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New Applications of Faber Polynomials and q-Fractional Calculus for a New Subclass of m-Fold Symmetric bi-Close-to-Convex Functions

Mohammad Faisal Khan [1], Suha B. Al-Shaikh [2,*], Ahmad A. Abubaker [2] and Khaled Matarneh [2]

1 Department of Basic Sciences, College of Science and Theoretical Studies, Saudi Electronic University, Riyadh 11673, Saudi Arabia; f.khan@seu.edu.sa
2 Faculty of Computer Studies, Arab Open University, Riyadh 11681, Saudi Arabia; a.abubaker@arabou.edu.sa (A.A.A.); k.matarneh@arabou.edu.sa (K.M.)
* Correspondence: s.alshaikh@arabou.edu.sa

Abstract: Using the concepts of q-fractional calculus operator theory, we first define a (λ, q)-differintegral operator, and we then use m-fold symmetric functions to discover a new family of bi-close-to-convex functions. First, we estimate the general Taylor–Maclaurin coefficient bounds for a newly established class using the Faber polynomial expansion method. In addition, the Faber polynomial method is used to examine the Fekete–Szegö problem and the unpredictable behavior of the initial coefficient bounds of the functions that belong to the newly established class of m-fold symmetric bi-close-to-convex functions. Our key results are both novel and consistent with prior research, so we highlight a few of their important corollaries for a comparison.

Keywords: analytic functions; quantum (or q-) calculus; q-fractional derivative; close-to-convex functions; m-fold symmetric functions; Faber polynomial expansion

MSC: 05A30; 30C45; 11B65; 47B38

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1. Introduction

Let $\mathcal{A}$ stand for the family of analytic functions in $E = \{z \in \mathbb{C} : |z| < 1\}$ that are normalized when $\eta(0) = 0$ and $\eta'(0) = 1$ and express every $\eta \in \mathcal{A}$ that has the following series in the form shown below:

$$\eta(z) = z + \sum_{j=2}^{\infty} a_j z^j.$$

In addition, $\mathcal{S}$ is a subclass of $\mathcal{A}$, and members of $\mathcal{S}$ are univalent in E. The function $\eta \in \mathcal{S}$ is called a starlike $(\mathcal{S}^*)$ function in E (see [1]) if

$$Re\left(\frac{z\eta'(z)}{\eta(z)}\right) > 0, \ \ z \in E$$

and the function $\eta \in \mathcal{S}$ is called a convex $(\mathcal{C})$ function in E (see [2]) if

$$1 + Re\left(\frac{z\eta''(z)}{\eta'(z)}\right) > 0, \ \ z \in E.$$

The function $\eta \in \mathcal{S}$ is called a close-to-convex $(\mathcal{K})$ function in E (see [3]) if and only if $g \in \mathcal{S}^*$, such that

$$Re\left(\frac{z\eta'(z)}{g(z)}\right) > 0.$$

In [4], Noor introduced the class of functions $\eta \in \mathcal{S}$ that are called quasi-close-to-convex $(\mathcal{Q})$ functions in E if and only if $g \in \mathcal{K}$ exists, such that

$$Re\left(\frac{(z\eta'(z))'}{g'(z)}\right) > 0.$$

Among the subclasses of $\mathcal{S}$, the starlike $(\mathcal{S}^*)$ convex $(\mathcal{C})$ and close-to-convex $(\mathcal{K})$ functions are the most well known. To learn more about the well-known and extensive research of the starlike and convex function subclasses $\mathcal{S}$ and $\mathcal{C}$, see [5–7].

The idea of starlike and convex functions of order α was first presented by Robertson [8] in 1936 as follows:

For $0 \leq \alpha < 1$, the function $\eta \in \mathcal{S}$ is called a starlike $(\mathcal{S}^*(\alpha))$ function of order α in E (see [8]) if

$$Re\left(\frac{z\eta'(z)}{\eta(z)}\right) > \alpha$$

and for $0 \leq \alpha < 1$, the function $\eta \in \mathcal{S}$ is called a convex $(\mathcal{C}(\alpha))$ function of order α in E (see [8]) if

$$Re\left(\frac{(z\eta'(z))'}{\eta'(z)}\right) > \alpha.$$

For $\alpha = 0$,

$$\mathcal{S}^*(\alpha) = \mathcal{S}^*$$

and

$$\mathcal{C}(\alpha) = \mathcal{C}.$$

Let $0 \leq \alpha < 1$; the function $\eta \in \mathcal{S}$ is called a close-to-convex $(\mathcal{K}(\alpha))$ function of order α in E (see [3]) if and only if $g \in \mathcal{S}^*(\alpha) = \mathcal{S}^*$, such that

$$Re\left(\frac{z\eta'(z)}{g(z)}\right) > \alpha.$$

For more details, see [5].

Let $0 \leq \alpha < 1$; the function $\eta \in \mathcal{S}$ is said to be in the class of quasi-close-to-convex $(\mathcal{Q}(\alpha))$ functions if and only if $g \in \mathcal{K}$ exists, such that

$$Re\left(\frac{(z\eta'(z))'}{g'(z)}\right) > \alpha.$$

For $\alpha = 0$,

$$\mathcal{K}(\alpha) = \mathcal{K}$$

and

$$\mathcal{Q}(\alpha) = \mathcal{Q}.$$

We present the well-known class $\mathcal{P}$ (see [6]) of analytic functions p in E, which satisfy the following conditions:

$$Re(p(z)) > 0$$

and

$$p(0) = 1.$$

For $\eta_1, \eta_2 \in \mathcal{A}$, and η_1 subordinate to η_2 in E, denoted by (see [9])

$$\eta_1(z) \prec \eta_2(z), \ \ z \in E,$$

suppose that an analytic function w_0, such that $|w_0(z)| < 1$ and $w_0(0) = 0$, and

$$\eta(z) = \eta_2(w_0(z)),\ z \in E.$$

Each function $\eta \in \mathcal{S}$ has an inverse $\eta^{-1} = F$ that may be written as

$$F(\eta(z)) = z,\ \ z \in E$$

and

$$\eta(F(w)) = w,\ |w| < r_0(\eta),\ r_0(\eta) \geq \frac{1}{4}.$$

The series of the inverse function is given by

$$F(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots. \tag{1}$$

An analytic function η is called bi-univalent in E if η and $\eta^{-1}$are univalent in E, and Σ stands for the class of all bi-univalent functions. Here, we give some examples of bi-univalent functions below:

$$\eta_1(z) = \frac{z}{1-z},\ \eta_2(z) = -\log(1-z),\ \eta_3(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),\ z \in E.$$

The famous Koebe function

$$k(z) = z(1-z)^{-2}, \quad \text{for all } z \in E,$$

is not in class Σ.

Lewin [10] introduced the concept of class Σ and established $|a_2| < 1.51$ for every $\eta \in \Sigma$. Following that, Brannan and Clunie [11] demonstrated that $|a_2| \leq \sqrt{2}$. Subsequently, Netanyahu [12] showed that $\max|a_2| = \frac{4}{3}$, and Styer and Wright [13] showed the existence of $\eta \in \Sigma$, for which $|a_2| < \frac{4}{3}$. Furthermore, Tan [14] demonstrated that, for functions in Σ, $|a_2| < 1.485$. Since class Σ was first introduced, many scholars have attempted to establish the connection between the geometric features of the functions inside it and the coefficient bounds. As a matter of fact, authors Lewin [10], Brannan and Taha [11], Srivastava et al. [15], and others [16–20] built a solid framework for the study of bi-univalent functions. In these more recent publications, the initial coefficients were only estimated using non-sharp methods, and the coefficient estimates for the general class of analytic bi-univalent functions were also discovered in [21]; however, Atshan [22] utilized the quasi-subordination characteristics and obtained some results for new bi-univalent function subclasses. A new subclass of m-fold bi-univalent functions was defined by Oros and Cotirla [23], who also found the coefficient estimates of the Fekete–Szegö problem. More recently, the integral operator based on the Lucas polynomial was used to estimate coefficients for general subclasses of analytic bi-univalent functions [24]. Numerous authors looked into the bounds for various m-fold bi-univalent function subclasses [25–30]. The sharp coefficient bound for $|a_m|$, $(m = 3, 4, 5, \ldots)$ is still an unsolved problem.

Gong [31] discussed the uses and significance of the Faber polynomial methods that Faber [32] introduced. The coefficient bounds $|a_j|$ for $j \geq 3$ were recently determined by Hamidi and Jahangiri [33,34] using the Faber polynomial expansion method. The Faber polynomial expansion approach has been used to introduce and study a number of new bi-univalent function subclasses. Bult introduced a few new subclasses of bi-univalent functions in References [35–37], and she implemented the Faber polynomial method to discover the general coefficient bounds $|a_j|$ for $j \geq 3$. She also discussed how the initial coefficient bounds have unpredictable behavior. In [38,39], new subclasses of meromorphic bi-univalent functions were studied using the Faber polynomial. Recently, the subordination features and the method of generating Faber polynomials were also used to derive the general coefficient bounds $|a_j|$ for $j \geq 3$ of analytic bi-univalent functions [40]. Altinkaya and Yalcin [41] addressed the unusual behavior of coefficient bounds for novel subclasses of bi-univalent functions using a similar methodology. Additionally, numerous authors used the Faber polynomial technique and obtained some intriguing findings for bi-univalent functions (see [42–47] for additional information).

Let $m \in \mathbb{N}$. If a rotation of a domain E with an angle of $2\pi/m$ at its origin maps that domain onto itself, then the domain is said to be m-fold symmetric.

Following that, it is demonstrated that an analytic η in E, being m-fold symmetric, satisfies the following requirement:

$$\eta\left(e^{\frac{2\pi i}{m}}z\right) = e^{\frac{2\pi i}{m}}\eta(z)$$

and $\mathcal{S}_m$ in E represents m-fold symmetric univalent functions. The function $\eta \in \mathcal{S}_m$ has the following form:

$$\eta(z) = z + \sum_{j=1}^{\infty} a_{mj+1}z^{mj+1}. \tag{2}$$

Srivastava et al. [48,49] gave an additional boost to the study of the family Σ_m, which has led to a large number of works on subclasses of Σ_m. Then, for a new subclass of Σ_m, Srivastava et al. [50] explored the initial coefficient bounds. Note that $\Sigma_1 = \Sigma$. Sakar and Tasar [51] developed further subclasses of m-fold bi-univalent functions and derived the initial coefficient bounds for the functions belonging to these families. In [52], coefficient bounds were established for new subclasses of analytic and m-fold symmetric bi-univalent functions. Recently, Swamy et al. [29] defined a new family of m-fold symmetric bi-univalent functions by ensuring that they satisfied the subordination requirement. References [53–58] presented interesting results on the initial coefficient bounds and the Fekete–Szegö functional problem for some subfamilies of Σ_m.

Recent work by Srivastava et al. [59] shows the series expansion for η^{-1} to be as follows:

$$F(w) = \eta^{-1}(w) = w - a_{m+1}w^{m+1} + A_m w^{2m+1} - B_m w^{3m+1}, \tag{3}$$

where

$$\begin{aligned} A_m &= (m+1)a_{m+1}^2 - a_{2m+1}, \\ B_m &= \frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1} \end{aligned}$$

For $m = 1$, Equation (3) coincides with Equation (1). Here, we provide examples of an insignificant number of m-fold symmetric bi-univalent functions:

$$\begin{aligned} \eta_4(z) &= \left(\frac{z^m}{1-z^m}\right)^m, \eta_5(z) = [\log(1-z^m)]^{\frac{-1}{m}}, \\ \eta_6(z) &= \log\sqrt{\frac{1+z^m}{1-z^m}}, \quad z \in E \end{aligned}$$

and their inverse functions are

$$\begin{aligned} F_7(z) &= \left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}, \ F_8(z) = \left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}}, \\ F_9(z) &= \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}}. \end{aligned}$$

Many new classes of analytic functions have been built and studied by scholars in the field of Geometric Function Theory (GFT) using q-calculus and fractional q-calculus. In 1909, Jackson [60] developed the q-calculus (D_q) operator, and in [61], Ismail et al. utilized this operator for the first time to build a class of q-starlike functions in E. See [62–65] for more reading on q-calculus and analytic functions.

The Faber polynomial is one such subject, and it has become more important in mathematics and other sciences in recent years. This article is divided into three parts. In Section 1, we quickly review some elementary concepts from the theory of geometric functions since they are essential to our primary discovery. These elements are all standard fare, and we appropriately reference them. In Section 2, we introduce the Faber polynomial method, give a few illustrations, define some key terms, and present some preliminary lemmas. In Section 3, we present the new (λ, q)-differintegral operator for m-fold symmetric functions, and, considering this operator, we define a new class of close-to-convex functions and investigate the main results. Section 4 offers some final remarks.

2. Preliminaries

Addressing the basic definitions and notions of q-fractional calculus is now necessary in order to construct some new subclasses of m-fold symmetric bi-univalent functions.

Definition 1 ([66]). *Let us define the q-shifted factorial* $(\gamma, q)_j$ *as*

$$(\gamma, q)_j = \prod_{j=0}^{j-1}\left(1 - \gamma q^j\right), \quad (j \in \mathbb{N}, \ \gamma, q \in \mathbb{C}). \tag{4}$$

If $\gamma \neq q^{-m}$, $(m \in \mathbb{N}_0 = \{0, 1, 2, \dots\})$, *then it can be written as*

$$(\gamma, q)_\infty = \prod_{j=0}^{\infty}\left(1 - \gamma q^j\right), \quad (\gamma \in \mathbb{C} \text{ and } |q| < 1). \tag{5}$$

Remark 1. *When* $\gamma \neq 0$ *and* $q \geq 1$, $(\gamma, q)_\infty$ *diverges. Thus, if and when this occurs* $(\gamma, q)_\infty$, *then we will assume* $|q| < 1$.

Remark 2. *When* $q \to 1-$ *in (4), then we obtain the Pochhammer symbol* $(\gamma)_j$ *defined as*

$$(\gamma)_j = \prod_{l=0}^{j-1}(\gamma + l), \ \text{if} \ j \in \mathbb{N}.$$

If $j = 0$, *then* $(\gamma)_j = 1$.

Definition 2 ([60]). *The expression for the q-factorial* $[j]_q$ *is*

$$[j]_q! = \prod_{l=1}^{j}[l]_q, \quad (l \in \mathbb{N}), \tag{6}$$

where

$$[j]_q = \frac{1 - q^j}{1 - q}.$$

If $j = 0$, *then*

$$[j]_q! = 1.$$

Definition 3 ([66]). $(\gamma, q)_j$ *in (4) can be precise in terms of the q-Gamma function as follows:*

$$\digamma_q(\gamma) = \frac{(1 - q)^{1-\gamma}(q, q)_\infty}{(q^a, q)_\infty}, \qquad (0 < q < 1),$$

or

$$(q^\gamma, q)_j = \frac{(1 - q^j)\digamma_q(\gamma + j)}{\digamma_q(\gamma)}, \quad (j \in \mathbb{N}).$$

For analytic functions, Jackson [60] presented the q-difference operator as follows:

Definition 4 ([60])**.** *For $\eta \in \mathcal{A}$, the q-difference operator is defined as*

$$D_q\eta(z) = \frac{\eta(z) - \eta(qz)}{z(1-q)}, \quad z \in E.$$

Note that

$$D_q(z^j) = [j]_q z^{j-1}, \; D_q\left(\sum_{j=1}^{\infty} a_j z^j\right) = \sum_{j=1}^{\infty} [j]_q a_j z^{j-1}.$$

Definition 5. *Pochhammer's generalized symbol for q is denoted by*

$$[\gamma]_{q,j} = \frac{\digamma_q(\gamma + j)}{\digamma_q(\gamma)}, \; j \in \mathbb{N}, \; \gamma \in \mathbb{C}.$$

Remark 3. *When $q \to 1-$, $[\gamma]_{q,j}$ simplifies to $(\gamma)_j = \frac{\Gamma(\gamma+j)}{\Gamma(\gamma)}$.*

Definition 6 ([67])**.** *For $\lambda > 0$, the fractional q-integral operator is defined by*

$$I_q^{\lambda}\eta(z) = \frac{1}{\digamma_q(\lambda)} \int_0^z (z - tq)_{\lambda-1}\eta(t) d_q(t), \tag{7}$$

where the definition of the q-binomial function $(z - tq)_{\lambda-1}$ is

$$(z - tq)_{\lambda-1} = z^{\lambda-1} {}_1\Phi_0\left(q^{-\lambda+1}, -, q, tq^{\lambda}/z\right).$$

The series ${}_1\Phi_0$ is given by

$${}_1\Phi_0(a, -, q, z) = 1 + \sum_{j=1}^{\infty} \frac{(a,q)_j}{(q,q)_j} z^j, \quad (|q| < 1, |z| < 1).$$

This final equivalence is known as the q-binomial theorem (for reference, see [68]). For more details, see [67,69].

Definition 7 ([68,70])**.** *For an analytic function η, the fractional q-derivative operator D_q^{λ} is defined by*

$$\begin{aligned} D_q^{\lambda}\eta(z) &= D_q I_q^{1-\lambda}\eta(z) \\ &= \frac{1}{\digamma_q(1-\lambda)} D_q \int_0^z (z - tq)_{-\lambda}\eta(t) d_q(t), \;\; (0 \le \lambda < 1). \end{aligned}$$

Definition 8 ([67,68])**.** *For k to be the smallest integer, the extended fractional q-derivative D_q^{λ} of order λ is defined by*

$$D_q^{\lambda}\eta(z) = D_q^k\left(I_q^{k-\lambda}\eta(z)\right). \tag{8}$$

We find from (8) that

$$D_q^{\lambda} z^j = \frac{\digamma_q(j+1)}{\digamma_q(j+1-\lambda)} z^{j-\lambda}, \quad (0 \le \lambda, \; j > -1).$$

***Note that** D_q^{λ} represents the fractional q-integral of order λ when $-\infty < \lambda < 0$ and the fractional q-derivative of order λ when $0 \le \lambda < 2$.*

Definition 9 ([71])**.** *Selvakumaran et al. defined the* (λ, q)*-differintegral operator* $\Omega_q^\lambda : \mathcal{A} \to \mathcal{A}$ *as follows:*

$$\begin{aligned} \Omega_q^\lambda \eta(z) &= \frac{F_q(2-\lambda)}{F_q(2)} z^\lambda D_q^\lambda \eta(z) \\ &= z + \sum_{j=2}^{\infty} \frac{F_q(2-\lambda) F_q(j+1)}{F_q(2) F_q(j+1-\lambda)} a_j z^j, \ \ z \in E, \end{aligned}$$

where

$$0 \le \lambda < 2, \text{ and } 0 < q < 1.$$

Consider the following:

$$\lim_{\lambda \to 1} \Omega_q^\lambda \eta(z) = \Omega_q \eta(z) = z D_q \eta(z).$$

Definition 10. *For k to be the smallest integer, the extended fractional q-derivative* $D_q^{\lambda,m}$ *of order* λ *is defined for m-fold symmetric functions as follows:*

$$D_q^{\lambda,m} \eta(z) = D_q^k \left(I_q^{k-\lambda} \eta(z) \right); \tag{9}$$

we find from (9) *that*

$$D_q^{\lambda,m} z^j = \frac{F_q(mj+2)}{F_q(mj+2-\lambda)} z^{mj+1-\lambda}, \quad (0 \le \lambda,\ j > -1,\ m \in \mathbb{N}).$$

The Faber Polynomial Expansion Method and Its Applications

The coefficients of the inverse map F may be expressed using the Faber polynomial method applied to the analytic functions (see [72,73]).

$$F(w) = \eta^{-1}(w) = w + \sum_{j=2}^{\infty} \frac{1}{j} Q_{j-1}^j (a_2, a_3, \ldots, a_j) w^j,$$

where

$$\begin{aligned} Q_{j-1}^{-j} = & \frac{(-j)!}{(-2j+1)!(j-1)!} a_2^{j-1} + \frac{(-j)!}{[2(-j+1)]!(j-3)!} a_2^{j-3} a_3 \\ & + \frac{(-j)!}{(-2j+3)!(j-4)!} a_2^{j-4} a_4 \\ & + \frac{(-j)!}{[2(-j+2)]!(j-5)!} a_2^{j-5} \left[a_5 + (-j+2) a_3^2 \right] \\ & + \frac{(-j)!}{(-2j+5)!(j-6)!} a_2^{j-6} [a_6 + (-2j+5) a_3 a_4] \\ & + \sum_{i \ge 7} a_2^{j-i} Q_i, \end{aligned}$$

and for $7 \leq i \leq j$, Q_i is a homogeneous polynomial in $a_2, a_3, \ldots a_j$. To be more specific, the first three terms of Q_{j-1}^{-j} are

$$\begin{aligned} \frac{1}{2}Q_1^{-2} &= -a_2, \ \frac{1}{3}Q_2^{-3} = 2a_2^2 - a_3, \\ \frac{1}{4}Q_3^{-4} &= -(5a_2^3 - 5a_2a_3 + a_4). \end{aligned}$$

The usual form of the expansion of Q_j^r for $r \in \mathbb{Z}$ ($\mathbb{Z} := 0, \pm 1, \pm 2, \ldots$ and $j \geq 2$ is

$$Q_j^r = ra_j + \frac{r(r-1)}{2}\mathcal{V}_j^2 + \frac{r!}{(r-3)!3!}\mathcal{V}_j^3 + \cdots + \frac{r!}{(r-j)!(j)!}\mathcal{V}_j^j,$$

where

$$\mathcal{V}_j^r = \mathcal{V}_j^r(a_2, a_3 \ldots .)$$

and according to [72], we have

$$\mathcal{V}_j^v(a_2, \ldots, a_j) = \sum_{j=1}^{\infty} \frac{v!(a_2)^{\mu_1} \ldots (a_j)^{\mu_j}}{\mu_1!, \ldots, \mu_j!}, \ \text{for } a_1 = 1 \text{ and } v \leq j.$$

The sum takes over all non-negative integers $\mu_1, \ldots, \mu_j$, which satisfies

$$\begin{aligned} \mu_1 + \mu_2 + \cdots + \mu_j &= v, \\ \mu_1 + 2\mu_2 + \cdots + j\mu_j &= j. \end{aligned}$$

Clearly,

$$\mathcal{V}_j^j(a_1, \ldots, a_j) = \mathcal{V}_1^j$$

and the first and last polynomials are

$$\mathcal{V}_j^j = a_1^j, \text{ and } \mathcal{V}_j^1 = a_j.$$

Lemma 1 ([5])**.** *If $p(z) = 1 + \sum_{j=1}^{\infty} c_j z^j \in \mathcal{P}$ and $Re(p(z) > 0$, then*

$$|c_j| \leq 2.$$

In this section, we define the (λ, q)-differintegral operator for m-fold symmetric functions, consider this operator, and define a new class of close-to-convex functions. Then, we obtain our main results by using the technique of Faber polynomial expansion.

3. Main Results

By using the same technique as Selvakumaran et al. [71], we define the (λ, q)-differintegral operator for m-fold symmetric functions as follows:

Definition 11. *For $m \in \mathbb{N}$, the (λ, q)-differintegral operator for m-fold symmetric functions $\Omega_q^{\lambda,m} : \mathcal{S}_m \to \mathcal{S}_m$ is defined as follows:*

$$\begin{aligned} \Omega_q^{\lambda,m}\eta(z) &= \frac{F_q(2-\lambda)}{F_q(2)} z^{\lambda} D_q^{\lambda,m}\eta(z) \\ &= z + \sum_{j=1}^{\infty} \frac{F_q(2-\lambda)F_q(mj+2)}{F_q(2)F_q(mj+2-\lambda)} a_{mj+1} z^{mj+1}, \ z \in E, \end{aligned}$$

where

$$0 \leq \lambda < 2, \text{ and } 0 < q < 1.$$

Taking motivation from [33] and considering the (λ, q)-differintegral operator, we define a new class of close-to-convex bi-univalent functions of class Σ_m.

Definition 12. *The function $f \in \Sigma_m$ belongs to class $C_\Sigma^{\lambda,q}(\alpha, m)$ if and only if there exists a function $g \in \mathcal{S}^*$ satisfying*

$$Re\left(\frac{D_q\left(\Omega_q^{\lambda,m}\eta(z)\right)}{g(z)}\right) > \alpha$$

and

$$Re\left(\frac{D_q\left(\Omega_q^{\lambda,m}F(w)\right)}{G(w)}\right) > \alpha,$$

where $0 \leq \alpha < 1, 0 \leq \lambda < 1, m \in \mathbb{N}, z, w \in E$ and $F = \eta^{-1}$.

The Faber polynomial method is applied to Definition 12 in order to derive the j^{th} coefficient bounds, $|a_{mj+1}|$, and the initial coefficient bounds, $|a_{m+1}|$, $|a_{2m+1}|$, as well as the Feketo–Szegö problem $|a_{2m+1} - \mu a_{m+1}^2|$.

Theorem 1. *Let $\eta \in C_\Sigma^{\lambda,q}(\alpha, m)$ be given by (2) if $a_{mk+1} = 0$, and $1 \leq k \leq j-1$. Then,*

$$|a_{mj+1}| \leq \frac{\digamma_q(2)\digamma_q(mj+2-\lambda)(3-2\alpha+mj)}{[mj+1]_q\digamma_q(2-\lambda)\digamma_q(mj+2)}, \quad \text{for} \quad j \geq 2.$$

Proof. Since $\eta \in C_\Sigma^{\lambda,q}(\alpha, m)$, then, by definition and using the Faber polynomial,

$$\begin{aligned} &\frac{D_q\left(\Omega_q^{\lambda}\eta(z)\right)}{g(z)} \\ = \;& 1 + \sum_{j=1}^{\infty}\left[K_1(q,m,j,\lambda)\sum_{l=1}^{j-1}Q_l^{-1}(b_{m+1}, b_{m+2}, \ldots b_{ml+1}) \times K_2(q,m,j,\lambda)\right]z^{mj}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} &K_1(q,m,j,\lambda) \\ = \;& \left([mj+1]_q\frac{\digamma_q(2-\lambda)\digamma_q(mj+2)}{\digamma_q(2)\digamma_q(mj+2-\lambda)}a_{mj+1} - b_{mj+1}\right) \\ &K_2(q,m,j,\lambda) \\ = \;& \left(([mj+1]_q - ml)\frac{\digamma_q(2-\lambda)\digamma_q(mj-ml+2)}{\digamma_q(2)\digamma_q(mj-ml+2-\lambda)}a_{mj+1-ml} - b_{mj+1-ml}\right). \end{aligned}$$

For the inverse map $F = \eta^{-1}$ and $G = g^{-1}$, we obtain

$$\begin{aligned} &\frac{D_q\left(\Omega_q^{\lambda}F(w)\right)}{G(w)} \\ = \;& 1 + \sum_{j=2}^{\infty}\left[K_3(q,m,j,\lambda)\sum_{l=1}^{j-1}Q_l^{-1}(B_{m+1}, B_{m+2}, \ldots B_{ml+1}) \times K_4(q,m,j,\lambda)\right]w^{mj}, \end{aligned} \tag{11}$$

where

$$
\begin{aligned}
&K_3(q,m,j,\lambda) \\
= &\left([mj+1]_q \frac{F_q(2-\lambda)F_q(mj+2)}{F_q(2)F_q(mj+2-\lambda)} A_{mj+1} - B_{mj+1}\right) \\
&K_4(q,m,j,\lambda) \\
= &\left(([mj+1]_q - ml)\frac{F_q(2-\lambda)F_q(mj-ml+2)}{F_q(2)F_q(mj-ml+2-\lambda)} A_{mj+1-ml} - B_{mj+1-ml}\right).
\end{aligned}
$$

As opposed to that, $Re\frac{D_q\left(\Omega_q^\lambda \eta(z)\right)}{g(z)} > \alpha$ in E, and

$$p(z) = 1 + \sum_{j=1}^{\infty} c_{mj} z^{mj};$$

therefore,

$$
\begin{aligned}
\frac{D_q\left(\Omega_q^\lambda \eta(z)\right)}{g(z)} &= 1 + (1-\alpha)p(z) \\
&= 1 + (1-\alpha)\sum_{j=1}^{\infty} c_{mj} z^{mj}. \qquad (12)
\end{aligned}
$$

Similarly, $Re\frac{D_q\left(\Omega_q^\lambda F(w)\right)}{G(w)} > \alpha$ in E, and there exists the function

$$s(w) = 1 + \sum_{j=1}^{\infty} d_{mj} w^{mj}$$

so that

$$
\begin{aligned}
\frac{D_q\left(\Omega_q^\lambda F(w)\right)}{G(w)} &= 1 + (1-\alpha)s(w) \\
&= 1 + (1-\alpha)\sum_{j=1}^{\infty} d_{mj} w^{mj}. \qquad (13)
\end{aligned}
$$

Evaluating the coefficients of Equations (10) and (12), for any $j \geq 2$, yields

$$\left\{K_1(q,m,j,\lambda)Q_l^{-1}(b_{m+1}, b_{m+2}, \ldots b_{ml+1}) \times K_2(q,m,j,\lambda)\right\} = (1-\alpha)c_{mj}. \qquad (14)$$

Evaluating the coefficients of Equations (11) and (13), for any $j \geq 2$, yields

$$K_3(q,m,j,\lambda)\sum_{l=1}^{j-1} Q_l^{-1}(B_{m+1}, B_{m+2}, \ldots B_{ml+1}) \times K_4(q,m,j,\lambda) = (1-\alpha)d_{mj}. \qquad (15)$$

For the special case $j = 1$, from Equations (14) and (15), we obtain

$$\frac{[m+1]_q F_q(2-\lambda)F_q(m+2)}{F_q(2)F_q(m+2-\lambda)} a_{m+1} - b_{m+1} = (1-\alpha)c_m$$

and

$$\frac{[m+1]_q F_q(2-\lambda)F_q(m+2)}{F_q(2)F_q(m+2-\lambda)} A_{m+1} - B_{m+1} = (1-\alpha)d_m.$$

By utilizing Lemma 1 and solving a_{m+1} in absolute values, we achieve

$$|a_{m+1}| \leq \frac{F_q(2)F_q(m+2-\lambda)}{[m+1]_q F_q(2-\lambda)F_q(m+2)}(3-2\alpha+m).$$

However, under this assumption, $a_{mk+1} = 0$ and $1 \leq k \leq j-1$ both yield

$$A_j = -a_j.$$

Therefore,

$$[mj+1]_q \frac{F_q(2-\lambda)F_q(mj+2)}{F_q(2)F_q(mj+2-\lambda)} a_{mj+1} - b_{mj+1} = (1-\alpha)c_{mj} \tag{16}$$

and

$$-[mj+1]_q \frac{F_q(2-\lambda)F_q(mj+2)}{F_q(2)F_q(mj+2-\lambda)} a_{mj+1} - B_{mj+1} = (1-\alpha)d_{mj}. \tag{17}$$

By solving Equations (16) and (17) for a_j and determining the absolute values, and by using Lemma 1, we obtain

$$|a_{mj+1}| \leq \frac{F_q(2)F_q(mj+2-\lambda)(3-2\alpha+mj)}{[mj+1]_q F_q(2-\lambda)F_q(mj+2)},$$

upon noticing that

$$|b_{mj+1}| \leq mj+1 \text{ and } |B_{mj+1}| \leq mj+1.$$

This completes Theorem 1. □

Corollary 1. *Let $\eta \in C_{\Sigma}^{\lambda,q}(\alpha,1)$ be given by (2) if $a_{k+1} = 0$, and $1 \leq k \leq j-1$. Then,*

$$|a_{j+1}| \leq \frac{F_q(2)F_q(j+2-\lambda)(3-2\alpha+j)}{[j+1]_q F_q(2-\lambda)F_q(j+2)}, \quad \textit{for} \quad j \geq 2.$$

Corollary 2. *Let $\eta \in C_{\Sigma}^{0,q}(\alpha,m)$ be given by (2) if $a_{mk+1} = 0$, and $1 \leq k \leq j-1$. Then,*

$$|a_{mj+1}| \leq \frac{(3-2\alpha+mj)}{[mj+1]_q}, \quad \textit{for} \quad j \geq 2.$$

Corollary 3. *Let $\eta \in C_{\Sigma}^{\lambda,1}(\alpha,m)$ be given by (2) if $a_{mk+1} = 0$, and $1 \leq k \leq j-1$. Then,*

$$|a_{mj+1}| \leq \frac{F(mj+2-\lambda)(3-2\alpha+mj)}{[mj+1]F(2-\lambda)F(mj+2)}, \quad \textit{for} \quad j \geq 2.$$

Corollary 4. *Let $\eta \in C_{\Sigma}^{0,1}(\alpha,m)$ be given by (2) if $a_{mk+1} = 0$, and $1 \leq k \leq j-1$. Then,*

$$|a_{mj+1}| \leq \frac{(3-2\alpha+mj)}{[mj+1]_q}, \quad \textit{for} \quad j \geq 2.$$

When we set $\lambda = 0$, $m = 1$, and $q \to 1-$, we have a well-established corollary, which is proven in [33].

Corollary 5 ([33]). *Let $\eta \in C_{\Sigma}(\alpha)$ if $a_{k+1} = 0$, $1 \leq k \leq j$. Then,*

$$|a_j| \leq 1 + \frac{2(1-\alpha)}{j}, \quad \textit{for } j \geq 3.$$

The following theorem is obtained given the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$, as well as the Feketo–Szegö problem $|a_{2m+1} - a_{m+1}^2|$ in $C_{\Sigma}(m,\alpha,q)$.

Theorem 2. *Let $\eta \in C_{\Sigma}^{\lambda,q}(\alpha, m)$ be given by (2). Then,*

$$|a_{m+1}| \leq \sqrt{\frac{2F_q(2)F_q(m+2-\lambda)F_q(2m+2)(1-\alpha)}{F_q(2-\lambda)\{K_5(q,m,j,\lambda) - K_6(q,m,j,\lambda)\}}}$$

for $0 \leq \alpha < 1 - \phi(q,\lambda)$.

$$|a_{m+1}| \leq \frac{2F_q(2)F_q(m+2-\lambda)(1-\alpha)}{[m+1]_q F_q(2-\lambda)F_q(m+2) - F_q(2)F_q(m+2-\lambda)},$$

for $1 - \phi(q,\lambda) \leq \alpha < 1$

$$|a_{2m+1}| \leq \frac{2F_q(2)F_q(2m+2-\lambda)(1-\alpha)}{[2m+1]_q F_q(2m+2)F_q(2-\lambda) - F_q(2)F_q(2m+2-\lambda)} \times K_7(q,m,j,\lambda),$$

where

$$\begin{aligned} &\phi(q,\lambda) \\ = \;& K_9(q,m,j,\lambda) \times \left(F_q(2)F_q(2m+2-\lambda)\{Q_1(q,m,\lambda)\}^2\right) \end{aligned}$$

and

$$\begin{aligned} K_9(q,m,j,\lambda) &= \frac{1}{2F_q(m+1-\lambda)F_q(2)Q_2(q,m,\lambda)} \\ Q_1(q,m,\lambda) &= [m+1]_q F_q(2-\lambda)F_q(m+2) - F_q(2)F_q(m+2-\lambda) \\ Q_2(q,m,\lambda) &= \{K_5(q,m,j,\lambda)F_q(m+1-\lambda) - K_6(q,m,j,\lambda)F_q(2-\lambda)\}. \end{aligned}$$

Now,

$$\left|a_{2m+1} - a_{m+1}^2\right| \leq \frac{2F_q(2)F_q(2m+2-\lambda)(1-\alpha)}{[2m+1]_q F_q(2-\lambda)F_q(2m+2) - F_q(2)F_q(2m+2-\lambda)}.$$

where $K_5(q,m,j,\lambda)$, $K_6(q,m,j,\lambda)$, and $K_7(q,m,j,\lambda)$ are given by (18)–(20).

Proof. In the proof of Theorem 1, we obtain $a_{mj} = -b_{mj}$ for the function $g(z) = \Omega_q^{\lambda}\eta(z)$. For $j = 1$, (14) and (15) respectively yield

$$\begin{aligned} a_{m+1}\left(\frac{[m+1]_q F_q(2-\lambda)F_q(m+2)}{F_q(2)F_q(m+2-\lambda)} - 1\right) &= (1-\alpha)c_m \\ a_{m+1}\left(-\frac{[m+1]_q F_q(2-\lambda)F_q(m+2)}{F_q(2)F_q(m+2-\lambda)} + 1\right) &= (1-\alpha)d_m. \end{aligned}$$

Any one of these two equations, when taken at its absolute value, gives

$$|a_{m+1}| \leq \frac{2F_q(2)F_q(m+2-\lambda)(1-\alpha)}{[m+1]_q F_q(2-\lambda)F_q(m+2) - F_q(2)F_q(m+2-\lambda)}.$$

For $j = 2$, Equations (14) and (15) respectively yield

$$\begin{aligned} & \left(\frac{[2m+1]_q F_q(2-\lambda) F_q(2m+2)}{F_q(2) F_q(2m+2-\lambda)} - 1\right) a_{2m+1} \\ & -\left(\frac{[m+1]_q F_q(2-\lambda) F_q(m+2)}{F_q(2) F_q(m+2-\lambda)} - 1\right) a_{m+1}^2 \\ = \;& (1-\alpha) c_{2m} \end{aligned}$$

and

$$\begin{aligned} & \left(2a_{m+1}^2 - a_{2m+1}\right)\left(\frac{[2m+1]_q F_q(2-\lambda) F_q(2m+2)}{F_q(2) F_q(2m+2-\lambda)} - 1\right) \\ & -\left(\frac{[m+1]_q F_q(2-\lambda) F_q(m+2)}{F_q(2) F_q(m+2-\lambda)} - 1\right) a_{m+1}^2 \\ = \;& (1-\alpha) d_{2m}. \end{aligned}$$

Combining the two equations and solving $|a_{m+1}|$ yield

$$\left|a_{m+1}^2\right| = \frac{F_q(2) F_q(m+2-\lambda) F_q(2m+2)(1-\alpha)|d_{2m} + c_{2m}|}{2F_q(2-\lambda)\{K_5(q,m,j,\lambda) - K_6(q,m,j,\lambda)\}},$$

where

$$K_5(q,m,j,\lambda) = [2m+1]_q F_q(2m+2) F_q(m+2-\lambda) \tag{18}$$

$$K_6(q,m,j,\lambda) = [m+1]_q F_q(m+2) F_q(2m+2-\lambda). \tag{19}$$

By applying Carathéodory's Lemma 1, we obtain

$$|a_{m+1}| \le \sqrt{\frac{2F_q(2) F_q(m+2-\lambda) F_q(2m+2)(1-\alpha)}{F_q(2-\lambda)\{K_5(q,m,j,\lambda) - K_6(q,m,j,\lambda)\}}}.$$

As a result, we obtain the estimate

$$\begin{aligned} & \sqrt{\frac{2F_q(2) F_q(m+2-\lambda) F_q(2m+2)(1-\alpha)}{F_q(2-\lambda)\{K_5(q,m,j,\lambda) - K_6(q,m,j,\lambda)\}}} \\ < \;& \frac{2F_q(2) F_q(m+2-\lambda) F_q(2m+2)(1-\alpha)}{F_q(2-\lambda)\{K_5(q,m,j,\lambda) - K_6(q,m,j,\lambda)\}}. \end{aligned}$$

By substituting

$$a_{m+1} = \frac{c_m(1-\alpha) F_q(2) F_q(m+2-\lambda)}{[m+1]_q F_q(2-\lambda) F_q(m+2) - F_q(2) F_q(m+2-\lambda)}$$

in (4), we obtain

$$\begin{aligned} a_{2m+1} = \;& \frac{F_q(2) F_q(2m+2-\lambda)(1-\alpha)}{[2m+1]_q F_q(2m+2) F_q(2-\lambda) - F_q(2) F_q(2m+2-\lambda)} \\ & \times\left\{c_{2m} + \frac{(1-\alpha) F_q(2) F_q(m+2-\lambda)}{[m+1]_q F_q(2-\lambda) F_q(m+2) - F_q(2) F_q(m+2-\lambda)} c_m^2\right\}. \end{aligned}$$

Using the modulus and Carathéodory's Lemma 1, we may prove the following:

$$|a_{2m+1}| \leq K_7(q,m,j,\lambda)\left(\frac{2F_q(2)F_q(2m+2-\lambda)(1-\alpha)}{[2m+1]_qF_q(2m+2)F_q(2-\lambda)-F_q(2)F_q(2m+2-\lambda)}\right),$$

where

$$\begin{aligned} & K_7(q,m,j,\lambda) \\ = \; & K_8(q,m,j,\lambda)\Big([m+1]_qF_q(2-\lambda)F_q(m+2)-W(q,m,\lambda)\Big), \end{aligned} \tag{20}$$

$$W(q,m,\lambda) = F_q(2)F_q(m+2-\lambda)+2(1-\alpha)F_q(2)F_q(m+2-\lambda)$$

and

$$\begin{aligned} & K_8(q,m,j,\lambda) \\ = \; & \frac{1}{[m+1]_qF_q(2-\lambda)F_q(m+2)-F_q(2)F_q(m+2-\lambda)}. \end{aligned}$$

Lastly, by subtracting Equation (4) from Equation (5), we obtain

$$\left|a_{2m+1}-a_{m+1}^2\right| \leq \frac{2F_q(2)F_q(2m+2-\lambda)(1-\alpha)}{[2m+1]_qF_q(2-\lambda)F_q(2m+2)-F_q(2)F_q(2m+2-\lambda)}.$$

□

Corollary 6. *Let $\eta \in C_\Sigma^{\lambda,q}(\alpha,1)$ be given by (2). Then,*

$$|a_2| \leq \sqrt{\frac{2F_q(2)F_q(3-\lambda)F_q(4)(1-\alpha)}{F_q(2-\lambda)\Big\{[3]_qF_q(4)F_q(3-\lambda)-[2]_qF_q(3)F_q(4-\lambda)\Big\}}}$$

for $0 \leq \alpha < 1-\phi(q,\lambda)$ *and*

$$|a_2| \leq \frac{2F_q(2)F_q(3-\lambda)(1-\alpha)}{[2]_qF_q(2-\lambda)F_q(3)-F_q(2)F_q(3-\lambda)}$$

for $1-\phi(q,\lambda) \leq \alpha < 1$ *and*

$$\begin{aligned} & |a_3| \\ \leq \; & \frac{2F_q(2)F_q(4-\lambda)(1-\alpha)}{[3]_qF_q(4)F_q(2-\lambda)-F_q(2)F_q(4-\lambda)} \\ & \times\left\{\frac{[2]_qF_q(2-\lambda)F_q(3)-F_q(2)F_q(3-\lambda)+2(1-\alpha)F_q(2)F_q(3-\lambda)}{[2]_qF_q(2-\lambda)F_q(3)-F_q(2)F_q(3-\lambda)}\right\} \end{aligned}$$

and

$$\left|a_3-a_2^2\right| \leq \frac{2F_q(2)F_q(4-\lambda)(1-\alpha)}{[3]_qF_q(2-\lambda)F_q(4)-F_q(2)F_q(4-\lambda)},$$

where

$$\begin{aligned} \phi(q,\lambda) &= \\ &= \frac{F_q(4-\lambda)\{[2]_qF_q(2-\lambda)F_q(3)-F_q(2)F_q(3-\lambda)\}^2}{2F_q(2-\lambda)W_1(q,\lambda)} \end{aligned}$$

and

$$W_1(q,\lambda) = \big([3]_q F_q(4) F_q(2-\lambda) F_q(3-\lambda) - [2]_q F_q(3) F_q(2-\lambda) F_q(4-\lambda)\big).$$

Corollary 7. *Let $\eta \in C_{\Sigma}^{0,q}(\alpha, m)$ be given by (2). Then,*

$$|a_{m+1}| \leq \sqrt{\frac{2(1-\alpha)}{\left\{[2m+1]_q - [m+1]_q\right\}}}$$

for $0 \leq \alpha < 1 - \phi(q,0)$. *Now,*

$$|a_{m+1}| \leq \frac{2(1-\alpha)}{[m+1]_q - 1}$$

for $1 - \phi(q,0) \leq \alpha < 1$.

$$|a_{2m+1}| \leq \frac{2(1-\alpha)}{[2m+1]_q - 1} \left\{ \frac{[m+1]_q - 1 + 2(1-\alpha)}{[m+1]_q - 1} \right\}$$

and

$$\left| a_{2m+1} - a_{m+1}^2 \right| \leq \frac{2(1-\alpha)}{[2m+1]_q - 1},$$

where

$$\phi(q,0) = \frac{F_q(m+2)\left\{[m+1]_q F_q(2) - F_q(2)\right\}^2}{2F_q(m+1)\left\{[2m+1]_q F_q(m+1) - [m+1]_q F_q(2)\right\}}.$$

Corollary 8. *Let $\eta \in C_{\Sigma}^{0,1}(\alpha, m)$ be given by (2). Then,*

$$|a_{m+1}| \leq \sqrt{\frac{2(1-\alpha)}{m}}$$

for $0 \leq \alpha < 1 - \phi(1,0)$. *Now,*

$$|a_{m+1}| \leq \frac{2(1-\alpha)}{m}$$

for $1 - \phi(1,0) \leq \alpha < 1$.

$$|a_{2m+1}| \leq \frac{1-\alpha}{m} \times \left\{ \frac{m + 2(1-\alpha)}{m} \right\}$$

and

$$\left| a_{2m+1} - a_{m+1}^2 \right| \leq \frac{1-\alpha}{m},$$

where

$$\phi(1,0) = \frac{m}{2}.$$

The well-known corollary for $\lambda = 0$, $m = 1$, and $q \to 1-$ is proven in [33].

Corollary 9 ([33]). *Let $\eta \in C_{\Sigma}(\alpha)$ be given by (2). Then,*

$$|a_2| \leq \begin{cases} \sqrt{2(1-\alpha)} & \text{if } 0 \leq \alpha < \frac{1}{2}, \\ 2(1-\alpha) & \text{if } \frac{1}{2} \leq \alpha < 1, \end{cases}$$

and

$$|a_3| \leq \begin{cases} 2(1-\alpha) & \text{if } 0 \leq \alpha < \frac{1}{2}, \\ (1-\alpha)(3-2\alpha) & \text{if } \frac{1}{2} \leq \alpha < 1. \end{cases}$$

4. Conclusions

In this paper, we introduced the (λ, q)-differintegral operator for m-fold symmetric functions given in (11) and discussed its applications for a class of m-fold symmetric bi-close-to-convex functions that is defined in (12). We applied the Faber polynomial technique and investigated the jth coefficient bounds, the initial coefficients, and the Fekete–Szegö functional for this newly defined class of m-fold symmetric functions. This research also shows how current discoveries and other improvements may be made via careful parameter specialization.

This article has three parts. Since the basics of geometric function theory are necessary to understand our major discovery, we briefly cover them in Section 1. These elements are all well recognized, and we appropriately reference them. The Faber polynomial method, several related applications, and some preliminary lemmas are presented in Section 2. In Section 3, we discuss our results. Researchers may create many other classes of m-fold symmetric bi-univalent functions by using different extended q-operators in place of the (λ, q)-differintegral operator in their future investigations. Researchers may also explore the behavior of coefficient estimations for newly defined subclasses of m-fold symmetric bi-univalent functions using the Faber polynomial approach.

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