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Discrete Analogue of Fishburn's Fractional-Order Stochastic Dominance

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Abstract: A stochastic dominance (SD) relation can be defined by two different perspectives: One from the view of distributions, and the other one from the view of expected utilities. In the early days, Fishburn investigated SD from the view of distributions, and we refer this perspective as Fishburn's SD. One of his many results was the development of fractional-order SD for continuous distributions. However, discrete fractional-order SD cannot be directly generalized, because some properties of fractional calculus may not possess a discrete counterpart. In this paper, we develop a discrete analogue of fractional-order SD for discrete utilities from the view of distributions. We generalize the order of SD by Lizama's fractional delta operator, show the preservation of SD hierarchy, and formulate the utility classes that are congruent with our SD relations. This work brings a message that some results of discrete SD cannot be directly generalized from continuous SD. We characterize the difference between discrete and continuous fractional-order SD, as well as the way to handle it for further applications in mathematics and computer science.

Keywords: fractional-order stochastic dominance; discrete stochastic dominance; discrete utility; fractional sum

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1. Introduction

Stochastic dominance (SD) is a tool to analyze risky decisions under uncertainty [1,2], analyze risk attitudes [3,4], formulate robust utility maximization [5,6], etc. It is a kind of partial ordering on two probability distributions that can be viewed from two different perspectives. The first perspective is solely based on the probability distributions that are usually associated with risky investments or portfolios. The second one is based on the expected utilities of the probability distributions in the sense of von Neumann–Morgenstern's utility [7], i.e., the higher the expected utility, the better the decision. Although many applications used only first- and second-order SD, higher-order risk attitudes, e.g., mixed risk aversion [8], have their economic implications and they can be analyzed by higher-order SD [9].

The two perspectives are equivalent for first- and second-order SD only; therefore, extra constraints are imposed on either of the perspectives to formulate an equivalent

definition [10]. The convention nowadays is to impose restrictions on the distributions so that higher-order SD works for all utility functions that are monotone (in alternative signs) up to a certain order of derivatives. More specifically, it requires a consistent numerical order on the end points of the higher-order cumulative distributions. One reason is that the exact formulation of the utility to be used may not be known in practice when analyzing risks, hence we desire a definition that works for all utilities with a simple assumption. Beyond risk management, there are applications that only concern some specific utilities only, e.g., in the analysis of transmission schemes in batched network codes [11,12]. Therefore, it is also a practical problem to impose constraints on the utility classes instead of the distributions.

In the early days, researchers also investigated the other way that considers a specific class of utilities so that higher-order SD works without imposing restrictions on the probability distributions. The class of utilities is usually a convex cone in function space in practice [13]. Fishburn is one of the scholars who investigated SD from this point of view, and he obtained many cornerstone results, e.g., the relation between SD and moments [14], which inspired many later works in related disciplines. In this paper, we denote the definition of SD that imposes constraints on the utility classes by Fishburn's SD.

One of Fishburn's important results is the development of the continuum of SD rules that extends the orders of SD from positive integers to real numbers no less than one [15,16]. Fractional-order SD is currently still an active research topic, e.g., [17–19], as integral-order SD is simply too coarse in some applications. For example, first-order SD models the insatiable individuals with increasing utilities, and second-order SD models the insatiable and risk-averse individuals with increasing concave utilities, but the cases between the substantial gap of the two orders cannot be well captured. Similar to what happens in fractional calculus, Fishburn's fractional-order SD, which is based on the Riemann–Liouville integral, is only one of the many ways to formulate fractional-order SD. The conventional restriction on the end points of non-integral-order cumulative distributions is not well defined; thus, it suffices to develop the relation between fractional-order SD and utilities.

On the other hand, most of the literature considered continuous distributions for SD. However, discrete distributions are often used in experimental tests due to the fact that the number of empirical data points is finite [20–22]. This also happens in pedagogical presentations, e.g., [23,24], because discretized small examples are easier to be understood. Most (integral-order) SD results on continuous distributions also work for discrete distributions with continuous utilities, as one can unify the cases by using Lebesgue integrals or Riemann–Stieltjes integrals [25]. For discrete utilities, these results still work but we need to directly cope with the discreteness, because finite differences are not sharing the same meaning as derivatives for real functions [26].

When we come to discrete fractional-order SD, it is a different story. For example, Fishburn's (continuous) fractional-order SD (on bounded distributions) is based on the observation that the definition of higher-order SD matches with the form in the Cauchy formula for repeated integration. However, there is no such elegant analogue for summations. In other words, the form of the discrete analogue (for discrete utilities) is not the same as that in the continuous case.

In this paper, we develop a discrete analogue of Fishburn's continuous fractional-order SD on bounded distributions and discrete utilities, i.e., we follow Fishburn's perspective and do not impose restrictions on the probability distributions. As a preliminary result to show that the discrete analogue of fractional-order SD is not trivial, we consider a sequence of consecutive integers $0, 1, \ldots, b$ as the domain of the utilities. This assumption is in fact natural in applications that use discrete distributions and discrete utilities. In Section 2, we give a brief introduction on SD and also discuss the fundamental reason for the disagreement between the conventional and Fishburn's definitions. We also discuss the definition of SD when the utilities are discrete. Then, we present our main results in Section 3. In Section 3, we first rewrite the definition of discrete SD (for discrete utility) as a single summation in terms of factorials. By doing so, we can generalize it to fractional-order SD,

which is in terms of Lizama's fractional delta operator [27], when the factorials are replaced by gamma functions. Under this definition, we show that the hierarchy of SD is preserved, i.e., a smaller-order SD implies a larger-order SD. After that, we define a class of utility for each order so that it is congruent with our fraction-order SD definition. At last, we show that our utility classes for integral orders are consistent with the traditional utilities that are monotone (in alternative signs) up to the corresponding order of derivative. Finally, we conclude this paper in Section 4 and outline some potential future research directions.

2. Integral-Order Stochastic Dominance

Stochastic dominance (SD) is a partial ordering for two probability distributions. In this section, we give a brief background on integral-order SD. In Sections 2.1 and 2.2, we give the definitions of SD with both perspectives from the distributions and the (continuous) utilities. We also discuss the fundamental reason on the disagreement between the conventional and Fishburn's definitions in Section 2.2. Last, we discuss the definitions of SD when the utilities are discrete in Section 2.3 and describe an example that uses discrete utilities. We leave the discussion on fractional-order SD and the explanation on why the discrete extension is not trivial in the next section.

Throughout this paper, we adopt the following set notations. Let $\mathbb{Z}_{>0}$ be the set of positive integers. Define $\mathbb{N}_c := \{0, 1, \dots, c\}$ for any non-negative integer c. Denote by $\mathbb{R}$ and $\mathbb{R}_{>0}$ the sets of real numbers and positive real numbers, respectively.

2.1. First- and Second-Order Stochastic Dominance

Let $F(x)$ and $G(x)$ be two cumulative distributions, and $f(x)$ and $g(x)$ be their probability density functions (which are continuous), respectively. Define

$$F_{n+1}(x) = \int_{-\infty}^{x} F_n(t)dt, \qquad G_{n+1}(x) = \int_{-\infty}^{x} G_n(t)dt$$

for all $n \in \mathbb{Z}_{>0}$, with $F_1(x) = F(x)$ and $G_1(x) = G(x)$.

Definition 1 (First-Order SD)**.** *F dominates G in the first order if and only if $F_1(x) \leq G_1(x)$ for all x in the union of the supports of $f(x)$ and $g(x)$. The dominance is strict if and only if the strict inequality holds for some x.*

As a remark, both continuous and discrete distributions can be written in the same Riemann–Stieltjes integral form; thus, all existing results for continuous integral-order SD can be extended to their discrete analogues without much effort.

From this definition, we can see that the cumulative distribution of F is always no larger than the cumulative distribution of G. Therefore, the expectation of F is no smaller than that of G. If we view x as the reward and $f(x), g(x)$ as the chances of getting the reward, we prefer a larger expected value on the reward. That is, we prefer F more than G, which is one of the basic explanations of why F dominates G in the first order. As we discuss below, first-order SD has a stronger meaning when compared with the above explanation. However, this explanation is one of the most useful properties that can be applied in other fields, e.g., in modeling communication channels with packet loss [12].

In a more general sense, the von Neumann–Morgenstern utility is an extension of the theory of consumer preferences. According to this theory, the optimal decision is the one that maximizes the expected utility derived from the choice made, i.e., the distribution that maximizes the expected utility. The simplest case is a monotonically increasing utility, which means that we prefer the largest reward.

From such perspective, we have another definition of first-order SD. Denote by $\mathbb{E}$ the expectation operator, and let U_1 be the set of all monotonically increasing and differentiable utilities.

Definition 2 (First-Order SD (Utility))**.** *F dominates G in the first order if and only if $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$ for all $u \in U_1$. The dominance is strict if and only if the strict inequality holds for at least one $u \in U_1$.*

In fact, the two definitions are equivalent. We see that in the discussion later. Notice that not all distributions may have a first-order SD relation.

We may further refine the relation to capture more information, such as which one returns a better mean while it involves relatively less risk. This is the tendency to prefer outcomes with a lower uncertainty than those with a high uncertainty, which is known as risk aversion. This relation is the second-order SD. Similarly, we have two equivalent definitions. Let U_2 be the set of all monotonically increasing, twice-differentiable and concave utilities.

Definition 3 (Second-Order SD)**.** *F dominates G in the second order if and only if $F_2(x) \leq G_2(x)$ for all x in the union of the supports of $f(x)$ and $g(x)$. The dominance is strict if and only if the strict inequality holds for some x.*

Definition 4 (Second-Order SD (Utility))**.** *F dominates G in the second order if and only if $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$ for all $u \in U_2$. The dominance is strict if and only if the strict inequality holds for at least one $u \in U_2$.*

2.2. Higher-Order Stochastic Dominance

In a similar manner, we can define higher-order SD. First, we define the SD used by Fishburn in his works, which is a direct extension of the above definition.

Definition 5 (Fishburn's nth-Order SD)**.** *F dominates G in the nth order if and only if $F_n(x) \leq G_n(x)$ for all x in the union of the supports of $f(x)$ and $g(x)$. The dominance is strict if and only if the strict inequality holds for some x.*

For the utility-based definition, let

$$U_n = \left\{ u(x) \colon (-1)^{j+1} \frac{d^j u(x)}{dx^j} \geq 0, j \in \{1, 2, \ldots, n\} \right\}$$

be the set of all increasing utilities such that up to the nth derivative (which exists), all odd derivatives are non-negative, and all even derivatives are nonpositive.

Definition 6 (nth-Order SD (Utility))**.** *F dominates G in the nth order if and only if $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$ for all $u \in U_n$. The dominance is strict if and only if the strict inequality holds for at least one $u \in U_n$.*

However, these two definitions are not equivalent. The textbook [25] shows a derivation for bounded continuous distributions. The evaluation for unbounded distributions can be found in [28,29]. To briefly describe the reason of this nonequivalence, we consider the result obtained in [25]. Let f and g be two density functions whose supports are subsets of $[a, b]$. Their cumulative distributions are $F = F_1$ and $G = G_1$, respectively. Note that $F_1(b) = G_1(b) = 1$ and $F_n(a) = G_n(a) = 0$ for all n. By repeatedly using integration by parts, we can eventually achieve

$$\mathbb{E}_F[u] - \mathbb{E}_G[u] = \sum_{j=1}^{n-1} (-1)^{j+1} \frac{d^j u(x)}{dx^j}\bigg|_{x=b} (G_{j+1}(b) - F_{j+1}(b)) + (-1)^{n+1} \int_a^b (G_n(x) - F_n(x)) \frac{d^n u(x)}{dx^n} dx. \tag{1}$$

For the first-order SD, Equation (1) equals $\mathbb{E}_F[u] - \mathbb{E}_G[u] = \int_a^b (G_1(x) - F_1(x)) u'(x) dx$. Since $G_1(x) \geq F_1(x)$ and $u'(x) \geq 0$ for all x, we know that $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$. For the second-order SD, Equation (1) equals $u'(b)(G_2(b) - F_2(b)) - \int_a^b (G_2(x) - F_2(x)) u''(x) dx$. This time,

we have $G_2(x) \geq F_2(x)$, $u'(x) \geq 0$ and $u''(x) \leq 0$ for all x; therefore, we can conclude that $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$.

The issue arises when we consider the third-order SD [10], where Equation (1) equals $u'(b)(G_2(b) - F_2(b)) - u''(b)(G_3(b) - F_3(b)) + \int_a^b (G_3(x) - F_3(x))u'''(x)dx$. We have $G_3(x) \geq F_3(x)$, $u'(x) \geq 0$, $u''(x) \leq 0$, and $u'''(x) \geq 0$ for all x. However, we cannot conclude whether $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$ or not due to the term $u'(b)(G_2(b) - F_2(b))$. Similarly, for the nth-order SD, we have the problematic terms $(-1)^{j+1} \frac{d^j u(x)}{dx^j}\Big|_{x=b} (G_{j+1}(b) - F_{j+1}(b))$ for $j = 1, 2, \ldots, n-2$.

In risk management, we concern ourselves with all possible utilities in U_n. To ensure that $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$, the convention is to assume $G_j(b) - F_j(b) \geq 0$ for all $j \in \{1, 2, \ldots, n-1\}$, so that all problematic terms become non-negative. This leads to the conventional definition of the nth-order SD below, which is equivalent to the utility-based definition.

Definition 7 (Conventional nth-Order SD)**.** *F dominates G in the nth order if and only if*

1. $F_n(x) \leq G_n(x)$ *for all* $x \in [a, b]$*; and*
2. $F_j(b) \leq G_j(b)$ *for all* $j \in \{1, 2, \ldots, n-1\}$.

The dominance is strict if and only if there is at least one strict inequality.

Recall that all the above definitions are applicable for both conventional continuous and discrete SD [25], because both continuous and discrete distributions can be written in the same Riemann–Stieltjes integral form. On the other hand, the SD hierarchy is also a well-known result: the first-order SD implies the second-order SD, and so on and so forth.

Note that this conventional definition is a compromise to impose the restrictions on the distributions such that the definition of utility is fulfilled. If we start from Fishburn's SD, we need to restrict the choices of utilities to a specific utility class in the corresponding equivalent definition. The choice of definition depends on the application. For example, if we are only concerned with whether the expectation of F is no less than that of G, then we do not need to use the stronger definition that works for utilities other than the identity function, thus Fishburn's definition is sufficient for this goal.

2.3. Discrete Utilities

In most scenarios, the aforementioned (continuous) utility classes are sufficient in risk management. However, discrete utilities can be useful in other fields. One example is the network coding theory, where the rank of a matrix formed by juxtaposing random linear combinations of some vectors over a finite field with certain chance to be erased could measure the performance of the network coding scheme in communication [11,12]. The corresponding utility, which is a key component to designing the transmission policy [30], is the affine combination of the expectation of such rank, where the coefficients are arising from a discrete distribution [31]. In this manner, both the probability and the utility are discrete.

For discrete distributions, we are only concerned with certain points in the utility, thus the utility can be a discrete function. However, this raises a problem: what are the "derivatives" of such a utility? This problem was tackled by Courtault et al. [26]. For discrete utilities on $\{x_1, x_2, \ldots, x_m\}$ in ascending order, the characterization becomes

$$U_n = \left\{ u(x) \colon (-1)^{j+1} u^{(j)}(i) \geq 0, i \in \{1, 2, \ldots, m-j\}, j \in \{1, 2, \ldots, n\} \right\}.$$

where

$$u^{(k)}(i) = \begin{cases} \frac{u^{(k-1)}(x_{i+1}) - u^{(k-1)}(x_i)}{x_{i+1} - x_i} & \text{if } k = 1, 2, \ldots, m-1, \text{ and } i = 1, 2, \ldots, m-k, \\ u(x_i) & \text{if } k = 0 \text{ and } i = 1, 2, \ldots, m. \end{cases}$$

The nth-order SD in the sense of the utility remains, i.e., F dominates G in the nth order if and only if $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$ for all $u \in U_n$. To cope with this utility characterization, Courtault et al. [26] proposed a twist on the cumulative functions for maintaining the form of SD rules in the conventional definition. Define

$$F_{n+1}(x_i) = \sum_{j=1}^{i} F_n(x_j)(x_{j+1} - x_j), \qquad G_{n+1}(x_i) = \sum_{j=1}^{i} G_n(x_j)(x_{j+1} - x_j)$$

for $i = 1, 2, \ldots, m-1$ and $n = 2, 3, \ldots, m$, where $F_1 = F$ and $G_1 = G$. (In [26], an equivalent form of $D_1(x_i) = G_1(x_i) - F_1(x_i)$ and $D_{n+1}(x_i) = \sum_{j=1}^{i} D_n(x_j)(x_{j+1} - x_j)$ were considered instead of the above separated summations). When the order is higher, the domain of the "derivative" of the utility is smaller. Therefore, the equivalent conventional SD definition for discrete utilities has a subtle difference. Specifically, F dominates G in the nth order if and only if $F_n(x_i) \leq G_n(x_i)$ for all $i = 1, 2, \ldots, m-n+1$ and $F_r(x_{m-r+1}) \leq G_r(x_{m-r+1})$ for all $r = 2, 3, \ldots, n-1$.

When the domain is a sequence of consecutive integers, we can simplify $u^{(k)}(i)$ as $\Delta^n u(i)$, where Δ^n is the nth-order forward difference operator defined by

$$\Delta^n u(i) = \begin{cases} \Delta^{n-1} u(i+1) - \Delta^{n-1} u(i) & \text{if } n > 1 \\ u(i+1) - u(i) & \text{if } n = 1. \end{cases}$$

From the definition, we can see that when the order is too high, the nth-order forward difference is undefined if the utility has a bounded domain. There is no such issue when the domain is not right-bounded.

If we further assume the distributions take values on $0, 1, \ldots, b$, then we have

$$F_{n+1}(x) = \sum_{j=0}^{x} F_n(j), \qquad G_{n+1}(x) = \sum_{j=0}^{x} G_n(j).$$

This is one of the simplest domains for investigation, and it naturally appears in some applications, e.g., the matrix rank distribution we described at the beginning of this subsection.

Note that it is not guaranteed for the existence of n that either F dominates G or G dominates F in some integral order in the conventional definition, even when the definition is extended to the infinite order [32]. The necessary and sufficient condition for such existence in conventional SD was shown in [32], where negative moments and Bernstein's theorem were applied on the totally monotone utilities. Surprisingly, such n exists for discrete distributions on $\{0, 1, \ldots, b\}$ in the sense of Fishburn's SD for discrete utilities, i.e., we omit the constraint $F_r(x_{m-r+1}) \leq G_r(x_{m-r+1})$. For simplicity, we write $F \succeq_n G$ to denote that F dominates G in the nth order using the definition of Fishburn's SD for discrete utilities. When the dominance is strict, i.e., at least one x such that $F_n(x) < G_n(x)$, we write $F \succ_n G$.

Theorem 1. *For any two bounded discrete cumulative distributions F, G on $\{0, 1, \ldots, b\}$, there exists an $n \in \mathbb{Z}_{>0}$ such that either $F \succeq_n G$ or $G \succeq_n F$. Furthermore, if the two distributions are distinct, then there exists an $n \in \mathbb{Z}_{>0}$ such that $F \succ_n G$ if $f(\lambda) < g(\lambda)$, or $G \succ_n F$ if $g(\lambda) < f(\lambda)$, where λ is the smallest value such that $f(\lambda) \neq g(\lambda)$.*

Proof. See Appendix A. □

The idea of the proof is that if $F_j(i) > G_j(i)$ for some $i > \lambda$ and some $j > 1$, we can keep accumulating the sum by increasing the order of summations. Eventually, we reach $F_{j'}(i) < G_{j'}(i)$ for some $j' > j$ since $f(\lambda) < g(\lambda)$.

3. Discrete Fishburn's Fractional-Order Stochastic Dominance

As a preliminary result to show that the discrete analogue of fractional-order SD is not trivial, we consider the following assumptions in this paper:

1. The domains of the probability mass functions are $\{0, 1, \ldots, b\}$, unless otherwise specified.
2. The utility functions are discrete.

Regarding the first assumption, we can define $f(x) = 0$ (or $g(x) = 0$) when $x \in \{0, 1, \ldots, b\}$ is not in the support. These assumptions simplify the divided differences of utilities into nth-order forward differences, and it is used in certain discrete SD applications in practice, e.g., in [11,12].

3.1. Definition

To begin with, we first revisit how Fishburn defined his fractional-order SD in [15] for continuous distributions on $[0, b]$. By expanding the recursive relation, we have the nth repeated integral

$$F_n(x) = \int_0^x F_{n-1}(t)dt = \int_0^x \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t_n)dt_n \cdots dt_2 dt_1.$$

This form matches with the Cauchy formula for repeated integration, which gives

$$F_n(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t)dt, \tag{2}$$

where the order n is now a value in the formula. By writing the factorial in a gamma function, i.e., $(n-1)! = \Gamma(n)$, we have the Riemann–Liouville integral

$$F_n(x) = \frac{1}{\Gamma(n)} \int_0^x (x-t)^{n-1} f(t)dt, \tag{3}$$

which extends the order of SD into a continuum. To distinguish fractional-order SD from integral-order SD, we use $\alpha \in [1, \infty)$ as the notation for the order, i.e., αth-order SD.

Now, we come to our problem on discrete distributions. Recall that we consider that the supports of f and g are subsets of $\{0, 1, \ldots, b\}$ for simplicity. In a similar manner, we have

$$F_n(x) = \sum_{i_n=0}^{x} \sum_{i_{n-1}=0}^{i_n} \cdots \sum_{i_1=0}^{i_2} f(i_1).$$

Although one may think that we can write $F_n(x) = \frac{1}{\Gamma(n)} \sum_{t=0}^{x} (x-t)^{n-1} f(t)$ because the integration and the summation are the same in the sense of the Riemann–Stieltjes integral, it is not true due to the fact that there is no discrete analogue of the Cauchy formula for repeated integration that gives a similar form as Equation (2) in a single summation.

We give an example to illustrate this fact. Consider the following probability masses p_i, where $i = 0, 1, 2, 3$. In Table 1, we also calculated $F_1(i)$ and $F_2(i)$ for $i = 0, 1, 2, 3$ using the definition in Equation (3).

Table 1. An example to show that a discrete analogue of fractional-order SD is not trivial.

i	0	1	2	3
p_i	0.1	0.2	0.3	0.4
$F_1(i) = \sum_{t=0}^{x} p_t$	0.1	0.3	0.6	1
$F_2(i) = \sum_{t=0}^{x} F_1(t)$	0.1	0.4	1	2

Now, we evaluate $\frac{1}{\Gamma(2)}\sum_{t=0}^{x}(x-t)^{n-1}p_t = \sum_{t=0}^{x}(x-t)p_t$ at $x=2$.

$$\sum_{t=0}^{2}(2-t)p_t = 2\cdot 0.1 + 1\cdot 0.2 + 0\cdot 0.3 = 0.4 \neq 1 = F_2(2).$$

In other words, we cannot directly convert the integral of Equation (2) into a summation as a discrete extension.

The following theorem states that after we merge the multiple summations into a single one, the factor is no longer $\frac{1}{\Gamma(n)}$.

Theorem 2. *$F_n(x) = \sum_{i=0}^{x}\binom{n-1+x-i}{n-1}f(i)$ for $x = 0, 1, \ldots, b$ and $n \in \mathbb{Z}_{>0}$.*

Proof. We expand the recursive form of $F_n(x)$ and group the terms as follows:

$$F_n(x) = \sum_{i_n=0}^{x}\sum_{i_{n-1}=0}^{i_n}\cdots\sum_{i_1=0}^{i_2} f(i_1) = \sum_{i=0}^{x} N(n,x,i)f(i),$$

where

$$N(n,x,i) = |\{(i_n, i_{n-1}, \ldots, i_2, i_1) \in \mathbb{Z}_{>0} \colon x \geq i_n \geq i_{n-1} \geq \ldots \geq i_2 \geq i_1 = i\}|.$$

Note that in the definition of $N(n,x,i)$, i_1 is fixed as $i_1 = i$, thus $N(n,x,i)$ is equivalent to a stars and bars counting problem with $x-i$ stars and $n-1$ bars. That is, we have $N(n,x,i) = \binom{n-1+x-i}{n-1}$. □

We can see that the form for the discrete case is totally different from the Riemann–Liouville integral. To extend the form into a continuum, consider

$$\binom{n-1+x-i}{n-1} = \frac{(n-1+x-i)!}{(n-1)!(x-i)!} = \frac{\Gamma(n+x-i)}{\Gamma(n)\Gamma(1+x-i)}.$$

Define

$$k^{\alpha}(j) = \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)\Gamma(j+1)}.$$

Then, the αth-order sum becomes

$$F_{\alpha}(x) = \sum_{j=0}^{x} k^{\alpha}(x-j)f(j), \tag{4}$$

which has the same form as Lizama's fractional delta operator [27]. This discrete fractional sum notation can be obtained via the translation from other notations such as [33–35].

After obtaining the definition in Equation (4), we need to verify its consistency with the properties of integral-order SD. That is, we need to show

- The preservation of the SD hierarchy: for any $\nu > 0$ and $\alpha \in [1, \infty)$, $F \succeq_{\alpha} G$ implies $F \succeq_{\alpha+\nu} G$;
- An equivalent definition by utilities: find the utility classes that are congruent with the αth-order SD;
- The monotonicity of utility classes: for integral nth-order SD, show that every u in the utility class that we found satisfies $(-1)^{p+1}\Delta^p u(i) \geq 0$ for all $p = 1, 2, \ldots, n$.

As a remark, due to the totally different forms of fractional-order SD for continuous and discrete distributions, the fractional-order SD between a continuous distribution and a discrete distribution remains an open problem. However, in practical applications that compare the risk (or performance) of two schemes, they should usually be in the same nature, i.e., either both are continuous distributions, or both are discrete distributions.

3.2. Stochastic Dominance Hierarchy

The proof for showing the preservation of the SD hierarchy applies a few mathematical tools. The first one is Abel's lemma, which is the discrete version of "integration by parts". Let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ be two real sequences. Abel's lemma states that

$$\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i} a_j \right) (b_i - b_{i+1}) + \left(\sum_{j=1}^{n} a_j \right) b_n.$$

Next, we also use the beta function $B(x, y) := \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, which has the property that

$$B(x+1, y) = \frac{xB(x, y)}{x+y}.$$

Moreover, when $x, y > 0$, we have $\Gamma(x) > 0$ and $\Gamma(y) > 0$, thus $B(x, y) > 0$.

Theorem 3. *For any $\nu > 0$ and $\alpha \in [1, \infty)$,*

- *$F \succeq_\alpha G$ implies $F \succeq_{\alpha+\nu} G$;*
- *$F \succ_\alpha G$ implies $F \succ_{\alpha+\nu} G$.*

Proof. First, $F \succeq_\alpha G$ implies that $G_\alpha(n) - F_\alpha(n) \geq 0$ for all $n \in \mathbb{N}_b$. Consider $n > 0$. For any $j \in \mathbb{N}_{n-1} \subset \mathbb{N}_b$, we have $n > n - 1 \geq j$, hence we know that

$$\frac{\nu}{\alpha + \nu + n - j - 1} \left(\prod_{k=1}^{j} \frac{\alpha + n - k}{\alpha + \nu + n - k} \right) > 0.$$

Therefore, we have

$$\begin{aligned} &\sum_{j=0}^{n-1} (G_\alpha(j) - F_\alpha(j)) \frac{\nu}{\alpha + \nu + n - j - 1} \left(\prod_{k=1}^{j} \frac{\alpha + n - k}{\alpha + \nu + n - k} \right) \geq 0 \\ \implies &\sum_{j=0}^{n-1} (G_\alpha(j) - F_\alpha(j)) \left(\prod_{k=1}^{j} \frac{\alpha + n - k}{\alpha + \nu + n - k} - \prod_{k=1}^{j+1} \frac{\alpha + n - k}{\alpha + \nu + n - k} \right) \geq 0. \end{aligned} \tag{5}$$

On the other hand, we know that

$$(G_\alpha(n) - F_\alpha(n)) \prod_{k=1}^{n} \frac{\alpha + n - k}{\alpha + \nu + n - k} \geq 0. \tag{6}$$

By summing up Section 3.2 and Equation (6), and expressing

$$G_\alpha(j) - F_\alpha(j) = \sum_{i=0}^{j} k^\alpha (n - i)(g(i) - f(i))$$

for all $j \in \mathbb{N}_n$, we obtain

$$\begin{aligned} &\sum_{j=0}^{n-1} \left(\sum_{i=0}^{j} k^\alpha (n - i)(g(i) - f(i)) \right) \left(\prod_{k=1}^{j} \frac{\alpha + n - k}{\alpha + \nu + n - k} - \prod_{k=1}^{j+1} \frac{\alpha + n - k}{\alpha + \nu + n - k} \right) \\ &\quad + \left(\sum_{i=0}^{n} k^\alpha (n - i)(g(i) - f(i)) \right) \prod_{k=1}^{n} \frac{\alpha + n - k}{\alpha + \nu + n - k} \geq 0. \end{aligned} \tag{7}$$

By Abel's lemma, we have

$$\sum_{j=0}^{n} k^{\alpha}(n-j)(g(j)-f(j))\prod_{k=1}^{j}\frac{\alpha+n-k}{\alpha+\nu+n-k} \geq 0.$$

Next, note that $\frac{B(\alpha,\nu)}{B(\alpha+n,\nu)} > 0$, and we have

$$\frac{B(\alpha,\nu)}{B(\alpha+n,\nu)}\sum_{j=0}^{n} k^{\alpha}(n-j)(g(j)-f(j))\prod_{k=1}^{j}\frac{\alpha+n-k}{\alpha+\nu+n-k} \geq 0.$$

By the property of the beta function, we know that

$$B(\alpha+n,\nu) = B(\alpha+n-1,\nu)\frac{\alpha+n-1}{\alpha+\nu+n-1} = \ldots = B(\alpha+n-j,\nu)\prod_{k=1}^{j}\frac{\alpha+n-k}{\alpha+\nu+n-k}.$$

Then, we have

$$\begin{aligned} & B(\alpha,\nu)\sum_{j=0}^{n}\frac{k^{\alpha}(n-j)(g(j)-f(j))}{B(\alpha+n-j,\nu)} \geq 0 \\ \Longrightarrow \quad & \sum_{j=0}^{n}\frac{\Gamma(\alpha)\Gamma(\nu)}{\Gamma(\alpha+\nu)}\frac{\Gamma(\alpha+n-j)}{\Gamma(\alpha)\Gamma(n-j+1)}\frac{g(j)-f(j)}{B(\alpha+n-j,\nu)} \geq 0. \end{aligned}$$

Finally, note that

$$B(\alpha+n-j,\nu) = \frac{\Gamma(\alpha+n-j)\Gamma(\nu)}{\Gamma(\alpha+\nu+n-j)};$$

thus, we have

$$\begin{aligned} & \sum_{j=0}^{n}\frac{\Gamma(\alpha+\nu+n-j)}{\Gamma(\alpha+\nu)\Gamma(n-j+1)}(g(j)-f(j)) \geq 0 \\ \Longrightarrow \quad & \sum_{j=0}^{n} k^{\alpha+\nu}(g(j)-f(j)) \geq 0. \end{aligned}$$

That is, we have $G_{\alpha+\nu}(n) - F_{\alpha+\nu}(n) \geq 0$ for all $n \in \mathbb{N}_b$, which implies that $F \succeq_{\alpha+\nu} G$.

For $F \succ_{\alpha} G$, we have $G_{\alpha}(n) - F_{\alpha}(n) \geq 0$ for all $n \in \mathbb{N}_b$, and there is some $n' \in \mathbb{N}_b$ such that $G_{\alpha}(n') - F_{\alpha}(n') > 0$. The proof for this case is almost the same as that for $F \succeq_{\alpha} G$. First, at least one of Section 3.2 and Equation (6) has a strict inequality. Therefore, the inequality in Equation (7) is strict, and so are the remaining inequalities. As a result, we have $\sum_{j=0}^{n} k^{\alpha+\nu}(g(j)-f(j)) > 0$, which means that $G_{\alpha+\nu}(n) - F_{\alpha+\nu}(n) \geq 0$ for all $n \in \mathbb{N}_b$, and there is some $n' \in \mathbb{N}_b$ such that $G_{\alpha+\nu}(n') - F_{\alpha+\nu}(n') > 0$. In other words, $F \succ_{\alpha+\nu} G$. □

3.3. Utility Classes

We first define the utility class for each $\alpha \in [1,\infty)$, then show that this utility class is congruent with the αth-order SD. Define

$$\begin{aligned} U_{\alpha} &:= \left\{ u\colon u(i) = -\sum_{x=i}^{b} t(x)k^{\alpha}(x-i) + c; c \in \mathbb{R}; t\colon \mathbb{N}_b \to \mathbb{R}_{>0} \cup \{0\} \right\}, \\ U'_{\alpha} &:= \left\{ u\colon u(i) = -\sum_{x=i}^{b} t(x)k^{\alpha}(x-i) + c; c \in \mathbb{R}; t\colon \mathbb{N}_b \to \mathbb{R}_{>0} \right\}. \end{aligned}$$

Fix an α. Now, we define the corresponding notations for utilities:

- $F \succeq_u G$ means that $\sum_{x=0}^{b} u(x)f(x) \geq \sum_{x=0}^{b} u(x)g(x)$ for all $u \in U_\alpha$;
- $F \succ_u G$ means that $\sum_{x=0}^{b} u(x)f(x) > \sum_{x=0}^{b} u(x)g(x)$ for all $u \in U'_\alpha$.

Note that our definition of $F \succ_u G$ is a very strong one: we require all utilities, not just some, to achieve the strict inequality.

Theorem 4. *$\succeq_\alpha \cong \succeq_u$ and $\succ_\alpha \cong \succ_u$ for each $\alpha \in [1, \infty)$.*

Proof. Fix an α. We first prove the relation $\succ_\alpha \cong \succ_u$. For any F, G, it is sufficient to prove that $F \succ_\alpha G$ implies $F \succ_u G$ for all $u \in U'_\alpha$, and also prove that if $F \succ_\alpha G$ is false, then there exists a $u \in U'_\alpha$ such that $\sum_{i=0}^{b} u(i)g(i) \geq \sum_{i=0}^{b} u(i)f(i)$.

We start with the first part. Given that $F \succ_\alpha G$, we have $F_\alpha(x) \leq G_\alpha(x)$ for all $x \in \mathbb{N}_b$, and $F_\alpha(x) < G_\alpha(x)$ for some $x \in \mathbb{N}_b$. As $t(x) > 0$ for all $x \in \mathbb{N}_b$, we have

$$\begin{aligned} & \sum_{x=0}^{b} -t(x)F_\alpha(x) > \sum_{x=0}^{b} -t(x)G_\alpha(x) \qquad (8) \\ \Longrightarrow \quad & \sum_{x=0}^{b} -t(x) \sum_{i=0}^{x} k^\alpha(x-i)f(i) > \sum_{x=0}^{b} -t(x) \sum_{i=0}^{x} k^\alpha(x-i)g(i) \\ \Longrightarrow \quad & \sum_{i=0}^{b} \left(-\sum_{x=i}^{b} t(x)k^\alpha(x-i) \right) f(i) > \sum_{i=0}^{b} \left(-\sum_{x=i}^{b} t(x)k^\alpha(x-i) \right) g(i). \end{aligned}$$

For any $c \in \mathbb{R}$, we have $\sum_{i=0}^{b} c \cdot f(i) = \sum_{i=0}^{b} c \cdot g(i) = c$, hence the above inequality becomes

$$\sum_{i=0}^{b} u(i)f(i) > \sum_{i=0}^{b} u(i)g(i)$$

for any $u \in U'_\alpha$.

We now prove the second part. When $F \succ_\alpha G$ is false, we have two cases. The first case is $F = G$, which gives $\sum_{i=0}^{b} u(i)f(i) = \sum_{i=0}^{b} u(i)g(i)$ for all $u \in U'_\alpha$. The second case is that there exists some $i \in \mathbb{N}_b$ such that $F_\alpha(i) > G_\alpha(i)$.

For the second case, let $S := \{x \colon F_\alpha(x) > G_\alpha(x)\}$. Pick an $x' \in S$. Let $F_\alpha(x') - G_\alpha(x') > \lambda$ for some $\lambda > 0$. As we only need to find a $u \in U'_\alpha$ that gives $\sum_{i=0}^{b} u(i)g(i) \geq \sum_{i=0}^{b} u(i)f(i)$, we consider a $u \in U'_\alpha$ where $c = 0$ and

$$t(x) = \begin{cases} 1 & \text{if } x = x', \\ \delta & \text{otherwise} \end{cases}$$

for some $\delta > 0$. For this u, we have

$$\begin{aligned} \sum_{i=0}^{b} u(i)f(i) &= \sum_{i=0}^{b} \left(-\sum_{x=i}^{b} t(x)k^\alpha(x-i) \right) f(i) \\ &= -\sum_{x=0}^{b} t(x) \sum_{i=0}^{x} k^\alpha(x-i)f(i) \\ &= -\sum_{x=0}^{b} t(x)F_\alpha(x) \\ &= -F_\alpha(x') - \delta \sum_{\substack{x=0 \\ x \neq x'}}^{b} F_\alpha(x). \end{aligned}$$

Similarly, we have

$$\sum_{i=0}^{b} u(i)g(i) = -G_\alpha(x') - \delta \sum_{\substack{x=0 \\ x\neq x'}}^{b} G_\alpha(x).$$

Now, consider

$$\begin{aligned}\sum_{i=0}^{b} u(i)(g(i)-f(i)) &= F_\alpha(x') - G_\alpha(x') + \delta \sum_{\substack{x=0 \\ x\neq x'}}^{b} (F_\alpha(x) - G_\alpha(x)) \\ &> \lambda + b\delta \min_{x\in\mathbb{N}_b\setminus\{x'\}} (F_\alpha(x) - G_\alpha(x)).\end{aligned}$$

For a sufficiently small $\delta > 0$, the last equation is positive. Therefore, there exists a $u \in U'_\alpha$ such that $\sum_{i=0}^{b} u(i)g(i) \geq \sum_{i=0}^{b} u(i)f(i)$, where the equality case is from the first case where $F = G$. This finishes the proof for $\succ_\alpha \cong \succ_u$.

Now, we prove the relation $\succeq_\alpha \cong \succeq_u$. The proof is similar as that for $\succ_\alpha \cong \succ_u$. For any F, G, it is sufficient to prove that $F \succeq_\alpha G$ implies $F \succeq_u G$ for all $u \in U_\alpha$, and also prove that if $F \succeq_\alpha G$ is false, then there exists a $u \in U_\alpha$ such that $\sum_{i=0}^{b} u(i)g(i) > \sum_{i=0}^{b} u(i)f(i)$.

For the first part, this time we have $F_\alpha(x) \leq G_\alpha(x)$ and $t(x) \geq 0$ for all $x \in \mathbb{N}_b$. Therefore, we have a $\geq$ instead of a $>$ in Section 3.3, as well as all inequalities below it. That is, we achieve

$$\sum_{i=0}^{b} u(i)f(i) \geq \sum_{i=0}^{b} u(i)g(i)$$

for any $u \in U_\alpha$.

For the second part, when $F \succeq_\alpha G$ is false, the only case is when there exists some $i \in \mathbb{N}_b$ such that $F_\alpha(i) > G_\alpha(i)$. We can construct the same u as used in the proof for $\succ_\alpha \cong \succ_u$, but this time this u is in U_α. Using the same argument, we can conclude that there exists a $u \in U_\alpha$ such that $\sum_{i=0}^{b} u(i)g(i) > \sum_{i=0}^{b} u(i)f(i)$. This finishes the proof for $\succeq_\alpha \cong \succeq_u$. □

Next, we want to show that our utility classes are consistent with the conventional (integral-order) SD in the sense of monotonicity, i.e., we want to know whether our utilities maintain the monotonicity when α is a positive integer. In other words, we aim to show that for any $n \in \mathbb{Z}_{>0}$, every $u \in U_n \cup U'_n = U_n$ satisfies $(-1)^{p+1}\Delta^p u(i) \geq 0$ for all $p = 1, 2, \ldots, n$. We can also show that the inequality is strict when $u \in U'_n$. A direct way is to find the exact formula of $\Delta^n u(i)$.

In the following theorem, we treat binomial coefficients as "generalized binomial coefficients", i.e., $\binom{n}{r} := \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}$, where $n \in \mathbb{R}$ and r is a non-negative integer. As a convention in combinatorics, define $\binom{-1}{0} = 1$ and $\binom{j-1}{j} = 0$ for all $j \in \mathbb{Z}_{>0}$.

Theorem 5. *For any $n \in \mathbb{Z}_{>0}$ and $u \in U_\alpha \cup U'_\alpha$,*

$$\Delta^n u(i) = (-1)^{n+1}\left(\sum_{j=0}^{n-1}\binom{\alpha-n+j-1}{j} t(i+j) + \prod_{j=1}^{n}(\alpha-j)\sum_{x=i+n}^{b}\frac{t(x)k^\alpha(x-i-n)}{\prod_{\ell=1}^{n}(x-i+1-\ell)}\right).$$

Proof. See Appendix B. □

We can now show the monotonicity result.

Corollary 1. *For any $n \in \mathbb{Z}_{>0}$, every $u \in U_n \cup U'_n$ satisfies $(-1)^{p+1}\Delta^p u(i) \geq 0$ for all $p = 1, 2, \ldots, n$. The inequality is strict if $u \in U'_n$.*

Proof. Substituting $\alpha = n$ into Theorem 5. Our goal is to show that

$$(-1)^{p+1}\Delta^p u(i) = \sum_{j=0}^{p-1} \binom{n-p+j-1}{j} t(i+j) + \prod_{j=1}^{p}(n-j) \sum_{x=i+p}^{b} \frac{t(x)k^n(x-i-p)}{\prod_{\ell=1}^{p}(x-i+1-\ell)} \geq 0$$

for all $p = 1, 2, \ldots, n$.

Note that the empty summation is defined to be 0. First, for every p, we have

$$\prod_{j=1}^{p}(n-j) \sum_{x=i+p}^{b} \frac{t(x)k^n(x-i-p)}{\prod_{\ell=1}^{p}(x-i+1-\ell)} \geq 0$$

as every term on the left side is non-negative. Next, for $1 \leq p < n$, we have $n - p \geq 1$, thus

$$\sum_{j=0}^{p-1} \binom{n-p+j-1}{j} t(i+j) \geq 0.$$

The inequality is strict if $u \in U_n'$ (and $n > 1$) because $t > 0$. The last piece of the puzzle is the case $1 \leq p = n$. We have

$$\sum_{j=0}^{p-1} \binom{n-p+j-1}{j} t(i+j) = \binom{-1}{0} t(i) + \sum_{j=1}^{p-1} \binom{j-1}{j} t(i+j) = t(i) \geq 0.$$

The inequality is strict if $u \in U_n'$, because $t > 0$. Combining all cases in the above discussion, the proof is completed. □

As a remark, the above discussion is for Fishburn's SD, so that when $\alpha = n \in \mathbb{Z}_{>0}$, the utility class does not cover all valid utilities for the conventional SD. However, this is what we need in applications that are concerned about arbitrary distributions but only use a certain utility for quantifying the overall performance.

4. Conclusions

In this paper, we discussed the diverse perspectives of higher-order stochastic dominance (SD) from the view of distributions and the view of utilities. The conventional SD makes a compromise to restrict the choices of distributions so that the utility-based definition must work for all utilities in the class. This is very useful in risk analysis as the exact form of the utility may not be exactly known. Other than risk management, some applications are only concerned with some specific utilities. In this way, we prefer imposing restrictions on the utility classes instead. We denoted this type of SD as "Fishburn's SD" in this paper.

Our motivation stems from the practical needs of developing SD between integral orders and applying discrete distributions and discrete utilities. The existing SD development mostly focuses on continuous distributions, where the results may be generalized to discrete distributions with ease. However, when we investigate fractional-order SD, there may not be a straightforward analogue between fractional calculus and fractional sum. This led to the investigation of this paper: if we consider the natural extension of discrete SD using a fractional sum, can there be a candidate of fractional-order SD that satisfies some requirements? We proved that our fractional-order SD definition was a candidate: the SD hierarchy was preserved, and it was congruent to certain utility classes that are consistent with integral-order SD.

There are some future directions related to this research. First, we crafted some specific utility classes to suit our needs. However, this construction only considered discrete utilities. A question would be whether there are any utility classes of mixed continuous and discrete utilities that are congruent with the discrete fractional-order SD. Second, we considered bounded distributions in this paper as preliminary results. However, how can

we extend these results to unbounded distributions? We suspect that some properties may not be valid in the unbounded case. These two points can be considered as an extension to general supports. Third, if we consider conventional SD but discrete utilities, how can one formulate the corresponding fractional-order SD? Fourth, how can one formulate the fractional-order SD when comparing a continuous distribution with a discrete distribution, or distributions that contain both continuous and discrete components? Fifth, although the formulations and the proofs are different for continuous and discrete fractional-order SD, is there a way to unify them in a general form? Sixth, besides risk management and communication theory, can SD be applied to other fields such as video and image processing [36]? At last, the investigation of continuity properties with respect to the fractional order, and the subsequent clarifications for the utility classes involved are also potential theoretical directions. All these directions may help us understand more about the properties of discrete SD and its practical usage, as well as its limitations.

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Abbreviations

The following abbreviation is used in this manuscript:

SD Stochastic dominance

Appendix A. Proof of Theorem 1

If F and G are identical, then we have $F \succeq_1 G$ by definition. Now consider distinct F and G. Without loss of generality, assume that $f(\lambda) < g(\lambda)$. The case $g(\lambda) < f(\lambda)$ can be proved by similar arguments.

Define $F \succ_K^m G$ if and only if $F_K(x) \leq G_K(x)$ for all $0 \leq x \leq m$ and there is at least one strict inequality. Note that $F \succ_K^b G$ is equivalent to $F \succ_K G$ as the distributions of F and G are bounded on $\{0, 1, \ldots, b\}$. When $F \succ_K^m G$, we also have

$$F_{K+1}(i) = \sum_{x=0}^{i} F_K(x) \leq \sum_{x=0}^{i} G_K(x) = G_{K+1}(i)$$

for all $0 \leq i \leq m$, and the equality between the summations does not hold when $i = m$. Therefore, $F \succ_K^m G$ implies $F \succ_{K+1}^m G$.

We use induction to prove the proposition: for any $\lambda \leq m \leq b$, there exists an integer K_m such that $F \succ_{K_m}^m G$.

We start with $m = \lambda$. By the definition of λ, we have $f(\lambda) < g(\lambda)$ and $f(i) = g(i)$ for all $0 \leq i < \lambda$. Hence, we have

$$G_1(i) - F_1(i) = \sum_{x=0}^{i} (g(x) - f(x)) \geq 0$$

for all $0 \leq i \leq \lambda$, where the equality does not hold when $i = \lambda$. That is, $F \succ_1^\lambda G$.

If $\lambda = b$, then we do not need the induction step below. For $\lambda \neq b$, we assume that there exists an integer K such that $F \succ_K^{k-1} G$, where $\lambda \leq k - 1 < b$. Consider $m = k$. If $F_K(k) \leq G_K(k)$, then it is clear that $F \succ_K^k G$.

Suppose $F_K(k) > G_K(k)$. Define

$$\omega_K^k = \sum_{i=0}^{k-1} (G_K(i) - F_K(i)) > 0.$$

Note that

$$\begin{aligned}\omega_{K+1}^k &= \sum_{i=0}^{k-1} \sum_{i'=0}^{i} (G_K(i') - F_K(i')) \\ &= \omega_K^k + \sum_{i=0}^{k-2} \sum_{i'=0}^{i} (G_K(i') - F_K(i')) > \omega_K^k.\end{aligned}$$

Similarly, we have $\omega_{K+j}^k > \omega_K^k > 0$ for all $j \in \mathbb{Z}_{>0}$.

Define $\varsigma_K^k = F_K(k) - G_K(k)$ and $N = \lfloor \varsigma_K^k / \omega_K^k \rfloor + 1$, which results in $\varsigma_K^k < N\omega_K^k$. Then, we have

$$F_{K+N}(k) - G_{K+N}(k) = \varsigma_K^k - \sum_{j=K}^{K+N-1} \omega_j^k \leq \varsigma_K^k - N\omega_K^k < 0.$$

As $F \succ_K^{k-1} G$ implies $F \succ_{K+N}^{k-1} G$, we have $F_{K+N}(i) \leq G_{K+N}(i)$ for all $0 \leq i \leq k$ with at least one strict inequality, i.e., $F \succ_{K+N}^{k} G$.

By induction, we have $F \succ_{K_b}^{b} G$ for some integer K_b. The proof is now completed.

Appendix B. Proof of Theorem 5

In this proof, we apply the extended Pascal rule. Let $n \in \mathbb{R}$ and $k \in \mathbb{N}$. Then, the rule is that

$$\begin{aligned}\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{\Gamma(n)}{\Gamma(k+1)\Gamma(n-k)} + \frac{\Gamma(n)}{\Gamma(k)\Gamma(n-k+1)} \\ &= \Gamma(n)\left(\frac{n-k}{\Gamma(k+1)\Gamma(n-k+1)} + \frac{k}{\Gamma(k+1)\Gamma(n-k+1)}\right) \\ &= \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \\ &= \binom{n}{k}.\end{aligned}$$

Now, we begin the proof by fixing a $u \in U_\alpha$ or $u \in U'_\alpha$. Write this u as

$$u(i) = -\sum_{x=i}^{b} t(x)k^\alpha(x-i) + c$$

for some t and c. We prove this theorem by induction. For $n = 1$, we have

$$\begin{aligned}
\Delta^1 u(i) &= u(i+1) - u(i) \\
&= -\sum_{x=i+1}^{b} t(x)k^{\alpha}(x-(i+1)) + \sum_{x=i}^{b} t(x)k^{\alpha}(x-i) \\
&= t(i)k^{\alpha}(0) + \sum_{x=i+1}^{b} t(x)(k^{\alpha}(x-i) - k^{\alpha}(x-i-1)) \\
&= t(i) + (\alpha-1)\sum_{x=i+1}^{b} \frac{t(x)k^{\alpha}(x-i-1)}{x-i} \\
&= (-1)^{(1)+1}\left(\sum_{j=0}^{(1)-1}\binom{\alpha-(1)+j-1}{j} t(i+j) + \prod_{j=1}^{(1)}(\alpha-j)\sum_{x=i+(1)}^{b}\frac{t(x)k^{\alpha}(x-i-(1))}{\prod_{\ell=1}^{(1)}(x-i+1-\ell)}\right).
\end{aligned}$$

Assume that the statement we want to prove holds for some $n \in \mathbb{Z}_{>0}$. As an induction step, we consider $\Delta^{n+1}u(i)$.

$$\begin{aligned}
\Delta^{n+1}u(i) &= \Delta^n u(i+1) - \Delta^n u(i) \\
&= (-1)^{n+1}\left(\sum_{j=0}^{n-1}\binom{\alpha-n+j-1}{j}(t(i+1+j)-t(i+j)) - \prod_{j=1}^{n}(\alpha-j)\frac{t(i+n)k^{\alpha}(0)}{\prod_{\ell=1}^{n}(n+1-\ell)}\right. \\
&\qquad \left. + \prod_{j=1}^{n}(\alpha-j)\sum_{x=i+n+1}^{b} t(x)\left(\frac{k^{\alpha}(x-i-1-n)}{\prod_{\ell=1}^{n}(x-i-\ell)} - \frac{k^{\alpha}(x-i-n)}{\prod_{\ell=1}^{n}(x-i+1-\ell)}\right)\right).
\end{aligned}$$

We separate the formulation into two parts. First, we consider

$$\begin{aligned}
&\sum_{j=0}^{n-1}\binom{\alpha-n+j-1}{j}(t(i+1+j)-t(i+j)) - \prod_{j=1}^{n}(\alpha-j)\frac{t(i+n)k^{\alpha}(0)}{\prod_{\ell=1}^{n}(n+1-\ell)} \\
&= \sum_{j=1}^{n-1}\left(\binom{\alpha-n+(j-1)-1}{j-1} - \binom{\alpha-n+j-1}{j}\right)t(i+j) + \binom{\alpha-n+(n-1)-1}{n-1}t(i+n) \\
&\qquad -\binom{\alpha-n-1}{0}t(i) - \binom{\alpha-(n+1)+n}{n}t(i+n) \\
&= -\sum_{j=1}^{n-1}\binom{\alpha-n+(j-1)-1}{j}t(i+j) + \binom{\alpha-(n+1)+n-1}{n-1}t(i+n) - t(i) - \binom{\alpha-(n+1)+n}{n}t(i+n) \\
&= -\left(\sum_{j=1}^{n-1}\binom{\alpha-(n+1)+j-1}{j}t(i+j) + \binom{\alpha-(n+1)+n-1}{n}t(i+n) + t(i)\right) \\
&= -\sum_{j=0}^{(n+1)-1}\binom{\alpha-(n+1)+j-1}{j}t(i+j).
\end{aligned}$$

Next, we consider

$$
\begin{aligned}
&\prod_{j=1}^{n}(\alpha-j)\sum_{x=i+n+1}^{b}t(x)\left(\frac{k^{\alpha}(x-i-1-n)}{\prod_{\ell=1}^{n}(x-i-\ell)}-\frac{k^{\alpha}(x-i-n)}{\prod_{\ell=1}^{n}(x-i+1-\ell)}\right)\\
&=\prod_{j=1}^{n}(\alpha-j)\sum_{x=i+n+1}^{b}t(x)k^{\alpha}(x-i-1-n)\left(\frac{1}{\prod_{\ell=1}^{n}(x-i-\ell)}-\frac{\alpha+x-i-1-n}{(x-i-n)\prod_{\ell=1}^{n}(x-i+1-\ell)}\right)\\
&=\prod_{j=1}^{n}(\alpha-j)\sum_{x=i+n+1}^{b}t(x)k^{\alpha}(x-i-1-n)\left(\frac{1}{\prod_{\ell=2}^{n+1}(x-i+1-\ell)}-\frac{\alpha+x-i-1-n}{\prod_{\ell=1}^{n+1}(x-i+1-\ell)}\right)\\
&=\prod_{j=1}^{n}(\alpha-j)\sum_{x=i+n+1}^{b}t(x)k^{\alpha}(x-i-1-n)\frac{-(\alpha-(n+1))}{\prod_{\ell=1}^{n+1}(x-i+1-\ell)}\\
&=-\prod_{j=1}^{n+1}(\alpha-j)\sum_{x=i+n+1}^{b}\frac{t(x)k^{\alpha}(x-i-(n+1))}{\prod_{\ell=1}^{n+1}(x-i+1-\ell)}.
\end{aligned}
$$

Combining the two parts, we have

$$
\Delta^{n+1}u(i)=(-1)^{(n+1)+1}\left(\sum_{j=0}^{(n+1)-1}\binom{\alpha-(n+1)+j-1}{j}t(i+j)+\prod_{j=1}^{(n+1)}(\alpha-j)\sum_{x=i+(n+1)}^{b}\frac{t(x)k^{\alpha}(x-i-(n+1))}{\prod_{\ell=1}^{n+1}(x-i+1-\ell)}\right).
$$

The proof is thus completed by induction.

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