac{1}{2}\left({}^{\alpha}R - \frac{b^kb^l}{b^2}\,{}^{\alpha}Ric_{kl}\right) + \frac{1}{2b^6}\left(r^m_{\ m}b^2 - r\right)^2 + \frac{(n-1)}{2b^2}r^k_{\ m}s^m_{\ k} + \frac{5}{2b^4}r^m r_m - \frac{(n-1)}{b^4}s^m r_m \right.\\
& \left. -\frac{1}{2b^6}r^2 + \frac{b^m}{2b^2}r^k_{\ k|m} - \frac{1}{2b^2}r^m_{\ |m} - \frac{(n-2)}{2b^4}s^m s_m + \frac{(n-2)}{2b^2}s^m_{\ |m} - \frac{(n-2)b^kb^l s_{k|l}}{2b^4}\right]\beta^3 \\
& + \left[\frac{(3n^2+n-7)}{6b^2}s^m_{\ 0}s_m + \frac{(n+1)}{2b^2}(r^m_{\ m}s_0 - s^m r_{m0} - s^m_{\ 0}r_m - \frac{rs_0}{b^2} + b^m s_{0|m} - b^2 s^m_{\ 0|m})\right]\beta^2 \\
& + \frac{(n-1)}{3}s^m_{\ 0}s_{m0}\beta,
\end{aligned}
$$

$$\Gamma_2 = \Big\{ {}^{\alpha}Ric + \frac{2}{b^2}b^k y^l \, {}^{\alpha}Ric_{kl}\beta + \frac{1}{b^4}(r^m_{\ m}b^2 - r)r_{00} + \frac{1}{b^2}b^m r_{00|m} + \frac{(3n^2+5n-3)}{3b^2}s^m_{\ 0}r_{m0}$$
$$+ \frac{r_0^2}{b^4} - \frac{(3n^2+13n-21)}{3b^4}r_0 s_0 - \frac{2(2n-3)}{3b^4}s_0^2 - \frac{1}{b^2}r_{0|0} + \frac{2(2n-3)}{3b^2}s_{0|0}$$
$$- \frac{(n-5)}{b^4}r^m_{\ m}r_0\beta + \frac{(n-1)}{b^4}r^m_{\ m}s_0\beta - \frac{(2n-1)}{b^4}r^m r_{m0}\beta - \frac{(2n-3)}{b^4}s^m r_{m0}\beta$$
$$+ \frac{(n-2)}{b^4}s^m_{\ 0}r_m\beta + \frac{4(n-2)}{b^6}r(r_0 - s_0)\beta - \frac{1}{b^2}r^m_{\ m|0}\beta + \frac{(n-1)}{b^2}r^m_{\ 0|m}\beta - \frac{(n-2)}{b^4}b^m r_{0|m}\beta$$
$$- \frac{(n-2)}{b^4}s^m_{\ 0}s_m\beta + \frac{(n-2)}{b^4}b^m s_{0|m}\beta \Big\}\beta^3,$$

$$\Gamma_3 = \Big\{ -\frac{1}{b^2}\,{}^{\alpha}Ric\beta - \frac{4(n-1)}{b^4}r_{00}r_0 + \frac{2(n-1)}{b^4}r_{00}s_0 + \frac{(n-1)}{b^2}r_{00|0} + \frac{(n-3)}{b^4}r^m_{\ m}r_{00}\beta$$
$$+ \frac{(n-1)}{b^4}r^m_{\ 0}r_{m0}\beta - \frac{2(n-2)}{b^4}s^m_{\ 0}r_{m0}\beta - \frac{4(n-2)}{b^6}r_{00}r\beta + \frac{(n-2)}{b^4}b^m r_{00|m}\beta$$
$$- \frac{7(n-2)}{b^6}r_0^2\beta + \frac{6(n-2)}{b^6}r_0 s_0\beta + \frac{1}{b^4}r_{0|0}\beta + \frac{(n-2)}{b^6}s_0^2\beta - \frac{(n-2)}{b^4}s_{0|0}\beta \Big\}\beta^4,$$

$$\Gamma_4 = \Big[-\frac{(n-1)(n-8)}{b^4}r_{00}^2 + \frac{40(n-2)}{b^6}r_{00}r_0\beta - \frac{8(n-2)}{b^6}r_{00}s_0\beta - \frac{4(n-2)}{b^4}r_{00|0}\beta \Big]\beta^5,$$

$$\Gamma_5 = -\frac{24(n-2)}{b^6}r_{00}^2\beta^7.$$

By (3), we have that α^2 divides Γ_5. Thus, there exists a scalar function $c = c(x)$ such that $r_{00} = c\alpha^2$, which is the second formula of (1). Thus, we deduce that

$$r_{ij} = ca_{ij};\ r_{i0} = cy_i;\ r_{ij|m} = c_m a_{ij};\ r_{i0|m} = c_m y_i;\ r_{i0|0} = c_0 y_i;$$
$$r_{00|k} = c_k\alpha^2;\ r_{00|0} = c_0\alpha^2;\ r^k_{\ k} = nc;\ r^k_{\ k|0} = nc_0;\ r_i = cb_i;$$
$$r_0 = c\beta;\ r_{i|j} = c_j b_i + cs_{ij} + c^2 a_{ij};\ r = cb^2;\ r_{i|0} = c_0 b_i + cs_{i0} + c^2 y_i;$$
$$r_{0|j} = c_j\beta + cs_{0j} + c^2 y_j;\ r_{0|0} = c_0\beta + c^2\alpha^2;\ r^k_{\ |k} = c_b + nc^2,$$

where $c_i = \frac{\partial c}{\partial x^i}$, $c_b = c_i b^i$, $c_0 = c_i y^i$.

Substituting the above equations into (3) yields

$$\alpha^6\Delta_0 + \alpha^4\Delta_1 + \alpha^2\Delta_2 + \Delta_3 = 0, \tag{4}$$

where

$$\Delta_0 = -n\Big[(n-1)\kappa(x) + \frac{1}{2b^2}s^m s_m + \frac{1}{4}s^m_{\ t}s^t_{\ m}\Big]\beta - \frac{n(1-b^2)}{12b^2}s^m_{\ 0}s_m,$$
$$\Delta_1 = \frac{1}{2b^4}\{b^2(b^2\,{}^{\alpha}R - b^k b^l\,{}^{\alpha}Ric_{kl}) + (n+1)b^2[(n-2)c^2 + c_b]$$
$$-(n-2)(2s^m s_m - b^2 s^m_{\ |m})\}\beta^3,$$

$$\Delta_2 = \frac{(n-2)}{b^4}(c_b - c^2)\beta^5 + \frac{1}{b^4}\Big\{2b^2 b^k y^l\,{}^{\alpha}Ric_{kl} - \frac{2(8n-15)}{3}cs_0 + (n-2)[2b^2 c_0$$
$$+ b^m s_{0|m} - s^m_{\ 0}s_m]\Big\}\beta^4 + \frac{1}{b^4}\Big[{}^{\alpha}Ric + \frac{2(2n-3)}{3}(b^2 s_{0|0} - s_0^2)\Big]\beta^3,$$
$$\Delta_3 = \frac{2}{b^6}\{-b^4\,{}^{\alpha}Ric + (n-2)[c^2\beta^2 + (2cs_0 - b^2 c_0)\beta + (s_0^2 - b^2 s_{0|0})]\}\beta^5.$$

By (4), we have that α^2 divides Δ_3, i.e., there exists a scalar function $f = f(x)$ such that

$$f\alpha^2 = -b^4\,{}^{\alpha}Ric + (n-2)[c^2\beta^2 + (2cs_0 - b^2c_0)\beta + (s_0^2 - b^2 s_{0|0})]. \tag{5}$$

Differentiating the above equation with respect to y^iy^j yields

$$\begin{aligned}2fa_{ij} = &-2b^4\,{}^{\alpha}Ric_{ij} + (n-2)[2c^2b_ib_j + 2c(s_ib_j + s_jb_i) - b^2(c_ib_j + c_jb_i)\\ &+ 2s_is_j - b^2(s_{i|j} + s_{i|j})].\end{aligned}$$

Contracting this formula with b^ib^j or a^{ij} yields, respectively,

$$\begin{aligned}f &= -b^2b^ib^j\,{}^{\alpha}Ric_{ij} + (n-2)b^2c^2 - (n-2)b^2c_b - (n-2)s^ms_m,\\ nf &= -b^4\,{}^{\alpha}R + (n-2)b^2c^2 - (n-2)b^2c_b + (n-2)(s^ms_m - b^2s^m_{\ |m}).\end{aligned}$$

Combining the above two formulas, we obtain

$$f = \frac{b^2}{n-1}\Big(b^ib^j\,{}^{\alpha}Ric_{ij} - b^2\,{}^{\alpha}R\Big) + \frac{n-2}{n-1}(2s^ms_m - b^2s^m_{\ |m}) \tag{6}$$

and

$$s^m_{\ |m} = -\frac{1}{n-2}(b^2\,{}^{\alpha}R - nb^ib^j\,{}^{\alpha}Ric_{ij}) + (n-1)(c_b - c^2) + \frac{(n+1)}{b^2}s^ms_m. \tag{7}$$

Substituting (5)–(7) into (4), we obtain

$$\alpha^4\Theta_0 + \alpha^2\Theta_1 + \Theta_2 = 0, \tag{8}$$

where

$$\begin{aligned}\Theta_0 &= -n\left[(n-1)\kappa(x) + \frac{1}{2b^2}s^ms_m + \frac{1}{4}s^m_{\ t}s^t_{\ m}\right]\beta - \frac{n(1-b^2)}{12b^2}s^m_{\ 0}s_m,\\ \Theta_1 &= \frac{1}{6b^4}\left[n(3n+5)\left(\frac{b^2b^kb^l}{n-2}\,{}^{\alpha}Ric_{kl} + s^ms_m + b^2c_b\right) - 2nb^2c^2\right]\beta^3\\ &\quad + \frac{1}{6}\left[-\frac{6(n+1)}{n-2}b^ky^l\,{}^{\alpha}Ric_{kl} + \frac{3n^2+4n-4}{b^2}s^m_{\ 0}s_m - 3(n+1)(s^m_{\ 0|m} - \frac{n}{b^2}cs_0 + c_0)\right]\beta^2\\ &\quad + 2(n-1)s^m_{\ 0}s_{m0}\beta,\\ \Theta_2 &= \frac{n}{3b^4}\left(-\frac{b^4}{n-2}\,{}^{\alpha}Ric + c^2\beta^2 + c\beta s_0 - b^2c_0\beta\right)\beta^3.\end{aligned}$$

By (8), we have that α^2 divides Θ_2. Then, there exists a scalar function $g = g(x)$ such that

$$g\alpha^2 = -\frac{b^4}{n-2}\,{}^{\alpha}Ric + c^2\beta^2 + c\beta s_0 - b^2c_0\beta, \tag{9}$$

which is the first formula of (1). Differentiating the above equation with respect to y^i or y^iy^j, respectively, we obtain

$$2gy_i = -\frac{2b^4}{n-2}y^l\,{}^{\alpha}Ric_{il} + 2c^2\beta b_i + cb_is_0 - b^2c_0b_i + c\beta s_i - b^2c_i\beta, \tag{10}$$

$$2ga_{ij} = -\frac{2b^4}{n-2}\,{}^{\alpha}Ric_{ij} + 2c^2b_ib_j + c(b_is_j + b_js_i) - b^2(c_ib_j + c_jb_i). \tag{11}$$

Contracting (10) with b^i yields

$$2g\beta = -\frac{2b^4}{n-2}b^i y^l\ {}^{\alpha}Ric_{il} + b^2(2c^2 - c_b)\beta + b^2 c s_0 - b^4 c_0. \tag{12}$$

Contracting (11) with $b^i b^j$ or a^{ij}, respectively, we obtain

$$g = -\frac{b^2}{n-2}b^i b^j\ {}^{\alpha}Ric_{ij} + b^2(c^2 - c_b), \tag{13}$$

$$ng = -\frac{b^4}{n-2}\ {}^{\alpha}R + b^2(c^2 - c_b). \tag{14}$$

Comparing (13) and (14) yields

$$g = \frac{b^2}{(n-1)(n-2)}\left(b^i b^j\ {}^{\alpha}Ric_{ij} - b^2\ {}^{\alpha}R\right) \tag{15}$$

and

$$c_b = c^2 + \frac{1}{(n-1)(n-2)}\left(b^2\ {}^{\alpha}R - nb^i b^j\ {}^{\alpha}Ric_{ij}\right). \tag{16}$$

Substituting (15) and (16) into (12) yields

$$c_0 = -\frac{2b^i y^j}{n-2}\ {}^{\alpha}Ric_{ij} + \frac{\beta}{n-1}\left[\frac{1}{n-2}\ {}^{\alpha}R + \frac{b^i b^j}{b^2}\ {}^{\alpha}Ric_{ij} + \frac{n-1}{b^2}c^2\right] + \frac{c}{b^2}s_0. \tag{17}$$

Combining (5), (6), (9), and (15), we obtain the third formula of (1).
Substituting (9), (15)–(17) into (8) yields

$$\alpha^2\Omega_0 + \Omega_1 = 0, \tag{18}$$

where

$$\begin{aligned}\Omega_1 = \frac{1}{6}\Bigg\{&\frac{3(n+1)}{b^2}\left[\frac{1}{(n-2)}(b^2\ {}^{\alpha}R - 2b^k b^l\ {}^{\alpha}Ric_{kl}) + (n-1)c^2 + \frac{n(3n+5)}{3(n+1)b^2}s^m s_m\right]\beta^2\\ &+\frac{1}{b^2}\left[(3n^2+4n-4)s^m_{\ 0}s_m - 3(n+1)b^2 s^m_{\ 0|m} + 3(n-1)(n+1)cs_0\right]\beta\\ &+2(n-1)s^m_{\ 0}s_{m0}\Bigg\}\beta,\\ \Omega_0 = -n\Bigg[&(n-1)\kappa(x) + \frac{1}{2b^2}s^m s_m + \frac{1}{4}s^m_{\ t}s^t_{\ m}\Bigg]\beta - \frac{n(1-b^2)}{12b^2}s^m_{\ 0}s_m.\end{aligned}$$

By (18), we have that α^2 divides Ω_1. Then, there exists a scalar function $h = h(x)$ such that

$$\begin{aligned}h\alpha^2 = \ &\tfrac{3(n+1)}{b^2}\left[\tfrac{1}{n-2}(b^2\ {}^{\alpha}R - 2b^k b^l\ {}^{\alpha}Ric_{kl}) + (n-1)c^2 + \tfrac{n(3n+5)}{3(n+1)b^2}s^m s_m\right]\beta^2\\ &+\tfrac{1}{b^2}\left[(3n^2+4n-4)s^m_{\ 0}s_m - 3(n+1)b^2 s^m_{\ 0|m} + 3(n-1)(n+1)cs_0\right]\beta\\ &+2(n-1)s^m_{\ 0}s_{m0},\end{aligned} \tag{19}$$

which is the fourth formula of (1). Differentiating the above equation with respect to $y^i y^j$ yields

$$
\begin{aligned}
2ha_{ij} = \;& \tfrac{6(n+1)}{b^2}\Big[\tfrac{1}{n-2}(b^2\,{}^{\alpha}R - 2b^k b^l\,{}^{\alpha}Ric_{kl}) + (n-1)c^2 + \tfrac{n(3n+5)}{3(n+1)b^2}s^m s_m\Big]b_i b_j \\
& + \tfrac{1}{b^2}\Big[(3n-2)(n+2)s_m(s^m_{\ j}b_i + s^m_{\ i}b_j) + 3(n-1)(n+1)c(s_j b_i + s_i b_j)\Big] \\
& - 3(n+1)(s^m_{\ j|m}b_i + s^m_{\ i|m}b_j) + 4(n-1)s^m_{\ i}s_{mj}.
\end{aligned} \tag{20}
$$

Contracting (20) with $b^i b^j$ or a^{ij}, respectively, we have

$$h = \frac{3(n+1)}{(n-2)}\Big[b^2\,{}^{\alpha}R - 2b^i b^j\,{}^{\alpha}Ric_{ij}\Big] + \frac{(3n+2)}{b^2}s^m s_m + 3(n+1)[(n-1)c^2 + s^m_{\ |m} + s^m_{\ t}s^t_{\ m}], \tag{21}$$

$$nh = \frac{3(n+1)}{(n-2)}\Big[b^2\,{}^{\alpha}R - 2b^i b^j\,{}^{\alpha}Ric_{ij}\Big] + \frac{(n+4)}{b^2}s^m s_m + 3(n+1)[(n-1)c^2 + s^m_{\ |m}] + (n+5)s^m_{\ t}s^t_{\ m}. \tag{22}$$

Comparing (21) and (22) yields

$$h = -2\left(\frac{s^m s_m}{b^2} + s^m_{\ t}s^t_{\ m}\right), \tag{23}$$

$$s^m_{\ |m} = \frac{1}{n-2}[2b^i b^j\,{}^{\alpha}Ric_{ij} - b^2\,{}^{\alpha}R] - (n-1)c^2 - \frac{3n+4}{3(n+1)b^2}s^m s_m - \frac{3n+5}{3(n+1)}s^m_{\ t}s^t_{\ m}. \tag{24}$$

By (7) and (24), we obtain

$$c_b = -\frac{1}{n-1}\left[b^i b^j\,{}^{\alpha}Ric_{ij} + \frac{3n^2+9n+7}{3(n+1)b^2}s^m s_m + \frac{3n+5}{3(n+1)}s^m_{\ t}s^t_{\ m}\right]. \tag{25}$$

Therefore, by (16), we have

$$c^2 = \frac{2b^i b^j\,{}^{\alpha}Ric_{ij} - b^2\,{}^{\alpha}R}{(n-1)(n-2)} - \frac{1}{3(n+1)(n-1)}\left[\frac{3n^2+9n+7}{b^2}s^m s_m + (3n+5)s^m_{\ t}s^t_{\ m}\right]. \tag{26}$$

Substituting (26) into (17) yields

$$
\begin{aligned}
c_0 = \;& \frac{\beta}{(n-1)b^2}\left\{\frac{nb^i b^j\,{}^{\alpha}Ric_{ij}}{(n-2)} - \frac{1}{3(n+1)}\left[\frac{3n^2+9n+7}{b^2}s^m s_m + (3n+5)s^m_{\ t}s^t_{\ m}\right]\right\} \\
& - \frac{2b^i y^j}{n-2}\,{}^{\alpha}Ric_{ij} + \frac{cs_0}{b^2}.
\end{aligned} \tag{27}
$$

Substituting (23) into (18) yields

$$R = n(n-1)\kappa(x) = -\frac{n(1-b^2)}{12b^2\beta}s^m_{\ 0}s_m - \frac{3n+2}{6b^2}s^m s_m - \frac{3n+4}{12}s^m_{\ t}s^t_{\ m}.$$

"Sufficiency". It is obviously true.
This completes the proof of Theorem 1. □

5. Other Related Results

In this section, we consider $s_0 = 0$ in Theorem 1.

Corollary 1. *Let F be a Kropina metric on an $n(\geq 3)$-dimensional manifold M. Assume $s_0 = 0$. Then F is of isotropic scalar curvature if and only if*

$$\begin{cases} b^4\, {}^{\alpha}Ric = \dfrac{1}{n-1}(b^2\, {}^{\alpha}R - b^i b^j\, {}^{\alpha}Ric_{ij})(b^2\alpha^2 - \beta^2) - b^i b^j\, {}^{\alpha}Ric_{ij}\beta^2 - 2b^2 b^i y^j\, {}^{\alpha}Ric_{ij}\beta, \\ r_{00} = c(x)\alpha^2, \\ 0 = s^m_{\ t} s^t_{\ m}(2\alpha^2 - \dfrac{3n+5}{b^2}\beta^2) - 3(n+1)s^m_{\ 0|m}\beta + 2(n-1)s^m_{\ 0}s_{m0}. \end{cases} \tag{28}$$

In this case, $R = -\frac{3n+4}{12}s^t_{\ m}s^m_{\ t}$.

Proof. Sufficiency is obviously true. Next we prove necessity. Assume that F is of isotropic scalar curvature, i.e., $R = n(n-1)\kappa(x)$. By Theorem 1, obviously, $r_{00} = c(x)\alpha^2$, (5), (19), (26) and (27) are true. When $s_0 = 0$, (26) and (27) can be simplified as

$$c^2 = \frac{(2b^i b^j\, {}^{\alpha}Ric_{ij} - b^2\, {}^{\alpha}R)}{(n-1)(n-2)} - \frac{(3n+5)}{3(n+1)(n-1)}s^m_{\ t}s^t_{\ m}, \tag{29}$$

$$c_0 = \frac{\beta}{(n-1)b^2}\left[\frac{nb^i b^j\, {}^{\alpha}Ric_{ij}}{(n-2)} - \frac{(3n+5)}{3(n+1)}s^m_{\ t}s^t_{\ m}\right] - \frac{2}{n-2}b^i y^j\, {}^{\alpha}Ric_{ij}. \tag{30}$$

Substituting $s_0 = 0$, (29) and (30) into (5), we obtain

$$f\alpha^2 = -b^4\, {}^{\alpha}Ric - 2b^2 b^i y^j\, {}^{\alpha}Ric_{ij}\beta - \frac{b^2\, {}^{\alpha}R + (n-2)b^i b^j\, {}^{\alpha}Ric_{ij}}{n-1}\beta^2,$$

where $f = \frac{b^2}{n-1}(b^i b^j\, {}^{\alpha}Ric_{ij} - b^2\, {}^{\alpha}R)$. This is the first formula of (28).
Substituting $s_0 = 0$ and (29) into (19), we obtain

$$h\alpha^2 = -\frac{3n+5}{b^2}s^m_{\ t}s^t_{\ m}\beta^2 - 3(n+1)s^m_{\ 0|m}\beta + 2(n-1)s^m_{\ 0}s_{m0},$$

where $h = -2s^m_{\ t}s^t_{\ m}$. This is the third formula of (28).
By Theorem 1, in this case, $R = -\frac{3n+4}{12}s^t_{\ m}s^m_{\ t}$. □

Corollary 2. *Let F be a Kropina metric on an $n(\geq 3)$-dimensional compact manifold M. Then F is of isotropic scalar curvature if and only if*

$$\begin{cases} b^4\, {}^{\alpha}Ric = \dfrac{1}{n-1}(b^2\, {}^{\alpha}R - b^i b^j\, {}^{\alpha}Ric_{ij})(b^2\alpha^2 - \beta^2) - b^i b^j\, {}^{\alpha}Ric_{ij}\beta^2 - 2b^2 b^i y^j\, {}^{\alpha}Ric_{ij}\beta, \\ r_{00} = c(x)\alpha^2, \\ 0 = (2\alpha^2 - \dfrac{3n+5}{b^2}\beta^2)s^m_{\ t}s^t_{\ m} - 3(n+1)s^m_{\ 0|m}\beta + 2(n-1)s^m_{\ 0}s_{m0}. \end{cases}$$

In this case, $R = -\frac{3n+4}{12}s^t_{\ m}s^m_{\ t}$.

Proof. Sufficiency is obviously true. Next we prove necessity. Assume that F is of isotropic scalar curvature, i.e., $R = n(n-1)\kappa(x)$. By Theorem 1, (24) and (26) are true. Substituting (26) into (24), we obtain

$$s^m_{\ |m} = \frac{n+1}{b^2}s^m s_m.$$

Using the divergence theorem, when M is a compact manifold, $s_0 = 0$. By Corollary 1, Corollary 2 is true. □

Based on Lemma 2 and Theorem 1, we obtain the following result.

Theorem 2. *Let a Kropina metric F be of isotropic scalar curvature. Then, F is of isotropic S curvature if and only if $S = 0$.*

Proof. Assume that F is of isotropic scalar curvature. By Theorem 1, we know that $r_{00} = c\alpha^2$. By Lemma 2, the result is obviously true. □

Lemma 3 ([6])**.** *For a Finsler metric or a spray on a manifold M, $R_{i\ mj}^{\ m} = R_{j\ mi}^{\ m}$ if and only if $\chi_i = 0$.*

Remark 1. *Li–Shen defined $\chi = \chi_i dx^i$ with the S curvature in [11], where $\chi_i := \frac{1}{2}\{S_{.i|m}y^m - S_{|i}\}$. Based on Theorem 2, we know that χ_i for a Kropina metric with isotropic scalar curvature vanishes, i.e., $R_{i\ mj}^{\ m} = R_{j\ mi}^{\ m}$. This means that $Ric_{ij} = \overline{Ric}_{ij}$.*

6. Conclusions

In this paper, we study the Kropina metric with isotropic scalar curvature. Firstly, we obtain the expressions of Ricci curvature tensor and scalar curvature. Based on these, we characterize Kropina metrics with isotropic scalar curvature by tensor analysis in Theorem 1. In Corollary 2, we discuss the case of a compact manifold. Kropina metrics with isotropic scalar curvature deserve further study by the navigation method.

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