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Review

Solutions for Some Mathematical Physics Problems Issued from Modeling Real Phenomena: Part 1

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Abstract: This paper brings together methods to solve and/or characterize solutions of some problems of mathematical physics equations involving p-Laplacian and p-pseudo-Laplacian. Using surjectivity or variational approaches, one may obtain or characterize weak solutions for Dirichlet or Newmann problems for these important operators. This article details three ways to use surjectivity results for a special type of operator involving the duality mapping and a Nemytskii operator, three methods starting from Ekeland's variational principle and, lastly, one with a generalized variational principle to solve or describe the above-mentioned solutions. The relevance of these operators and the possibility of their involvement in the modeling of an important class of real phenomena determined the author to group these seven procedures together, presented in detail, followed by many applications, accompanied by a general overview of specialty domains. The use of certain variational methods facilitates the complete solution of the problem via appropriate numerical methods and computational algorithms. The exposure of the sequence of theoretical results, together with their demonstration in as much detail as possible has been fulfilled as an opportunity for the complete development of these topics.

Keywords: modeling real phenomena; mathematical physics problems; p-Laplacian; p-pseudo-Laplacian; surjectivity methods; variational methods; Dirichlet problem; Neumann problem

MSC: 35A01; 35A15; 35J35; 35J40; 47J30

Citation: Meghea, I. Solutions for Some Mathematical Physics Problems Issued from Modeling Real Phenomena: Part 1. *Axioms* **2023**, *12*, 532. https://doi.org/10.3390/axioms12060532

Academic Editor: Feliz Manuel Minhós

Received: 8 April 2023
Revised: 9 May 2023
Accepted: 18 May 2023
Published: 29 May 2023

1. Introduction

Problems for partial differential equations involving the p-Laplacian and p-pseudo-Laplacian are mathematical models that often occur in studies on the p-Laplace or p-pseudo-Laplace equation, generalized reaction–diffusion theory, non-Newtonian fluid theory, non-Newtonian filtration, the turbulent flow of a gas in a porous medium, glaciology, non-Newtonian rheology, etc. These fractional order operators are very important mathematical models describing a multitude of anomalous dynamic behaviors in applied sciences. In the non-Newtonian fluid theory, the quantity p is a medium characteristic. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are pseudoplastics. If $p = 2$, they are Newtonian fluids. The p-Laplacian appears in the study of flow through porous media in turbulent regime at Diaz et al. [1,2] or glacier ice when treated as a non-Newtonian fluid with a nonlinear relationship between the rate deformation tensor and the deviatoric stress tensor, as described by Glowinski et al. [3]. It is also used in the Helle-Shaw approximation for a moving boundary problem by King et al. [4] and also for "power-law fluids" at Aronson et al. [5]. The p-Laplacian also appears in the study of flow in porous media ($p = 3/2$, at Schowalter et al. [6]) or glacial sliding ($p \in (1, 4/3]$, at Péllissier [7]). Quasilinear problems with a variable coefficient appear in the mathematical model of the torsional creep (elastic for $p = 2$, plastic for $p \to \infty$; see in Bhattacharya et al. [8] and at Kawohl [9]). A nonlinear field equation in Quantum Mechanics involving the p-Laplacian for $p = 6$ was proposed by Benci et al. [10].

Surjectivity methods to solve and/or characterize the solutions for Dirichlet problems involving the p-Laplacian and the p-pseudo-Laplacian have been previously used by the author in [11], together with other variational results, in [12] where two solving methods are displayed, in [13] which discusses some Fredholm alternative types, in [14] with solutions for the p-pseudo-Laplacian treated following two approaches, and in [15] involving surjectivity methods. Solving such problems via results obtained with Ekeland variational principle and other generalized variational principles has been the goal of other works of the author where some Dirichlet or Newmann problems have been studied as in [12,14,16] with the use of a perturbed variational principle, in [17] with variational procedures, as also in [18]. Mountain pass theorem variants and applications involved in modeling real phenomena are performed by the author in [19], while other applied variational methods are capitalized on by the author in [20,21], and several variational principles, together with generalizations and variants, have been compared and analyzed in the monograph [22].

The fractional differential equations are frequently used as modeling tools for processes implied in anomalous diffusion or spatial heterogeneity [23]. Also, in water resources, fractional models have been used to design chemical or contaminant transport in heterogeneous aquifers. In the field of magnetic resonance, fractional models of the Bloch–Torrey equation for drawing anomalous diffusion have been considered. Concerning the domain of cell biology, anomalous diffusion has been measured in fluorescence photo-bleaching recovery and fractional-in-time models have been created for simple types of chemical reaction–diffusion equations and for the simulation of microscale diffusion in the cell wall lining of plants. Similar problems appear in models of chemical reactions, heat transfer, population dynamics and so on [24]. Power law diffusion equations with p-Laplacian having constant and/or variable p are significantly related to researches on non-Newtonian fluids, turbulence modeling phase transitions, data clustering, machine learning and image processing. Many studies devoted to power-law diffusions call for the development of efficient numerical methods for solving elliptic partial differential equations with nonlinearities of p-type gradient [25].

The interest in these kinds of operators is topical both for mathematical approaches and results, as at Mukherjee et al. [26], Zhang et al. [27], Benedikt et al. [28], Lafleche et al. [29], Cellina [30], Khan [31], Xu [32], Akagi et al. [33], Gulsen et al. [34] and Lee et al. [35], and also for applications in many models in various fields, as in Rasouli [36], Yang et al. [37], Elmoataz et al. [38], Gupta et al. [39], Liero et al. [40] and Silva [41].

In this paper, several sequences of results are proposed, starting from the most general and abstract theory, passing step by step through many concrete stages until their applications in models issued from design of real phenomena. The focus falls on a very extended justification, containing all the necessary details and displaying all the theoretical arguments. Some of the results presented in this paper are introduced with an integer justification background and they are used to obtain and/or characterize the solutions of equations of mathematical physics proposed by other authors for similar problems for which they gave solutions via different methods. For such problems resulting from mathematical modeling, we came up with original solving methods following the involved abstract frame. From these original approaches stems the novelty of this work. Our interest is now focused on such kinds of problems and their solutions, and this is one of the initial studies for future developments to find a mathematical model (there is none available) that can be applied for describing diffusion phenomena involved in the micro-emulsification of disperse systems in tight connection with surface properties at the interface in self-organized systems.

This article is the first part of review type of the work under this title containing whole theoretical support detailed in a complete exposition of the arguments, while many applications of the results presented here represent the aim of another original paper under the same title, Part two.

2. Surjectivity for the Operators $\lambda J_\varphi - S$: Applications to Partial Differential Equations

2.1. Surjectivity of the Operators of the Form $\lambda T - S$

In this section, a generalization of a theorem from [42] (Theorem 1.1) is presented and in this result the author used: normed space instead of Banach space, bijection with continuous inverse instead of homeomorphism. Two corollaries of this statement, obtained in [12,14], have also been presented.

Firstly, to have a short expression, we introduce the following.

Definition 2.1. *T: $X \to Y$, where X and Y are normed spaces, is (K, L, a), where $K > 0$, L > 0, a > 0, if*

$$K\,\|x\|^a \le \|Tx\| \le L\,\|x\|^a\ \forall x \text{ from } X.$$

Proposition 2.1. *Let X and Y be real normed spaces, T: $X \to Y$ (K, L, a) odd bijection with continuous inverse and S: $X \to Y$ odd compact operator. For any $\lambda \neq 0$, if*

$$\lim_{||x|| \to +\infty} ||\lambda Tx - Sx|| = +\infty,$$

then $\lambda T - S$ is surjective.

Proof. Let z_0 be from Y; we state that:

$$\exists\ x_0 \text{ in } X \text{ e.g., } \lambda T x_0 - S x_0 = z_0. \tag{2.1}$$

Take $R > 0$ with the property (see the hypothesis)

$$\|x\| \ge R \Rightarrow \|\lambda Tx - Sx\| > \|z_0\| \tag{2.2}$$

and the open ball from Y $\sigma := B(0, r)$, $r := |\lambda|\, LR^a$. If $y \in \partial\sigma$ and $y = \lambda Tx$, then $\|x\| \ge R$, and hence,

$$\|\ \lambda Tx - Sx\ \| \overset{(2.2)}{>} ||z_0|| \tag{2.3}$$

Let be the operator A: $Y \to Y$,

$$Ay = ST^{-1}\left(\frac{y}{\lambda}\right)$$

A is compact, odd and $Ay \neq y$ when $y \in \partial\sigma$ (ad absurdum, put y in the form λTx and take into account (2.3), i.e., $0 \notin (I - A)(\partial\sigma)$). Applying Borsuk theorem, the Leray-Schauder degree $d(I - A, \sigma, 0)$ is odd. However,

$$H:\ [0,\ 1]\ \times\ \overline{\sigma}\ \to\ Y,\ H(t,\ y) = Ay + tz_0$$

being a homotopy of compact transformations on $\overline{\sigma}$, we have

$$d(I - H(0,\cdot), \sigma, 0) = d(I - H(1,\cdot), \sigma, 0), \text{ i.e.,}$$
$$d(I - A, \sigma, 0) = d(I - A - z_0, \sigma, 0),$$

consequently, $d(I - A, \sigma, 0)$ is an odd number, particularly different from zero, therefore $\exists y_0$ in σ, e.g., $(I - A - z_0)(y_0) = 0$ and it remains only to take x_0 in X with $y_0 = \lambda T x_0$ to obtain (2.1). □

Corollary 2.1. *Let X, Y be real normed spaces, T: X → Y odd (K, L, a) bijection with continuous inverse, S: X → Y odd compact operator and* $\alpha := \overline{\lim}_{||x|| \to +\infty} \frac{\|Sx\|}{\|x\|^a} < +\infty$. *If*

$$|\lambda| > \frac{\alpha}{K}, \lambda \in \mathbf{R},$$

then $\lambda T - S$ *is surjective.*

Explanations [43], Volume 5, V, §5, 11.9_1, p. 372: $f: X \to Y$, X and Y normed spaces,

$$\overline{\lim}_{||x|| \to +\infty} \| f(x) \| \overset{def}{=} \inf_{\rho > 0} \sup_{\substack{x \in X \\ ||x|| \geq \rho}} \| f(x) \| = \lim_{\rho \to +\infty} \sup_{\substack{x \in X \\ ||x|| \geq \rho}} \| f(x) \| .$$

If $\alpha = \overline{\lim}_{||x|| \to +\infty} \| f(x) \|$, then $x_n \in X$ $\forall n$ from **N**, and $\|x_n\| \to +\infty$ implies $\overline{\lim}_{n\to\infty} \| f(x_n) \| \leq \alpha$.

If $\alpha = \overline{\lim}_{n\to\infty} \| f(x_n) \|$, for any (x_n) with x_n from X and $\|x_n\| \to +\infty$, then $\alpha = \overline{\lim}_{||x|| \to +\infty} \|f(x)\|$.

Proof. It remains to prove:

$$\overline{\lim}_{||x|| \to +\infty} ||\lambda Tx - Sx|| = +\infty, \tag{2.4}$$

Assuming, ad absurdum, the contrary, obtain $\rho > 0$ and a sequence $(x_n)_{n \geq 1}$, $x_n \in X$, $\|x_n\| \to +\infty$, e.g.,

$$\|\lambda Tx_n - Sx_n\| \leq \rho \ \forall n \geq 1. \tag{2.5}$$

From (2.5),

$$\lim_{n\to\infty} \left\| \frac{\lambda Tx_n}{||x_n||^a} - \frac{Sx_n}{||x_n||^a} \right\| = 0,$$

hence, $\lim_{n\to\infty} \left[\frac{|\lambda|\|Tx_n\|}{\|x_n\|^a} - \frac{\|S(x_n)\|}{\|x_n\|^a} \right] = 0$, and as $\overline{\lim}_{n\to\infty} \frac{\|S(x_n)\|}{\|x_n\|^a} \leq \alpha$, it results

$$\overline{\lim}_{n\to\infty} \frac{|\lambda|\|Tx_n\|}{\|x_n\|^a} \leq \alpha. \tag{2.6}$$

But the condition (K, L, a) imposes:

$$K \leq \overline{\lim}_{n\to\infty} \frac{\|Tx_n\|}{\|x_n\|^a}. \tag{2.7}$$

From (2.6) and (2.7), we obtain $K \leq \frac{\alpha}{|\lambda|}$. If $\alpha \neq 0$, then $|\lambda| \leq \frac{\alpha}{K}$, which contradicts the hypothesis, and if $\alpha = 0$, then $K = 0$, also in contradiction with the hypothesis, and consequently, (2.4). □

Corollary 2.2. *Under the conditions of Corollary 2.1, if* $\alpha = 0$, *then* $\lambda T - S$ *is surjective for* λ *any in* $\mathbf{R} \backslash \{0\}$.

2.2. Surjectivity for Operators of the Form $\lambda J_\varphi - S$, J_φ *Duality Map*

2.2.1. Preliminaries—Duality Map

Let X be a real Banach space, X^* its dual (), x^* an element any in X^*, 2^{M} the set of subsets of M.

Definition 2.2. φ: $\mathbf{R}_+ \to \mathbf{R}_+$ *continuous and strictly increasing with* $\varphi(0) = 0$ *and* $\lim_{r\to+\infty}$

φ (see, for instance [43], Volume 2, p. 27r) = +∞ is, by definition, weight or normalization function.

The multiple-valued map $J_\varphi : X \to 2^{X^*}$, φ weight,

$$\begin{cases} J_\varphi 0_X = \{0_{X^*}\}, \\ x \neq 0 \Rightarrow J_\varphi x = \varphi(||x||)\{x^* \in X^* : \ ||x^*|| = 1, x^*(x) = \ ||x||\}, \end{cases}$$

equivalently,

$$x \in X \Rightarrow J_\varphi\, x = \{x^* \in X^*\colon \|x^*\| = \varphi(\|x\|),\, x^*(x) = \varphi(\|x\|)\|x\|\},$$

is, by definition, *the duality map* (*on X*) *relative to* φ. (Beurling-Livingstone).

Its name becomes *normalized duality map* when $\varphi(t) = t$ (in this case, $x^* \in J_\varphi\, x \Longleftrightarrow x^*(x) = \|x\|^2 = \|x^*\|^2$).

Definition 2.3. *Let X be a real normed space. A nonempty set β of bounded subsets of X, having the properties:*

$$1^{\text{o}} \bigcup_{A \in \beta} A = X,\ 2^{\text{o}}\ A \in \beta \Rightarrow -A \in \beta, \lambda A \in \beta(\lambda > 0) \text{ and}$$

3° β is filtered to the right related to the inclusion "⊂", i.e.,

for any A, B in β ∃ C in β, e.g., $A \subset C$ and $B \subset C$,

is called bornology on X.

Let β be a bornology on X. The function $f : X \to \overline{\mathbf{R}}$, which is locally finite in the point a (i.e., there exists a neighborhood of a on which f is finite), is, by definition, β-*differentiable in a* if there exists φ in the topological dual X^* such that for every S in β, we have:

$$\lim_{\substack{t \to 0 \\ h \in S}} u \frac{f(a + th) - f(a)}{t} = \varphi(h) \text{ (uniform limit on } S \text{ for } t \to 0).$$

φ is the β-*derivative of* f *in* a, and it is denoted by

$$\nabla_\beta f\,(a).$$

If β is the set β_G of the finite symmetrical parts of X or the set β_F of the bounded symmetrical parts of X, the β-derivative coincides with the Gâteaux derivative and Fréchet derivative, respectively.

Proposition 2.2. 1° $J_\varphi\, x \neq \emptyset$ *∀x in X;*

2° *For any x in X,* $J_\varphi\, x$ *is a convex, closed, bounded part of* X^* (*it is contained in the sphere of the equation* $\|y\|_{X^*} = \varphi(\|x\|)$);

3° $J_\varphi\, x = \partial\psi\,(x)$, *the subdifferential in x, where*

$$\psi(u) = \int_0^{\|u\|} \varphi(t)dt$$

(*apply Asplund theorem; since ψ is continuous and convex, ψ is Gâteaux differentiable in x iff* $\partial\psi(x)$ *has a single element, and then* $\psi'(x) = \partial\psi(x)$*; consequently,* J_φ is *uni-valued iff ψ is Gâteaux differentiable, and in this case,* $J_\varphi\, x = \psi'(x)$);

4° J_φ *is odd* ($J_\varphi\,(-x) = -J_\varphi\, x$ *∀x in X);*

5° $J_\varphi(\lambda x) = \frac{\varphi(\lambda\|x\|)}{\varphi(\|x\|)} Jx$, *∀x in* $X \backslash \{0\}$, $\forall \lambda \geq 0$;

6° *J_φ is monotonous; more precisely*: $\forall x, y$ *in* X *and* $\forall x^*, y^*$ *from* $J_\varphi\ x, J_\varphi\ y$,

$$(x^* - y^*)(x - y) \geq [\varphi(\|x\|) - \varphi(\|y\|)](\|x\| - \|y\|) \geq 0;$$

7° *The normalized duality map is linear iff X is a Hilbert space;*
8° *If $x^* \in J_\varphi\ x$, then $x \in J_{\varphi^{-1}}(x^*)$ (duality map on X^*);*
9° J_φ, J_{φ_1} *being duality maps on* X, *there exists* μ: $[0, +\infty] \to [0, +\infty)$ *such that*

$$J\varphi\ x = \mu(\|\ x\ \|)J_{\varphi_1}\ \forall x \text{ in } X;$$

10° *Let x, y be in X. Then,* $\|x\| \leq \|x + \lambda y\|\ \forall\lambda \geq 0$ *iff there exists x^* in Jx, J the normalized duality map on X, with the property* $x^*(y) \geq 0$ (*Kato*).

Furthermore, retain that if J_φ is uni-valued, then it is coercive since:

$$\lim_{\|x\|\to+\infty} \frac{\langle J_\varphi x, x\rangle}{\|\ x\ \|} = \lim_{\|x\|\to+\infty} \varphi(\|\ x\ \|) = +\infty.$$

The Banach space X is, by definition, *smooth* (*Krein*) if

$$\forall x \neq 0 \text{ there exists } x_0^* \text{ unique in } X*, \text{ e.g., } \|\ x_0^*\| = 1,\ x_0^*(x)\ = \|\ x\ \|\ .$$

Thus, any duality map on a smooth space is uni-valued and reciprocal.
Since x_0^* from X^* is sub-gradient in $x_0 \neq 0$ for $x \to \|x\|$, iff $\|x_0^*\| = 1$ and $x_0^*(x) = \|x\|$ and $x \to \|x\|$ is convex and continuous.

Proposition 2.3. *X is smooth space iff the norm of X is Gâteaux differentiable on* $X\backslash\{0\}$.

From this, combined with Proposition 2.2, 3°, we obtain:

$$X \text{ smooth} \Rightarrow J_\varphi\ x = \varphi(\|x\|)\|x\|',\ x \neq 0.\ (*)$$

To validate this formula, we prove the following:

Proposition 2.4. *Let X and Y be real normed spaces and $f: X \to Y$ be Gâteaux differentiable, F: $Y \to$ **R** of Gâteaux C^1 class. Then, $g := F \circ f$ is Gâteaux differentiable, and $g'(x) = F'(f(x)) \circ f'(x)$ [43].*

Proof. We use the formula of finite increases [43], Volume 9, p. 93:
Let β be a bornology on X and $f: X \to$ **R** β-differentiable on the segment $[a, b]$ from X. There exists θ in (0,1) such that:

$$f(b) - f(a) = \nabla_\beta f(a + \theta(b - a))(b - a).$$

Standard justification. Take F: $[0,1] \to$ **R**, $F(t) = f(a + t(b - a))$. With δ being small,

$$\frac{F(t+\delta) - F(t)}{\delta} = \frac{1}{\delta}[f(a + t(b-a) + \delta(b-a)) - f(a + t(b-a))],$$

let A be in β with $b - a \in A$ (property 1° from the definition of the bornology), and take the limit for $\delta \to 0$, $F'(t) = \nabla_\beta f(a + t(b - a))(b - a)$, $F(1) - F(0) = F'(\theta)$, with θ in (0,1), etc. □

We continue the proof of this proposition. Let x_0 and h be in X.

$$\frac{1}{t}[g(x_0+th)-g(x_0)] = \frac{1}{t}\big[F\big(f(x_0+th)\big)-F\big(f(x_0)\big)\big] =$$
$$F'(f(x_0)+\theta(t)[f(x_0+th)-f(x_0)])\left(\frac{f(x_0+th)-f(x_0)}{t}\right),$$
$$\lim_{t\to 0}\frac{f(x_0+th)-f(x_0)}{t} = f'(x_0)(h), \lim_{t\to 0}[f(x_0+th)-f(x_0)] = 0,$$
$$\lim_{t\to 0}\theta(t)[f(x_0+th)-f(x_0)] = 0$$

and hence the conclusion, since:

$$x_n \to a \text{ and } y_n \to b \Rightarrow F'(y_n)(x_n) \to F'(b)(a):$$
$$F'(y_n)(x_n)-F'(b)(a) = [F'(y_n)(x_n)-F'(y_n)(a)]+[F'(y_n)(a)-F'(b)(a)],$$
$$\| F'(y_n)(x_n)-F'(y_n)(a) \| = \| F'(y_n)(x_n-a) \| \leq \| F'(y_n) \| \| x_n-a \| \leq \| x_n-a \|$$

(the sequence ($\| F'(y_n) \|$) is bounded, being convergent),

$$\| F'(y_n)(a)-F'(b)(a)\| = \| \langle F'(y_n)-F'(b), a\rangle \| \leq \| a \| \| F'(y_n)-F'(b)\|. \square$$

Finally, we introduce the following:

Definition 2.4. *F: $X \to 2^{X^*}$ is upper semicontinuous in x_0 if, for any neighborhood V of $F(x_0)$ in the $*$-weak topology on X^*, there exists U neighborhood of x_0, e.g., $F(U) \subset V$.*

Proposition 2.5. *Any duality map J_φ on X is upper semicontinuous on X (Browder).*

Definition 2.5. *A Banach space X is called strictly convex (Clarkson) if one of the following equivalent properties is fulfilled:*
1° $x \neq y$, $\|x\| = \|y\| = 1 \Rightarrow \|\lambda x + (1-\lambda)y\| < 1$ $\forall\lambda$ *in* $(0, 1)$;
2° $x \neq y$, $\|x\| = \|y\| = 1 \Rightarrow \|x + y\| < 2$;
3° $\|x + y\| = \|x\| + \|y\|$, $y \neq 0 \Rightarrow \exists\, \lambda \geq 0$ *with* $x = \lambda y$;
4° $\|x\| = \|y\| = 1$ *and* $\|x + y\| = \|x\| + \|y\| \Rightarrow x = y$;
5° *The sphere* $\{x \in X: \|x\| = 1\}$ *does not contain any segment;*
6° *Any x^* from X^* attains its inferior upper bound on the unity ball of X in at least one point;*
7° *The function* $x \to \|x\|^2$ *is strictly convex.*

Proposition 2.6. *If X^* is smooth (respectively strictly convex), then X is strictly convex (respectively smooth). The reciprocal assertions are true when X is reflexive.*

Proposition 2.7. *If the Banach space X is reflexive, there exists a norm on X equivalent strictly convex such that the dual norm is also strictly convex. (Lindenstrauss, Asplund).*

Definition 2.6. *A Banach space is uniformly convex (Clarkson) if*

$$\forall\varepsilon > 0\ \exists\delta > 0 \text{ such that } \|x\| = \|y\| = 1 \text{ and } \|x - y\| \geq \varepsilon \Rightarrow \|x + y\| \leq 2(1 - \delta),$$

equivalently

$$\| x_n \| = \| y_n \| = 1\ \forall n \text{in } \mathbf{N} \text{ and } \| x_n + y_n \| \to 2 \Rightarrow \lim_{n\to\infty}(x_n - y_n) = 0.$$

A Banach space is local uniformly convex (*Lovaglia*) *if*

$$\forall \varepsilon > 0 \text{ and } \forall x \text{ with } \|x\| = 1\ \exists \delta > 0 \text{ e.g., } \|y\| = 1 \text{ and } \|x - y\| \geq \varepsilon \Rightarrow \|x + y\| \leq 2(1 - \delta),$$

equivalently,

$$\| x \| = \| x_n \| = 1\ \forall n \text{in } \mathbf{N} \text{ and } \| x_n + x \| \to 2 \Rightarrow \lim_{n\to\infty} x_n = x.$$

Proposition 2.8. *X uniformly convex ⇒ locally uniformly convex ⇒ X is strictly convex.*

Proposition 2.9. *X uniformly convex ⇒ x is reflexive. (Milman).*

Proposition 2.10. *If X is local uniformly convex, then, for any sequence $(x_n)_{n\geq 1}$, $x_n \in X$,*

$$x_n \overset{w}{\to} x \text{ and } \| x_n \| \to \| x \| \Rightarrow x_n \to x.$$

Here is a characterization of the uniform convexity.

Proposition 2.11. *X and X* are uniformly convex iff the norm on X* and X, respectively, is uniformly Fréchet differentiable.*

Clarification. f is *uniformly Fréchet differentiable* $\overset{def}{\Leftrightarrow}$ $\forall \varepsilon > 0\ \exists \delta > 0$ with the property

$$\|h\| \leq \delta \Rightarrow \| f(x + h) - f(x) - f'(x)(h)\| \leq \varepsilon\|h\|\ \forall x \text{ with } \|x\| < 1;$$

$x \to \|x\|$ is uniformly Fréchet differentiable iff it is Fréchet differentiable and the Fréchet derivative is uniformly continuous on the unity ball.

Proposition 2.12. *If X is local uniformly convex and reflexive, then the norm on X* is Fréchet differentiable.*

Proposition 2.13. *For any reflexive Banach space, there exists on this an equivalent norm for which it becomes local uniformly convex. (Troianski).*

Proposition 2.14. *For any reflexive Banach space X, there exists an equivalent norm on X for which X and X* are locally uniformly convex. (Asplund).*

Thus, the equivalent norm and its dual norm are Fréchet differentiable (Proposition 2.12).

Proposition 2.15. *If X is smooth, for any weight φ, J_φ is uni-valued and continuous from X, with the strong topology on X* endowed with the ∗-weak topology.*

Proof. Take Proposition 2.5 into account. □

Corollary 2.3. *If X is smooth and reflexive, any duality map J on X is semicontinuous.*

Proof. Let $x_n \to x$. Setting $x_n^*: = x_n$, $x^* : = Jx$, we have $x_n^*(u) \to x^*(u)\ \forall u$ from X, i.e., denoting $u^{**} : = \Phi u$, Φ is the canonical embedding in bidual, $u^{**}(x_n^*) \to u^{**}(x^*)$, hence $Jx_n \overset{W}{\to} Jx$. □

The strict convexity can be characterized by the duality map.

Proposition 2.16. *X is strictly convex iff any duality map on X is strictly monotonous. (Petryshin).*

Consequently,

Proposition 2.17. *If X is strictly convex and smooth, then any duality map on X is uni-valued and injective.*

Proposition 2.18. *Let X be smooth and local uniformly convex, J duality map on X and* $(x_n)_{n\geq 1}$ *,* $x_n \in X$*. If* $(Jx_n - Jx)(x_n - x) \to 0$*, then* $x_n \to x$*.*

Proposition 2.19. *Let X be Banach space. A duality map on X is uni-valued and continuous in the topologies of the norms iff the norm on X is Fréchet differentiable.*

Proof. *Necessary*. X is smooth, and the norm is Gâteaux differentiable (Proposition 2.3), even of the C^1 Gâteaux class in accordance with $(*)$. *Sufficient*. This results from a Cudia theorem. □

Proposition 2.20. *Let X be a Banach space and* J_φ *a duality map on X.* X^* *is uniformly convex iff* J_φ *is uni-valued and uniformly continuous on the bounded sets of X.*

Proof. *Necessary*. X^* is uniformly convex $\overset{\text{Proposition 2.11}}{\Rightarrow}$ $x \to \|x\|$ is Fréchet differentiable $\Rightarrow$ $Jx = \varphi(\|x\|)\|x\|'$, $x \to \|x\|'$ is uniformly continuous on the closed unity ball. *Sufficient*. J uni-valued $\overset{\text{Proposition 2.3}}{\Rightarrow}$ $x \to \|x\|$ is Gâteaux differentiable $\overset{\text{Proposition 2.11}}{\Rightarrow}$ $x \to \|x\|$ is uniformly Fréchet differentiable $\overset{\text{Proposition 2.11}}{\Rightarrow}$ X^* is uniformly convex. □

Place here the characterization of the reflexivity of a Banach space.

Proposition 2.21. *The Banach space X is reflexive iff any* x^* *in* X^* *attains* $\underset{\|x\|\leq 1}{sup}\ x^*(x)$.

In the following, there is a characterization of the reflexivity of the duality map.

Proposition 2.22. *Let X be a Banach space and* J_φ *duality map on X. X is reflexive iff*

$$X^* = \bigcup_{x\in X} J_\varphi x.$$

Proof. *Necessary*. Let x_0^* be in X^*. There exists x_0 in X with $\|x_0\| = 1$ and $x_0^*(x_0) = \|x_0^*\|$; take $t_0 > 0$, e.g., $\varphi(t_0) = \|x_0^*\|$, then $x_0^* \in J_\varphi\ (t_0\ x_0)$. *Sufficient*. Use Proposition 2.21. Let x_0^* be any in X^*, $\exists\ x_0$ in X, e.g., $x_0^* \in J_\varphi\ x_0$, i.e., $\|x_0^*\| = \varphi(\|x_0\|)$, $x_0^*(x_0) = \varphi(\|x_0\|)\|x_0\|$. For y_0: $= \frac{x_0}{\|x_0\|}$, we have $\|y_0\| = 1$, $x_0^*(y_0) = \|x_0^*\| = \sup_{\|x\|\leq 1} x_0^*(x)$. □

Corollary 2.4. *Let X be a smooth space and J a duality map on X.*
$1°$ *X is reflexive* $\Longleftrightarrow$ *J is surjective;*
$2°$ *X is reflexive and strictly convex* $\Longleftrightarrow$ *J is bijective.*

Proof. Take Proposition 2.22 and Proposition 2.16 into account. □

Proposition 2.23. *Let X be a reflexive and smooth space and* J_φ *a duality map on X. Then* $J^{-1} : X^* \to 2^X$, $J^{-1}\ x^* = \{x \in X: J_\varphi\ x = x^*\}$ *coincides with* $J^*_{\varphi^{-1}}$*, the duality map on* X^* *coincides with the weight* φ^{-1} *(identification via canonical embedding* Φ *in the bidual). If X is also strictly convex, then* J^{-1} *is uni-valued, and the formula holds true:*

$$J_\varphi^{-1} = \Phi^{-1}\ J^*_{\varphi^{-1}}$$

Proof. The statement is, in accordance with Corollary 2.4, correct. *First assertion*. Let x^* be any in X^*. $x \in J^{-1}\, x^* \Longleftrightarrow x^* = J_\varphi\, x \Longleftrightarrow \|x^*\| = \varphi(\|x\|)$ and $x^*(x) = \varphi(\|x\|)\|x\| \Longleftrightarrow \|x\| = \varphi^{-1}(\|x^*\|)$ and $x(x^*) = \varphi^{-1}\,(\|x^*\|)\|x^*\| \Longleftrightarrow x \in J^*_{\varphi^{-1}}\, x^*$. *Second assertion*. X^* is smooth (Proposition 2.6). □

Pass to continuity properties of the duality map. A previous result is represented by Corollary 2.3.

Definition 2.7. *A Banach space has the property (h) if*

$$x_n \overset{w}{\to} x \text{ and } \| x_n \| \to \| x \| \Rightarrow x_n \to x.$$

A Banach space has the *property* (H) if it is strictly convex, and it has the property (h).
For instance, any local uniformly convex space has the property (H) (Propositions 2.8 and 2.9).

Proposition 2.24. *If X is reflexive and X* has the property (H), then any duality map J on X is uni-valued, surjective and continuous relative to the strong topologies on X and X*.*

Proof. J is indeed uni-valued (Proposition 2.6) and surjective (Corollary 2.4). Let $x_n \to x$. Then, $Jx_n \overset{w}{\to} Jx$ (Corollary 2.3); moreover, $\|Jx_n\| \to \|Jx\|$ since $\|Jx_n\| = \varphi(\|x_n\|)$ and $\|Jx\| = \varphi(\|x\|)$. Therefore, $Jx_n \to Jx$. □

Combining Proposition 2.24 with Corollary 2.4, one obtains the following.

Proposition 2.25. *If X is reflexive and strictly convex and X* has the property (H), then any duality map on X is uni-valued, bijective and continuous relative to the strong topologies on X and X*.*

We next proceed to the continuity of the inverse of the duality map.

Proposition 2.26. *If X is reflexive, smooth and has the (H) property, then any duality map J_φ on X is bijective with J_φ^{-1} continuous relative to the strong topologies on X and X^*.*

Proof. J_φ is bijective (Corollary 2.4), and X^* is reflexive, smooth and strictly convex (Proposition 2.6). Let $x_n^* \to x_0$. Then, $J^*_{\varphi^{-1}} x_n^* \overset{w}{\to} J^*_{\varphi^{-1}} x^*$ (Corollary 2.3), and the formula from Proposition 2.23 imposes $J^*_{\varphi^{-1}} x_n^* \to J^*_{\varphi^{-1}} x^*$ (see (H)). □

Combining Propositions 2.25 and 2.26, one obtains the following.

Proposition 2.27. *If X is reflexive and X and X* have the property (H), then any duality map on X is a homeomorphism of X in X* relative to the strong topologies.*

We finish this subsection with the following result:

Proposition 2.28. *Any uni-valued duality map J_φ on a local uniformly convex space X has the property S_+:*

$$x_n \overset{w}{\to} x \text{ and } \overline{\lim_{n\to\infty}} \langle J_\varphi x_n - J_\varphi x_0, x_n - x_0 \rangle \le 0 \Rightarrow x_n \to x.$$

Proof. The hypothesis implies

$$\overline{\lim_{n\to\infty}} \langle J_\varphi x_n - J_\varphi x_0, x_n - x_0 \rangle \le 0,$$

but

$$0 \leq [\varphi(\| x_n \|) - \varphi(\| x_0 \|)](\| x_n \| - \| x_0 \| \leq \langle J_\varphi x_n - J_\varphi x_0, x_n - x_0 \rangle,$$

consequently,

$$\lim_{n\to\infty}[\varphi(\| x_n \|) - \varphi(\| x_0 \|)](\| x_n \| - \| x_0 \|) = 0. \tag{2.8}$$

We denote t_n: = $\| x_n \|$, t_0 : = $\| x_0 \|$. Somehow,

$$\lim_{n\to\infty} t_n = t_0, \tag{2.9}$$

we apply Proposition 2.11 and obtain $x_n \to x$. Assume, ad absurdum, that $t_n \nrightarrow t_0$. Then there exists (t_{kn}) subsequence of (t_n) e.g., for instance, $t_{kn} \geq \rho > t_0\ \forall n$. (t_{kn}) is bounded, otherwise it has a subsequence (t_{ik_n}), $t_{ik_n} \to +\infty$, which implies $\varphi(t_{ik_n}) \to +\infty$, $[\varphi(t_{ik_n}) - \varphi(t_0)](t_{ik_n} - t_0) \to +\infty$, in contradiction with (2.8). Thus, (t_{kn}) has a convergent subsequence (t_{lk_n}), $t_{lk_n} \to t_0' \neq t_0$. Consequently, since $\varphi(t_{lk_n}) \to \varphi(t_0') \neq \varphi(t_0)$, from a rank on, we have, with $\delta > 0$,

$$|t_{lk_n} - t_0| \ \geq\ \delta,\ |\varphi(t_{lk_n}) - \varphi(t_0)| \ \geq\ \delta$$

and one obtains a final contradiction with (2.8). Here is another justification for (2.9). (t_n) is bounded (see above); let t^* be an adherence value any for this sequence (a fortiori $t^* \in$ **R**) and (t_{j_n}) subsequence with $\lim_{n\to\infty} t_{j_n} = t^*$. Then, from (2.8), $[\varphi(t^*) - \varphi(t_0)](t^* - t_0)$, which implies $t^* = t_0$ (ad absurdum !) and, consequently, (2.9). □

2.2.2. Main Results

Proposition 2.29. *Let X be a real reflexive Banach space, smooth and having the property (H), J_φ duality map on X with φ being (K, L, a) function, S: $X \to X^*$ odd compact operator and*

$$\alpha\ := \overline{\lim_{||x||\to+\infty}} \frac{\|Sx\|}{\|x\|^a} < +\infty.$$

Then

1° $\alpha > 0 \Rightarrow \lambda J_\varphi - S$ *is surjective* $\forall\lambda$ *with* $|\lambda| > \frac{\alpha}{K}$;
2° $\alpha = 0 \Rightarrow \lambda J_\varphi - S$ *is surjective* $\forall\lambda \neq 0$.

Proof. J_φ is odd and bijective with a continuous inverse (Proposition 2.26). Moreover, since

$$Kt^a \leq \varphi(t) \leq Lt^a\ \forall t \geq 0,$$

we have

$$K\|x\|^a \leq \varphi(\|x\|) = \|J_\varphi x\| \leq L\|x\|^a\ \forall x \text{ from } X,$$

i.e., J_φ is (*K, L, a*). We apply Corollaries 2.1 and 2.2 from Section 2.1. □

Proposition 2.30. *Let X be a real reflexive Banach space, smooth with the property (H), and J_φ the duality map on X with $\varphi(t) = t^{p-1}$, $p \in (1, +\infty)$. Suppose that X is compactly embedded by the linear injection i in a Banach space Z,*

$$\|i(u)\| \leq c_0\ \|u\|\ \forall u \text{ from } X \tag{2.10}$$

and N: $Z \to Z^*$ is an odd semicontinuous operator with the property

$$\|Nx\| \leq c_1\|x\|^{q-1} + c_2\ \forall x \text{ from } Z,\ c_1, c_2 \geq 0,\ q \in (1, p). \tag{2.11}$$

Then $\lambda J_\varphi - N$ is surjective for any $\lambda \neq 0$.

Explanation. N is the short notation for the operator, which acts from X to X^*, i' o N o i, *i' being the adjoint of i.*

Proof. It follows to state that

$$\forall h \text{ from } X^* \ \exists u \text{ in } X, \text{ e.g., } \lambda J_\varphi \ u - (i' \circ N \circ i)\, u = h. \tag{2.12}$$

Apply Proposition 2.1 with $T = J_\varphi$ (correctly, as φ is (K, L, a) with $K = L = 1$, $a = p - 1$), $S = i' \circ N \circ i$. S is obviously odd and also compact: let $(x_n)_{n\in\mathbf{N}}$, $x_n \in X$, be a bounded sequence $(i(x_n))_{n\geq 1}$ has a convergent subsequence $(x_{k_n})_{n\geq 1}$; let $i(x_{k_n}) \to \gamma$, $\gamma \in Z$, then $N(i(x_{k_n})) \xrightarrow{w} N(\gamma)$ and, consequently, $i'(N(i(x_{k_n}))) \to i'(N(\gamma))$ since i' is also compact (Schauder theorem). So, to obtain the conclusion, it remains to prove

$$\overline{\lim_{\|u\|\to+\infty}} \frac{\|(i\prime \circ N \circ i)u\|}{\|u\|^{p-1}} = 0. \tag{2.13}$$

$\|i'(N(i(u)))\| \leq \|i'\| \ \|N(i(u))\| \overset{(2.10),(2.11)}{\leq} c_0\,(c_1\,\|i(u)\|^{q-1} + c_2) \leq c_0\,(c_0^{q-1}\,c_1\,\|u\|^{q-1} + c_2)$, from which it results (2.13) ($\|i'\| \leq \|i\| \leq c_0$ has been used) [11,12,15]. □

In the following, we search for the surjectivity of the operator $\lambda J_\varphi - N$, when N verifies the growth condition (2.11), where $q = p$, i.e.,

$$\|Nx\| \leq c_1\,\|x\|^{p-1} + c_2 \ \forall x \text{ from } Z,\ c_1, c_2 \geq 0. \tag{2.14}$$

For this reason, we present the statement:

Proposition 2.31. *Let X be a real reflexive Banach space compactly embedded by the linear injection i in the Banach space Z,*

$$\|\ i(u)\ \| \leq\ c_0\ \|\ u\ \|\ \ \forall u \text{ from } X. \tag{2.15}$$

If

$$\lambda_1 := \inf\left\{\frac{\|u\|^p}{\|i(u)\|^p} : u \in X\backslash\{0\}\right\},\ p\ \in\ (1,+\infty),$$

then

1° λ_1 *is attained and nonzero;*

2° $\lambda_1^{-\frac{1}{p}}$ *is optimal for (2.15) (i.e.,* $\lambda_1^{-\frac{1}{p}} \leq c_0$ *for any* c_0*);*

3° *If X and Z are smooth and* $J_{XX^*} : X \to X^*$, $J_{ZZ^*} : Z \to Z^*$ *are duality maps relative to the same weight* φ: $\varphi(t) = t^{p-1}$, *then* λ_1 *is the smallest eigenvalue of the couple* (J_{XX^*}, J_{ZZ^*}) *[44].*

Clarification. λ is, by definition, *eigenvalue* for the couple (J_{XX^*}, J_{ZZ^*}) if there exists $u_0 \neq 0$ in Z, e.g., $\lambda\,(i' \circ J_{XX^*} \circ i)\,u_0 = J_{ZZ^*}\,u_0$. In this case, u_0 is, by definition, an *eigenvector*.

Proof. The set from the statement is correctly defined: $u \neq 0 \Rightarrow i(u) \neq 0$.

1° We have

$$\lambda_1 = \inf\{\|v\|^p : v \in X,\ \|i(v)\| = 1\}$$

(the two sets coincide, as $\left\| i\left(\frac{u}{i(u)}\right)\right\| = 1$). Let $(v_n)_{n\geq 1}$, $v_n \in X$, be with $\|i(v_n)\| = 1$ and $\|v_n\| \to \lambda_1^{\frac{1}{p}}$.

X being reflexive, $(v_n)_{n\geq 1}$ has a subsequence (we use the same notation for it) that is weakly convergent in X, $v_n \xrightarrow{w} v$ (Kakutani theorem [43], Volume 3, p. 155). Then,

$$\|\ v\ \| \leq \underline{\lim_{n\to\infty}}\ \|\ v_n\ \|,$$

$$\|v\|^p \leq \lambda_1. \tag{2.16}$$

On the other hand, since i is compact, we have $i(v_n) \to i(v)$, which implies $\|i(v_n)\| \to \|i(v)\|$, $\|i(v)\| = 1$ and hence $\|v\|^p \geq \lambda_1$, $\|v\|^p \overset{(2.16)}{=} \lambda_1$, and λ_1 is attained and, a fortiori, nonzero.

2° We take the definitions of λ_1 and 1^0 into account.

3° Firstly, we show that λ_1 is an eigenvalue for the couple (J_{ZZ^*}, J_{XX^*}), i.e., $\exists u_0 \neq 0$ in X e.g.,

$$\lambda_1 \, (i' \circ J_{ZZ^*} \circ i)u_0 = J_{XX^*}\, u_0. \tag{2.17}$$

Taking the functional Φ: $X \to \mathbf{R}$,

$$\Phi(u) = \frac{1}{p}\| \; u \; \|^p - \frac{\lambda_1}{p} \| \; i(u) \; \|^p \;.$$

$\Phi(u) \geq 0 \; \forall u$ in X (see the definition of λ_1) and, for $u_0 \neq 0$ e.g., $\lambda_1 \overset{1^0}{=} \left(\frac{\|u_0\|}{\|i(u_0)\|}\right)^p$, $\Phi(u_0) = 0$, which imposes (taking into account Proposition 2.3)

$$\Phi'(u_0) = 0 \text{ (Gâteaux derivative)}. \tag{2.18}$$

Then (the formulae: $(*)$ and that from Proposition 2.4), $\forall u$ in X,

$$0 \overset{(2.18)}{=} \Phi'(u_0)(u) = \left\langle \|u_0\|^{p-1} \| \cdot \|' (u_0), u \right\rangle - \lambda_1 \left\langle \|i(u_0)\|^{p-1} \| \cdot \|' (i(u_0)), i(u) \right\rangle = (J_{XX^*} u_0)(u) - \lambda_1 (J_{ZZ^*}(i(u_0))(i(u))) =$$
$$\left\langle J_{XX^*} u_0 - \lambda_1 (i' \circ J_{ZZ^*} \circ i)(u_0), u \right\rangle, \text{ i.e. (2.17)}.$$

Let λ now be an eigenvalue for the couple (J_{ZZ^*}, J_{XX^*}) and u be a corresponding eigenvector. Then

$$\| \; u \; \|^p = (J_{XX^*} \; u)(u) = \lambda \langle J_{ZZ^*}(i(u)), i(u) \rangle = \lambda \; \| \; i(u) \; \|^p,$$

hence

$$\lambda = \frac{\|u\|^p}{\|i(u)\|^p} \geq \; \lambda_1. \; \square$$

We can now state the following.

Proposition 2.32. *Let X be a real reflexive Banach space, smooth and have the property (H),* J_φ *duality map on X with* $\varphi(t) = t^{\,p-1}$, $p \in (1, +\infty)$. *Suppose that X is compactly embedded with the linear injection i in the Banach space Z and let* N: $Z \to Z^*$ *be an odd semicontinuous operator with:*

$$\|Nx\| \leq c_1 \; \|x\|^{p-1} + c_2 \; \forall x \text{ from } Z, c_1, c_2 \geq 0.$$

Then, for any λ*, if*

$$|\lambda| \; > \overline{\lim_{\|u\| \to +\infty}} \frac{||(i' \circ N \circ i)u||}{||u||^{p-1}}, \text{ a fortiori if } |\lambda| \; > c_1 \lambda_1^{-1},$$

where

$$\lambda_1 := \inf\left\{ \frac{\|u\|^p}{\|i(u)\|^p} : u \in X \backslash \{0\} \right\},$$

then $\lambda J_\varphi - N$ *is surjective (N means* $i' \circ N \circ i$*).*

Proof. The statement is correct, $\lambda_1 \neq 0$ (Proposition 2.31, 1^0). We apply, as for Proposition 2.30, Proposition 2.29 with $T = J_\varphi$, $S = i' \circ N \circ i$. We prove

$$\overline{\lim_{\|u\| \to +\infty}} \frac{||(i' \circ N \circ i)u||}{||u||^{p-1}} \leq c_1 \lambda_1^{-1} \tag{2.19}$$

using 2° from Proposition 2.31, which will be sufficient to impose the conclusion. $\|(i\,' \circ N \circ i)u\| \leq \|i\,'\| \, \|N(i(u))\| \leq \lambda_1^{-\frac{1}{p}} \, (c_1\lambda_1^{\frac{1-p}{p}} \, ||u||^{p-1} + c_2) \, (||i\prime|| \leq ||i|| \leq \lambda_1^{-\frac{1}{p}})$, and (2.19) becomes obvious. □

Remark 2.1. *Propositions 29, 30 and 32 have been briefly presented by the author in [11,12,15].*

2.3. Existence of the Solutions of the Problems

Consider the problems

$$(*) \begin{cases} -\lambda \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(\,\cdot\,, u(\,\cdot\,)) + h, \; x \in \Omega \\ u|\partial\Omega = 0 \end{cases}, \; p \in (1, +\infty)$$

and

$$(**) \begin{cases} -\lambda \sum\limits_{\mathrm{i}=1}^{\mathrm{n}} \frac{\partial}{\partial x_{\mathrm{i}}} \left(\left| \frac{\partial u}{\partial x_{\mathrm{i}}} \right|^{\mathrm{p}-2} \frac{\partial u}{\partial x_{\mathrm{i}}} \right) = f(\,\cdot\,, u(\,\cdot\,)) + h, \; x \in \Omega \\ u|\partial\Omega = 0 \end{cases}, \; p \in [2, +\infty)$$

In this subsection, we apply (the idea originates in [45]) the results from Section 2.2 to partial differential equations (weak solutions).

2.3.1. Preliminaries for Sobolev Spaces

To prepare the framework for this subsection and to be coherent and understandable, we start with a theoretical recapitulation for Sobolev spaces.

The spaces $L^p(\Omega)$ (Lebesgue integral in $\mathbf{R}^{\mathrm{N}}$).

Let Ω be a nonempty open set in $\mathbf{R}^{\mathrm{N}}$.

$$p \in [1, +\infty) \Rightarrow L^p(\Omega) := \{u : \Omega \to \mathbf{R} : u \text{measurable}, \; |u|_p \text{ Lebesgue integrable}\},$$

$$\| u \|_{L^p(\Omega)} := \left(\int\limits_{\Omega} |u|^{\mathrm{p}} dx \right)^{\frac{1}{\mathrm{p}}}.$$

This norm is Gâteaux differentiable on $L^p(\Omega)\backslash\{0\}$ for $p > 1$. It is even of Fréchet C^1 class ([43,46]).

$$p = +\infty \Rightarrow L^\infty(\Omega) := \{u : \Omega \to \mathbf{R} : u \text{ measurable}, \; \exists\, c > 0 \text{ e.g. } |u(x)| \leq c \text{ on } \Omega \text{ a.e.}\},$$

$$\|u\|_{L^\infty(\Omega)} := \inf c.$$

If $\mu(\Omega) < +\infty$, then $\| u \|_{L^p(\Omega)} \leq [\mu(\Omega)]^{\frac{1}{p}-\frac{1}{q}} \| u \|_{L^q(\Omega)}$, $1 \leq p \leq q \leq +\infty$.

$$p \in [1, +\infty] \Rightarrow \| u \|_{0,p} := \| u \|_{L^p(\Omega)}.$$

$L^p(\Omega)$, modulo the known factorization, is a Banach space for $p \in [1, +\infty]$, $L^p(\Omega)$ is uniformly convex and hence reflexive (Proposition 2.9) for $p \in (1, +\infty)$, $L^1(\Omega)$ and $L^\infty(\Omega)$ are not reflexive, $L^p(\Omega)$ is separable for $p \in [1, +\infty)$, and $L^\infty(\Omega)$ is not separable.

For p in $[1, +\infty]$, p' is defined by:

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Then,

$$p \in (1, +\infty) \Rightarrow (L^p(\Omega))* = L^{p'}(\Omega), \; (L^1(\Omega))* = L^\infty(\Omega), \; (L^\infty(\Omega))* \underset{\neq}{\supset} L^1(\Omega).$$

$L^1_{\mathrm{loc}}(\Omega) := \{u \colon \Omega \to \mathbf{R} \colon u$ integrable on any compact part of $\Omega\}$.

At the end of this part of the exposure, we prove ([43], Ω is a Lebesgue measurable set):

Proposition 2.33. *When $p \in (1, +\infty)$, $u \to \| u \|_{L^p(\Omega)}$ is of Fréchet C^1 class on $L^p(\Omega) \backslash \{0\}$.*

Proof. For the following calculus, we use the inequalities [47]:

$$\xi, \zeta \in \mathbf{R}^N, p \in (1, +\infty) \Rightarrow \| \|\xi\|^{p-2}\xi - \|\zeta\|^{p-2}\zeta\| \le \begin{cases} c \| \xi - \zeta \| (\| \xi \| + \| \zeta \|)^{p-2}, p > 2 \\ c \| \xi - \zeta \|^{p-1}, p \in (1,2] \end{cases},$$

$$c \text{ independent by } \xi, \zeta. \tag{2.20}$$

Let

$$\Phi(u) := \| u \|_{0,p},\ \Psi(u) := \frac{1}{p}||u||^p_{0,p},$$

hence,

$$\Phi(u) = p^{\frac{1}{p}}[\Psi(u)]^{\frac{1}{p}}. \tag{2.21}$$

Ψ is Gâteaux differentiable on $L^p(\Omega)$ and [48]

$$\Psi\prime(u)(h) = \int_\Omega |u|^{p-1}(\mathrm{sgn} u)h\, dx,\ \forall h \text{ in } L^p(\Omega)$$

We prove that $u \to \Psi'(u)$ is continuous on $L^p(\Omega)$ and then (2.21) will impose the conclusion, taking into account Proposition 2.4.

Let

$$u_n \ \to\ u_0 \text{ in } L^p(\Omega). \tag{2.22}$$

It follows to prove

$$\lim_{n\to\infty} \Psi'(u_n) = \Psi'(u_0) \text{ in } (L^p(\Omega))^*, \text{ i.e.}$$

$$\lim_{n\to\infty} \sup_{\|h\|_{0,p}=1} |\langle \Psi'(u_n) - \Psi'(u_0), h\rangle| = 0. \tag{2.23}$$

$\|h\|_{0,p} = 1 \Rightarrow |\langle \Psi'(u_n) - \Psi'(u_0), h\rangle| = |\int_\Omega (|u_n|^{p-1}\mathrm{sgn} u_n - |u_0|^{p-1}\mathrm{sgn} u_0)h\, dx| \le \int_\Omega |(|u_n|^{p-1}\mathrm{sgn} u_n - |u_0|^{p-1}\mathrm{sgn} u_0)h|\, dx \overset{\text{Hölder}}{\le} A_n,\ A_n := \| |u_n|^{p-1} \mathrm{sgn}\ u_n - |u_0|^{p-1} \mathrm{sgn}\ u_0||_{L^{p'}(\Omega)}$, since $||h||_{L^p(\Omega)} = 1$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

However, $| \ |u_n|^{p-1} \mathrm{sgn}\ u_n - |u_0|^{p-1} \mathrm{sgn}\ u_0 \ |^{p'} = | \ |u_n|^{p-2} u_n - |u_0|^{p-2} u_0 \ |^{p'} \overset{(2.20)}{\le}$

$$\begin{cases} c_0|u_n - u_0|^{p'}(|u_n| + |u_0|)^{p'(p-2)}, p > 2 \\ c_0|u_n - u_0|^{p}, p \in (1,2] \end{cases},$$

hence $A_n^{p'} = \int_\Omega | \, |u_n|^{p-1}\mathrm{sgn} u_n - |u_0|^{p-1}\mathrm{sgn} u_0|^{p'}\, dx \le$

$$\begin{cases} c_0 \int_\Omega |u_n - u_0|^{p'}(|u_n| + |u_0|)^{p'(p-2)} dx, p > 2 \\ c_0 \int_\Omega |u_n - u_0|^{p} dx = c_0||u_n - u_0||^p_{0,p}, p \in (1,2] \end{cases},$$

therefore, when $p \in (1, 2]$, $\lim_{n\to\infty} A_n^{p'} = 0$ (see (2.22)), i.e., $\lim_{n\to\infty} A_n = 0$ and hence (2.23), and when $p > 2$,

$$\lim_{n\to\infty} A_n^{p'} = 0, \tag{2.24}$$

i.e., $\lim\limits_{n\to\infty} A_n = 0$ and hence (2.23). (2.24) is proved as follows:

$$\int\limits_{\Omega} |u_n - u_0|^{p'}(|u_n| + |u_0|)^{p'(p-2)}dx \overset{\text{Hölder}}{\leq} \| |u_n - u_0|^{p'} \|_{0,\frac{p}{p'}} \| (|u_n| + |u_0|)^{p'(p-2)} \|_{0,\frac{p}{p'(p-2)}}$$

$= (||u_n - u_0||_{0,p})^{p'} \| \,|u_n| + |u_0|\, \|_{0,p}^{p'(p-2)}$, $\lim\limits_{n\to\infty} \|u_n - u_0\|_{0,p}^{p'} \overset{(2.22)}{=} 0$, and the second factor is bounded, since $\| \,|u_n| + |u_0|\, \|_{0,p} \leq \|u_n\|_{0,p} + \|u_0\|_{0,p}$ and $\|u_n\|_{0,p} \overset{(2.22)}{\to} \|u_0\|_{0,p}$. □

Corollary 2.5. *When $p \in (1, +\infty)$, any duality map on $L^p(\Omega)$ is a homeomorphism of $L^p(\Omega)$ on $L^{p'}(\Omega)$.*

Proof. $L^p(\Omega)$ is smooth (Proposition 2.3) and uniformly convex, hence local uniformly convex (Proposition 2.8) and, in particular, has the (H) property, even $L^{p'}(\Omega)$ has the same properties (by applying Proposition 2.27). □

Let Ω be a nonempty open set from $\mathbf{R}^N$ and p in $[1, +\infty]$.

$W^{1,p}(\Omega)$ designates the real vector space of the functions u from $L^p(\Omega)$ for which there exists $g_1, \ldots, g_N$ in $L^p(\Omega)$ e.g.,

$$\forall \varphi \text{ in } C_c^\infty(\Omega) \int_\Omega u\frac{\partial\varphi}{\partial x_i}dx = -\int_\Omega g_i\varphi dx,\ i = \overline{1, N}.$$

In the definition, $C_c^\infty(\Omega)$ can be replaced by $C_c^1(\Omega)$. We denote, for each i, $1 \leq i \leq N$,

$$\frac{\partial u}{\partial x_i} := g_i, \textit{ the weak derivative.}$$

These are uniquely determined.

Remark 2.2. *By weak differentiation, one remains in $L^p(\Omega)$.*

Additionally, for u from $W^{1,p}(\Omega)$,

$$\nabla u = \operatorname{grad} u := \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right), \text{the weak gradient}$$

$$|\nabla u| := \left[\sum_{i=1}^{N}\left(\frac{\partial u}{\partial x_i}\right)^2\right]^{\frac{1}{2}},$$

$$\operatorname{div} u := \sum_{i=1}^{N}\frac{\partial u}{\partial x_i}, \text{the weak divergence.}$$

$$|\nabla u| \in L^p(\Omega) : |\nabla u| \leq g := \sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right| \in L^p(\Omega), |\nabla u|^p \leq g^p.$$

Moreover, for $p \in (1, +\infty)$ and p' the conjugated coefficient,

$$|\nabla u|^{p-2}\frac{\partial u}{\partial x_i} \in L^{p'}(\Omega), i = \overline{1, N},$$

$$\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) \in L^{p'}(\Omega), i = \overline{1, N}.$$

To justify the above two relations, for

$$p \neq 2, \left(|\nabla u|^{p-2} \left| \frac{\partial u}{\partial x_i} \right| \right)^{p'} = |\nabla u|^{p'(p-2)} \left| \frac{\partial u}{\partial x_i} \right|^{p'}, |\nabla u|^{p'(p-2)} \in L^{\frac{p}{p'(p-2)}}(\Omega),$$
$$\left| \frac{\partial u}{\partial x_i} \right|^{p'} \in L^{\frac{p}{p'}}(\Omega), \frac{p'(p-2)}{p} + \frac{p'}{p} = 1,$$

and for $p = 2$—obviously since $p' = 2$.

We define:

$$\| u \|_{W^{1,p}(\Omega)} := \| u \|_{L^p(\Omega)} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)},$$

a norm on $W^{1,p}(\Omega)$.

Sometimes, when $p \in [1, +\infty)$, one takes the equivalent norm:

$$\left(\| u \|^p_{L^p(\Omega)} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$

$W^{1,p}(\Omega)$ is a Banach space for $p \in [1, +\infty]$, it is uniformly convex and reflexive for $p \in (1, +\infty)$ and separable for $p \in [1, +\infty)$. When $p > N$, any function from $W^{1,p}(\Omega)$ is Fréchet differentiable on Ω a.e. ([49], Chapter VIII).

We now provide some clarifications related to the weak derivative, $p \in [1, +\infty]$. If $u \in C^1(\Omega) \cap L^p(\Omega)$ and $\frac{\partial u}{\partial x_i} \in L^p(\Omega)$, $i = \overline{1, N}$ (derivatives in the usual meaning), then $u \in W^{1,p}(\Omega)$ and the weak derivatives coincide with them in the usual sense. In particular, if Ω is bounded, then $C^1(\overline{\Omega}) \subset W^{1,p}(\Omega)$. Reciprocally, if $u \in W^{1,p}(\Omega) \cap C(\Omega)$ and $\frac{\partial u}{\partial x_i} \in C(\Omega)$, $i = \overline{1, N}$ (weak derivatives), then $u \in C^1(\Omega)$.

Let p be in $[1, +\infty)$.

$$W_0^{1,p}(\Omega) := \overline{C_c^1(\Omega)}^{W^{1,p}(\Omega)} = \overline{C_c(\Omega)}^{W^{1,p}(\Omega)}.$$

$W_0^{1,p}(\Omega)$ with the norm induced is a separable Banach space. It is reflexive if $p \in (1, +\infty)$.

Since $C_c^1(\mathbf{R}^N)$ is dense in $W^{1,p}(\mathbf{R}^N)$, $W_0^{1,p}(\mathbf{R}^N) = W^{1,p}(\mathbf{R}^N)$. However, when $\Omega \neq \mathbf{R}^N$, in general, $W_0^{1,p}(\Omega) \neq W^{1,p}(\Omega)$.

Proposition 2.34. *For any p in $[1, +\infty)$,*

$$W^{1,p}(\Omega) \cap C_c(\Omega) \subset W_0^{1,p}(\Omega).$$

The following considerations strongly imply $W_0^{1,p}(\Omega)$ in the theory of partial differential equations.

Definition 2.8. *We define, by local charts, $W^{1,p}(\Gamma)$ with $p \in [1, +\infty)$, Γ regular manifold, for instance $\Gamma = \partial\Omega$, Ω open set of C^1 class with $\partial\Omega$ bounded. In this situation there exists a unique continuous linear operator γ: $W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega)$, the trace, such that γ is surjective and*

$$u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \Rightarrow \gamma(u) = u \mid \partial\Omega.$$

By the way,

Proposition 2.35. *Let Ω be of C^1 class and u in $W^{1,p}(\Omega) \cap C(\overline{\Omega})$, $p \in [1, +\infty)$. Then*

$$u \in W_0^{1,p}(\Omega) \iff u \mid \partial\Omega = 0.$$

Here is another characterization of the spaces $W_0^{1,p}(\Omega)$.

Proposition 2.36. *Let* Ω *be of* C^1 *class and* u *in* $L^p(\Omega)$, $p \in (1,+\infty)$. *Then*

$$u \in W_0^{1,p}(\Omega) \Longleftrightarrow \exists c > 0 \text{ such that } \left|\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx\right| \le c \| \varphi \|_{L^{p'}(\Omega)}, i = \overline{1,N}, \forall \varphi \text{ in } C_c^1(\Omega).$$

Suppose $\mu(\Omega) < +\infty$. Then $u \to \||\nabla u|\|_{L^p(\Omega)}$ is a norm on $W_0^{1,p}(\Omega)$ *equivalent* (and hence also *complete*) to $u \to u_{W^{1,p}(\Omega)}$.

We denote:

$u \in W_0^{1,p}(\Omega) \Rightarrow \|u\|_{1,p:} = \||\nabla u|\|_{L^p(\Omega)}$ $(= \||\nabla u|\|_{0,p}$, when confusion cannot appear).

This norm, when $p \in (1,+\infty)$, is Gâteaux differentiable on $W_0^{1,p}(\Omega)\backslash\{0\}$ (combine Proposition 2.15 and Proposition 2.4), which assures uni-valued duality maps (Proposition 2.3).

Proposition 2.37. $(W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$, $p \in (1,+\infty)$, *is uniformly convex and hence particularly reflexive.*

Proof ([50]). *The case* $p \in [2,+\infty)$. We use the following inequality: ([51], Euclidean norm)

$$\xi, \zeta \in \mathbf{R}^n, n \ge 1 \Rightarrow \left\|\frac{\xi+\eta}{2}\right\|^p + \left\|\frac{\xi-\eta}{2}\right\|^p \le \frac{1}{2}(\|\xi\|^p + \|\eta\|^p) \tag{2.25}$$

Let ε be from $(0,2]$ and u, v from $W_0^{1,p}(\Omega)$ with

$$\|u\|_{1,p} = \|v\|_{1,p} = 1, \|u-v\|_{1,p} \ge \varepsilon. \tag{2.26}$$

Then

$$\left\|\frac{u+v}{2}\right\|_{1,p}^p + \left\|\frac{u-v}{2}\right\|_{1,p}^p = \int_\Omega \left(\left|\frac{\nabla u + \nabla v}{2}\right|^p + \left|\frac{\nabla u - \nabla v}{2}\right|^p\right) dx \overset{(2.25)}{\le}$$

$$\frac{1}{2}\int_\Omega (|\nabla u|^p + |\nabla v|^p) dx = \frac{1}{2}(\|u\|_{1,p}^p + \|v\|_{1,p}^p) \overset{(2.26)}{=} 1, \left\|\frac{u+v}{2}\right\|_{1,p}^p \overset{(2.26)}{\le} 1 - \left(\frac{\varepsilon}{2}\right)^p.$$

We take $\delta > 0$ defined by

$$\left[1 - \left(\frac{\varepsilon}{2}\right)^p\right]^{\frac{1}{p}} = 1 - \delta.$$

The case $p \in (1,2)$. We use

$$\xi, \zeta \in \mathbf{R}^n, n \ge 1 \Rightarrow \left\|\frac{\xi+\eta}{2}\right\|^{p'} + \left\|\frac{\xi-\eta}{2}\right\|^{p'} \le \left[\frac{1}{2}(\|\xi\|^p + \|\eta\|^p)\right]^{\frac{1}{p-1}}, \tag{2.27}$$

p' the coefficient conjugated with p ([51]).

We remark that, for u in $W_0^{1,p}(\Omega)$,

$$|\nabla u|^{p'} \in L^{p-1}(\Omega), \|u\|_{1,p}^{p'} = \| |\nabla u|^{p'} \|_{0,p-1}. \tag{2.28}$$

Let ε be in $(0,2]$ and u, v in $W_0^{1,p}(\Omega)$. Then $|\nabla u|^{p'}, |\nabla v|^{p'} \overset{(2.27)}{\in} L^{p-1}(\Omega)$ and as

$$\| |\nabla u|^{p'} \|_{0,p-1} + \| |\nabla v|^{p'} \|_{0,p-1} \le \| |\nabla u|^{p'} + |\nabla v|^{p'} \|_{0,p-1}. \tag{2.29}$$

Since $0 < p - 1 < 1$, it results:

$$\left\|\frac{u+v}{2}\right\|_{1,\mathrm{p}}^{\mathrm{p}'} + \left\|\frac{u-v}{2}\right\|_{1,\mathrm{p}}^{\mathrm{p}'} \overset{(2.28),(2.29),(2.27)}{\leq} \left[\frac{1}{2}(\|u\|_{1,\mathrm{p}}^{\mathrm{p}} + \|v\|_{1,\mathrm{p}}^{\mathrm{p}})\right]^{\frac{1}{\mathrm{p}-1}},$$

consequently, if $\|u\|_{1,\mathrm{p}} = \|v\|_{1,\mathrm{p}} = 1$ and $\|u - v\|_{1,\mathrm{p}} \geq \varepsilon$, one obtains

$$\left\|\frac{u+v}{2}\right\|_{1,\mathrm{p}}^{\mathrm{p}'} \leq 1 - \left(\frac{\varepsilon}{2}\right)^{\mathrm{p}'}$$

and hence the conclusion. □

The dual of $W_0^{1,\mathrm{p}}(\Omega)$, p in $[1, +\infty)$, is denoted

$$W^{-1,\mathrm{p}'}(\Omega),$$

p' the coefficient conjugated to p.

If Ω is bounded,

$$\frac{2N}{N+2} \leq p < +\infty \Rightarrow W_0^{1,\mathrm{p}}(\Omega) \subset L^2(\Omega) \subset W^{-1,\mathrm{p}'}(\Omega)$$

with continuous injections and dense and, if Ω is not bounded,

$$\frac{2N}{N+2} \leq p \leq 2 \Rightarrow W_0^{1,\mathrm{p}}(\Omega) \subset L^2(\Omega) \subset W^{-1,\mathrm{p}'}(\Omega).$$

The elements of $W^{-1,\mathrm{p}'}(\Omega)$ can be characterized by the following.

Proposition 2.38. *Let F be from $W^{-1,p'}(\Omega)$. There exists $f_0, \ldots, f_N$ in $L^{p'}(\Omega)$* such that

$$F(u) = \int_\Omega f_0 u dx + \sum_{\mathrm{i}=1}^{\mathrm{N}} \int_\Omega f_{\mathrm{i}} \frac{\partial u}{\partial x_{\mathrm{i}}} dx \ \forall u \text{ in } W_0^{1,\mathrm{p}}(\Omega)$$

and

$$\| F \| = \max_{1 \leq i \leq N} \| f_{\mathrm{i}} \|_{\mathrm{L}^{\mathrm{p}'}(\Omega)}.$$

When Ω is bounded, one can take $f_0 = 0$.

2.3.2. The Operators $-\Delta_{\mathrm{p}}$, $-\Delta_{\mathrm{p}}^{\mathrm{s}}$ and N_{f}

$-\boldsymbol{\Delta}_{\mathbf{p}}$, $p \in (1, +\infty)$, the p-Laplacian

Let Ω be an open set, with the *finite Lebesgue measure*, from $\mathbf{R}^{\mathrm{N}}$, $N \geq 2$. The norm on $W_0^{1,p}(\Omega)$ will be $u \to \|u\|_{1,p}$.

Consider the operator $-\Delta_{\mathrm{p}} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$,

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u). \tag{2.30}$$

This acts according to [45]:

$$\langle -\Delta_{\mathrm{p}} u, v \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \mathrm{d}x \forall u, v \text{ in } W_0^{1,p}(\Omega). \tag{2.31}$$

Taking into account the following, the next property of the p-Laplacian—the identification with a particular duality map—is the most important.

Proposition 2.39. *Let* Ψ: $W_0^{1,p}(\Omega) \to \mathbf{R}$,

$$\Psi(u) = \frac{1}{p}||u||_{1,p}^{p}.$$

Then Ψ *is Gâteaux differentiable on* $W_0^{1,p}(\Omega)\backslash\{0\}$, *and*

$$\Psi\prime(u) = -\Delta_p\, u = J_\varphi\, u \forall_u from\; W_0^{1,p}(\Omega),$$

where $\varphi(t) = t^{p-1}$ ([50]).

Proof. Since $\Psi(u) = \int_0^{||u||_{1,p}} \varphi(t)dt$, we have

$$J_\varphi\, u = \partial\, \Psi(u)\ \forall u \text{in}\ W_0^{1,p}(\Omega)\ \text{(Proposition 2.2)},$$

thus, it remains to prove that Ψ is Gâteaux differentiable and

$$\Psi\prime(u) = -\Delta_p\, u\ \forall u \text{in}\ W_0^{1,p}(\Omega). \tag{2.32}$$

$\Psi = g \circ f$, where g: $L^p(\Omega) \to \mathbf{R}$, $g(u) = \frac{1}{p}\, ||u||_{0,p}^{p}$, $f : W_0^{1,p}(\Omega) \to L^p(\Omega)$, $f\,(u) = |\nabla u\,|$.

From now on, the proof is continued as in [11] and has been proposed by the author.

g is of Fréchet C^1 class on $L^p(\Omega)$ (Proposition 2.33). f is Gâteaux differentiable on $W_0^{1,p}(\Omega)\backslash\{0\}$ and $f\,'(u)(h) = \frac{\nabla u \cdot \nabla h}{|\nabla u|}\ \forall h$ in $W_0^{1,p}(\Omega)$ [50]. Applying Proposition 2.4, $u \neq 0$ and $h \in W_0^{1,p}(\Omega) \Rightarrow \Psi'(u)(h) = \int_\Omega |\,\nabla u|^{p-1}\frac{\nabla u \cdot \nabla h}{|\nabla u|}\, dx = \int_\Omega |\,\nabla u|^{p-2}\nabla u \cdot \nabla h\, dx \overset{(2.31)}{=} \langle -\,\Delta_p\, u, h\rangle$ and the case $u = 0$ remains to finish with (2.32).

However, $\Psi'(0)(h) = \lim\limits_{t\to 0}\frac{1}{t}\Psi(th) = \lim\limits_{t\to 0}\frac{t^{p-1}}{p}\,||h||_{1,p}^{p} = 0 = \langle -\,\Delta_p\, 0, h\rangle$. □

Remark 2.3. Ψ *has even the Fréchet* C^1 *class on* $W_0^{1,p}(\Omega)$ [44,52].

Corollary 2.6. $u \to \|u\|_{1,p}$ *is Gâteaux differentiable on* $W_0^{1,p}(\Omega)\backslash\{0\}$ *and* $W_0^{1,p}(\Omega)$ *is smooth.*

Proof. For the first assertion, apply Proposition 2.4 considering $\varphi(u) = \|u\|_{1,p} = p^{\frac{1}{p}}\,[\Psi(u)]^{\frac{1}{p}}$; for the second assertion, we use Proposition 2.3. □

Proposition 2.40. *The operator* $-\,\Delta_p$: $W_0^{1,p}(\Omega) \to W^{-1,p\prime}(\Omega)$ *is bijective with monotonous inverse, bounded and continuous.*

Proof. $-\Delta_p = J_\varphi$ (Proposition 2.39), and $W_0^{1,p}(\Omega)$ is uniformly convex; apply Proposition 2.26 and take into account the formula $J_\varphi^{-1} = \Phi^{-1}\, J_{\varphi-1}$, Φ is the canonical embedding in bidual (Proposition 2.23). □

$-\boldsymbol{\Delta}_{\mathbf{p}}^{\mathbf{s}}$, $p \in (1, +\infty)$, the p-Pseudo-Laplacian

Let Ω be an open set of finite Lebesgue measure from $\mathbf{R}^N$, $N \geq 2$, and p in $(1, +\infty)$.

$$\|u\|_{1,p} := \left(\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}^{p}\right)^{\frac{1}{p}}$$

is a norm on $W_0^{1,p}(\Omega)$ since

$$\left(\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}+\frac{\partial v}{\partial x_i}\right\|_{0,p}^{p}\right)^{\frac{1}{p}} \leq \left[\sum_{i=1}^{N}\left(\left\|\frac{\partial u}{\partial x_i}\right\|_{0,\mathrm{p}}^{p}+\left\|\frac{\partial v}{\partial x_i}\right\|_{0,\mathrm{p}}^{p}\right)^{p}\right]^{\frac{1}{p}},$$

applying Minkovski inequality.

The dual of $(W_0^{1,p}(\Omega), \| \cdot \|_{1,\mathrm{p}})$ is also designated by $W^{-1,\mathrm{p}'}(\Omega)$, where p' is the exponent conjugated with p.

$\| \cdot \|_{1,\mathrm{p}}$ is equivalent to the norm $| u |_{1,\mathrm{p}} := \sum_{\mathrm{i}=1}^{\mathrm{N}} \left\| \frac{\partial u}{\partial x_{\mathrm{i}}} \right\|_{\mathrm{L^p}(\Omega)}$:

$$\sum_{\mathrm{i}=1}^{\mathrm{N}}\left\|\frac{\partial u}{\partial x_{\mathrm{i}}}\right\|_{\mathrm{L^p}(\Omega)} \leq N\|u\|_{1,\mathrm{p}} \leq N\sum_{\mathrm{i}=1}^{\mathrm{N}}\left\|\frac{\partial u}{\partial x_{\mathrm{i}}}\right\|_{\mathrm{L^p}(\Omega)}.$$

However, $| \cdot |_{1,\mathrm{p}}$ is equivalent to $\| \cdot \|_{1,\mathrm{p}}$ since $\|u\|_{1,\mathrm{p}} \leq | u |_{1,\mathrm{p}} \leq N \|u\|_{1,\mathrm{p}}$. Consequently,

Proposition 2.41. $(W_0^{1,p}(\Omega), \| \cdot \|_{1,\mathrm{p}})$, $\mathrm{p} \in (1, +\infty)$, *is Banach space.*

Furthermore,

Proposition 2.42. $(W_0^{1,p}(\Omega), \| \cdot \|_{1,\mathrm{p}})$, $\mathrm{p} \in [2, +\infty)$*, is uniformly convex.*

Proof. The following proof was proposed by the author in [11]. Use the inequality (2.25): $\xi, \eta \in \mathbf{R}^n$, $n \geq 1 \Rightarrow \left\|\frac{\xi+\eta}{2}\right\|^{\mathrm{p}} + \left\|\frac{\xi-\eta}{2}\right\|^{\mathrm{p}} \leq \frac{1}{2}(\|\xi\|^{\mathrm{p}} + \|\eta\|^{\mathrm{p}})$ with the Euclidean norm [51].

Let ε be in $(0, 2]$ and define u, v with $\|u\|_{1,\mathrm{p}} = \|v\|_{1,\mathrm{p}} = 1$, $\|u - v\|_{1,\mathrm{p}} \geq \varepsilon$. Suppose $p \in [2, +\infty)$. $\left\|\frac{u+v}{2}\right\|^{\mathrm{p}} + \left\|\frac{u-v}{2}\right\|^{\mathrm{p}} = \sum_{\mathrm{i}=1}^{\mathrm{N}} \int_{\Omega} \left(\left|\frac{\frac{\partial u}{\partial x_{\mathrm{i}}}+\frac{\partial v}{\partial x_{\mathrm{i}}}}{2}\right|^{\mathrm{p}} + \left|\frac{\frac{\partial u}{\partial x_{\mathrm{i}}}-\frac{\partial v}{\partial x_{\mathrm{i}}}}{2}\right|^{\mathrm{p}}\right) dx \leq \sum_{\mathrm{i}=1}^{\mathrm{N}} \int_{\Omega} \frac{1}{2}\left(\left|\frac{\partial u}{\partial x_{\mathrm{i}}}\right|^{\mathrm{p}} + \left|\frac{\partial v}{\partial x_{\mathrm{i}}}\right|^{\mathrm{p}}\right) dx = 1$,

and hence, $\left\|\frac{u+v}{2}\right\|_{1,\mathrm{p}}^{\mathrm{p}} \leq 1 - \left(\frac{\varepsilon}{2}\right)^{\mathrm{p}}$, take δ defined by $1 - \delta = \left[1 - \left(\frac{\varepsilon}{2}\right)^{\mathrm{p}}\right]^{\frac{1}{\mathrm{p}}}$. □

Let Ω be an open set in $\mathbf{R}^{\mathrm{N}}$, $N \geq 2$, of the finite Lebesgue measure and p in $(1, +\infty)$. Considering the operator $-\Delta_{\mathrm{p}}^{\mathrm{s}}$: $W_0^{1,\mathrm{p}}(\Omega) \to W^{-1,p\prime}(\Omega)$,

$$-\Delta_{\mathrm{p}}^{\mathrm{s}} u = -\sum_{\mathrm{i}=1}^{n} \frac{\partial}{\partial x_{\mathrm{i}}}\left(\left|\frac{\partial u}{\partial x_{\mathrm{i}}}\right|^{\mathrm{p}-2}\frac{\partial u}{\partial x_{\mathrm{i}}}\right).$$

This acts according to [45]:

$$\langle -\Delta_{\mathrm{p}}^{\mathrm{s}} u,\, h\rangle = \sum_{\mathrm{i}=1}^{\mathrm{N}}\int_{\Omega}\left|\frac{\partial u}{\partial x_{\mathrm{i}}}\right|^{\mathrm{p}-2}\frac{\partial u}{\partial x_{\mathrm{i}}}\frac{\partial h}{\partial x_{\mathrm{i}}}dx,\ \forall u,\ h \text{ in } W_0^{1,\mathrm{p}} \tag{2.33}$$

Proposition 2.43. *The function* Ψ*:* $\Psi(u) = \frac{1}{p}\|u\|_{1,\mathrm{p}}^{\mathrm{p}}$, $\mathrm{p} \in (1, +\infty)$, *is Gâteaux differentiable on* $W_0^{1,p}(\Omega)\backslash\{0\}$*, and*

$$\Psi\prime(u) = -\Delta_{\mathrm{p}}^{\mathrm{s}} u = J\varphi\ u,\ \varphi(t) := t^{p-1.}$$

Proof ([11]). Fix the index i, $1 \le i \le N$, and let g: $W_0^{1,p}(\Omega) \to \mathbf{R}$, $g(u) = \left\| \frac{\partial u}{\partial x_i} \right\|_{0,p}^{p}$. We have $g = F \circ f$, $f: W_0^{1,p}(\Omega) \to L^p(\Omega)$, $f(u) = \frac{\partial u}{\partial x_i}$, F: $L^p(\Omega) \to \mathbf{R}$, $F(v) = \|v\|_{0,p}^{p}$. As $f'(u)(h) = \frac{\partial h}{\partial x_i}$ and $F'(v)(h) = p\int\limits_{\Omega} |v|^{p-2} vh dx$ ([48]), $g'(u) = F'(f(u)) \circ f'(u)$ (formula from Proposition 2.4),

$$g\prime(u)(h) = p\int\limits_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial h}{\partial x_i} dx \tag{2.34}$$

and hence

$$\Psi\prime(u)(h) = \sum_{i=1}^{N} \left(\left\| \frac{\partial u}{\partial x_i} \right\|_{0,p}^{p} \right)'(h) \overset{(2.34),(2.33)}{=} \left\langle -\Delta_p^s u,\, h \right\rangle.$$

The rest of the proof is the same as for Proposition 2.39. □

Corollary 2.7. $u \to \|u\|_{1,p}$, $p \in (1, +\infty)$, *is Gâteaux differentiable on* $W_0^{1,p}(\Omega)\backslash\{0\}$. *Consequently,* $(W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$ *is a smooth space.*

Proof ([11]). Taking $\Phi(u) = \|u\|_{1,p}$, we have

$$\Phi(u) = p^{\frac{1}{p}} (\Psi(u))^{\frac{1}{p}}.$$

We apply the formula from Proposition 2.4. For the second assertion, we take Proposition 2.3 into account. □

Nemytskii Operator N_f

In the following, some statements from [53] are necessary to develop some results.

Definition 2.9. *Let* Ω *be a nonempty open Lebesgue measurable (L.m.) set from* $\mathbf{R}^N$, $N \ge 1$, μ *the Lebesgue measure in* $\mathbf{R}^N$ *and* $M(\Omega) := \{u: \Omega \to \mathbf{R}: u \text{ L.m.}\}$. *By definition,* $f: \Omega \times \mathbf{R} \to \mathbf{R}$ *is a Carathéodory function if:*

1° $f(\cdot, s)$ *is* L.m. $\forall s$ *in* $\mathbf{R}$*;*

2° $f(x, \cdot)$ *is continuous* $\forall x$ *in* $\Omega\backslash A$, $\mu(A) = 0$.

Proposition 2.44. *If* $f: \Omega \times \mathbf{R} \to \mathbf{R}$ *is a Carathéodory function, then, for any* u *in* $M(\Omega)$, $x \to f(x, u(x))$ *is L.m.*

Proof. Let $(\varphi_n)_{n\ge1}$ be a sequence of real functions, simple, L.m., with

$$\varphi_n(x) \underset{x\in\Omega}{\to} u(x).$$

$F_n : F_n(x) := f(x, \varphi_n(x))$ is L.m. on $A_1, \ldots, A_p$, where $\varphi_n \mid A_k$ = constant, $k = \overline{1,p}$ (1^0 from Definition 2.9), hence F_n is L.m. on Ω. $\forall x$ in $\Omega\backslash A$, and since $\varphi_n(x) \to u(x)$, we have $F_n(x) \to f(x, u(x))$ (2^0 from Definition 2.9), which implies, since the Lebesgue measure is complete, $x \to f(x, u(x))$ L.m. □

Definition 2.10. *Thus, one may consider the Nemytskii operator:*

$$N_f : M(\Omega) \to M(\Omega),\ (N_f u)x = f(x, u(x)).$$

Proposition 2.45. *Suppose* $\mu(\Omega) < +\infty$. *Then*

$$u_n(x) \underset{x\in\Omega}{\overset{\mu}{\to}} u_0(x) \Rightarrow N_f u(x) \underset{x\in\Omega}{\overset{\mu}{\to}} N_f u_0(x).$$

Proof ([54])**.** It is sufficient to show the proof in the case $f\,(x, 0) = 0\ \forall x$ in Ω and $u_n\,(x) \underset{x\in\Omega}{\overset{\mu}{\rightarrow}} 0$; thus, it remains to prove that:

$$\forall\varepsilon, \eta > 0\ \exists\, N \text{ in } \mathbf{N}, \text{e.g., } n \geq N \Rightarrow \mu(\{x \in \Omega\!: \mid f\,(x, u_n\,(x)) \mid \, \geq \varepsilon\}) \leq \eta. \tag{2.35}$$

Set $\Omega_0 := \Omega\backslash A$. For $k \in \mathbf{N}$, $\Omega_{\mathrm{k}} := \{x \in \Omega_0 : \mid s \mid < \frac{1}{k} \Rightarrow \mid f(x, s) \mid < \varepsilon\}$ (nonempty set for sufficiently big k; $f\,(x, \cdot)$ is continuous in 0), we have $\Omega_{\mathrm{k}} \subset \Omega_{\mathrm{k+1}}\ \forall k$ and $\Omega_0 = \bigcup\limits_{\mathrm{k}=1}^{\infty} \Omega_{\mathrm{k}}$, hence $\mu(\Omega_0) = \lim\limits_{k\to\infty} \mu(\Omega_k)$ and hence $\exists\, k_0$ in $\mathbf{N}$, e.g., $\mu(\Omega_0\backslash\Omega_{k_0}) \leq \frac{\eta}{2}$. Let $A_{n:} = \{x \in \Omega_0 : \mid u_n\,(x) \mid < \frac{1}{k_0}\}$, $\exists\, N$ in $\mathbf{N}$, e.g., $n \geq N \Rightarrow \mu(\Omega_0\backslash A_n) \leq \frac{\eta}{2}$. Setting $B_{n:} = \{x \in \Omega_0 : \mid f\,(x, u_n\,(x)) \mid < \varepsilon\}$, we have, since $A_n \cap \Omega_{\mathrm{k}_0} \subset B_n$, $n \geq N \Rightarrow \mu(\Omega_0\backslash B_n) \leq \mu(\Omega_0\backslash A_n) + \mu(\Omega_0\backslash\Omega_{\mathrm{k}_0}) \leq \eta$, which implies (2.35). □

Proposition 2.46. *If the Carathéodory function f verifies the growth condition:*

$$\mid f(x, s) \mid \, \leq c \mid s \mid^{\mathrm{r}} + \beta(x), \forall x \in \Omega\backslash A \text{ with } \mu(A) = 0\ \forall s \in \mathbf{R},$$
$$\text{where } c \geq 0, r > 0, \beta \in L^p(\Omega), 1 \leq \mathrm{p} \geq +\infty,$$

then

1° $N_f\,(L^{pr}(\Omega)) \subset L^p(\Omega)$;
2° N_f *is continuous* ($p < +\infty$) *and bounded on* $L^{pr}(\Omega)$.

Clarification. A map between metric spaces is *bounded* if the image of any bounded set is bounded.

Proof. 1° Let u be in $L^{pr}(\Omega)$.

$$\mid f\,(x, u(x)) \mid \, \leq c\, \mid u(x) \mid^{\mathrm{r}} + \beta(x), x \in \Omega\backslash A, s \in \mathbf{R}, \tag{2.36}$$

but $\mid u \mid^{\mathrm{r}} \in L^p(\Omega)$, $\beta \in L^p(\Omega)$, hence $N_{\mathrm{f}}\,u \in L^p(\Omega)$ (when $p < +\infty$, taking the power p, the first member is integrable and measurable (Proposition 2.44); when $p = +\infty$, the justification is obvious).

2° From (2.36),

$$||N_{\mathrm{f}}\ u||_{L^{\mathrm{p}}(\Omega)} \leq\ ||c\ |u|^{\mathrm{r}} + \beta||_{L^{\mathrm{p}}(\Omega)} \leq\ c\ ||\ |u|^{\mathrm{r}}\ ||_{L^{\mathrm{p}}(\Omega)} + ||\beta||_{L^{\mathrm{p}}(\Omega)} = c||u||^{r}_{L^{\mathrm{p}}(\Omega)}\ +\ ||\beta||_{L^{\mathrm{p}}(\Omega)},$$

and hence N_{f} is bounded on $L^{pr}(\Omega)$.

We proceed to the continuity. Suppose $f\,(x, 0) = 0\ \forall x$ in Ω, and let

$$u_n\ \rightarrow\ 0 \text{ in } L^{pr}(\Omega). \tag{2.37}$$

We will prove:

$$N_{\mathrm{f}}\ u_n\ \rightarrow\ 0 \text{ in } L^p(\Omega). \tag{2.38}$$

For (2.38), it is sufficient to prove that any subsequence ($N_{\mathrm{f}}\ u_{k_n}$) has a subsequence ($N_{\mathrm{f}}\ u_{l_{k_n}}$) convergent to 0 in $L^p(\Omega)$ (if (x_n), the sequence in the metric space X has the property that any subsequence (x_{k_n}) has a subsequence $(x_{l_{k_n}})$ with $x_{l_{k_n}} \rightarrow x_0$, and then $x_n \rightarrow x_0$ — ad absurdum justification).

Since $u_{k_n} \rightarrow 0$ in $L^{pr}(\Omega)$, $\exists\,(u_{l_{k_n}})$ a subsequence with

$$u_{l_{k_n}}(x)\ \rightarrow\ 0, x\ \in\ \Omega\backslash B,\ \mu(B) = 0$$

and

$$|u_{l_{k_n}}(x)|\ \leq\ g(x),\ g \in\ L^{pr}(\Omega).$$

From (2.36),

$$|\, f\,(x,\, u_{l_{k_n}}(x))| \;\leq\; c(g(x))^{\mathrm{r}} \;+\; \beta(x),\; x \;\in\; \Omega\backslash(A \;\cup\; B). \tag{2.39}$$

Taking the power p in (2.39), the second member is integrable, and as $x \in \Omega\backslash(A \cup B) \Rightarrow f\,(x, u_{l_{k_n}}(x)) \to 0$, it results in (Lebesgue theorem of dominated convergence)

$$\int\limits_{\Omega} |N_{\mathrm{f}} u_n(x)|^{\mathrm{p}} dx \;\to 0,$$

i.e., (2.38).

Pass to the general case and let $u_n \to u_0$ in $L^{pr}(\Omega)$. g: $\Omega \times \mathbf{R} \to \mathbf{R}$,

$$g(x, s) = f\,(x, s + u_0\,(x)) - f\,(x, u_0\,(x)),$$

is a Carathéodory function. Since $g(x, 0) = 0\;\forall x$ in Ω and $u_n - u_0 \to 0$ in $L^{pr}(\Omega)$, we obtain $N_{\mathrm{g}}\,(u_n - u_0) \to 0$ in $L^p(\Omega)$ and, hence $N_{\mathrm{f}}\,u_n \to N_{\mathrm{f}}\,u_0$ in $L^p(\Omega)$. □

Remark 2.4. *Retain the inequality:*

$$||N_{\mathrm{f}} u||_{\mathbf{L}^{\mathrm{p}}(\Omega)} \leq\; c||u||^{\mathrm{r}}_{\mathbf{L}^{\mathrm{pr}}(\Omega)} + ||\beta||_{\mathbf{L}^{\mathrm{p}}(\Omega)}.$$

2.3.3. The Problem

Consider the problem

$$(*)\begin{cases} -\lambda\,\Delta_{\mathrm{p}}\,u = f(\,\cdot\,,\,u(\,\cdot\,)) + h,\; x \in \Omega,\; \lambda \in \mathbf{R} \\ \qquad\qquad u|\partial\Omega = 0 \end{cases}$$

The next two propositions were obtained by the author and are given in [11,12,15].

Proposition 2.47. *Let Ω be an open bounded set of the C^1 class from $\mathbf{R}^{\mathrm{N}}$, $N \geq 2$, $p \in (1, +\infty)$, h be from $W^{-1,p'}(\Omega)$ and $f : \Omega \times \mathbf{R} \to \mathbf{R}$ a Carathéodory function with the properties*

1° $f\,(x, -\,s) = -\,f\,(x, s)\;\forall s$ from $\mathbf{R}$, $\forall x$ from Ω,

2° $|f\,(x, s)| \leq c_1\,|s|^{q-1} + \beta(x)\;\forall s$ from $\mathbf{R}$, $\forall x$ from $\Omega\backslash A$, $\mu(A) = 0$,

where $c_1 \geq 0$, $q \in (1, p)$, $\beta \in L^{q'}(\Omega)$, $\frac{1}{q} + \frac{1}{q'} = 1$.

Then, for any $\lambda \neq 0$, the problem $(*)$ has a solution in $W_0^{1,p}(\Omega)$ in the sense of $W^{-1,p'}(\Omega)$.

Explanations. The relationship $u \mid \partial\Omega$ from $(*)$ is in the sense of the trace (Definition 2.8). Moreover, $\gamma^{-1}(0) = W_0^{1,p}(\Omega)$. $f\,(\cdot,\, u) = N_f\, u$, where N_f is the Nemytskii operator (see Section 2.3.2 above), and so the equation from $(*)$ can be written as

$$-\,\lambda\,\Delta_{\mathrm{p}}\,u = N_{\mathrm{f}}\,u + h. \tag{2.40}$$

From 2° of the last assertion, it is determined (via Proposition 2.46) that N_f maps $L^q(\Omega)$ on $L^{q'}(\Omega)$, and it is continuous and bounded. Moreover (Proposition 2.46),

$$\|\; N_{\mathrm{f}}\;u\;\|_{0,\mathrm{q}'} \leq\; c_1\;||u||^{\mathrm{q}-1}_{0,\mathrm{q}} + c_2\;,\; c_2 := ||\beta||_{0,\mathrm{q}'}\;,\;\forall u \text{in } L^q(\Omega). \tag{2.41}$$

Since $q \in (1, p)$ and $q < p^*$ (the Sobolev conjugated exponent), $(W_0^{1,p}(\Omega), \|\cdot\|_{1,\,\mathrm{p}})$ is compactly embedded in $L^q(\Omega)$. Let i (linear injection) be such an embedding,

$$\|\; i(u)\;\|_{0,\mathrm{q}} \leq\; c_{0,\mathrm{q}}\;\;\|\;u\;\|_{1,\mathrm{q}}\;\;\forall u \text{in } W_0^{1,p}$$

(using the Rellich-Kondrashev theorem and taking into account that the norms $\|\cdot\|_{1,\,\mathrm{p}}$ and $\|\cdot\|_{W^{1,p}(\Omega)}$ are equivalent).

Let i': $L^{q'}(\Omega) \to \mathrm{W}^{-1,\mathrm{p}'}(\Omega)$ be the adjoint of i (as $(L^q(\Omega))^* = L^{q'}(\Omega)$!).

u_0 from $W_0^{1,p}(\Omega)$ is a solution for $(*)$ in the sense of $W^{-1,p'}(\Omega)$ if

$$-\lambda\,\Delta_{\mathrm{p}}\,u_0 = (i' \text{ o } N_{\mathrm{f}} \text{ o } i)u_0 + h. \tag{2.42}$$

We proceed to the proof of Proposition 2.47.

Proof. $-\Delta_{\mathrm{p}} \overset{\text{Proposition2.39}}{=} J_\varphi$, where J_φ is the duality map with $\varphi(t) = t^{\mathrm{p}-1}$; the Banach space $(W_0^{1,p}(\Omega), \|\cdot\|_{1,\mathrm{p}})$ is uniformly convex (Proposition 2.37) and, consequently, has the (H) property, and it is reflexive (uniformly convex $\Rightarrow$ reflexive). It is also smooth (its norm being Gâteaux differentiable on $W_0^{1,p}(\Omega)\backslash\{0\}$ (Proposition 2.3)). Thus, one can apply Proposition 2.30 with $X = W_0^{1,p}(\Omega)$, $Z = L^q(\Omega)$, $N = N_{\mathrm{f}}$ − odd continuous operator (Proposition 2.46) and $Z^* = L^{q'}(\Omega)$ and take (2.41) into account; the operator $\lambda(-\Delta_{\mathrm{p}}) - S$: $W_0^{1,p}(\Omega) \to W^{-1,\mathrm{p}'}(\Omega)$, where $S = i' \text{ o } N_{\mathrm{f}} \text{ o } i$ is surjective, a fortiori the operator $-\lambda\Delta_{\mathrm{p}} - S - h$ is surjective (commutative group) and hence $\exists u_0$ in $W_0^{1,p}(\Omega)$ which verifies (2.42). □

By replacing q with p in Proposition 2.47, 2^0, and by applying Proposition 2.32, we obtain the following:

Proposition 2.48. *Let Ω be an open bounded set of the C^1 class from $\mathbf{R}^{\mathrm{N}}$, $N \geq 2$, $p \in (1, +\infty)$, h from $W^{-1,p'}(\Omega)$ and $f : \Omega \times \mathbf{R} \to \mathbf{R}$ Carathéodory function having the properties*

1° $f(x, -s) = -f(x, s)$ $\forall x$ from Ω, $\forall s$ from $\mathbf{R}$,

2° $|f(x, s)| \leq c_1 |s|^{\mathrm{p}-1} + \beta(x)$ $\forall s$ from $\mathbf{R}$, $\forall x$ from $\Omega\backslash A$, $\mu(A) = 0$,

where $c_1 \geq 0$, $\beta \in L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$.

Finally, let $i : W_0^{1,p}(\Omega) \to L^p(\Omega)$ be linear compact embedding. Then, for any λ, if

$$|\lambda| \;>\; c_1\lambda_1^{-1},\; \lambda_1 \;:= \inf\left\{ \frac{||u||_{1,p}^p}{||i(u)||_{0,p}^p} : u \in W_0^{1,p}(\Omega)\backslash\{0\} \right\}$$

then the problem $(*)$ has solution in $W_0^{1,p}(\Omega)$ in the sense of $\mathrm{W}^{-1,\mathrm{p}'}(\Omega)$.

Proof. The statement is correct since $(W_0^{1,p}(\Omega), \|\cdot\|_{1,\,\mathrm{p}})$ is compactly embedded in $L^p(\Omega)$ (Rellich-Kondrashev theorem). Apply Proposition 2.32. □

Remark 2.5. *The condition from Proposition 2.48 can be replaced (Proposition 2.32 allows this) by:*

$$|\lambda| \;>\; \overline{\lim_{||u||\to+\infty}} \frac{||(i\prime \circ N_f \circ i)u||}{||u||^{p-1}}.$$

Attention to λ_1: $\lambda_1^{-\frac{1}{p}}$ is optimal for the inequality from the statement; it is attained and nonzero, and it is the smallest eigenvalue of the couple $(J_{L^q L^{q'}}, J_{W_0^{1,p} W^{-1,p'}})$ (see Proposition 2.31).

2.3.4. The Problem

Consider the problem

$$(**)\begin{cases} -\lambda\,\Delta_{\mathrm{p}}^{\mathrm{s}} u = f(\,\cdot\,, u(\,\cdot\,)) + h,\; x \in \Omega,\; \lambda \in \mathbf{R} \\ \qquad\qquad u|\partial\Omega \;=\; 0 \end{cases}$$

The following two statements are obtained by the author and given in [11,12,15].

Proposition 2.49. *Let Ω be an open bounded set of C^1 class from $\mathbf{R}^N$, $N \geq 2$, $p \in [2, +\infty)$, h from $W^{-1,p'}(\Omega)$ and $f : \Omega \times \mathbf{R} \to \mathbf{R}$ Carathéodory function with the properties*

1° *$f(x, -s) = -f(x, s)$ $\forall x$ from Ω, $\forall s$ from* $\mathbf{R}$,

2° $|f(x, s)| \leq c_1 |s|^{q-1} + \beta(x)$ *$\forall s$ from* $\mathbf{R}$, *$\forall x$ from* $\Omega \backslash A$, $\mu(A) = 0$,

where $c_1 \geq 0$, $q \in (1, p)$, $\beta \in L^{q'}(\Omega)$, $\frac{1}{q} + \frac{1}{q'} = 1$.

Then, for any $\lambda \neq 0$, the problem $(**)$ has solution in $W_0^{1,p}(\Omega)$ in the sense of $W^{-1,p'}(\Omega)$.

Explanations (similar to those for Proposition 2.47): The relationship $u \mid \partial\Omega$ from $(**)$ is in the sense of the trace. $f(\cdot, u) = N_f u$, N_f Nemytskii operator, and so the equation from $(**)$ can be written as

$$-\lambda\Delta_p^s u = N_f\, u + h. \tag{2.43}$$

Now, the norm that endows $W_0^{1,p}(\Omega)$ is $\|\cdot\|_{1,p}$, and $(W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$ is a Banach space that is compactly embedded in $L^q(\Omega)$ since $\|\cdot\|_{1,p}$ *and* $\|\cdot\|_{1,p}$, and hence also $||\cdot||_{W^{1,p}(\Omega)}$, are equivalent (see above). Let i be the embedding.

Let i': $L^{q'}(\Omega) \to W^{-1,p'}(\Omega)$ be the adjoint of i (as $(L^q)^* = L^{q'}$). u_0 from $W_0^{1,p}(\Omega)$ is a solution for $(**)$ in the sense of $W^{-1,p'}(\Omega)$ if

$$-\lambda\Delta_p^s\, u_0 = (i' \circ N_f \circ i)u_0 + h. \tag{2.44}$$

Proof. $-\Delta_p^s \overset{\text{Prop.2.43}}{=} J_\varphi$, J_φ duality map with $\varphi(t) = t^{p-1}$, the Banach space $(W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$ being uniformly convex (see above, proposition 2.42). It is also smooth (its norm being Gâteaux differentiable on $W_0^{1,p}(\Omega)\backslash\{0\}$). So, we apply Proposition 2.30 with $X = W_0^{1,p}(\Omega)$, $Z = L^q(\Omega)$, $N = N_f$ — odd continuous operator, $Z^* = L^{q'}(\Omega)$, take into account $\|N_f u\|_{0,q'} \leq c_1||u||_{0,q}^{q-1} + c_2$, $c_2 := \|\beta\|_{0,q'}$, $\forall u$ from $L^q(\Omega)$ the operator $\lambda\,(-\Delta_p^s) - S$: $W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$, where $S = i' \circ N_f \circ i$ is surjective, a fortiori the operator $-\lambda\Delta_p^s - S - h$ is surjective (commutative group) and hence $\exists u_0$ in $W_0^{1,p}(\Omega)$ which verifies (2.44). □

Replacing q with p in 2° from Proposition 2.49 and applying Proposition 2.32, obtain:

Proposition 2.50. *Let Ω be an open bounded set of C^1 class from $\mathbf{R}^N$, $N \geq 2$, $p \in [2, +\infty)$, h from $W^{-1,p'}(\Omega)$ and $f : \Omega \times \mathbf{R} \to \mathbf{R}$ Carathéodory function having the properties*

1° *$f(x, -s) = -f(x, s)$ $\forall x$ from Ω, $\forall s$ from* $\mathbf{R}$,

2° $|f(x, s)| \leq c_1 |s|^{p-1} + \beta(x)$ *$\forall s$ from* $\mathbf{R}$, *$\forall x$ from* $\Omega \backslash A$, $\mu(A) = 0$,

where $c_1 \geq 0$, $\beta \in L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$.

Finally, let i: $W_0^{1,p}(\Omega) \to L^p(\Omega)$ *be a linear compact embedding. Then, for any λ, if*

$$|\lambda| > c_1\lambda_1^{-1},\ \lambda_1 := \inf\left\{\frac{\|u\|_{1,p}^p}{||i(u)||_{0,p}^p} : u \in W_0^{1,p}(\Omega)\backslash\{0\}\right\},$$

the problem $(**)$ has solution in $W_0^{1,p}(\Omega)$ in the sense of $W^{-1,p'}(\Omega)$.

Proof. The proof is the same as for Proposition 2.48. □

Remark 2.6. *The condition from Proposition 2.49 can be replaced (see Proposition 2.48) by:*

$$|\lambda| > \overline{\lim_{||u||\to+\infty}} \frac{||(i' \circ N_f \circ i)u||}{\|u\|^{p-1}}.$$

Attention to $\lambda_1 : \lambda_1^{-\frac{1}{p}}$ *is optimal for the inequality from the statement; it is attained and nonzero, and it is the smallest eigenvalue of the couple* $(J_{L^q L^{q'}}, J_{W_0^{1,p} W^{-1,p'}})$ *(see Proposition 2.31).*

Remark 2.7. *The above results will be used to provide solutions, together with their characterizations, for particular problems from glaciology [55–57], for nonlinear elastic membrane [41,58], for the pseudo-torsion problem [9,59] for nonlinear elastic membrane with the p-pseudo-Laplacian as in [60]. They are presented in the second part of this paper.*

3. Results of the Fredholm Alternative Type for Operators $\lambda J_\varphi - S$

3.1. Important Results

In this section, we continue with results that complete the previous theory provided in Section 2. The statements from this section originate from the generalization due to the author in [11,13] of a theorem of Nečas [42,61] in which normed space is used instead of Banach space and the goal function is a bijection with continuous inverse instead of homeomorphism. The results mentioned above have been obtained based on this theorem and also on propositions of the author and presented in the previous section.

Definition 3.1. *Let X, Y be real normed spaces and F: X* $\to$ *Y,* $F_0 : X \to Y$*. F is strongly closed and strongly continuous, respectively, if*

$$x_n \overset{w}{\to} a \text{ and } F(x_n) \to \alpha \Rightarrow \alpha = F(a),$$

and, respectively,

$$x_n \overset{w}{\to} a \text{ and } F(x_n) \to F(a).$$

For instance, any linear compact operator between Banach spaces is strongly continuous [62].

Definition 3.2. *Let a be a real, strictly positive number. F is, by definition, a-homogeneous if* $F(tu) = t^a F(u)$, $\forall u$ *in X,* $\forall t \geq 0$.

F is *a-quasi-homogeneous relative to* F_0, and F_0 is *a*-homogeneous if

$$t_n \downarrow 0,\ u_n \overset{w}{\to} u_0 \text{ and } t_n^a\ F\left(\frac{u_n}{t_n}\right) \to \gamma \Rightarrow \gamma = F_0\ (u_0).$$

F is a-strongly quasi-homogeneous relative to F_0*,* F_0 *a-homogeneous* if

$$t_n \downarrow 0,\ u_n \overset{w}{\to} u_0 \text{ and } t_n^a\ F\left(\frac{u_n}{t_n}\right) \to F_0\ (u_0).$$

Proposition 3.1. *Let F be an a-homogeneous and strongly closed (respectively, strongly continuous), then F is a-quasi-homogeneous (respectively a-strongly quasi-homogeneous) relative to F.*

Proof. Observe, in both cases, that $t_n^a\ F\left(\frac{u_n}{t_n}\right) \to F_0\ (u_0)$. □

Proposition 3.2. *If F is a-strongly quasi-homogeneous relative to* F_0*, then* F_0 *is a-homogeneous and strongly continuous.*

Proof. *First assertion.* $t > 0$. Let u_0 be arbitrarily fixed from X and $t_n \downarrow 0$, $u_n \overset{w}{\to} u_0$. Then $t_n^a F\left(\frac{u_n}{t_n}\right) \to F_0\ (u_0)$, hence $(tt_n)^a\ F\left(\frac{u_n}{t_n}\right) \to t^a\ F_0\ (u_0)$, but $(tt_n)^a\ F\left(\frac{tu_n}{tt_n}\right) \to F_0\ (tu_0)$ since $tu_n \overset{w}{\to} tu_0$, $t^a\ F_0\ (u_0) = F_0\ (tu_0)$. $t = 0$. It should be shown that $F_0\ (0) = 0$. Take $t_n \downarrow 0$ and $(u_n)_{n \geq 1}$ with $u_n = 0\ \forall n$. Then, $t_n^a\ F\left(\frac{u_n}{t_n}\right) \to F_0\ (0)$, but $F\left(\frac{u_n}{t_n}\right) = 0$, etc.

Second assertion. From the hypothesis,

$$\lim_{t\to 0+} t^{a} F\left(\frac{u}{t}\right) = F_0(u) \forall u \text{ in } X. \tag{3.1}$$

Suppose, ad absurdum, that F_0 is not strongly continuous. Then there exists u_n, $u_n \overset{w}{\to} u_0$, and

$$F_0(u_n) \nrightarrow F_0(u_0). \tag{3.2}$$

$\varepsilon > 0$ being arbitrarily fixed, from (3.2), there exists a subsequence of (u_n), it is denoted in the same manner, e.g.,

$$\|F_0\,(u_n) - F_0\,(u_0)\| \geq \varepsilon\ \forall n \geq 1. \tag{3.3}$$

Furthermore, from (3.1), for any n from $\mathbf{N}$ $\exists\ t_n$, $0 < t_n \leq \frac{1}{n}$, for which

$$\|\ F_0\,(u_n) - t_n^{a}\ F\left(\frac{u_n}{t_n}\right)\| \leq\ \frac{\varepsilon}{2}. \tag{3.4}$$

Then

$$\varepsilon \overset{(3.3)}{\leq} \|\ F_0\,(u_0) - F_0\,(u_n)\ \| \leq \|\ F_0\,(u_0) - t_n^{a}\ F\left(\frac{u_n}{t_n}\right)\| \ +\ \|\ t_n^{a}\ F\left(\frac{u_n}{t_n}\right) - F_0\,(u_n)\ \| \overset{(3.4)}{\leq} \frac{\varepsilon}{2} + \|\ F_0\,(u_0) - t_n^{a}\ F\left(\frac{u_n}{t_n}\right)\ \|,$$

and taking the limit for $n \to \infty$, we obtain $\varepsilon \leq \frac{\varepsilon}{2}$, which is a contradiction. □

Proposition 3.3. *The uni-valued duality map J_φ is a-homogeneous iff φ is a-homogeneous.*

Proof. *Necessary*. $\forall u \neq 0, \forall t \geq 0$, $J_\varphi\,(tu) = \frac{\varphi(t\|u\|)}{\varphi(\|u\|)}\,J_\varphi\,u$ (Proposition 2.2, 5°), $J_\varphi\,(tu) = t^{a}\,J_\varphi\,u$, and by taking the norm, $\varphi(t\|u\|) = t^{a}\ \varphi(\|u\|)$, taking into account that $u \to \|u\|$ takes all the values from $\mathbf{R}_+$. *Sufficient*. Use the same formula. □

Thus, for $p \in (1, +\infty)$, $-\Delta_{\mathrm{p}}$ and $-\Delta_p^s$ are $(p-1)$-homogeneous maps on $W_0^{1,p}(\Omega)$ (Propositions 2.39 and 2.43).

Proposition 3.4. *If the Banach space is reflexive, smooth and has the property (H), then any duality map J_φ on X is strongly closed.*

Proof. Let $u_n \overset{w}{\to} u_0$ and $J_\varphi\ u_n \to \gamma$. We have

$$\langle J_\varphi\ u_n - J_\varphi\ u_0, u_n - u_0\rangle \to 0,$$

but

$$\langle J_\varphi\ u_n - J_\varphi\ u_0, u_n - u_0\rangle \geq [\varphi(\|u_n\|) - \varphi(\|u_0\|)](\|u_n\| - \|u_0\|) \geq 0 \text{ (Proposition 2.2)},$$

hence $\lim_{n\to\infty}[\varphi(\|\ u_n\ \|) - \varphi(\|u_0\|)](\|u_n\| - \|u_0\|) = 0$, which implies $\|u_n\| \to \|u_0\|$ (see the proof for Proposition 2.28). With X having the property (H), we obtain $u_n \to u_0$, which implies (X is smooth reflexive $\Rightarrow J_\varphi$ is semicontinuous, Corollary 2.3) that $J_\varphi\ u_n \overset{w}{\to} J_\varphi\ u_0$ and hence $J_\varphi\ u_0 = \gamma$. □

Corollary 3.1. *If the Banach space X is reflexive, smooth and has the property (H), any duality map on X J_φ that is a-homogeneous is a-quasi-homogeneous related to J_φ.*

Proof. Combine Propositions 3.1 and 3.4. □

We proceed to the basis proposition of this section. The conditions are slightly weakened. Previously:

Definition 3.3. *The map $f: X \to Y$, where X and Y are normed spaces, is regularly surjective if it is surjective and* $\forall R > 0\ \exists r > 0$ *e.g.,*

$$\| f(x) \| \leq R \Rightarrow \|x\| \leq r.$$

Proposition 3.5. *Fredholm alternative. Let X and Y be real normed spaces, T: $X \to Y$ a(K, L, a) and a-homogeneous bijection, odd with a continuous inversei and S: $X \to Y$ an odd compact a-homogeneous operator. Theni for any $\lambda \neq 0$, $\lambda T - S$ is regularly surjective iff λ is not an eigenvalue for the couple (T, S).*

Proof. *Necessary*. Let, ad absurdum, $x_0 \neq 0$ be from X such that

$$\lambda T(x_0) - S(x_0) = 0. \tag{3.5}$$

Multiplying (3.5) by t^{a}, we obtain

$$\lambda T(tx_0) - S(tx_0) = 0 \tag{3.6}$$

and as $\lim\limits_{t \to +\infty} \|tx_0\| = +\infty$, (3.6) imposes (ad absurdum!) the conclusion that $\lambda T - S$ is not regularly surjective, which is a contradiction.

Sufficient. Firstly, we prove that:

$$\rho := \inf_{||x||=1} \| \lambda T(x) - S(x) \| > 0. \tag{3.7}$$

Assume, ad absurdum,

$$\rho = 0. \tag{3.8}$$

With (3.8), we obtain a sequence $(x_n)_{n \in \mathbf{N}}$, $x_n \in X$,

$$\|x_n\| = 1 \tag{3.9}$$

and

$$\lim_{n \to \infty} [\lambda T(x_n) - S(x_n)] = 0. \tag{3.10}$$

The sequence (x_n) being bounded, $(S(x_n))_{n \in \mathbf{N}}$ has a subsequence (x_{k_n}) convergent in Y, and let $\gamma = \lim\limits_{n \to \infty} S(x_{k_n})$. However, T is surjective and $\lambda \neq 0$, so $\exists\ x_0 \in X$ such that $\lambda T(x_0) = \gamma$, and then, from (3.10),

$$\lim_{n \to \infty} \lambda T(x_{k_n}) = \lambda T(x_0). \tag{3.11}$$

From (3.11), T having a continuous inverse, we obtain

$$\lim_{n \to \infty} x_{k_n} = x_0. \tag{3.12}$$

(3.12) imposes, on the one hand, $\|x_0\| \overset{(3.9)}{=} 1$ and, on the other hand, $\lim\limits_{n \to \infty} S(x_{k_n}) = S(x_0)$, which, combined with (3.10) and (3.11), gives $\lambda T(x_0) - S(x_0) = 0$, which is a contradiction, and hence (3.7).

Thus, from (3.7),

$$\left\| \lambda T\left(\frac{x}{||x||}\right) - S\left(\frac{x}{||x||}\right) \right\| \geq \rho\ \forall x \in X \backslash \{0\},$$

so

$$\rho \|x\|^{a} \leq \|\lambda T(x) - S(x)\|\ \forall x \in X \backslash \{0\}. \tag{3.13}$$

From (3.13),

$$\lim_{||x||\to+\infty} \| \lambda T(x) - S(x) \| = +\infty, \tag{3.14}$$

from which one concludes that $\lambda T - S$ is surjective (see Proposition 2.1).

This surjectivity is regular. Indeed, assuming, ad absurdum, the contrary, we obtain $R > 0$ such that $\forall n \in \mathbf{N}\ \exists x'_n \in X$, $\|x'_n\| > n$ and

$$\| \lambda T(x'_n) - S(x'_n) \| \leq R. \tag{3.15}$$

However, since $\lim_{n\to\infty} \|x'_n\| = +\infty$, we have $\lim_{n\to\infty} \|\lambda T(x'_n) - S(x'_n)\| \overset{(3.14)}{=} +\infty$ and obtain a contradiction with (3.15). □

Proposition 3.6. *Let X be a real reflexive Banach space, smooth and having the property (H), which is compactly embedded in the real Banach space Z, and N: $Z \to Z^*$ an a-homogeneous odd semicontinuous operator. Then the operator $\lambda J_\varphi - N$, with J_φ being the duality map on X with $\varphi(t) = t^a$, $\lambda \neq 0$, is regularly surjective iff λ is not an eigenvalue for the couple (J_φ, N) ([11]).*

Explanation. In the expressions $\lambda J_\varphi - N$ and (J_φ, N), N is actually i' o N o i, i: $X \to Z$ linear compact injection, i': $Z^* \to X^*$ is its adjoint.

Proof ([11])**.** We apply Proposition 3.5 with $T := J_\varphi$, $S := i'$ o N o i, correctly, as J_φ is (K, L, a) with $K = L = 1$, bijective with continuous inverse (Proposition 2.26), odd and S is odd, a-homogeneous and compact (see the proof of Proposition 2.30). □

3.2. Applications

3.2.1. Application for the p-Laplacian and p-Pseudo-Laplacian

In Proposition 3.6, we now take $(X, \|\cdot\|_X) = (W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$, where $p \in (1, +\infty)$ and Ω is an open bounded set of C^1 class from $\mathbf{R}^n$, $n \geq 2$ (hence $J_\varphi = -\Delta_p$, $\varphi(t) = t^{p-1}$, Proposition 2.39), $(Z, \|\cdot\|_Z) = (L^p(\Omega), \|\cdot\|_{0,p})$, N: $L^p(\Omega) \to L^{p'}(\Omega)$, $\frac{1}{p}+\frac{1}{p'} = 1$, $Nu = |u|^{p-2}$ u.

$W_0^{1,p}(\Omega)$ is uniformly convex (Proposition 2.37) and hence also reflexive (Proposition 2.9) with the property (H), with its norm being Gâteaux differentiable (Corollary 2.6) and hence smooth (Proposition 2.3). It is compactly embedded in $L^p(\Omega)$. For the last assertion, we can mention the following:

Theorem 3.1. *Let Ω be a bounded set of the C^1 class. Then*

$$p < n \Rightarrow W^{1,p}(\Omega) \subset L^q(\Omega)\ \forall q \text{ in } [1, p^*],\ \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n},$$

$$p = n \Rightarrow W^{1,p}(\Omega) \subset L^q(\Omega)\ \forall q \text{ in } [1, +\infty),$$

$$p > n \Rightarrow W^{1,p}(\Omega) \subset C(\overline{\Omega}),$$

in all cases with compact injections (*Rellich-Kondrashev*).

Concerning N, it is the duality map on $L^p(\Omega)$ relative to the weight $t \to t^{p-1}$ (see the following Proposition 3.8); consequently, N is a homeomorphism of $L^p(\Omega)$ on $L^{p'}(\Omega)$ (Corollary 2.5), odd and $(p-1)$-homogeneous.

We apply Proposition 3.6 in order to obtain the following statement [11,13].

Proposition 3.7. *Let p be from* $(1, +\infty)$ *and* $\lambda \neq 0$. *If*

$$\lambda(-\Delta p\ u) = |u|^{p-2}\ u$$

does not have a nonzero solution in $W_0^{1,p}(\Omega)$*, then, for any h from* $W^{-1,p'}(\Omega)$*, the equation*

$$\lambda(-\Delta p\ u) = |u|^{p-2}\ u + h$$

has solution in $W_0^{1,p}(\Omega)$ *in the sense of* $W^{-1,p'}(\Omega)$.

Explanation. The term $|u|^{p-2}\ u$ from (3.16) and (3.17) is actually considered to be its image through a compact embedding of $L^{p'}(\Omega)$ in $W^{-1,p'}(\Omega)$ (use Schauder's theorem).

Regarding the operator N, we can complete it with the following result.

Proposition 3.8. *The duality map on* $L^p(\Omega)$*,* $p \in (1, +\infty)$*, of weight* $\varphi(t) = t^{p-1}$ *is*

$$J\varphi\ u = |u|^{p-1}\ \mathrm{sgn}\ u,\ u\ \in\ L^p(\Omega)$$

i.e.,

$$\langle J_\varphi\ u,\ h\rangle = \int_\Omega |u|^{p-1}(\mathrm{sgn} u)h\mathrm{d}x\ \forall h\ \in\ L^p(\Omega)$$

Proof. Let Ψ: $\Psi(u) = \frac{1}{p}||u||_{0,p}^p$. As $\Psi(u) = \int_0^{||u||_{0,p}} \varphi(t)dt$, we have (Proposition 2.2, 3°)

$$J_\varphi\ (u) = \partial\Psi(u).$$

However, $\Psi'(u)(h) = \int_\Omega |u|^{p-1}(\mathrm{sgn} u)hdx\ \forall h$ from $L^p(\Omega)$ ([46]), and thus the conclusion. □

Proposition 3.9. *In the statement of Proposition 3.7, if* $p \in [2, +\infty)$*, then* $-\Delta_p$ can be replaced by $-\Delta_p^s$ *([11,13]).*

Proof ([11,13])**.** In Proposition 3.6, we take $(X, \|\cdot\|) = (W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$ (see above), and $(Z, \|\cdot\|_Z) = (L^p(\Omega), \|\cdot\|_{0,p})$, N: $L^p(\Omega) \to L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$, $Nu = |u|^{p-2}\ u$, and take into account that $(W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$ is uniformly convex (see also Proposition 2.42 and Corollary 2.7 above). The compact embedding of $W_0^{1,p}(\Omega)$ in $L^p(\Omega)$ is given by the equivalence of the norms $\|\cdot\|_{1,p}$ and $\|\cdot\|_{1,p}$ since $\|\cdot\|_{1,p}$ is equivalent to the norm $|\cdot|_{1,p}$ (see the p-pseudo-Laplacian in Section 2.3.2). □

3.2.2. Another Application for p-Laplacian

Here, in Proposition 3.6, we take $(X, \|\cdot\|_X) = (W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$, where Ω is an open bounded set of C^1 class in $\mathbf{R}^n$, $n \geq 2$, $(Z, \|\cdot\|_Z) = (L^p(\Omega), \|\cdot\|_{0,p})$, N: $L^p(\Omega) \to L^{p'}(\Omega)$, $\frac{1}{p}+\frac{1}{p'} = 1$, $Nu = N_f\ u$, N_f is the Nemytskii operator, with $f: \Omega \times \mathbf{R} \to \mathbf{R}$ a Carathéodory function which verifies

1° $|f(x, s)| \leq c_1\ |s|^{p-1} + \beta(x)\ \forall s \in \mathbf{R}, \forall x \in \Omega\backslash A, \mu(A) = 0$, where $c_1 \geq 0$, $\beta \in L^{p'}(\Omega)$;

2° f is odd and $(p-1)$-homogeneous in the second variable.

Then, N_f is odd, $(p-1)$-homogeneous and continuous (Proposition 2.46). We apply Proposition 3.6 (see also Section 3.2.1 above) and obtain the following:

Proposition 3.10. *Let p be from* $(1, +\infty)$ *and* $\lambda \neq 0$*. If*

$$\lambda(-\Delta_p\ u) = N_f\ u$$

has no nonzero solution in $W_0^{1,p}(\Omega)$ *in the sense of* $W^{-1,p'}(\Omega)$*, then, for any h from* $W^{-1,p'}(\Omega)$*, the equation*

$$\lambda(-\Delta_p\ u) = f(\cdot, u(\cdot)) + h,\ x \in \Omega$$

has a solution in $W_0^{1,p}(\Omega)$ in the sense of $W^{-1,p'}(\Omega)$ ([11,13]).

Remark 3.1. *This statement can be compared with Proposition 2.48 above.*

Remark 3.2. *Applications to real phenomena regarding the nonlinear elastic membrane with p-Laplacian and with p-pseudo-Laplacian will be developed in the second part of this article.*

4. Surjectivity to Different Homogeneity Degrees

The propositions in this section originate from another assertion of the author [11,15], which generalizes a theorem of Fučik [42], but they are also based on other propositions obtained by the author. Applications to partial differential equations (weak solutions) are also given.

4.1. Theoretical Results

Proposition 4.1. *Let X and Y be real normed spaces, X complete and reflexive, T: X → Y (K, L, a) bijection odd with continuous inverse and S: X → Y odd compact operator b-strongly quasi-homogeneous relative to S_0, b < a. For any $\lambda \neq 0$, the operator $\lambda T - S$ is surjective.*

Remark 4.1. *The author proposed this weakened version of the theorem from [42], i.e., with normed space instead of Banach space, and bijection with continuous inverse instead of homeomorphism.*

Proof. According to Corollary 2.2, it is sufficient to prove:

$$\overline{\lim_{||x||\to+\infty}} \frac{||Sx||}{||x||^a} \left(= \lim_{||x||\to+\infty} \frac{||Sx||}{||x||^a} \right) = 0. \tag{4.1}$$

Supposing, ad absurdum, the contrary, we obtain a sequence $(x_n)_{n\in\mathbf{N}}$, $x_n \in X\backslash\{0\}$, $\lim_{n\to\infty} \|x_n\| = +\infty$, for which

$$\frac{||S(x_n)||}{||x_n||^a} \geq \varepsilon_0 \ \forall n \text{ in } \mathbf{N}, \tag{4.2}$$

where $\varepsilon_0 > 0$. With X being complete and reflexive, the bounded sequence $(y_n)_{n\in\mathbf{N}}$, $y_n := \frac{x_n}{||x_n||}$ has a weakly convergent subsequence, one denotes this identically, $y_n \overset{w}{\to} y_0$. Then,

$$\lim_{n\to\infty} \frac{S(||x_n||y_n)}{||x_n||^b} = S_0\ (y_0)$$

and as $\lim_{n\to\infty} \frac{||x_n||^b}{||x_n||^a} = 0$, we obtain

$$\lim_{n\to\infty} \frac{||S(x_n)||}{||x_n||^a} = 0,$$

in contradiction with (4.2), and hence (4.1). □

An immediate consequence:

Proposition 4.2. *Let X be a real reflexive Banach space and smooth with the property (H) which is compactly embedded in the real Banach space Z and N: Z→ Z* odd semicontinuous and b-homogeneous operator.*
Then, for any $\lambda \neq 0$,

$$\lambda J_\varphi - N,$$

J_φ the duality map on X with $\varphi(t) = t^a$, a > b, is surjective ([11,15]).

Clarification. In the expression $\lambda J_\varphi - N$, N is actually (abbreviation!) the operator i' o N o i, i' is the adjoint of i.

Proof ([11,15]). Applying Proposition 4.1, with $T = J_\varphi$ ($K = L = 1$ is odd bijective with continuous inverse (Proposition 2.26)), $S := i' \circ N \circ i$ is odd, compact and b-homogeneous. It remains only to prove that S is b-strongly quasi-homogeneous relative to S. Let $t_n \downarrow 0$ and $u_n \xrightarrow{w} u_0$, then $t_n^b S\left(\frac{u_n}{t_n}\right) = S(u_n)$ and $S(u_n) \to S(u_0)$: $u_n \xrightarrow{w} u_0 \overset{i \text{ compact}}{\Rightarrow} i(u_n) \to i(u_0) \overset{N \text{ semicontinuous}}{\Rightarrow} N(i(u_n)) \xrightarrow{w} N(i(u_0)) \overset{i' \text{ compact}}{\Rightarrow} S(u_n) \to S(u_0)$. □

4.2. Applications

4.2.1. First Application

We now take $(X, \|\cdot\|_X) = (W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$ with $p \in (1, +\infty)$ and $(X, \|\cdot\|_X) = (W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$ with $p \in [2, +\infty)$, respectively, and Ω is open bounded set of C^1 class in $\mathbf{R}^n$, $n \geq 2$, $\varphi(t) = t^{p-1}$; hence $J_\varphi = -\Delta_p$ (Proposition 2.39) and $J_\varphi = -\Delta_p^s$ (Proposition 2.43), respectively, $(Z, \|\cdot\|_Z) = (L^q(\Omega), \|\cdot\|_{0,q})$ with $q \in (1, p)$, N: $L^q(\Omega) \to L^{q'}(\Omega)$, $Nu = |u|^{q-2} u$, $\frac{1}{q} + \frac{1}{q'} = 1$. N is odd, continuous (an even homeomorphism; see Proposition 3.8 and Corollary 2.5 above) and $(q-1)$-homogeneous, $q - 1 < p - 1$. Applying Proposition 4.2, we obtain the following.

Proposition 4.3. *Under the above conditions, for any $\lambda \neq 0$ and for any h from $W^{-1,p'}(\Omega)$, there exists u_0 in $W_0^{1,p}(\Omega)$ such that*

$$\lambda(-\Delta_p)u_0 = (i' \circ N \circ i)u_0 + h$$

and

$$\lambda(-\Delta_p^s)u_0 = (i' \circ N \circ i)u_0 + h$$

respectively ([11,15]).

4.2.2. Second Application

This second application of Proposition 4.2 is made by replacing the operator N from Proposition 4.3 with N_f, the Nemytskii operator. More precisely, we take N: $L^q(\Omega) \to L^{q'}(\Omega)$, $N = N_f$, with $f : \Omega \times \mathbf{R} \to \mathbf{R}$ odd Carathéodory function and $(q-1)$-homogeneous in the second variable, which verifies the growth condition

$$|f(x, s)| \leq c_1 |s|^{q-1} + \beta(x) \ \forall s \text{ in } \mathbf{R}, \forall x \text{ in } \Omega \backslash A, \mu(A) = 0,$$

where $c_1 \geq 0$, $\beta \in L^{q'}(\Omega)$.

Then, N_f is odd, $(q-1)$-homogeneous and continuous (Proposition 2.46), and one can apply Proposition 4.2 to obtain:

Proposition 4.4. *Under the above conditions, for any $\lambda \neq 0$ and for any h in $W^{-1,p'}(\Omega)$, there exists u_0 in $W_0^{1,p}(\Omega)$ such that*

$$\lambda(-\Delta_p)u_0 = (i' \circ N_f \circ i)u_0 + h$$

and

$$\lambda(-\Delta_p^s)u_0 = (i' \circ N_f \circ i)u_0 + h$$

respectively ([11,15]).

Remark 4.2. *Applications to models of real phenomena involving a nonlinear elastic membrane and a nonlinear elastic membrane with p-Laplacian and the p-pseudo-Laplacian will be provided in the second part of this work.*

5. Weak Solutions Starting from Ekeland Variational Principle

5.1. Critical Points and Weak Solutions for Elliptic Type Equations

The theoretical results in the following two subsections were obtained by the author in [17].

5.1.1. Theoretical Support

In order to introduce the first result, theoretical support will be given, starting with:

Ekeland Principle. *Let (X, d) be a complete metric space and φ: $X \to (-\infty, +\infty]$ bounded from below, lower semicontinuous and proper. For any $\varepsilon > 0$ and u of X with*

$$\varphi(u) \leq \inf \varphi(X) + \varepsilon$$

and for any $\lambda > 0$, there exists v_ε in X such that

$$\varphi(v_\varepsilon) \; < \; \varphi(w) + \frac{\varepsilon}{\lambda} d(v_\varepsilon, \, w) \; \forall w \; \in \; X \backslash \{v_\varepsilon\}$$

and

$$\varphi(v_\varepsilon) \leq \varphi(u), \, d(v_\varepsilon, u) \leq \lambda$$

([22,63,64]).

We continue with the following:

Definition 5.1. *Let X be a real normed space, β a bornology (Definition 2.3) on X, and let φ: $X \to \mathbf{R}$. Let c be in $\mathbf{R}$ and F a nonempty subset of X. φ verifies the Palais-Smale condition on level c around F (or relative to F), $(\mathrm{PS})_{\mathrm{c,F}}$, with respect to β, when $\forall (u_n)_{n \geq 1}$ a sequence of points in X for which*

$$\lim_{n\to\infty} \varphi(u_n) = c, \; \lim_{n\to\infty} ||\nabla_\beta \varphi(u_n)|| \; = 0 \text{ and } \lim_{n\to\infty} \operatorname{dist}(u_n, \, F) = 0, \tag{5.1}$$

this sequence has a convergent subsequence.

To clarify the above notation, see Definition 2.3 regarding the β-derivative.

Let us introduce the definition of the metric gradient in order to provide other observations related to this central notion for the following statement.

Definition 5.2. *In a real normed space X, consider the Gâteaux-differentiable functional $f: X \to \mathbf{R}$. The metric gradient of f is the multiple-valued mapping:*

$$\nabla f \; : \; X \to \; \mathrm{P}\,(X), \; \nabla \, f(x) = i^{-1} J_* f'_w(x),$$

where J_*: $X^* \to \mathrm{P}\,(X^{**})$ is the duality mapping on X^* corresponding to the identity, and i is the canonical injection of X into X^{**}: $i(x) = x^{**}$, $\langle x^{**}, x^* \rangle = \langle x^*, x \rangle$, $\forall x^* \in X^*$.

Consequently, for any $x \in X$: $\nabla f\,(x) = \{y \in X$: $i(y) \in J_* f'_w(x)\} = \{y \in X$: $\langle i(y), f'_w(x) \rangle = \langle f'_w(x), y \rangle = \|f'_w(x)\|^2$, $\|i(y)\| = \|y\| = \|f'_w(x)\|\}$. If X is reflexive, for any $x \in X$, $\nabla f\,(x)$ is nonempty. X^{**} being strictly convex, J_* is single-valued. So, if X is reflexive and strictly convex, then $\nabla f: X \to X$, $\nabla f\,(x) = i^{-1} J_* f'_w(x)$, and the following equalities hold:

$$\langle f'_w\,(x), \, \nabla f\,(x) \rangle = ||f'_w(x)||^2, \; ||\nabla f\,(x)|| = ||f'_w(x)||.$$

Through the minimization of a functional on F (*minimization with constraints*), its global critical points may be obtained.

As a preliminary, we generalize some results from [65] by introducing Banach space instead of Hilbert space and Gâteaux differentiability instead of C^1-class Fréchet.

Proposition 5.1. *Let X be real reflexive strictly convex Banach space, let φ: X → **R** be lower semicontinuous and Gâteaux differentiable and let F be a closed subset of X such that for every u from F with the metric gradient* $\nabla \varphi\,(u) \neq 0$, *for sufficiently small r > 0,*

$$\left(u - \delta \frac{\nabla \varphi(u)}{||\nabla \varphi(u)||}\right) \in F, \forall \delta \in [0, r]. \tag{5.2}$$

Then, if φ is lower bounded on F, for every $(v_n)_{n\geq 1}$ *a minimizing sequence for φ on F, there exists a sequence* $(u_n)_{n\geq 1}$ *in F such that*

$$\|\varphi\prime(u_n)\| \ \leq \sqrt{\varepsilon_n}, \tag{5.3}$$

$$\varphi(u_n) \leq \varphi(v_n)\ \forall n \tag{5.4}$$

$$\lim_{n\to\infty} \|u_n - v_n\| \ = 0, \tag{5.5}$$

where $\varepsilon_n > 0$ and $\varepsilon_n \to 0$.

Remark 5.1. *This result is reported in [65] as Lemma 9 in the frame of Hilbert spaces having the function φ of the Fréchet* C^1 *class, but the condition (5.3) is more complicated due to another condition imposed on the set F.*

Proof. Denote $c := \inf \varphi(F)$ and let n be from **N**. For $\varepsilon_n := \varphi(v_n) - c + \frac{1}{n}$, hence $\varepsilon_n > 0$, we have $\varphi(v_n) < c + \varepsilon_n$. We apply the enounced *Ekeland principle* with $\lambda = \sqrt{\varepsilon_n}$, $\exists u_n$ in F, with known properties. Thus, we obtain the sequence $(u_n)_{n\geq 1}$ satisfying (5.4), (5.5) ($||u_n - v_n|| \leq \sqrt{\varepsilon_n}$, $\varepsilon_n \to 0$) and

$$\varphi(v) \geq \varphi(u_n) - \sqrt{\varepsilon_n}||v - u_n|| \ \ \forall v \in F. \tag{5.6}$$

Next, we verify (5.3). It is sufficient to work under the assumption that $||\varphi\prime_w(u_n)|| > 0$ $\forall n$. Thus, we apply the hypothesis made in the statement with respect to F with $u = u_n$ and, denoting, for $\delta \in (0, r]$, $v_\delta := u_n - \delta \frac{\nabla \varphi(u_n)}{||\nabla \varphi(u_n)||}$ ($\in F$), replace v_δ in (5.6) and find

$$\sqrt{\varepsilon_n}||v_\delta - u_n|| \geq \varphi(u_n) - \varphi(v_\delta),$$

multiply this inequality by $\frac{1}{\delta}$, $\delta > 0$, and take the limit for $\delta \to 0+$ in order to keep the sense of the inequality. We remark that $\lim\limits_{\delta\to 0} v_\delta = u_n$; $\lim\limits_{\delta\to 0} \frac{||v_\delta - u_n||}{\delta} = \lim\limits_{\delta\to 0} \frac{\delta \frac{||\nabla \varphi(u_n)||}{||\nabla \varphi(u_n)||}}{\delta} = 1$. Consider that the existence of the limit for $\delta \to 0$ implies the existence of the limit for $\delta \to 0\pm$, together with their equality, $\lim\limits_{\delta\to 0+} \frac{\varphi(u_n) - \varphi(v_\delta)}{\delta} = \lim\limits_{\delta\to 0+} \frac{\varphi\left(u_n - \delta \frac{\nabla \varphi(u_n)}{||\nabla \varphi(u_n)||}\right) - \varphi(u_n)}{-\delta} = \varphi_{\mathrm{w}}{}'(u_n)\left(\frac{\nabla \varphi(u_n)}{||\nabla \varphi(u_n)||}\right) = \frac{1}{||\nabla \varphi(u_n)||}\langle \varphi_{\mathrm{w}}{}'(u_n), \nabla \varphi(u_n)\rangle = \frac{1}{||\nabla \varphi(u_n)||}\,||\varphi\prime_w(u_n)||^2 = ||\varphi'_w(u_n)||$; taking into account the definition of the Gâteaux derivative and the above considerations on the metric gradient, (5.3) is also fulfilled. □

Remark 5.2. *The Gâteaux derivative from the above statement can be replaced by any β-derivative, and the result remains the same. In the case of the Fréchet derivative, the condition "φ lower semicontinuous" must be removed from the statement.*

Notation. φ: $X \to$ **R** is β-differentiable, $c \in$ **R** ⇒

$$K_c(\varphi) := \left\{x \in X : \varphi(x) = c,\ \nabla_\beta \varphi(x) = 0\right\}.$$

Proposition 5.2. *Let X be a real reflexive strictly convex Banach space and φ: X → **R** lower semicontinuous and Gâteaux differentiable and let F be a nonempty convex closed subset such that*

$(I - \nabla\varphi)(F) \subset F$, where I is the identity map. If φ is lower bounded on F, then for every $(v_n)_{n\geq 1}$, a minimizing sequence for φ on F, there is a sequence $(u_n)_{n\geq 1}$ in F such that

$$\varphi(u_n) \ \leq \ \varphi(v_n)\ \forall n,\ \lim_{n\to\infty}\|u_n - v_n\| \ = 0,\ \lim_{n\to\infty}\|\varphi\prime_{\mathrm{w}}(u_n)\| \ = 0.$$

Moreover, if φ satisfies $(\mathrm{PS})_{\mathrm{c,F}}$, where $c = \inf\varphi(F)$, then

$$F \cap K_c(\varphi) \neq \varnothing.$$

Proof. Applying Proposition 5.1, (5.2) is satisfied; indeed, if $u \in F$ and $\varphi\prime_w(u) \neq 0$, then, F being convex,

$$u - \delta\frac{\nabla\varphi(u)}{||\nabla\varphi(u)||} = \left(1 - \frac{\delta}{||\varphi'_{\mathrm{w}}(u)||}\right) u + \frac{\delta}{||\varphi'_{\mathrm{w}}(u)||}(I - \nabla\varphi)\ (u) \in F.$$

Let $(u_n)_{n\geq 1}$ be the sequence given by the statement. $c \leq \varphi(u_n) \leq \varphi(v_n)\ \forall n$, hence $\varphi(u_n) \to c$. $||\varphi'_{\mathrm{w}}(u_n)|| \overset{(5.3)}{\leq} \sqrt{\varepsilon_n}$, hence $||\varphi'_{\mathrm{w}}(u_n)|| \ \to 0$, clearly dist $(u_n, F) = 0$, and consequently, $(u_n)_{n\geq 1}$ has a convergent subsequence $(u_{k_n})_{n\geq 1}$, $u_{k_n} \to u_0 \in F$. This implies $||\varphi'_w(u_{k_n})|| \to ||\varphi'_w(u_0)|| = 0$ and thus u_0 is a global critical point of φ contained in F. □

5.1.2. Weak Solutions

Open set of C^1 *class in* $\mathbf{R}^{\mathrm{N}}$. We use the following notations (the norm is that Euclidean from $\mathbf{R}^{\mathrm{N}-1}$): $\mathbf{R}^{\mathrm{N}}_{+} = \{x = (x', x_{\mathrm{N}})$: $x_{\mathrm{N}} > 0\}$, $Q = \{x = (x',\ x_{\mathrm{N}}) : \|x'\| < 1,\ |x_{\mathrm{N}}| < 1\}$, $Q_+ = Q \cap \mathbf{R}^{\mathrm{N}}_{+}$, $Q_0 = \{x = (x', x_{\mathrm{N}})$: $\|x'\| < 1, x_{\mathrm{N}} = 0\}$. Let Ω be an open nonempty set in $\mathbf{R}^{\mathrm{N}}$, $\Omega \neq \mathbf{R}^{\mathrm{N}}$ and $\partial\Omega$ its boundary. By definition, Ω is of C^1 *class* if $\forall x$ from $\partial\Omega$ $\exists U$ is a neighborhood of x in $\mathbf{R}^{\mathrm{N}}$ and $f : Q \to U$ is bijective such that $f \in C^1(\overline{Q}), f^{-1} \in C^1(\overline{U}), f\ (Q_+) = U \cap \Omega$, and

$$f\ (Q_0) = U \cap \partial\Omega.$$

Weak solution. Let Ω be an open bounded nonempty set in $\mathbf{R}^{\mathrm{N}}$, $N > 1, f : \Omega \times \mathbf{R}^{\mathrm{N}} \to \mathbf{R}$, and $u_0 \in W_0^{1,p}(\Omega)$. Consider the problems:

$$(*)\ \begin{cases} -\Delta_{\mathrm{p}} u = f(x,u),\ x \in \Omega \\ \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

and

$$(**)\begin{cases} -\Delta^{\mathrm{s}}_{\mathrm{p}} u = f(x,u),\ x \in \Omega \\ \quad u = 0 \text{ on } \partial\Omega. \end{cases}$$

The equality $u = 0$ on $\partial\Omega$ for both problems is in the sense of the trace (Definition 2.8). $\overline{u}$ from $X = W_0^{1,p}(\Omega)$ is, by definition, a *weak solution* for $(*)$ and $(**)$ if $\overline{u} = 0$ on $\partial\Omega$ in the sense of the trace and

$$\int_\Omega |\nabla\overline{u}|^{\mathrm{p}-2}\nabla\overline{u}\cdot\nabla v dx - \int_\Omega f(x,\overline{u}(x))v dx = 0\ \forall v \in\ W_0^{1,p} \tag{5.7}$$

and

$$\sum_{\mathrm{i}=1}^{n}\int_\Omega\left|\frac{\partial\overline{u}}{\partial x_{\mathrm{i}}}\right|^{\mathrm{p}-2}\frac{\partial\overline{u}}{\partial x_{\mathrm{i}}}\frac{\partial v}{\partial x_{\mathrm{i}}}dx - \int_\Omega f(x,\overline{u}(x))v dx = 0\ \forall v \in\ W_0^{1,p} \tag{5.8}$$

respectively.

Remark 5.3. *Here, ∇w is the weak gradient (see* Section 2.3.1 *here and what follows). $X := W_0^{1,p}(\Omega)$ is endowed in the first case $(*)$ with the norm $\|\cdot\|_{1,\,\mathrm{p}}$ that was defined in Section 2.3.1,* *which is equivalent to the norm* $u \to \left(||u||^{p}_{L^p(\Omega)} + \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|^{p}_{L^p(\Omega)} \right)^{\frac{1}{p}}$ *(also highlighted there). For the second case* $(**)$*, equip the same vector space with the norm* $u \to \| u \|_{1,p} = \left(\sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|^{p}_{L^p(\Omega)} \right)^{\frac{1}{p}}$, *which is equivalent to* $u \to | u |_{1,p} = \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}$ *(Section 2.3.2).*

For the *Nemytskii operator*, see also Section 2.3.1 and suppose now that $\mu(\Omega) < +\infty$. Then,

$$u_n(x) \underset{x\in\Omega}{\overset{\mu}{\to}} u_0(x) \;\Rightarrow\; N_f u_n(x) \underset{x\in\Omega}{\overset{\mu}{\to}} N_f u_0(x).$$

Assume that f satisfies the *growth condition*:

$$|f(x,\, s)| \;\le\; c\,|s|^{p-1} + b(x),\ \forall x \;\in\; \Omega\backslash A \text{ with } \mu(A) = 0, \forall x \;\in\; \mathbf{R},$$

where $c \ge 0$, $p > 1$ and $b \in L^q(\Omega)$, $q \in [1, +\infty]$. Then $N_f\,(L^{q(p-1)}(\Omega)) \subset L^q(\Omega)$; N_f is continuous ($q < +\infty$) and bounded on $L^{(p-1)q}(\Omega)$ (Proposition 2.46). If Ω is bounded and $\frac{1}{p} + \frac{1}{q} = 1$, then $N_f(L^p(\Omega)) \subset L^q(\Omega)$ with N_f continuous; moreover, $N_F(L^p(\Omega)) \subset L^1(\Omega)$, with N_F continuous (*ibidem*), where $F(x,s) = \int_0^s f(x,t)dt$, and Φ: $L^p(\Omega) \to \mathbf{R}$, $\Phi(u) = \int_\Omega F(x,u(x))dx$ is of Fréchet C^1 class and $\Phi' = N_f$ [65], so it is also Gâteaux differentiable.

Theorem 5.1. *Let* Ω *be an open bounded nonempty set in* $\mathbf{R}^N$ *and* $f : \Omega \times \mathbf{R} \to \mathbf{R}$ *a Carathéodory function with the growth condition:*

$$|f\,(x,s)| \le c\,|s|^{p-1} + b(x), \tag{5.9}$$

where $c > 0$, $2 \le p \le \frac{2N}{N-2}$ *when* $N \ge 3$ *and* $2 \le p < +\infty$ *when* $N = 1, 2$, *and where* $b \in L^q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$.

Then, the energy functional $\varphi : W_0^{1,p}(\Omega) \to \mathbf{R}$, *and*

$$\varphi(u) = \frac{1}{p}||u||^{p}_{1,p} - \int_\Omega F(x,u(x))dx, \text{ for the problem } (*) \tag{5.10}$$

and

$$\varphi(u) = \frac{1}{p}\| u \|^{p}_{1,p} - \int_\Omega F(x,u(x))dx, \text{ for the problem } (**), \tag{5.11}$$

where $F(x,s) = \int_0^{s} f(x,t)dt$ *is Gâteaux differentiable on* $W_0^{1,p}(\Omega)\backslash\{0\}$ *and, respectively,*

$$\varphi'_w(u)(v) = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx - \int_\Omega f(x,u(x))v dx, \forall u,v \in W_0^{1,p}(\Omega) \tag{5.12}$$

and

$$\varphi'_w(u)(v) = \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx - \int_\Omega f(x,u(x))v dx, \forall u,v \in W_0^{1,p}(\Omega). \tag{5.13}$$

Proof. One may consider φ, in both cases, to be the sum of two other functions. The second of these functions being Gâteaux differentiable (see the above considerations), it is sufficient to remark that the maps $u \to \frac{1}{p}||u||^{p}_{1,p}$ and $u \to \frac{1}{p}\| u \|^{p}_{1,p}$ are also Gâteaux differentiable on $W_0^{1,p}(\Omega)\backslash\{0\}$ (Propositions 2.39 and 2.43) and then φ is Gâteaux differentiable on $W_0^{1,p}(\Omega)\backslash\{0\}$. □

Corollary 5.1. *Let Ω and f be as in Theorem 5.1 above. Then the weak solutions of* $(*)$ *and* $(**)$ *are precisely the critical points of the functional* $\varphi : W_0^{1,p}(\Omega) \to \mathbf{R}$, *respectively:*

$$\varphi(u) = \frac{1}{p}||u||_{1,p}^{p} - \int_\Omega F(x,u(x))dx,\, F(x,s) := \int_0^s f(x,t)dt$$

and

$$\varphi(u) = \frac{1}{\mathrm{p}}\| u \|_{1,\mathrm{p}}^{\mathrm{p}} - \int_\Omega F(x,u(x))dx,\, F(x,s) := \int_0^s f(x,t)dt.$$

Proof. Indeed, if $\overline{u}$ is a weak solution of $(*)$ and $(**)$, then $\varphi'_{\mathrm{w}}(\overline{u})(v) = 0\ \forall v \in W_0^{1,p}(\Omega)$ ((5.7) and (5.8) respectively (Theorem 5.1)), hence, $\varphi'_{\mathrm{w}}(\overline{u}) = 0$. The inverse assertion is obvious. □

Weak subsolutions and weak supersolutions of $(*)$ and $(**)$. Let Ω be an open bounded set of C^1 class in $\mathbf{R}^{\mathrm{N}}$, $N \geq 3, f : \Omega \times \mathbf{R} \to \mathbf{R}$ a Carathéodory function and let $\overline{u} \in W_0^{1,p}(\Omega)$. $\overline{u}$ is a *weak subsolution* and a *weak supersolution*, respectively, of $(*)$ or $(**)$ if

$$\overline{u} \leq\ 0 \text{ on } \partial\Omega \text{ and } \overline{u} \geq 0 \text{ on} \partial\Omega, \text{ respectively, and}$$

$$\begin{cases} \int\limits_\Omega |\nabla\overline{u}|^{\mathrm{p}-2}\nabla\overline{u}\cdot\nabla v dx \leq \int\limits_\Omega f(x,\overline{u}(x))v dx\ \forall v \in W_0^{1,\mathrm{p}}(\Omega),\ v \geq 0 \\ \qquad\qquad\qquad\text{respectively} \\ \int\limits_\Omega |\nabla\overline{u}|^{\mathrm{p}-2}\nabla\overline{u}\cdot\nabla v dx \geq \int\limits_\Omega f(x,\overline{u}(x))v dx\ \forall v\ \in W_0^{1,\mathrm{p}}(\Omega),\ v \geq 0. \end{cases} \tag{5.14}$$

or

$$\begin{cases} \sum\limits_{\mathrm{i}=1}^{n}\int\limits_\Omega \left|\frac{\partial\overline{u}}{\partial x_{\mathrm{i}}}\right|^{\mathrm{p}-2}\frac{\partial\overline{u}}{\partial x_{\mathrm{i}}}\frac{\partial v}{\partial x_{\mathrm{i}}}dx \leq \int\limits_\Omega f(x,\overline{u}(x))v dx\ \forall v \in W_0^{1,\mathrm{p}}(\Omega),\ v \geq 0 \\ \qquad\qquad\qquad\text{respectively} \\ \sum\limits_{\mathrm{i}=1}^{n}\int\limits_\Omega \left|\frac{\partial\overline{u}}{\partial x_{\mathrm{i}}}\right|^{\mathrm{p}-2}\frac{\partial\overline{u}}{\partial x_{\mathrm{i}}}\frac{\partial v}{\partial x_{\mathrm{i}}}dx \geq \int\limits_\Omega f(x,\overline{u}(x))v dx\ \forall v\ \in W_0^{1,\mathrm{p}}(\Omega),\ v \geq 0. \end{cases} \tag{5.15}$$

Proposition 5.3. *Let Ω be an open bounded set of* C^1 *class in* $\mathbf{R}^{\mathrm{N}}$, $N \geq 3$, *and let* $f : \Omega \times \mathbf{R} \to \mathbf{R}$ *be a Carathéodory function and* u_1 *and* u_2 *from* $W_0^{1,p}(\Omega)$ *bounded weak subsolution and weak supersolution of* $(*)$, *respectively, with* $u_1(x) \leq u_2(x)$ *a.e. on* Ω. *Suppose that* f *verifies* (5.9) *and there is* $\rho > 0$ *such that the function* g: $g(x, s) = f(x, s) + \rho s$ *is strictly increasing in* s *on* $[\inf u_1(\Omega), \sup u_2(\Omega)]$. *Then there is a weak solution* $\overline{u}$ *of* $(*)$ *in* $W_0^{1,p}(\Omega)$ *with the property*

$$u_1(x) \leq \overline{u}(x) \leq u_2(x) \text{ a.e. on } \Omega.$$

Proof. Taking the equivalent norm on $X = W_0^{1,p}(\Omega)$, we obtain

$$||u|| = \left(\rho||u||_{\mathrm{L^p}(\Omega)}^{\mathrm{p}} + \sum_{\mathrm{i}=1}^{n}\left\|\frac{\partial u}{\partial x_{\mathrm{i}}}\right\|_{\mathrm{L^p}(\Omega)}^{\mathrm{p}}\right)^{\frac{1}{\mathrm{p}}}.$$

Considering the functional φ: $W_0^{1,p}(\Omega) \to \mathbf{R}$,

$$\varphi(u) = \frac{1}{p}||u||^{\mathrm{p}} - \int\limits_\Omega G(x,u(x))dx, G(x,s) := \int\limits_0^{\mathrm{s}} g(x,t)dt. \tag{5.16}$$

where φ is Gâteaux differentiable, and its critical points are the weak solutions of $(*)$ (see Corollary 5.1 above). φ is also lower-bounded, with the norm on $L^p(\Omega)$ actually being of

Fréchet C^1 class (see, for instance, [43], Volume 2). Use Proposition 5.2, $(X, \|\cdot\|)$ being a reflexive strictly convex Banach space (see Section 2). Let

$$F := \{u \in W_0^{1,p}(\Omega) : u_1(x) \le u(x) \le u_2(x) \text{ a.e. on } \Omega\}.$$

F is closed convex. We also obtain

$$(I - \nabla\varphi)F \subset F.$$

Here, $\nabla\varphi$ denotes the metric gradient of φ. Since $(X, \|\cdot\|)$ is reflexive and strictly convex (see in Section 2), $\nabla\varphi$ is thus uni-valued, and it has the above-described properties. Indeed, let u be in F and $v := (I - \nabla\varphi)(u)$. We should prove that $v \in F$. $v = u - \nabla\varphi(u) \in W_0^{1,p}(\Omega)$ and $u_1(x) \le v(x) \le u_2(x)$. Since the definition relation of the subsolution for u_1 actually means $\varphi_w{}'(u_1)(w) \le 0$ $\forall w$ in $W_0^{1,p}(\Omega)$ with $w(x) \ge 0$ almost everywhere (a.e.) on Ω, and that of the supersolution for u_2 is $\varphi_w{}'(u_2)(w) \ge 0$ $\forall w$ in $W_0^{1,p}(\Omega)$, verifying $w(x) \ge 0$ a.e. on Ω, we will prove that $v(x) - u_1(x) \ge 0$ a.e. on Ω and $u_2(x) - v(x) \ge 0$ a.e. on Ω using the Gâteaux derivatives of φ in u_1 and u_2, respectively. $\varphi_w{}'(u_1)(v - u_1) = \varphi_w{}'(u_1)(u_1 - u) - \varphi_w{}'(u_1)(\nabla\varphi(u)) \le -\varphi_w{}'(u_1)(\nabla\varphi(u)) \le -\varphi_w{}'(u)(\nabla\varphi(u)) = -\|\varphi_w{}'(u)\|^2 \le 0$ (take into account that $u_1 \le u$, $\varphi_w{}'(u_1)$ is a linear map and some properties of the metric gradient). Also $\varphi_w{}'(u_2)(u_2 - v) = \varphi_w{}'(u_2)(u_2 - v) + \varphi_w{}'(u_2)(\nabla\varphi(u)) \ge \varphi_w{}'(u_2)(\nabla\varphi(u)) \ge \varphi_w{}'(u)(\nabla\varphi(u)) = \|\varphi_w{}'(u)\|^2 \ge 0$. φ is lower bounded on F, φ being continuous, actually (for this assertion, see Section 2). Until now, applying Proposition 5.2, for every $(v_n)_{n\ge1}$, a minimizing sequence for φ on F, there is a sequence $(u_n)_{n\ge1}$ in F such that $\varphi(u_n) \le \varphi(v_n)$ $\forall n$, $\lim_{n\to\infty} ||u_n - v_n|| = 0$, $\lim_{n\to\infty} ||\varphi'_w(u_n)|| = 0$. So $\lim_{n\to\infty} \varphi(u_n) = c$ and since $c := \inf \varphi(F)$, we have $\lim_{n\to\infty} ||\varphi'_w(u_n)|| = 0$ already, and the last property from the $(PS)_{c,F}$ condition is verified. To finish the proof, we once again apply Proposition 5.2. □

Example 1. *Consider the problem (Ω is an open bounded set of C^1 class in $\mathbf{R}^N$, $N \ge 3$)*

$$\begin{cases} -\Delta_p u = \alpha(x)\, u\, |u|^{p-2} \text{on } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.17}$$

where $p = \frac{2N}{N-2}$ and α is continuous with $1 \le \alpha(x) \le a < +\infty$ on Ω. Then $u_1 := 1$ is a weak subsolution, $u_2 := M$, $M > 1$ sufficiently big, is a weak supersolution, $|f(x, s)| \le a|s|^{p-1}$ (condition (5.9)), and $s \to \alpha(x)s|s|^{p-2} + s$ is increasing in s on $[1, M]$; consequently, according to Proposition 5.3, (5.17) has a weak solution $\bar{u}$ with $1 \le \bar{u}(x) \le M$ a.e. on Ω.

Proposition 5.4. *Let Ω be an open bounded set of C^1 class in $\mathbf{R}^N$, $N \ge 3$, and $f : \Omega \times \mathbf{R} \to \mathbf{R}$ a Carathéodory function and u_1, u_2 from $W_0^{1,p}(\Omega)$ bounded weak subsolution and weak supersolution of $(**)$, respectively, with $u_1(x) \le u_2(x)$ a.e. on Ω. Suppose that f verifies (5.9) and there is $\rho > 0$ such that the function g: $g(x, s) = f(x, s) + \rho s$ is strictly increasing in s on $[\inf u_1(\Omega), \sup u_2(\Omega)]$. Then there is a weak solution $\bar{u}$ of $(**)$ in $W_0^{1,p}(\Omega)$ with the property*

$$u_1(x) \le \bar{u}(x) \le u_2(x) \text{ a.e. on } \Omega.$$

Proof. We follow, step by step, the above proof for Proposition 5.3 considering the real reflexive strictly convex Banach space $X = W_0^{1,p}(\Omega)$ endowed with the norm $u \to \|u\|_{1,p} = \left(\sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}}$ or the equivalent norm $u \to u_{1,p} = \sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}$, which are both also equivalent to the other two norms used in Remark 5.3. The function φ is from (5.11), having the weak derivative given in Theorem 5.1. Using similar calculus, we obtain a similar conclusion. □

Example 2. *Consider the problem* (Ω *is an open bounded set of the* C^1 *class in* $\mathbf{R}^N$, $N \geq 3$)

$$\begin{cases} -\Delta_p^s u = \alpha(x)\, u\, |u|^{p-2} \text{on } \Omega, \\ u = 0 \text{ on } \partial\,\Omega\,, \end{cases} \tag{5.18}$$

where $p = \frac{2N}{N-2}$ *and* α *is continuous with* $1 \leq \alpha(x) \leq a < +\infty$ on Ω. *Then,* $u_1 := 1$ *is a weak subsolution,* $u_2 := M$, $M > 1$ *being sufficiently big, is a weak supersolution,* $|f(x,s)| \leq a\,|s|^{p-1}$ *(condition (5.9)), and* $s \to \alpha(x)s\,|s|^{p-2} + s$ *is increasing in* s *on* $[1, M]$; *consequently, according to Proposition 5.8, (5.18) has a weak solution* $\overline{u}$ *with* $1 \leq \overline{u}(x) \leq M$ *a.e. on* Ω.

Remark 5.4. *The results from Sections 5.1.1, 5.1.2 and 5.2.1 have been reported by the author in [17].*

Remark 5.5. *Applications to real phenomena, as well as an application in glaciology, a nonlinear elastic membrane, the pseudo-torsion problem and a nonlinear elastic membrane with the p-pseudo-Laplacian, will be presented in the second part of this article.*

5.2. Critical Points for Nondifferentiable Functionals

5.2.1. Theoretical Results

The meaning of the title is actually "not compulsory differentiable". We start this section with the following:

Definition 5.3. x_0 *is a critical point (in the sense of the Clarke subderivative) for the real function* f *if* $0 \in \partial f(x_0)$. *In this case,* $f(x_0)$ *is a critical value (in the sense of the Clarke subderivative) for* f. *To clarify this notion, the Clarke derivative should be introduced. Let* X *be a real normed space,* $E \subset X$, $f : E \to \mathbf{R}$, $x_0 \in \overset{o}{E}$ *and* $v \in X$. *We set*

$$f^0(x_0;\, v) := \overline{\lim_{\substack{x \to x_0 \\ t \to 0+}}} \frac{f(x+tv) - f(x)}{t}.$$

The upper limit obviously exists. $f^0(x_0; v)$ *is by definition the Clarke derivative* (*or the generalized directional derivative*) *of the function* f *at* x_0 *in the direction* v. *The functional* ξ *from* X^* *is by definition Clarke subderivative* (*or generalized gradient*) *of* f *in* x_0 *if* $f^0(x_0; v) \geq \xi(v)\ \forall v \in X$. *The set of these generalized gradients is designated as* $\partial f(x_0)$.

Here it is a generalization at p-Laplacian and p-pseudo-Laplacian of an application of this concept from [66].

Let Ω be a bounded domain of $\mathbf{R}^N$ with the smooth boundary $\partial\Omega$ (topological boundary). Consider the nonlinear boundary value problems $(*)$ and $(**)$ from Section 5.1.2 above, where $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a measurable function with *subcritical growth*, i.e.,

$$\text{(I)}\ |f(x,s)| \leq a + b\,|s|^{\sigma}\ \forall s \in \mathbf{R}, x \in \Omega \text{ a.e.},$$

where $a, b > 0$, $0 \leq \sigma < \frac{N+2}{N-2}$ for $N > 2$ and $\sigma \in [0, +\infty)$ for $N = 1$ or $N = 2$.

Set [67]:

$$\underline{f}(x,\, t) = \underline{\lim}_{s \to t} f(x,\, s), \overline{f}(x,\, t) = \overline{\lim_{s \to t}} f(x,\, s).$$

Suppose

$$\text{(II)}\ \underline{f} \text{ and } \overline{f} : \Omega \times \mathbf{R} \to \mathbf{R} \text{ are measurable with respect to } x.$$

We emphasize that (II) is verified in the following two cases:

1° f is independent of x;

2° f is Baire measurable and $s \to f(x, s)$ is decreasing $\forall x \in \Omega$, in which case we have:

$$\overline{f}(x, t) = \max\{ f(x, t+), f(x, t-)\}, \underline{f}(x, t) = \min\{ f(x, t+), f(x, t-)\}.$$

Definition 5.4. *u from $W_0^{1,p}(\Omega)$, $p > 1$, is* solution of $(*)$ *and* $(**)$ *if $u = 0$ on $\partial\,\Omega$ in the sense of the trace (see in* Section 2.3.2 *above) and*

$$-\Delta_{\mathrm{p}}\, u(x) \in [\underline{f}(x, u(x)), \overline{f}(x, u(x))] \text{ in } \Omega \text{ a.e.} \tag{5.19}$$

and

$$-\Delta^{\mathrm{s}}_{\mathrm{p}} u(x) \in [\underline{f}(x, u(x)), \overline{f}(x, u(x))] \text{ in } \Omega \text{ a.e.} \tag{5.20}$$

respectively.

Let $X := W_0^{1,p}(\Omega)$, but in the first case $(*)$, the norm endowing X is $\|\cdot\|_{1,\mathrm{p}}$ i.e., $\|u\|_{1,\mathrm{p}} \overset{\text{notation}}{=} ||u||_{W_0^{1,p}(\Omega)} = ||u||_{L^p(\Omega)} + \sum\limits_{\mathrm{i}=1}^{\mathrm{N}} \left\| \frac{\partial u}{\partial x_{\mathrm{i}}} \right\|_{\mathrm{L^p}(\Omega)}$, which is equivalent to the norm $u \to \left(||u||^{\mathrm{p}}_{\mathrm{L^p}(\Omega)} + \sum\limits_{\mathrm{i}=1}^{\mathrm{N}} \left\| \frac{\partial u}{\partial x_{\mathrm{i}}} \right\|^{\mathrm{p}}_{\mathrm{L^p}(\Omega)} \right)^{\frac{1}{\mathrm{p}}}$. For the second case $(**)$, equip (also as previously) the same set X with the norm $u \to |\!|\!| u |\!|\!|_{1,\mathrm{p}} = \left(\sum\limits_{\mathrm{i}=1}^{\mathrm{N}} \left\| \frac{\partial u}{\partial x_{\mathrm{i}}} \right\|^{\mathrm{p}}_{\mathrm{L^p}(\Omega)} \right)^{\frac{1}{\mathrm{p}}}$, which is equivalent to $u \to |u|_{1,\mathrm{p}} = \sum\limits_{\mathrm{i}=1}^{\mathrm{N}} \left\| \frac{\partial u}{\partial x_{\mathrm{i}}} \right\|_{\mathrm{L^p}(\Omega)}$.

Associate with $(*)$ the locally Lipschitz functional $\Phi : X \to \mathbf{R}$,

$$\Phi(u) = \frac{1}{p} \| u \|^{\mathrm{p}}_{1,\mathrm{p}} - \int\limits_{\Omega} F(x, u)dx,\ u \in X, \tag{5.21}$$

and associate with $(**)$

$$\Phi(u) = \frac{1}{p} |\!|\!| u |\!|\!|^{\mathrm{p}}_{1,\mathrm{p}} - \int\limits_{\Omega} F(x, u)dx,\ u \in X, \tag{5.22}$$

where $F(x, s) = \int\limits_0^{\mathrm{s}} f(x, t)dx$. Set

$$Q(u) := \frac{1}{p} \| u \|^{\mathrm{p}}_{1,\mathrm{p}},\ u \in X,\ \Psi_1(u) := \int\limits_{\Omega} F(x, u)dx,\ u \in X, \tag{5.23}$$

and

$$Q(u) := \frac{1}{p} |\!|\!| u |\!|\!|^{\mathrm{p}}_{1,\mathrm{p}},\ u \in X,\ \Psi_1(u) := \int\limits_{\Omega} F(x, u)dx,\ u \in X, \tag{5.24}$$

respectively, where F, a map defined on $\Omega \times \mathbf{R}$, taking values in $\mathbf{R}$, is locally Lipschitz (use (I)). The functional Ψ: $L^{\sigma+1}(\Omega) \to \mathbf{R}$, $\Psi(u) = \int\limits_{\Omega} F(x, u)dx$, is also locally Lipschitz (again (I)). Using the Sobolev embedding $X \subset L^{\sigma+1}(\Omega)$, we find that $\Psi_{1:} = \Psi \,|\, X$ is locally Lipschitz on X, which implies that Φ is locally Lipschitz on X, and consequently, according to a local extremum result for Lipschitz functions (if x_0 is a point of local extremum for f, then $0 \in \partial f(x_0)$), the critical points of Φ for Clarke subderivative can be taken into account. One may state:

Proposition 5.5. *Suppose* (I) *and* (II) *are satisfied. Then* Ψ *is locally Lipschitz on* $L^{\sigma+1}(\Omega)$ *and*

(*i*) $\partial \Psi(u) \subset [\underline{f}(x, u(x)), \overline{f}(x, u(x))]$ *in* Ω *a.e.*

(*ii*) *If* $\Psi_1 = \Psi \mid X$, *where* $X = W_0^{1,p}(\Omega)$ *endowed with the norm* $\| \cdot \|_{1,p}$ *for the problem* $(*)$ *and* $\| \cdot \|_{1,p}$ *for the problem* $(**)$, *respectively, then*

$$\partial \Psi_1(u) \subset \partial \Psi(u) \; \forall u \in X.$$

Proof. The proof for (*i*) can be found in [67], Theorem 2.1, which remains the same here, while the problem was solved for the Laplacian with $X = H_0^1(\Omega)$ only. In order to prove (*ii*), we use 2.2 from [67], observing for both cases (X is endowed with each one from those two norms) that X is reflexive and dense in $L^{\sigma+1}(\Omega)$, as can be seen, for instance, in Section 2, where they are summarized. □

Proposition 5.6. *If* (I) *and* (II) *are verified, every critical point of* Φ *is a solution for* $(*)$ *and* $(**)$, *respectively.*

Proof. *Problem* $(*)$. Let u_0 be a critical point for Φ. We have

$$0 \in \partial \Phi(u_0) \subset \partial Q(u_0) + \partial(-\Psi_1)(u_0) \tag{5.25}$$

since $\Phi \overset{(5.21)}{=} Q - \Psi_1$, and we apply some rules of subdifferential calculus concerning finite sums. $\partial Q(u_0) = \{Q'(u_0)\}$, where $Q'(u_0)(v) = \int_\Omega |\nabla u_0|^{p-2} \cdot \nabla v dx = \langle -\Delta_p u_0, v\rangle$ (Section 2).

Using (5.25) and a specific property of a function f Lipschitz around x_0 ($f^0(x_0; v) = \sup_{\xi \in \partial f(x_0)} \xi(v)$, $\forall v \in X$, f^0 the Clarke derivative of f), we find

$$0 \le \int_\Omega |\nabla u_0|^{p-2} \cdot \nabla v dx + (-\Psi_1)0\,(u_0; v).$$

However, $(-\Psi_1)^0(u_0; v) = \Psi_1^0(u_0; -v)$ (a property of the Clarke derivative; see [22]), and thus,

$$\int_\Omega |\nabla u_0|^{p-2} \cdot \nabla(-v) dx \le \Psi^0(u_0; -v) \; \forall v \in X;$$

that is,

$$\mu_0(v) := \int_\Omega |\nabla u_0|^{p-2} \cdot \nabla v dx \le \Psi_1^0(u_0; v) \; \forall v \in X,$$

$\mu_0 = -\Delta_p u_0 \in \partial \Psi_1(u_0)$ and, using Proposition 5.5, $-\Delta_p u_0 \in \partial \Psi(u_0)$. Since $\partial \Psi(u_0) \subset (L^{\sigma+1}(\Omega))^* = L^{(\sigma+1)/\sigma}(\Omega)$, we obtain $u_0 \in W^{2,(\sigma+1)/\sigma}(\Omega)$ and (5.19):

$$-\Delta_p u_0(x) \in [\underline{f}(x, u_0(x)), \overline{f}(x, u_0(x))] \text{ in } \Omega \text{ a.e.}$$

Problem $(**)$. Let u_0 be a critical point for Φ. We have

$$0 \in \partial \Phi(u_0) \subset \partial Q(u_0) + \partial(-\Psi_1)(u_0) \tag{5.26}$$

since $\Phi \overset{(5.22)}{=} Q - \Psi_1$, and we apply some rules of subdifferential calculus concerning finite sums (Section 2).

$$\partial Q(u_0) = \{Q'(u_0)\}, \text{ where } Q'(u_0)(v) = \sum_{i=1}^N \int_\Omega \left|\frac{\partial u_0}{\partial x_i}\right|^{p-2} \frac{\partial u_0}{\partial x_i}\frac{\partial v}{\partial x_i} dx = \left\langle \Delta_p^s u_0, v \right\rangle.$$

Using (5.26) and a mentioned property of a function f Lipschitz around x_0, we find

$$0 \le \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_0}{\partial x_i} \right|^{p-2} \frac{\partial u_0}{\partial x_i} \frac{\partial v}{\partial x_i} dx + (-\Psi_1)0\ (u_0;\ v).$$

However, $(-\Psi_1)^0\ (u_0; v) = \Psi_1^0(u_0; -v)$ (a property of the Clarke derivative, see [22]), and thus,

$$\sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_0}{\partial x_i} \right|^{p-2} \frac{\partial u_0}{\partial x_i} \frac{\partial (-v)}{\partial x_i} dx \ \le\ \Psi^0\ (u_0; -v)\ \forall v\ \in\ X;$$

that is

$$\mu_0\ (v) := \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_0}{\partial x_i} \right|^{p-2} \frac{\partial u_0}{\partial x_i} \frac{\partial v}{\partial x_i} dx\ \le \Psi_1^0(u_0,\ v)\ \forall v\ \in\ X,$$

$\mu_0 = -\Delta_p^s u_0 \in \partial\ \Psi_1(u_0)$ and, using Proposition 5.5, $-\Delta_p^s u_0 \in \partial\ \Psi(u_0)$. Since $\partial\ \Psi(u_0) \subset (L^{\sigma+1}(\Omega))^* = L^{(\sigma+1)/\sigma}(\Omega)$, we obtain $u_0 \in W^{2,(\sigma+1)/\sigma}(\Omega)$ and (5.20),

$$-\Delta_p^s u_0\ (x)\ \in\ [\underline{f}(x,\ u_0(x)), \overline{f}(x,\ u_0(x))] \text{ in } \Omega \text{ a.e.}$$

□

Remark 5.6. *Some applications to characterize the solution of the modeling given in [68] and solutions using this kind of definition for Dirichlet problems derived from the previously presented problems of the movement of glacier, nonlinear elastic membrane, pseudo-torsion problem or a nonlinear elastic membrane with the p-pseudo-Laplacian will be developed in Part two of this article.*

5.3. Other Solutions

The results from this section have been obtained by the author in [18].

5.3.1. Basic Results

Let us now consider the two problems $(*)$ and $(**)$ from the above two subsections, but the boundary condition is now by $Bu = 0$ instead of $u = 0$. Once again, we take the function f as in Section 5.2.1 with the corresponding $\underline{f}$ and $\overline{f}$, as was performed there.

Definition 5.5. *u from $W^{2,p}(\Omega)$, $p > 1$, is solution of $(*)$ and $(**)$ from this section if $Bu = 0$ on $\partial\Omega$ in the sense of the trace (whose meaning is introduced above) and*

$$-\Delta_p\ u(x)\ \in\ [\underline{f}(x,\ u(x)), \overline{f}(x,\ u(x))] \text{ in } \Omega \text{ a.e.} \tag{5.27}$$

and

$$-\Delta_p^s u(x)\ \in\ [\underline{f}(x,\ u(x)), \overline{f}(x,\ u(x))] \text{ in } \Omega \text{ a.e.} \tag{5.28}$$

respectively.

We now continue with some necessary results on Lipschitz functions and Palais-Smale type conditions. First, we provide some comments related to the Clarke derivative. From Definition 5.3, the Clarke derivative is:

$$f^0(x_0; v) = \inf_{\substack{V \in V(x_0) \\ r \in (0,+\infty)}} \ \sup_{\substack{x \in V \\ t \in (0,r)}} \frac{f(x+tv) - f(x)}{t}. \tag{5.29}$$

Proposition 5.7. *Let f be Lipschitz around x_0 with the constant L. Then*

1° *the function $v \to f^0(x_0; v)$ has values in* **R**, *is positive homogeneous and subadditive on X and*

$$\left| f^0(x_0; v) \right| \le L\|v\| \ \forall v \ \in \ X;$$

2° $f^0(x_0; -v) = (-f)^0(x_0; v)\ \forall v \in X$, $\lambda \ge 0 \Rightarrow (\lambda f)^0(x_0; v) = \lambda f^0(x_0; v)\ \forall v \in X$;

3° *$v \to f^0(x_0; v)$ is Lipschitz on X with the constant L [69].*

Proof. 1° $f^0(x_0; v) \in \mathbf{R}$. For x near x_0 and with t strictly positive near 0, we have

$$\left| \frac{f(x+tv) - f(x)}{t} \right| \le \frac{1}{t} L \parallel tv \parallel = L \parallel v \parallel . \tag{5.30}$$

From (5.30), we obtain

$$\left| f^0(x_0; v) \right| \le L \parallel v \parallel \text{ and so } f^0(x_0;\ v) \ \in \ \mathbf{R}. \tag{5.31}$$

Indeed, suppose ad absurdum that $f^0(x_0; v) > L\|v\|$, for instance. Then, with $\forall V$ from V (x_0) and $\forall r$ from $(0, +\infty)$, we have

$$\sup_{\substack{x \in V \\ t \in (0, r)}} \frac{f(x+tv) - f(x)}{t} > L \parallel v \parallel,$$

in contradiction with (5.30).

$v \to f^0(x; v)$ is positive homogeneous. Since $f^0(x_0; 0) = 0$, let $\lambda > 0$. Then, $f^0(x_0; \lambda v) = \lambda \overline{\lim_{\substack{x \to x_0 \\ \lambda t \to 0+}}} \frac{f(x+t\lambda v) - f(x)}{\lambda t}$, etc.

$v \to f^0(x; v)$ is subadditive.

$$f(x_0; v_1 + v_2) = \overline{\lim_{\substack{x \to x_0 \\ t \to 0+}}} \left[\frac{f((x+tv_1)+tv_2) - f(x+tv_1)}{t} + \frac{f(x+tv_1) - f(x)}{t} \right] \le$$
$$\overline{\lim_{\substack{x \to x_0 \\ t \to 0+}}} \frac{f((x+tv_1)+tv_2) - f(x+tv_1)}{t} + \overline{\lim_{\substack{x \to x_0 \\ t \to 0+}}} \frac{f(x+tv_1) - f(x)}{t}. \tag{5.32}$$

As $\lim_{\substack{x \to x_0 \\ t \to 0+}} (x + tv_1) = x$, in the third member of (5.32), the first term is equal to $f^0(x_0; v_2)$, and the second is equal to $f^0(x_0; v_1)$.

2° $f^0(x_0; -v) = \overline{\lim_{\substack{x \to x_0 \\ t \to 0+}}} \frac{f(x-tv) - f(x)}{t} \overset{u=x-tv}{=} \overline{\lim_{\substack{u \to x_0 \\ t \to 0+}}} \frac{(-f)(u+tv) - (-f)(u)}{t} = (-f)^0(x_0;$ $v)$, since $\lim_{\substack{x \to x_0 \\ t \to 0+}} (x - tv) = x_0$. For the second statement, we use Formula (5.29).

3° Let v and w be arbitrary in X. For x near x_0 and with t strictly positive near 0, we have

$$f(x_0 + tv) - f(x_0) \le f(x_0 + tw) - f(x_0) + Lt\|v - w\|.$$

Dividing by t and taking the upper limit for $x \to x_0$ and $t \to 0+$, one finds:

$$f^0(x_0; v) \le f^0(x_0; w) + L\|v - w\|.$$

Exchanging v with w, we come to the desired conclusion. □

Proposition 5.8. *Let f be locally Lipschitz on X. The function* $\Phi : X \times X \to \mathbf{R}$,

$$\Phi(x; v) = f^0\,(x_0; v)$$

is upper semicontinuous [69].

Proof. Let (x_0, v_0) be a point in $X \times X$ and $(x_n, v_n)_{n\geq 1}$ be an arbitrary sequence such that $(x_n, v_n) \to (x_0, v_0)$. We show that

$$\varlimsup_{n\to\infty} f^0(x_n, v_n) \leq f^0(x_0; v_0). \tag{5.33}$$

According to (5.29), we have $\forall m, p \in \mathbf{N}$

$$f^0(x_n; v_n) - \frac{1}{n} < \sup_{\substack{x \in B(x_n, \frac{1}{m}) \\ t \in (0, \frac{1}{p})}} \frac{f(x + tv_n) - f(x)}{t}. \tag{5.34}$$

We fix m, p in $\mathbf{N}$ and let k_n be in $\mathbf{N}$, $k_n > n$ such that

$$\frac{1}{m} + \frac{1}{p} < \frac{1}{k_n}. \tag{5.35}$$

With (5.34), we find y_{k_n} and t_{k_n} such that $\|y_{k_n} - x_n\| < \frac{1}{m}$ and $t_{k_n} \in \left(0, \frac{1}{p}\right)$, hence

$$\|y_{k_n} - x_n\| + t_{k_n} \overset{(5.35)}{<} \frac{1}{k_n} \tag{5.36}$$

and moreover (see (5.34)),

$$f^0(x_n; v_n) - \frac{1}{k_n} < \frac{f(y_{k_n} + t_{k_n} v_n) - f(y_{k_n})}{t_{k_n}}. \tag{5.37}$$

The second member of (5.37) is equal to

$$\frac{f(y_{k_n} + t_{k_n} v_0) - f(y_{k_n})}{t_{k_n}} + \frac{f(y_{k_n} + t_{k_n} v_n) - f(y_{k_n} + t_{k_n} v_0)}{t_{k_n}},$$

consequently passing in (5.37) to the upper limit for $n \to \infty$, and taking into account $y_{k_n} \to x_0$, $t_{k_n} \to 0$ (see (5.36)) and $\left|f(y_{k_n} + t_{k_n} v_n) - f(y_{k_n} + t_{k_n} v_0)\right| \leq L t_{k_n} \|v_n - v_0\|$, we find even (5.33), since $\varlimsup_{n\to\infty} \frac{f(y_{k_n} + t_{k_n} v_0) - f(y_{k_n})}{t_{k_n}} \leq f^0(x_0; v_0)$. □

Proposition 5.9. *If f is Lipschitz around* x_0*, and L is the constant, then*

1° $\partial f\,(x_0)$ *is nonempty, convex,* $*$*-weakly compact (for X complete) and*

$$\|\xi\| \leq L\ \forall \xi \in \partial f(x);$$

2° $f^0(x_0; v) = \sup_{\xi \in \partial f(x_0)} \xi(v)\ \forall v \in X$ *([69]).*

Proof. 1° With the function $v \to f^0\,(x_0; v)$ being positive homogeneous and subadditive (Proposition 5.7), there is a linear functional ξ such that

$$-f^0\,(x_0; -v) \leq \xi(v) \leq f^0\,(x_0; v)\ \forall v \in X.$$

However,

$$f^0(x_0; v) \overset{\text{Proposition 5.7}}{\leq} L\|v\|, -f^0(x_0; -v) \geq -L\|-v\|,$$

consequently $|\xi(v)| \leq L\|v\|\ \forall v \in X$, hence $\xi \in X^*$, $\xi \in \partial f\,(x_0)$ and $\|\xi\| \leq L$.

The convexity is obvious according to Definition 5.3.

For the remaining statement, i.e., $\partial f\,(x_0)$ is $*$-weakly compact, it is sufficient to show that $\partial f\,(x_0)$ is $*$-weakly closed (corollary of the Alaoglu-Bourbaki-Kakutani theorem [43], Volume 10, p. 144, 3.37$_1$). Let $\xi \in \overline{\partial f(x_0)}^{*-\text{weak}}$. Prove

$$f^0\,(x_0; v) \geq \xi(v)\ \forall v \in X. \tag{5.38}$$

Fix v in X and let $\varepsilon > 0$ be arbitrary. There is μ in $\partial f\,(x_0)$ such that $|(\xi - \mu)(v)| < \varepsilon$. Thus $\mu(v) > \xi(v) - \varepsilon$ and we have

$$f^0\,(x_0; v) \geq \mu(v) \geq \xi(v) - \varepsilon.$$

We pass to the limit for $\varepsilon \to 0$ and obtain (5.38).

2° Suppose, ad absurdum, that $\exists\, v_0 \in X$ such that $f^0(x_0; v_0) \neq \sup_{\xi \in \partial f(x_0)} \xi(v_0)$. Since f^0 $(x_0; v_0) \geq \xi(v_0)\ \forall \xi \in \partial f\,(x_0)$, this implies

$$f\,(x_0; v_0) > \xi(v_0)\ \forall \xi \in \partial f\,(x_0). \tag{5.39}$$

We take μ, a linear functional on X, with

$$\mu(v_0) = f^0\,(x_0; v_0) \tag{5.40}$$

and $-f^0\,(x_0; -v) \leq \mu(v) \leq f^0\,(x_0; v)\ \forall v \in X$ ([43], Volume 10, p. 91, 2.2). However, $\mu \in \partial f$ (x_0) (see the beginning of the proof), and consequently, (5.40) contradicts (5.39). □

Proposition 5.10. *Let X be a reflexive or separable Banach space and $f : X \to \mathbf{R}$ locally Lipschitz. For every x_0 from X and $\varepsilon > 0$, there is $\delta > 0$ such that, for every ξ in $\partial f\,(x)$ with $\|x - x_0\| \leq \delta$, there exists ξ' in $\partial f\,(x_0)$ having the property ([67], p. 105)*

$$|(\xi - \xi')(v)| \leq \varepsilon\ \forall v \in X.$$

Proof. Suppose, ad absurdum, the contrary, $\exists\, x_0, v_0 \in X$, $\varepsilon > 0$, and also the sequences $(x_n)_{n\geq 1}$ in X, $(\xi_n)_{n\geq 1}$, ξ_n in $\partial f\,(x_n)$ such that $\forall n \in \mathbf{N}$:

$$\|x_n - x_0\| \leq \frac{1}{n} \text{ and } |(\xi_n - \xi)(v_0)| > \varepsilon_0\ \forall \xi\ \in\ \partial f\,(x_0). \tag{5.41}$$

According to (5.41), x_n is, for $n \geq N$, in a neighborhood of x_0 as in Proposition 5.9, hence $n \geq N \Rightarrow \|\xi_n\| \leq L$, so there is a subsequence $(\xi_{k_n})_{n\geq 1}$ that is weakly convergent (Šmulian corollary [43], Volume 10, p. 171, 4.29) and, consequently, $*$-weakly convergent (X reflexive $\Leftrightarrow$ X^* reflexive (Pettis [43], Volume 10, p. 151, 4.7)). Let $\xi_{k_n} \overset{*-\text{weak}}{\to} \xi_0$, i.e.,

$$\xi_{k_n}(v) \to \xi_0(v)\ \forall v\ \in\ X \tag{5.42}$$

([43], Volume 10, p. 145, 3.39 or p.145, ex.9 or p. 164, 4.25). However, $\xi_0 \in \partial f\,(x_0)$. Indeed, with u being arbitrarily fixed in X, $\exists\, (h_n)_{n\geq 1}$, $h_n \to 0$ and $(t_n)_{n\geq 1}$, $t_n \downarrow 0$ such that (use Definition 5.3—the Clarke derivative)

$$\forall n\ \frac{1}{t_n}[f(x_{k_n} + h_n + t_n u) - f(x_{k_n} + h_n)] \geq \xi_{k_n}(u) - \frac{1}{n},$$

pass to the limit for $n \to \infty$, taking (5.42) into account, $f^0(x_0; u) \geq \xi_0(u)$, i.e., $\xi_0 \in \partial f(x_0)$. Thus, it can take $\xi = \xi_0$ in (5.41), and one obtains a contradiction with (5.42). □

Remark 5.7. *In [67], Proposition 5.10, alias 1, (6), has another formulation. Moreover, X must be reflexive or separable.*

Proposition 5.11. *Let X be a real reflexive space and* $f : X \to \mathbf{R}$ *be locally Lipschitz.*

$1°$ *For every* x_0 *in X, there is* ξ_0 *in* $\partial f(x_0)$ *such that*

$$\|\xi_0\| = \inf\{\|\xi\| : \xi \in \partial f(x_0)\}.$$

$2°$ *The function* $\mu: X \to \mathbf{R}$

$$\mu(x) = \inf\{\|\xi\| : \xi \in \partial f(x)\}$$

is lower semicontinuous ([67], p. 105).

Proof. $1°$ The application $\xi \to \|\xi\|$ of X^* in $\mathbf{R}$ is weakly lower semicontinuous and hence $*$-weakly lower semicontinuous ([43], vol 10, p. 147, (0)), since X^* is reflexive. On the other hand, $\partial f(x_0)$ is $*$-weakly compact (Proposition 5.9), hence the conclusion.

$2°$ Suppose, ad absurdum, the contrary and let x_0 and x_n, $n \in \mathbf{N}$, be from X, $x_n \to x_0$, with

$$\lim_{n\to\infty} \mu(x_n) < \mu(x_0). \tag{5.43}$$

We take ξ_n in $\partial f(x_n)$ with $\mu(x_n) = \|\xi_n\|$ (see $1°$). Proposition 5.10 yields, for every n from $\mathbf{N}$, ξ_n in $\partial f(x_n)$ and ξ_n^0 in $\partial f(x_0)$ such that

$$|(\xi_n - \xi_n^0)(v)| \leq \frac{1}{n} \, \forall v \in X. \tag{5.44}$$

With $\partial f(x_0)$ being $*$-weakly compact (Proposition 5.9), it is bounded, and hence, $(\xi_n^0)_{n\geq 1}$ has a weakly convergent subsequence and hence is $*$-weakly convergent, $\left(\xi_{k_n}^0\right)_{n\geq 1}$,

$$\xi_{k_n}^0 \overset{*-\text{weak}}{\to} \xi^0 \in \partial f(x_0). \tag{5.45}$$

Since

$$|(\xi_{k_n} - \xi^0)(v)| \leq |(\xi_{k_n} - \xi_{k_n}^0)(v)| + |(\xi_{k_n}^0 - \xi^0)(v)|,$$

from (5.44) and (5.45), we obtain $\xi_{k_n} \overset{*-\text{weak}}{\to} \xi^0$, and this attracts $\lim_{n\to\infty} \|\xi_{k_n}\| \geq \|\xi^0\|$ ([43], Volume 10, p. 145, 3.39, $3°$), i.e., $\lim_{n\to\infty} \mu(x_{k_n}) \geq \|\xi^0\| \geq \mu(x_0)$ and we obtain a contradiction with (5.43). □

Remark 5.8. *In [67], 1, (7) (here, Proposition 5.11), the condition "X reflexive" is lacking, and this is an error.*

Definition 5.6. *Let X be a real normed space,* $E \subset X$ and $x_0 \in \overline{E}$. *The vector v* of *X is by definition a possible direction for E according to* x_0, *if there is* $\rho > 0$ *such that* $x_0 + tv \in E \; \forall t$ *in* $(0, \rho)$. *In addition, let be* $f : E \to (-\infty, +\infty]$ *and* $x_0 \in \operatorname{dom} f$. *If*

$$\lim_{t\to 0+} \frac{f(x_0 + tv) - f(x_0)}{t}$$

exists and is finite, it is designated by

$$f\prime(x_0; v),$$

the derivative of f at x_0 according to the vector v or *according to the direction v* (*directional derivative*). Thus

$$f'(x_0;v) = \lim_{t\to 0+} \frac{f(x_0+tv)-f(x_0)}{t}.$$

Remark 5.9. *Suppose E convex, f convex and finite on a neighborhood V* of x_0. *Then there exists* $f'(x_0;v)$. *Indeed, one can suppose V an open ball with the center in* x_0, *let* δ *be its radius,* $\delta \le \rho\|v\|$. *Consider the function* $F : \left[0, \frac{\delta}{\|v\|}\right) \to \mathbf{R}$, $F(t) = f(x_0+tv)$ (*we supposed* $v \neq 0$, $f'(x_0;0) = 0$). *F is convex*: $t_1, t_2 \in \left[0, \frac{\delta}{\|v\|}\right)$ *and* $\lambda_1, \lambda_2 \ge 0$, $\lambda_1 + \lambda_2 = 1 \Rightarrow F(\lambda_1 t_1 + \lambda_2 t_2) = f((\lambda_1+\lambda_2)x_0 + (\lambda_1 t_1 + \lambda_2 t_2)v) = f(\lambda_1(x_0+t_1v) + \lambda_2(x_0+t_2v)) \le \lambda_1 f(x_0+t_1v) + \lambda_2 f(x_0+t_2v) = \lambda_1 F(t_1) + \lambda_2 F(t_2)$. *Thus the function* $\Phi : \left(0, \frac{\delta}{\|v\|}\right) \to \mathbf{R}$, $\Phi(t) = \frac{F(t)-F(0)}{t}$ *is increasing, that is the function* $t \to \frac{f(x_0+tv)-f(x_0)}{t}$ *has this property on* $\left(0, \frac{\delta}{\|v\|}\right)$, *consequently* $\lim_{t\to 0+} \frac{f(x_0+tv)-f(x_0)}{t}$ *is finite.*

Proposition 5.12. *Let E be a convex subset of a real Banach space and* $f : E \to \mathbf{R}$ *be convex. If f is Lipschitz around* $x_0 \in \overset{o}{E}$, *then for every v in X, we have*

$$f^0(x_0;v) = f'(x_0;v)$$

and the set of subderivatives in x_0 *coincides with the set of Clarke subderivatives in* x_0 ([69]).

Proof. It is sufficient to prove the first statement because the second is obtained by the formulae that are found in Proposition 5.9 and [22], I, 5.5. We obviously suppose that $v \neq 0$ and let $\varepsilon > 0$ be arbitrarily fixed. We set

$$\alpha := \inf_{\substack{V\in \mathrm{V}(x_0)\\ r\in(0,+\infty)}} \sup_{\substack{x\in V\\ t\in(0,r)}} \frac{f(x+tv)-f(x)}{t} \left(\overset{(5.29)}{=} f^0(x_0;v)\right),$$

$$\beta := \lim_{r\to 0+} \sup_{\|x-x_0\|<r\varepsilon} \sup_{t\in(0,r)} \frac{f(x+tv)-f(x)}{t}. \tag{5.46}$$

The suppositions $\alpha < \beta$ and $\beta < \alpha$ lead to a contradiction (attention to the definitions: limit, least upper bound and greatest lower bound), therefore $\alpha = \beta$. However, with the function $t \to \frac{f(x+tv)-f(x)}{t}$, x near x_0, increasing on an interval $(0,\eta)$ (see Remark 5.9), we obtain, via (5.46),

$$f^0(x_0;v) = \lim_{r\to 0+} \sup_{\|x-x_0\|<r\varepsilon} \frac{f(x+rv)-f(x)}{r}. \tag{5.47}$$

However, we have

$$\frac{f(x+rv)-f(x)}{r} = \left[\frac{f(x+rv)-f(x)}{r} - \frac{f(x_0+rv)-f(x_0)}{r}\right] + \frac{f(x_0+rv)-f(x_0)}{r}$$

and, via the Lipschitz condition verified on the open ball $B(x_0;r\varepsilon)$ for r near 0, $x \in B(x_0;r\varepsilon)$ $\Rightarrow \left|\frac{f(x+rv)-f(x)}{r} - \frac{f(x_0+rv)-f(x_0)}{r}\right| \le 2\varepsilon L$, and hence, using (5.47),

$$f^0(x_0;v) \le \lim_{r\to 0+} \frac{f(x_0+rv)-f(x_0)}{r} + 2\varepsilon L = f'(x_0;v) + 2\varepsilon L$$

and hence

$$f^0(x_0;v) \le f'(x_0;v).$$

We take the opposite inequality. We have, with $\forall t$ from $(0, r)$,

$$\frac{f(x_0+tv)-f(x_0)}{t} \leq \sup_{\substack{x\in V\\ t\in(0,r)}} \frac{f(x+tv)-f(x)}{t}, \text{ where } V \in \mathrm{V}\ (x_0);$$

hence, $\lim\limits_{t\to 0+} \frac{f(x_0+tv)-f(x_0)}{t} \leq \sup\limits_{\substack{x\in V\\ t\in(0,r)}} \frac{f(x+tv)-f(x)}{t}$ and, consequently, according to (5.29),

$$f'(x_0; v) \leq f^0\ (x_0; v).$$

□

Remark 5.10. *The second part of the proof uses only the hypothesis "there exists* $f'(x_0; v)$*".*

Proposition 5.13 (Local extremum). *Let f be Lipschitz around x_0. If x_0 is a point of the local extremum for f, we have ([69])*

$$0 \in \partial f\ (x_0).$$

Proof. x_0 *local minimum point*. Let v be arbitrary in X. We must prove that $f^0(x_0; v) \geq 0$. Suppose ad absurdum that $f^0(x_0; v) < 0$. There is, according to (5.29), V in $\mathrm{V}\ (x_0)$ and $r > 0$ such that $\sup\limits_{x\in V} \frac{f(x+tv)-f(x)}{t} < 0$; hence, in particular, we have $t \in (0, r) \Rightarrow f\ (x_0 + tv) - f\ (x_0) < 0$, which prevents x_0 from being a local minimum point for f, which is a contradiction.

x_0 *local maximum point*. In this case, x_0 is a local minimum point for $-f$, and hence $0 \in \partial(-f)(x_0) = -\partial f\ (x_0)$ ([22], *Rules of subdifferential calculus*, 5.26), $0 \in \partial f\ (x_0)$. □

5.3.2. Some Palais-Smale Type Conditions

We now present some results following from some ideas in [67] that are improved and generalized. These results contain conditions of Palais-Smale type suggested by Ekeland principle.

Let X be a complete metric space, φ: $X \to \mathbf{R}$ and $c \in \mathbf{R}$.

φ satisfies the $(\mathrm{PS})^*{}_{c,+}$ *condition* when, for every sequence $(u_n)_{n\geq 1}$, $u_n \in X$, $(\varepsilon_n)_{n\geq 1}$ and $(\delta_n)_{n\geq 1}$, ε_n, $\delta_n \in \mathbf{R}_+$, $\varepsilon_n \to 0$ and $\delta_n \to 0$, if

$$\varphi(u_n) \to c \tag{5.48}$$

and

$$\forall u \in X\ d(u_n, u) \leq \delta_n \Rightarrow \varphi(u_n) \leq \varphi(u) + \varepsilon_n \mathrm{d}(u_n, u), \tag{5.49}$$

then

$(u_n)_{n\geq 1}$ has a convergent subsequence.

By changing u_n and u to each other in (5.49), we obtain the $(\mathrm{PS})^*{}_{c,-}$ *condition*. Finally, $(\mathrm{PS})^*{}_{c}$ *condition* means $(\mathrm{PS})^*{}_{c,+}$ + $(\mathrm{PS})^*{}_{c,-}$.

When, with X being a real Banach space, the conclusion required by the hypothesis

"$(u_n)_{n\geq 1}$ has a convergent subsequence"

is replaced by

"$(u_n)_{n\geq 1}$ has a weak convergent subsequence",

we obtain, respectively, the conditions

$$(\mathrm{PS})^*{}_{c,\,w,\,+},\ (\mathrm{PS})^*{}_{c,\,w,\,-},\ (\mathrm{PS})^*{}_{c,\,w}.$$

Suppose that X is a real Banach space and φ is locally Lipschitz. φ satisfies the $[\mathrm{PS}]^*{}_{\mathrm{c},+}$ *condition* (obvious definition for $[\mathrm{PS}]^*{}_{\mathrm{c},-}$, $[\mathrm{PS}]^*{}_{\mathrm{c}}$) when the properties in (5.48) and (5.49) imply that

c *is a critical value of* φ (for the Clarke subderivative).

The definition is, according to Proposition 5.9, coherent. We have

$$(\mathrm{PS})^*{}_{\mathrm{c},+} \Rightarrow [\mathrm{PS}]^*{}_{\mathrm{c},+},$$

and the reciprocal assertion it is not true (consider $\varphi\colon \mathbf{R} \to \mathbf{R}$ Lipschitz, periodic and

$$c = \inf \varphi(\mathbf{R}) \text{ or } c = \sup \varphi(\mathbf{R})).$$

Finally, we provide the last version of the Palais-Smale condition according to Chang [67]. Let X be a real Banach space, $\varphi\colon X \to \mathbf{R}$ be locally Lipschitz and $c \in \mathbf{R}$. φ satisfies the $(\mathrm{PS})_{\mathrm{c}}^{\mathrm{ch}}$ *condition* when, for every sequence $(u_n)_{n\geq 1}$ from X, if

$$\varphi(u_n) \to c \tag{5.50}$$

and

$$\mu(u_n)\colon = \inf\{\|\xi_n\|\colon \xi_n \in \partial\, \varphi(u_n)\} \to 0, \tag{5.51}$$

then

$(u_n)_{n\geq 1}$ *has a convergent subsequence.*

The definition is correct, $\partial\varphi(u_n) \neq \varnothing\ \forall n$ (Proposition 5.9, see also Proposition 5.10). One can state the following.

Proposition 5.14. *Let X be a real Banach space and $\varphi\colon X \to \mathbf{R}$ locally Lipschitz and convex. Then,*

$$\varphi \text{ verifies } (\mathrm{PS})_{\mathrm{c}}^{\mathrm{ch}} \;\Rightarrow\; \varphi \text{ verifies } (\mathrm{PS})_{\mathrm{c},-}^{*}.$$

Proof. Let (u_n) be a sequence from X and let (ε_n) and (δ_n) be sequences from $\mathbf{R}_+$, $\varepsilon_n \to 0$, $\delta_n \to 0$, such that

$$\varphi(u_n) \to c$$

and

$$\|u_n - u\| \leq \delta_n \Rightarrow \varphi(u) \leq \varphi(u_n) + \varepsilon_n\, u_n - u\ \forall u \in X. \tag{5.52}$$

We must prove, for finding a convergent subsequence of (u_n), that

$$\mu(u_n)\colon = \inf\{\|\xi_n\|\colon \xi_n \in \partial\, \varphi(u_n)\} \to 0. \tag{5.53}$$

Take $u\colon = u_n + tv$, $\|v\| = 1$, $0 < t \leq \delta_n$. Since $\|u_n - u\| \leq \delta_n$, (5.52) gives:

$$\frac{\varphi(u_n + tv) - \varphi(u_n)}{t} \leq \varepsilon_n$$

and, passing to the limit for $t \to 0+$, we obtain (Proposition 5.12):

$$\varphi^0(u_n; v) = \varphi'(u_n; v) \leq \varepsilon_n,$$

and consequently, $\xi(v) \leq \varepsilon_n\ \forall \xi \in \partial\, \varphi(u_n)$ (Proposition 5.9); hence, changing v in $-v$,

$$\|\xi\| \leq \varepsilon_n\ \forall \xi \in \partial\, \varphi(u_n). \tag{5.54}$$

Let ξ_n be in $\partial\varphi(u_n)$ such that $\|\xi_n\| = \mu(u_n)$ (see Proposition 5.9). Then, taking (5.50) into account, we obtain

$$\mu(u_n) \leq \varepsilon_n,$$

which yields (5.49) by passing to the limit. □

Remark 5.11. *The last statement represents what the author recovered from Proposition 5.3 in [67], p. 475. The proof of this ([67], p. 483) contains, among other things, the implicit statement that* $v \to \Phi^0(u_0; v)$ *is not a subnorm, but an even linear functional.*

We now proceed to some propositions of [67].

Proposition 5.15. *Let X be a complete metric space,* $\varphi\colon X \to \mathbf{R}$ *lower bounded, lower semicontinuous and* $c := \inf \varphi(X)$. *c is attained when* φ *verifies* $(\mathrm{PS})^*_{c,+}$.

Proof. Let $(v_n)_{n\geq 1}$, $v_n \in X$, be a minimizing sequence for φ such that, for every n, $\varepsilon_n := \varphi(v_n) - c > 0$, and hence $\varepsilon_n \to 0$. Applying Ekeland principle with $\varepsilon = \varepsilon_n$, $\lambda = 1$, one finds $(u_n)_{n\geq 1}$, a sequence in X, with the properties

$$\varphi(u_n) \leq \varphi(v_n),$$
$$\varphi(u_n) \leq \varphi(u) + \varepsilon_n\, d(u_n, u)\ \forall u \in X.$$

Since $c \leq \varphi(u_n)$, we have $\varphi(u_n) \to c$; applying $(\mathrm{PS})^*_{c,+}$ and letting $(u_{k_n})_{n\geq 1}$ be a convergent subsequence, $u_{k_n} \to u_0$. However, $\varphi(u_0) \leq \lim_{n\to\infty} \varphi(u_{k_n}) = c$, and this imposes $\varphi(u_0) = c$. □

Proposition 5.16. *Let X be a real Banach space,* $\varphi\colon X \to \mathbf{R}$ *lower bounded, locally Lipschitz and* $c := \inf \varphi(X)$. *If* φ *satisfies* $(\mathrm{PS})^*_{c,+}$, *then* φ *has critical points (for the Clarke subderivative).*

Proof. Apply Proposition 5.15 combined with Proposition 5.13. □

Proposition 5.17. *Let X be a real Banach space,* $\varphi\colon X \to \mathbf{R}$ *lower bounded, locally Lipschitz and* $c := \inf \varphi(X)$. *If* φ *verifies* $[\mathrm{PS}]^*_{c,+}$, *then c is a critical value of* φ *(for the Clarke subderivative).*

Proof. Let $(\varepsilon_n)_{n\geq 1}$, $\varepsilon_n > 0$, $\varepsilon_n \to 0$. For every ε_n, take v_n such that $\varphi(v_n) \leq c + \varepsilon_n$ and apply Ekeland principle with $\lambda = 1$. $\exists\, u_n$ such that $\varphi(u_n) \leq \varphi(v_n)$,

$$\varphi(u_n) \leq \varphi(u) + \varepsilon_n \|u_n - u\|,\ \forall u \in X. \tag{5.55}$$

Since $\varphi(u_n) \to c$, (5.54) allows the application of the hypothesis $[\mathrm{PS}]^*_{c,+}$, where c is a critical value. □

As an application, we continue with the problems $(*)$ and $(**)$ in this subsection. However, firstly:

Proposition 5.18. *Let* $X := W^{1,p}(\Omega)$ and $\Phi\colon X \to \mathbf{R}$, $\Phi(u) = \frac{1}{p}\,||u||^p_{1,p} - \int_\Omega G(u)dx - \int_\Omega hudx$ and $\Phi(u) = \frac{1}{p}|\!|\!|u|\!|\!|^p_{1,p} - \int_\Omega G(u)\,dx - \int_\Omega hudx$, *respectively, where* $G : \mathbf{R} \to \mathbf{R}$ *has the period T and is Lipschitz,* $h \in L^{p'}(\Omega)$ and $\int_\Omega huhudx = 0$. *Then, for every c from* $\mathbf{R}$, Φ *verifies* $[\mathrm{PS}]^*_c$.

Clarification. On $X = \mathrm{W}^{1,\mathrm{p}}(\Omega)$, we can consider for this statement the following norms: $\|u\|_{1,\mathrm{p}} = \left(||u||^{\mathrm{p}}_{\mathrm{L^p}(\Omega)} + \sum_{\mathrm{i}=1}^{\mathrm{N}} \left\|\frac{\partial u}{\partial x_\mathrm{i}}\right\|^{\mathrm{p}}_{\mathrm{L^p}(\Omega)}\right)^{\frac{1}{\mathrm{p}}}$, which is equivalent to the norm $u \to ||u||_{L^p(\Omega)} + \sum_{\mathrm{i}=1}^{\mathrm{N}} \left\|\frac{\partial u}{\partial x_\mathrm{i}}\right\|_{\mathrm{L^p}(\Omega)}$ for $(*)$. For the second case $(**)$, we equip the same vector space with the norm $u \to |\!|\!|u|\!|\!|_{1,\mathrm{p}} = \left(\sum_{\mathrm{i}=1}^{\mathrm{N}} \left\|\frac{\partial u}{\partial x_\mathrm{i}}\right\|^{\mathrm{p}}_{\mathrm{L^p}(\Omega)}\right)^{\frac{1}{\mathrm{p}}}$, which is equivalent to $u \to |\,u\,|_{1,\mathrm{p}} = \sum_{\mathrm{i}=1}^{\mathrm{N}} \left\|\frac{\partial u}{\partial x_\mathrm{i}}\right\|_{\mathrm{L^p}(\Omega)}$ (see the considerations in the sections above).

Proof. It is sufficient to prove this for $[\mathrm{PS}]^*{}_{c,+}$. Let $(u_n)_{n\geq 1}$ be a sequence from X, $(\varepsilon_n)_{n\geq 1}$ and $(\delta_n)_{n\geq 1}$ sequences from $\mathbf{R}_+$ convergent to 0. Suppose $\Phi(u_n) \to c$ and

$$\|u_n - u\| \leq \delta_n \Rightarrow \Phi(u_n) \leq \Phi(u) + \varepsilon_n \,\|u_n - u\|. \tag{5.56}$$

We decompose X into the direct sum

$$X = X_0 \oplus X_1, \tag{5.57}$$

where X_1 is the vector space of constant functions, and $X_0 = X_1{}^{\perp}$, the vector subspace of functions from $\mathrm{W}^{1,\mathrm{p}}(\Omega)$ having a mean value equal to 0. Let $u_n = v_n + c_n$, $v_n \in X_0$, $c_n \in \mathbf{R}$ and $|G(s)| \leq M$ on $\mathbf{R}$. Hence $\left|\int_\Omega G(u_n)dx\right| \leq \int_\Omega |G(u_n)|dx \leq \int_\Omega Mdx = M\mu(\Omega)$, and since $\int_\Omega c_n h\,dx = 0$, we have, in the first case,

$$\Phi(u_n) = \frac{1}{p}\|u_n\|_{1,p}^p - \int_\Omega G(u_n)dx - \int_\Omega hu_n\,dx = \frac{1}{p}\|v_n + c_n\|_{1,p}^p - \int_\Omega G(v_n + c_n)dx -$$
$$\int_\Omega h(v_n + c_n)\,dx \geq \frac{1}{p}\|v_n + c_n\|_p^p + \frac{1}{p}\sum_{i=1}^N \left\|\frac{\partial v_n}{\partial x_i}\right\|_p^p - M\mu(\Omega) - \int_\Omega hv_n\,dx - c_n\int_\Omega h\,dx \geq$$
$$\frac{1}{p}\|v_n\|_p^p + \frac{1}{p}\sum_{i=1}^N \left\|\frac{\partial v_n}{\partial x_i}\right\|_p^p - M\mu(\Omega) - \|h\|_{p'}\|v_n\|_{1,p} = \frac{1}{p}\|v_n\|_{1,p}^p - M\mu(\Omega) - \|h\|_{p'}\|v_n\|_{1,p}$$
$$\Rightarrow \Phi(u_n) \geq \|v_n\|_{1,p}\left(\frac{1}{p}\|v_n\|_{1,p}^{p-1} - \|h\|_{p'} - M\mu(\Omega)\right) \tag{5.58}$$

since $\int_\Omega hv_n dx \leq \left|\int_\Omega hv_n dx\right| \leq \|h\|_{p'}\,\|v_n\|_p \leq \|h\|_{p'}\,\|v_n\|_{1,p}$, and, similarly, for the second case,

$$\Phi(u_n) = \frac{1}{p}|\!|\!|u_n|\!|\!|_{1,p}^p - \int_\Omega G(u_n)dx - \int_\Omega hu_n\,dx = \frac{1}{p}|\!|\!|v_n + c_n|\!|\!|_{1,p}^p - \int_\Omega G(v_n + c_n)dx -$$
$$\int_\Omega h(v_n + c_n)\,dx \geq \frac{1}{p}|\!|\!|v_n + c_n|\!|\!|_{1,p}^p - M\mu(\Omega) - \int_\Omega hv_n\,dx - c_n\int_\Omega h\,dx \geq \frac{1}{p}|\!|\!|v_n|\!|\!|_{1,p}^p -$$
$$M\mu(\Omega) - \|h\|_{p'}\|v_n\|_p = \frac{1}{p}|\!|\!|v_n|\!|\!|_{1,p}^p - M\mu(\Omega) - \alpha\,|\!|\!|v_n|\!|\!|_{1,p} \Rightarrow$$
$$\Phi(u_n) \geq |\!|\!|v_n|\!|\!|_{1,p}\left(\frac{1}{p}|\!|\!|v_n|\!|\!|_{1,p}^{p-1} - \alpha - M\mu(\Omega)\right) \tag{5.59}$$

since $\int_\Omega hv_n dx \leq \left|\int_\Omega hv_n dx\right| \leq \|h\|_{p'}\,\|v_n\|_p \leq \alpha\,|\!|\!|v_n|\!|\!|_{1,p}$.

As $(\Phi(u_n))_{n\geq 1}$ is bounded, (5.58) and (5.59) impose that $\left(\int_\Omega |\nabla v_n|^2 dx\right)_{n\geq 1}$ is bounded, and hence, $(\|v_n\|_{1,p})_{n\geq 1}$ and $(|\!|\!|v_n|\!|\!|_{1,p})_{n\geq 1}$, respectively, are also bounded.

Consider the sequence $(\tilde{u}_n)_{n\geq 1}$, $\tilde{u}_n = v_n + \tilde{c}_n$, where $\tilde{c}_n \equiv c_n$ (modulo T) and $\tilde{c}_n \in [0, T]$. Since Φ has the period T, (5.56) gives

$$\|\tilde{u}_n - (u + \tilde{c}_n - c_n)\| \leq \delta_n \Rightarrow \Phi(\tilde{u}_n) \leq \Phi(u) + \varepsilon_n \,\|(\tilde{u}_n - u) + (c_n - \tilde{c}_n)\|,$$

i.e.,

$$\|\tilde{u}_n - w\| \leq \delta_n \Rightarrow \Phi(\tilde{u}_n) \leq \Phi(w) + \varepsilon_n \,\|\tilde{u}_n - w\|. \tag{5.60}$$

However, (v_n) and $(\widetilde{c}_n)$ are bounded, and hence $(\widetilde{u}_n)$ is bounded, consequently, it has a weakly convergent subsequence $(\widetilde{u}_{k_n})_{n\geq 1}$, $\widetilde{u}_{k_n} \overset{weak}{\to} \widetilde{u}$ (Eberlein–Šmulian), whence the existence of a convergent subsequence of $(\widetilde{u}_{k_n})_{n\geq 1}$, and using the same notation,

$$\widetilde{u}_{k_n} \to \widetilde{u} \tag{5.61}$$

(the same proof as in Proposition 4 [67], p. 484). In (5.60), taking $w = \widetilde{u}_{k_n} + \delta_{k_n} v$, $\|v\| = 1$, we obtain $\Phi(\widetilde{u}_{k_n} + \delta_{k_n} v) - \Phi(\widetilde{u}_{k_n}) \geq -\varepsilon_{k_n}\delta_{k_n}$, $-\varepsilon_{k_n} \leq \frac{1}{\delta_{k_n}}[\Phi(\widetilde{u}_{k_n} + \delta_{k_n} v) - \Phi(\widetilde{u}_{k_n})]$, and passing to the limit, we find that, since $(\widetilde{u}_{k_n}, \delta_{k_n}) \overset{(5.61)}{\to} (\widetilde{u}, 0)$, $0 \leq \overline{\lim_{n\to\infty}} \frac{1}{\delta_{k_n}}[\Phi(\widetilde{u}_{k_n} + \delta_{k_n} v) - \Phi(\widetilde{u}_{k_n})] \leq \Phi^0(\widetilde{u}; v)$, $0 \leq \Phi^0(\widetilde{u}; v)$, $\|v\| = 1$, whence $0 \leq \Phi^0(\widetilde{u}; v)$ $\forall v \in X$ $(0 \leq \Phi^0(\widetilde{u}; \frac{v}{||v||}) = \frac{1}{||v||}\Phi^0(\widetilde{u}; v))$ (Proposition 5.7), i.e., $0 \in \partial\, \Phi(\widetilde{u})$. Moreover, $c = \Phi(\widetilde{u})$, since $\Phi(\widetilde{u}_{k_n}) = \Phi(\widetilde{u}_{k_n} - \widetilde{c}_{k_n} + c_{k_n}) = \Phi(\widetilde{u}_{k_n}) \to c$, and also $\Phi(\widetilde{u}_{k_n}) \overset{(5.61)}{\to} \Phi(\widetilde{u})$ (Φ is continuous, being locally Lipschitz), and c is a critical value for Φ. □

Now:

Proposition 5.19. *Nonlinear Neumann problems*

$$(\mathrm{N}) \begin{cases} -\Delta_p u = g(u) + h(x), x \in \Omega \\ \quad Bu = 0 \text{ on } \partial\Omega, \end{cases}$$

and

$$(\mathrm{N}') \begin{cases} -\Delta_p^s u = g(u) + h(x), x \in \Omega \\ \quad Bu = 0 \text{ on } \partial\Omega, \end{cases}$$

respectively, with the conditions

$$(\mathrm{III})\ g\ :\ \mathbf{R}\ \to\ \mathbf{R}\ boundedmeasurable\ T - periodic, \int_0^T g(s)ds = 0$$

and

$$(\mathrm{IV})\ h\ boundedmeasurable, \int_\Omega hdx = 0,$$

have solution in $W^{1,p}(\Omega)$ in the sense of (5.27) and (5.28), respectively.

Proof. We are in the presence of problems of types $(*)$ and $(**)$ (this subsection), respectively, with $f(x, u) = g(u) + h(x)$. Conditions III and IV imply (I) ($\sigma = 0$) and (II) from Section 5.2.1. The associated functionals are

$$\Phi(u) = \frac{1}{p}||u||_{1,p}^p - \int_\Omega G(u)\, dx - \int_\Omega hudx,\ u\ \in\ W^{1,p}(\Omega),$$

and

$$\Phi(u) = \frac{1}{p}\|u\|_{1,p}^p - \int_\Omega G(u)\, dx - \int_\Omega hudx,\ u\ \in\ W^{1,p}(\Omega),$$

respectively, where $G(u(x)) = \int_0^{u(x)} g(t)dt$. G is Lipschitz and has the period T (use (III)). Since (I) and (II) are satisfied, any critical point of Φ is, according to Proposition 2.6, a solution for the problems (N) and (N′), respectively. However, Φ verifies $[\mathrm{PS}]^*{}_c$ for every c in $\mathbf{R}$, particularly for $c = \inf_{u\in W^{1,p}(\Omega)} \Phi(u)$. This is correct since Φ is lower bounded (the same justification as for (5.58) and (5.59), respectively). It only remains to apply Proposition 5.18, c is a critical value, $c = \Phi(u_0)$, u_0 a critical point, u_0 is a solution for (N) or (N′), respectively. □

Remark 5.12. *The series of results in this subsection has been presented by the author in [18].*

Remark 5.13. *Other applications to the velocity problem under the assumption of solid friction, or to the problem studied in [70], for thermal transfer or another pseudo torsion problem are provided in the second part of this work.*

6. Weak Solutions Using a Perturbed Variational Principle

An Application of Ghoussoub-Maurey Linear Principle to p-Laplacian and to p-Pseudo-Laplacian

The results in this section have been partially reported by the author in [17]. We start with the statement of the generalized perturbed variational principle. To clarify the involved notions, we present the following:

Definition 6.1. *Let X be a real normed space, $f : X \to (-\infty, +\infty]$, and C a nonempty subset of X, $x_0 \in C$. f strongly exposes C from below in x_0 when $f(x_0) = \inf f(C) < +\infty$ and $x_n \in C$ $\forall n \geq 1$, $f(x_n) \to f(x_0) \Rightarrow x_n \to x_0$. "$f$ strongly exposes C from above in x_0" and has a similar definition. We remark that, taking $C = X$ in the given definition, we fall on the definition of strongly minimum point. And also, a set of G_δ type means a set that is a countable intersection of open sets. A set of the F_σ type means a set that is a countable union of closed sets.*

Ghoussoub-Maurey Linear Principle. *Let X be a reflexive separable space and φ: $X \to (-\infty$, $+\infty]$* lower semicontinuous and proper.

(I) *If φ is bounded from below on the closed bounded nonempty subset C, the set*

$$\{\xi \in X^* : \varphi + \xi \text{ strongly exposes } C \text{ from below}\}$$

is of G_δ type and everywhere dense.

(II) *If, for any ξ from X^*, $\varphi + \xi$ is bounded from below, the set*

$$\{\xi \in X^* : \varphi + \xi \text{ strongly exposes } X \text{ from below}\}$$

is of G_δ type and everywhere dense.

The above linear principle devolves (see the continuation to Theorem 6.1) from the more general Theorem 6.1, and we proceed to its preparation with definitions and some auxiliary propositions.

Definition 6.2. *Let X be a real normed space and C, D with $C \subset D$ nonempty subsets of X^*. C is strict w-H_δ set in D or strict w^*-H_δ set in D if*

$$D \backslash C = \bigcup_{n=1}^{\infty} K_n, \tag{6.1}$$

dist $(K_n, C) > 0$, and K_n convex and weakly compact or $*$-weakly compact, respectively.

For instance,

Proposition 6.1. *Any nonempty closed set C of a separable reflexive space X, $C \neq X$, is strict w-H_δ set in X. In particular, if φ: $X \to (-\infty, +\infty]$ is l. s. c. (lower semicontinuous) and proper, then the epigraph of φ in $X \times \mathbf{R}$ is strict w-H_δ set in $X \times \mathbf{R}$.*

Proof. Let $(x_n)_{n\geq 1}$ be a sequence with the set of the terms dense in $X \backslash C$ (open set). Take, for each n from $\mathbf{N}$, K_n the closed ball centered in x_n with the radius $r_n := \frac{1}{4}\text{dist}(x_n, C)$. K_n is convex, weakly compact (Kakutani–Šmulian theorem [43], Volume 10, p. 151) and dist$(K_n$,

$C) > r_n$: let x be from K_n, dist $(x, C) \geq$ dist $(x_n, C) -$ dist $(x, x_n) \geq 4r_n - r_n = 3r_n$. We take the greatest lower bound. Moreover, $X\backslash C = \bigcup_{n=1}^{\infty} K_n$: let x be from $X\backslash C$, and let $\exists\ (x_{p_n})_{n\geq 1}$ subsequence of $(x_n)_{n\geq 1}$ such that $x_{p_n} \to x$, which also implies that dist $(x_{p_n}, C) \to$ dist (x, C), and if u and v are taken such that $0 < u < v <$ dist (x, C), from a rank on, we have 4 dist $(x_{p_n}, x) < u$ but, on the other side, $4r_n =$ dist $(x_{p_n}, C) > v$, and hence $x \in K_{p_n}$. □

Let X be a reflexive space and C, D subsets of X^*, $C \subset D$. We set

$$M\,(C, D) := \{x \in X : \exists\ \xi \in C \text{ such that } Jx\,(\xi) \geq Jx\,(\eta)\ \forall \eta \in D\},$$

otherwise expressed, $M\,(C, D)$ is the set of x from X for which Jx is upper-bounded on D, and the least upper bound is attained at a point of C, J the Hahn embedding of X in X^{**}. So, with X being reflexive, J is an isomorphism of vector spaces that preserves the norms.

In the following, to abridge the writing, sometimes x designates Jx.

Retain that if C is $*$-weakly compact, $M\,(C, D)$ is closed.

Notations. $B_X\,(x_0, r) \equiv$ the closed ball centered in x_0 of radius r in the normed space X.

$$B_X \equiv B_X\,(0,1).$$

$\overline{E}^{\,*} \equiv$ the closure of the subset E from X^* for the $*$-weak topology.

We proceed to the auxiliary propositions.

Definition 6.3. *Let (X, d) be metric space and* (M, δ) *the metric space of real functions defined on X. For each nonempty subset A of X, we consider*

$$\mathrm{M_A} := \{f \in \mathrm{M} : f \text{ upper bounded on } A\}$$

and, for each f from $\mathrm{M_A}$ *and* $t > 0$, *the slice* of A,

$$S(A,\ f,\ t) \stackrel{\text{def}}{=} \{x\ \in\ A :\ f\ (x)\ >\ \sup f(A) - t\},$$

is a set that is obviously nonempty when $\mathrm{M_A} \neq \emptyset$.

Proposition 6.2. *Let X be a reflexive space, D ⊂ X* and K ⊂ D, and K be convex ∗-weakly compact. If*

$$B_X\,(x, \alpha) \subset M(K, D),$$

then, for any $\varepsilon > 0$,

$$S(D,\ Jx,\ \varepsilon)\ \subset\ K + \frac{\varepsilon}{\alpha} Bx*.$$

In particular, when $\mathrm{C} \subset \mathrm{D} \subset \overline{\text{conv}}^{\,*}\mathrm{C}$, we have

$$\text{dist}\,(K, C) = 0.$$

Proof. *First assertion*. This reverts to

$$\xi\ \notin\ K + \frac{\varepsilon}{\alpha} B_{X*}\ \Rightarrow\ \xi\ \notin\ S(D,\ x,\ \varepsilon). \tag{6.2}$$

Suppose ad absurdum that $\xi \in S(D, x, \varepsilon)$, i.e., (see Definition 3.3)

$$x(\xi) > \sup x(D) - \varepsilon\ (x \in M(K, D) \Rightarrow Jx\,(D) \text{ upper bounded}). \tag{6.3}$$

The first member of (6.2) gives

$$||\xi - \eta||_{X*}\ >\ \frac{\varepsilon}{\alpha}\ \forall \eta\ \in\ K. \tag{6.4}$$

Let z be from X such that

$$\|Jz\|_{X^{**}}\ (= \|z\|) = 1 \text{ and } Jz(\xi - \eta) = \|\xi - \eta\|_{X^*} \tag{6.5}$$

(Hahn lemma [43], Volume 10, p. 94).

Combining this with (6.4) and taking the least upper bound, one obtains

$$\sup z(K) \ \leq \ z(\xi) - \frac{\varepsilon}{2}. \tag{6.6}$$

However, $x + \alpha z \overset{(6.5)}{\in} B_X\ (x, \alpha) \subset M(K, D)$, hence $\exists \eta_0$ is in K such that

$$(x + \alpha z)(\eta_0) \geq (x + \alpha z)(\xi). \tag{6.7}$$

However,

$$(x + \alpha z)(\xi) = x(\xi) + \alpha z(\xi) \overset{(6.3),(6.6)}{>} [\sup x(D) - \varepsilon] + \alpha[\sup z(K) + \frac{\varepsilon}{\alpha}] \ \geq \ x(\eta_0) + z(\eta_0)$$

and we obtain a contradiction with (6.7); thus, (6.2) is validated.

Second assertion. This results from

$$\bigcap_{\varepsilon>0} S(D, x, \varepsilon) \subset \ K, \tag{6.8}$$

$$\bigcap_{\varepsilon>0} S(D, x, \varepsilon) \cap \ C \ \neq \ \varnothing. \tag{6.9}$$

For (6.8): $\xi \in \bigcap_{\varepsilon>0} S(D, x, \varepsilon) \overset{(6.2)}{\Rightarrow} \xi = \eta_\varepsilon + \frac{\varepsilon}{\alpha} u_\varepsilon, \eta_\varepsilon \in K, \|u_\varepsilon\| \leq 1$, and hence $\|\xi - \eta_\varepsilon\| \leq \frac{\varepsilon}{\alpha}$, ξ is strong adherent point of K, and the strong closure of K is included in the $*$-weak closure of this, which is in K. □

Proposition 6.3. *Let X be reflexive space, $C \subset X^*$ nonempty and $U \subset X$ nonempty open having the property*

$$\sup Jx\ (C) < +\infty\ \forall x \in U.$$

Then Jx, for any x from U, is upper bounded on $\overline{\text{conv}}\ {}^*C$ and attains its least upper bound.

Proof. Set $D := \overline{\text{conv}}\ {}^*C$. We have

$$\sup Jx\ (C) = \sup Jx\ (D)$$

$[\xi \in \text{conv}\ C \Rightarrow \xi = \lambda_1\ \xi_1 + \lambda_2\ \xi_2, \xi_1, \xi_2 \in C, \lambda_1 + \lambda_2 = 1, \lambda_1, \lambda_2 \geq 0 \Rightarrow Jx\ (\xi) = \lambda_1\ Jx\ (\xi_1) + \lambda_2\ Jx(\xi_2) \leq \sup Jx\ (C)$; $\xi \in D \Rightarrow \exists\ \xi_n \in \text{conv}\ C, \xi_n \overset{*-\text{weak}}{\rightarrow} \xi \Rightarrow \xi_n\ (x) \rightarrow \xi(x), \xi_n\ (x) \leq \sup Jx\ (C)$ $\forall n \geq 1$, hence $\xi(x) \leq \sup Jx\ (C)]$, and so,

$$\sup Jx\ (D) < +\infty\ \forall x \in U, \tag{6.11}$$

and the first assertion is proved.

We proceed to the second assertion. We fix x from U, $\exists \varepsilon > 0$ with $x + \varepsilon z \in U\ \forall z$ in B_X. Then

$$\sup J(x + \varepsilon z)(D) = \sup\ (Jx + \varepsilon Jz)(D) < +\infty\ \forall z \in B_X\ ((6.11)),$$

consequently, once again using (6.11),

$$\sup Jz\ (D) < +\infty\ \forall z \in B_X, \tag{6.12}$$

which implies

$$\sup Jz\ (D) < +\infty\ \forall z \in X \tag{6.13}$$

(for any fixed z, $z \neq 0$, replace z in (6.12) with $\frac{z}{||z||}$, $Jy(\xi) = \xi(y)$). Replacing z with $-z$ in (6.13), one finds

$$\inf Jz\,(D) > -\infty \;\forall z \in X. \tag{6.14}$$

However, X is reflexive, hence (6.13) and (6.14) express that D is weakly bounded, and consequently, D is even bounded. With D also being $*$-weakly closed, it is $*$-weakly compact ([43], Volume 10, p. 144), hence the conclusion by applying Weierstrass theorem. □

Remark 6.1. *Proposition 6.3 is Lemma 2.7 from [65], Ch.2. Here, an improved proof is proposed.*

Proposition 6.4. *Let X be a reflexive space, $C \subset X^*$ nonempty and U nonempty open from X such that*

$$\sup Jx\,(C) < +\infty \;\forall x \in U.$$

If C is a strict w^*-H_δ set in $D{:} = \overline{\text{conv}}\,^*C$, then the set

$$V := \{x \in U\text{: } Jx \text{ attains } \sup Jx\,(D) \text{ in } C\}$$

includes a set of G_δ type that is dense in U.

Proof. According to the definition

$$\mathrm{D}\backslash\mathrm{C} = \bigcup_{n=1}^{\infty} K_n, \tag{6.15}$$

K_n is convex $*$-weakly compact and dist $(K_n, C) > 0$. Every Jx, $x \in U$ is upper bounded on D and attains its supremum on this (Proposition 6.3). As dist $(K_n, C) > 0$, Proposition 6.2 prevents $M(K_n, \mathrm{D})$ from including any nonempty ball; in other words, $\forall n \geq 1$ int $M(K_n, D) = \varnothing$, and so $M(K_n, D)$, being also closed, is thin, and $U_n := X\backslash M(K_n, D)$ is open and dense in X. Then $\bigcap_{n=1}^{\infty} U_n$ is dense in X (Baire theorem), hence $\bigcap_{n=1}^{\infty}(U \cap U_n)$, a set of the G_δ type, is dense in U. We see that if $x \in \bigcap_{n=1}^{\infty}(U \cap U_n)$, then Jx, which is upper bounded on D, attains a fortiori its least upper bound on C, as $x \notin M(K_n, D)\; \forall n \geq 1$ and one takes into account (6.13). □

Proposition 6.5. *Let X be a reflexive space, C subset of X^* and separable, $D := \overline{\text{conv}}\,^*C$ and U nonempty open subset of X such that $\sup Jx\,(C) < +\infty \;\forall x \in U$. Suppose that $M(C, D)$ includes a dense and of G_δ-type subset of U. Then, for any $K \subset D$ $*$-weakly compact with $K \cap C = \varnothing$ and for any $\varepsilon > 0$, the set*

$$G(K,\, \varepsilon) := \left\{x \,\in\, U :\, \exists r \,>\, 0 \text{ such that } S^*(D,\, Jx,\, r) \,\cap\, K = \varnothing \text{ and } \operatorname{diam} S^*(D,\, Jx,\, r) < \varepsilon\right\}$$

is open and dense in U.

Proof. $G(K, \varepsilon)$ *is open*. We use the fact that, D being bounded (proof for Proposition 6.3), the subset $S(D, Jx, r)$ is also bounded.

$G(K, \varepsilon)$ *is dense in U*. Let $V \subset U$, V nonempty open arbitrary. C being separable, we can find a sequence $(C_n)_{n\geq 1}$, $C_n \subset D$, with C_n being convex $*$-weakly compact, with the properties

$$C \subset \bigcup_{n=1}^{\infty} C_n, \tag{6.16}$$

$$\text{dist}\,(C_n, K) > 0 \;\forall n, \tag{6.17}$$

$$\text{diam } Cn \leq \frac{\varepsilon}{2} \;\forall n. \tag{6.18}$$

We obviously have $V \supset \bigcap \bigcup_{n=1}^{\infty} M(C_n, D) \cap V \overset{(6.16)}{\supset} M(C, D) \cap V$. However, the last member includes, according to the hypothesis, a set of the G_δ type that is dense in V, which forces, via the Baire theorem, the second member to have at least one term, let this term be $M(C_{n_0}, D) \cap V$, with a nonempty interior. From this interior, we take a point x. It remains to show

$$x \in G(K, \varepsilon). \tag{6.19}$$

Applying Proposition 6.2, for every $\rho > 0$, there exists $r > 0$ such that

$$S(D, Jx, r) \subset C_{n_0} + \rho B_{X^*} \tag{6.20}$$

($\exists\ \alpha > 0$ such that $B_X(x, \alpha) \subset M(C_{n_0}, D)$, taking $r = \alpha\rho$). We take

$$\rho < \min\left\{\frac{\varepsilon}{4}, \text{dist}\,(C_{n_0}, K)\right\}. \tag{6.21}$$

(6.20) gives

$$\overline{S}^{\,*}(D, Jx, r) \subset \overline{C_{n_0} + \rho B_{X^*}}^{\,*} = C_{n_0} + \rho B_{X^*} \tag{6.22}$$

because the last term, being $*$-weakly compact, is also $*$-weakly closed. However,

$$(C_{n_0} + \rho B_{X^*}) \cap K = \varnothing : \tag{6.23}$$

let, ad absurdum, $\xi + \rho u = \zeta$, $\xi \in C_{n_0}$, $u \in B_{X^*}$, $\zeta \in K$, then $\rho = \|\xi - \zeta\| \geq \text{dist}\,(C_{n_0}, K)$, and we obtain a contradiction with (6.21). So, (6.22) and (6.23) give $\overline{S}^{*}(D, Jx, r) \cap K = \varnothing$. Moreover,

$$\text{diam}\,\overline{S}^{\,*}(D, Jx, r) \overset{(6.22)}{\leq} \text{diam}\, C_{n_0} + 2\rho \overset{(6.18),(6.21)}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which finishes the proof. □

Now:

Theorem 6.1. *Let X be a reflexive space, C separable subset of X*, which is a strict w*-H_δ set in D* : $= \overline{\text{conv}}^{\,*}C$, *and U be open subset of X such that* $\sup Jx\,(C) < +\infty\ \forall x \in U$. *Then*
(I) *The set*

$$\{x \in U\text{: } Jx \text{ strongly exposes } D \text{ from above at a point of } C\}$$

is of the G_δ type and dense in U.

(II) *If* $\varphi\colon C \to (-\infty, +\infty]$ *is proper lower semicontinuous and* $\varphi + Jx$ *is,* $\forall x$ *from X, bounded from below on C, then the set*

$$\{x \in X\text{: } \varphi + Jx \text{ strongly exposes } C \text{ from below}\}$$

is of G_δ type and dense in X (N. Ghoussoub, B. Maurey [65]).

Proof. (I). According to the hypothesis, $D \backslash C = \bigcup_{n=1}^{\infty} K_n$, K_n is convex $*$-weakly compact, dist $(K_n, D) > 0$. $M(C, D)$ includes a subset of the G_δ type dense in U (Proposition 6.4), but then, for each n from $\mathbf{N}$, the set $V_n := G(K_1 \cup K_2 \cup \ldots \cup K_n, \frac{1}{n})$ is open and dense in U (Proposition 6.5), consequently $\bigcap_{n=1}^{\infty} V_n$ is dense in U (relativized Baire theorem) and it remains only to observe that $\bigcap_{n=1}^{\infty} V_n = \{x \in X\text{: } Jx$ strongly exposes D from above at a point of $C\}$.

(II). $C \times \mathbf{R}$, separable subset of $X^* \times \mathbf{R}$, is a strict w^*-H_δ set in $D \times \mathbf{R}$ and then, φ being l. s. c., the epigraph epi φ in $C \times \mathbf{R}$ (nonempty set, φ is proper) is a strict w^*-H_δ set in $D \times \mathbf{R}$ and hence also in $\overline{\text{conv}}\,^*$epi φ. $W := \{(x, \alpha): x \in X, \alpha < 0\}$ is open in $X \times \mathbf{R}$, and sup (Jx, α)(epi $\varphi) < +\infty\ \forall (x, \alpha) \in W$ [(Jx, α), continuous linear functional, acts on $C \times \mathbf{R}$ by the rule $(Jx, \alpha)(\xi, \lambda) = Jx(\xi) + \alpha\lambda$]. Indeed, $Jx(\xi) + \alpha\lambda \leq Jx(\xi) + \alpha\varphi(\xi)$, $\exists\, a$ in $\mathbf{R}$, such that $(J\frac{x}{\alpha})(\xi) + \varphi(\xi) \geq a\ \forall \xi \in C$ (the hypothesis), hence $Jx(\xi) + \alpha\varphi(\xi) \leq \alpha a\ \forall \xi \in C$. We show that for each $\varepsilon > 0$, $\exists\, y_0$ with $\|y_0\| \leq 2\varepsilon$ and $\varphi + Jy_0$ strongly exposes C from below, which is enough to validate (II). Applying (I), $\exists\, (x_\varepsilon, \alpha_\varepsilon)$ in W such that

$$\|(x_\varepsilon, \alpha_\varepsilon) - (0, -1)\| \leq \varepsilon \tag{6.24}$$

($(0, -1) \in W$!) and $(Jx_\varepsilon, \alpha_\varepsilon)$ strongly exposes epi φ from above at a point (ξ_0, λ_0). Then, $\forall$ (ξ, λ) from epi φ with $\xi \neq \xi_0$, we have

$$Jx_\varepsilon\ (\xi_0) + \alpha_\varepsilon\ \lambda_0 > Jx_\varepsilon\ (\xi) + \alpha_\varepsilon\ \lambda,$$

consequently, taking $y_0 := \frac{x_\varepsilon}{\alpha_\varepsilon}$, we have, in particular,

$$\varphi(\xi_0) + Jy_0\ (\xi_0) < \varphi(\xi) + Jy_0\ (\xi)\ \forall\ \xi \in C\backslash\{\xi_0\},$$

ξ_0 is a strict global minimum point for $\varphi + Jy_0$. Moreover, as $\varepsilon < \frac{1}{2}$ can be supposed, we have $\|y_0\| = \frac{\|x_\varepsilon\|}{|\alpha_\varepsilon|} \leq 2\varepsilon$, because, via (6.24), $\|x_\varepsilon\| \leq \varepsilon$ and $|\alpha_\varepsilon + 1| \leq \varepsilon$, hence, $\alpha_\varepsilon \in \left(-\frac{3}{2}, -\frac{1}{2}\right)$.

Finally, let $(\xi_{\,n})_{n\geq 1}$ be a minimizing sequence for $\varphi + Jy_0$ on C, then $(\xi_n, \varphi(\xi_n))_{n\geq 1}$ is maximizing sequence for $(Jx_\varepsilon, \alpha_\varepsilon)$ which strongly exposes epi φ in (ξ_0, λ_0), which imposes $\xi_n \to \xi_0$. □

Proof of Ghoussoub-Maurey linear principle

Proof. (I). Set $Y := X^*$, a separable reflexive space ([43], Volume 10, p. 162). Then, $Y^* = X$ (identification via the Hahn embedding; X is reflexive). C is separable and strict w^*-H_δ set in Y^* (Proposition 6.1, the weak and $*$-weak topologies coincide) and hence also in $D := \overline{\text{conv}}^* C$ ($X\backslash C = \bigcup_{n=1}^{\infty} K_n$ with the properties from (6.1), take the intersection with D). Apply (II), Theorem 6.1 transcribed with Y replaced by X; this is correct, as $\varphi + \xi$, $\xi \in X^* = Y^{**}$, is bounded from below $|(\xi(x)| \leq \|\xi\|\ \|x\|$ and C is bounded).

(II). The epigraph epi φ of φ in $X \times \mathbf{R}$ is strict w-H_δ set in $X \times \mathbf{R}$ (Proposition 6.1). In the following, using the proof for (II), Theorem 6.1 beginning from (6.24), epi φ is that considered above. □

Corollary 6.1. *Let X be reflexive space, C a subset of X^* separable bounded strict w^*-H_δ set in $D := \overline{\text{conv}}^* C$ and φ: $X \to (-\infty, +\infty]$ bounded from below, l. s. c. and proper. For any $\varepsilon > 0$, there exists x_0 in X with $\|x_0\| \leq \varepsilon$ and ξ_0 in C such that*

1° $(\varphi + Jx_0)(\xi_0) < (\varphi + Jx_0)(\xi)\ \forall \xi \in C\backslash\{\xi_{\,0}\}$;

2° *Any minimizing sequence from C for $\varphi + Jx_0$ converges to ξ_0 ([65]).*

Proof. C bounded implies that $\varphi + Jx$ bounded from below $\forall x \in X$, consequently (II), Theorem 6.1 can intercede to obtain 1° and 2°. □

We imply this theorem in two generalizations of a minimization problem of the form [71]:

$$C_f := \min\{\int_\Omega \left[\frac{1}{p}\left(|u|^{\rm p} + \sum_{\rm i=1}^{\rm N}\left|\frac{\partial u}{\partial x_{\rm i}}\right|^{\rm p}\right) - f(u)\right] dx:\ u\ \in\ W_0^{1,p}(\Omega), \|\ u\ \|\ 2* = 1\}, \tag{6.25}$$

and

$$C_f := \min\{\int_{\Omega} \left(\frac{1}{p}\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^p - f(u)\right)dx\colon\ u \in W_0^{1,p}(\Omega), \| u \| 2* = 1\},\tag{6.26}$$

where Ω is open set of C^1 class in $\mathbf{R}^N$, $N \geq 3$, $f \in W^{-1,p\prime}(\Omega)$ (= $(W_0^{1,p}(\Omega))^*$), $\frac{1}{p}+\frac{1}{p'}$ = 1, 2* = $\frac{2N}{N-2}$—the critical exponent for the Sobolev embedding (for the necessary explanations, here and in the following, see Section 2).

Let Ω be an open bounded set of the C^1 class in $\mathbf{R}^N$, $N \geq 3$. Consider the problems $(*)$ and $(**)$ from Section 5.1.2, where $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function with the growth condition

$$|f\ (x,\ s)|\ \leq\ c|s|^{p-1}\ + b(x),\ >\ 0,\ 2\ \leq\ p\ \leq \frac{2N}{N-2},\ b\ \in\ L^{p'}(\Omega),\ \frac{1}{p}+\frac{1}{p'}\ = 1.\tag{6.27}$$

The functionals φ: $W_0^{1,p}(\Omega) \to \mathbf{R}$,

$$\varphi(u) = \int_{\Omega} \left[\frac{1}{p}\left(|u|^p + \sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^p\right) - F(x,u(x))\right] dx\tag{6.28}$$

and

$$\varphi(u) = \int_{\Omega} \left(\frac{1}{p}\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^p - F(x,u(x))\right) dx,\tag{6.29}$$

with $F(x, s) := \int_0^s f(x,t)\,dt$, are of the Fréchet C^1 class and their critical points are the weak solutions of the problems $(*)$ and $(**)$, respectively.

Problem $(*)$. Let λ_1 be the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$ with a homogeneous boundary condition. We have (see, for instance, Section 2)

$$\lambda_1\ = \inf\{\frac{||u||_{1,p}^p}{|i(u)|_{0,p}^p}\colon\ u \in\ W_0^{1,p}(\Omega)\backslash\{0\}\}\ (\text{the Rayleigh} - \text{Ritz quotient}).\tag{6.30}$$

Now, we provide an answer to (6.25). We use the norm $\|\cdot\|_{1,p}$ on $W_0^{1,p}(\Omega)$ (see above). We denote the dual of $(W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$ by $W^{-1,p\prime}(\Omega)$, where p' is the conjugate of p (i.e., $\frac{1}{p}$ + $\frac{1}{p'}$= 1).

Proposition 6.6. *Under the above assumptions and, in addition, the growth condition*

$$F(x,\ s)\ \leq\ c_1\ \frac{s^p}{p}\ +\ \alpha(x)s,\tag{6.31}$$

with $0 < c_1 < \lambda_1$, $\alpha \in L^{q\prime}(\Omega)$ *for some* $2 \leq q \leq \frac{2N}{N-2}$ *and* $f(x, -s) = -f(x, s)$, $\forall x$ *from* Ω, *the following assertions hold:*

(i) The set of functions h from $W^{-1,p\prime}(\Omega)$, *having the property that the functional* $\varphi_h : W_0^{1,p}(\Omega) \to \mathbf{R}$,

$$\varphi h\ (u) = \frac{1}{p}||u||_p^p - \int_{\Omega}\ (F(x,u(x)) + h(u(x)))\,dx\tag{6.32}$$

has an attained minimum in only one point and includes a G_δ *set that is everywhere dense.*

(ii) The set of functions h from $W^{-1,p\prime}(\Omega)$, having the property

$$\text{the problem} \begin{cases} -\Delta_p u = f(x,u) + h\ (u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\,\Omega \end{cases} \text{has solutions,}$$

includes a G_δ set that is everywhere dense.

(*iii*) *Moreover, if $s \to f(x,s)$ is increasing, then the set of functions h from $W^{-1,p\prime}(\Omega)$, having the property*

$$\text{the problem} \begin{cases} -\Delta_p u = f(x,u) + h(u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \text{has a unique solution,}$$

includes a G_δ set that is everywhere dense.

Remark 6.2. *This is a generalization applied to the p-Laplacian and at $W_0^{1,p}(\Omega)$ of Theorem 2.13 from [65].*

Proof. It is sufficient to justify (*i*). Consider, for each h from $W^{-1,p\prime}(\Omega)$, the functional ξ_h from $W^{-1,p\prime}(\Omega)$,

$$\xi_h\,(u) = -\int_\Omega h(u(x))\,dx. \tag{6.33}$$

One may see that $\varphi_h = \varphi + \xi_h$ (see (6.28)). Consequently, according to the Ghoussoub-Maurey linear principle, (II), if we show that φ_h is bounded from below for any h from $W^{-1,p\prime}(\Omega)$, then (*i*) is proven. However, taking into account the Sobolev embedding and (6.31), we have $\forall u \in W_0^{1,p}(\Omega)$,

$$(\varphi + \xi_h)(u) = \frac{1}{p}\|u\|_{1,p}^p - \int_\Omega F(x,u(x))\,dx - \int_\Omega h(u(x))dx \geq \frac{1}{p}\|u\|_{1,p}^p - c_1 \int_\Omega \frac{|u(x)|^p}{p}\,dx -$$
$$\int_\Omega \alpha(x)u(x))\,dx - \int_\Omega h(u(x))dx \geq \frac{1}{p}\|u\|_{1,p}^p - \frac{c_1}{\lambda_1 p}\|u\|_{1,p}^p - \|\alpha\|_{q'}\|u\|_q - \|h\|_{W^{-1,p'}}\|u\|_{1,p} \geq$$
$$\frac{1}{p}\left(1 - \frac{c_1}{\lambda_1}\right)\|u\|_{1,p}^p - r\|u\|_{1,p} = \|u\|_{1,p}\left[\frac{1}{p}\left(1 - \frac{c_1}{\lambda_1}\right)\|u\|_{1,p}^{p-1} - r\right],$$

$r \in \mathbf{R}$, and hence the conclusion since $1 - \frac{c_1}{\lambda_1} > 0$. To prove some of these inequalities,

$$\int_\Omega F(x,u(x))\,dx \leq \int_\Omega \left(\frac{|u(x)|^p}{p} + \alpha(x)u(x)\right)dx = \frac{1}{p}\|i(u)\|_{0,p}^p + \int_\Omega \alpha(x)u(x)\,dx \leq$$
$$\frac{1}{p\lambda_1}\|u\|_{1,p}^p + \|\alpha\|_{0,q'}\|u\|_q \leq \frac{1}{p\lambda_1}\|u\|_{1,p}^p + K\|u\|_{1,p}$$

(see q and properties of Sobolev spaces in Section 2) and

$$\int_\Omega h(u(x))\,dx = \langle h,\ u\rangle \ \leq\ ||h||_{W^{-1,p'}}||u||_{1,p} \text{ (the norm of the linear continuous map).}$$

□

Problem $(**)$. Let λ_1 be the first eigenvalue of $-\Delta_p^s$ in $W_0^{1,p}(\Omega)$ with a homogeneous boundary condition. We have (see Section 2)

$$\lambda_1 = \inf\left\{\frac{\|u\|_{1,p}^p}{|i(u)|_{0,p}^p} : u \in W_0^{1,p}(\Omega)\backslash\{0\}\right\} \text{ (the Rayleigh} - \text{Ritz quotient)}.$$

Now, we provide an answer to (6.26). We use the norm $\|\cdot\|_p$ on $W_0^{1,p}(\Omega)$ (see above). We denote the dual of $(W_0^{1,p}(\Omega), \|\cdot\|_p)$ also by $W^{-1,p'}(\Omega)$, where p' is the conjugate of p (i.e., $\frac{1}{p}+\frac{1}{p'}=1$).

Proposition 6.7. *Under the above assumptions and, in addition, the growth condition*

$$F(x, s) \le c1\,\frac{s^p}{p} + \alpha(x)s,$$

with $0 < c_1 < \lambda_1$, $\alpha \in L^{q'}(\Omega)$ *for some* $2 \le q \le \frac{2N}{N-2}$ *and* $f(x, -s) = -f(x, s)$, $\forall x$ *from* Ω, *the following assertions hold.*

(*i*) *The set of functions h from* $W^{-1,p'}(\Omega)$, *having the property that the functional* $\varphi_h : W_0^{1,p}(\Omega) \to \mathbf{R}$,

$$\varphi h\,(u) = \frac{1}{p}\|u\|_p^p - \int_\Omega (F(x, u(x)) + h(u(x)))\,dx$$

has an attained minimum in only one point and includes a G_δ *set that is everywhere dense.*

(*ii*) *The set of functions h from* $W^{-1,p'}(\Omega)$, *having the property*

$$\text{the problem}\begin{cases} -\Delta_p^s u = f(x,u) + h\ (u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\,\Omega \end{cases} \text{has solutions},$$

includes a G_δ *set that is everywhere dense.*

(*iii*) *Moreover, if* $s \to f(x, s)$ *is increasing, then the set of functions h from* $W^{-1,p'}(\Omega)$, *having the property*

$$\text{the problem}\begin{cases} -\Delta_p^s u = f(x,u) + h(u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \text{has a unique solution},$$

includes a G_δ *set that is everywhere dense.*

Remark 6.3. *This is a generalization applied to the p-pseudo-Laplacian and at* $W_0^{1,p}(\Omega)$ *of Theorem 2.13 from [65].*

Proof. The proof and the afferent calculus follow, step by step, those for Proposition 6.6. There, one replaces the norm $\|\cdot\|_p$ on $W_0^{1,p}(\Omega)$ with the norm $\|\cdot\|_p$, and the inequalities and considerations remain the same. □

Remark 6.4. *For the above results, there are some applications for particular problems from [72–74], together with others for the pseudo torsion problem.*

7. Conclusions

Seven methods to obtain and/or characterize weak solutions for some problems of mathematical physics equations involving Dirichlet or Neumann problems for the p-Laplacian and the p-pseudo-Laplacian have been developed. They were presented starting from the most general abstract framework, together with detailed proofs, and numerous auxiliary propositions are highlighted. The aim of this unfolding is to be applied to problems derived from the modeling of real phenomena.

The first three ways use surjectivity results (obtained from three generalizations due to the author of three surjectivity theorems of Fučik and Nečas) and they are applied to duality maps and Nemytskii operators. The novelty of this work consists of the presentation of all the proof details, together with examples, to use this theory to solve mathematical physics problems describing real phenomena. Many demonstration details are explained in accordance with the aim of this journal. We proposed solving methods, and the characterization of the solutions for problems derived from glaciology, a nonlinear elastic membrane either with the p-Laplacian or with the p-pseudo-Laplacian and the pseudo torsion problem will be provided in Part two of this work.

The fourth sequence of results starts with Ekeland variational principle to obtain theoretical propositions that generalize two statements given by Ghoussoub, in which the author replaced the real Hilbert space with a real reflexive uniformly convex Banach space and the Fréchet C^1 class of the goal function with the condition imposed that it be lower semicontinuous and Gâteaux differentiable. It is also worthwhile to underline that Gâteaux differentiability can be replaced by the property of β-differentiability, with β being a bornology any. These theoretical statements have been used to characterize weak solutions for the p-Laplacian and for the p-pseudo-Laplacian. Some adequate examples will be also given in Part two of this work. The novelty consists in using these results for the targeted modeling of real phenomena problems solved for glaciology, nonlinear elastic membrane with p-Laplacian and p-pseudo-Laplacian and pseudo torsion problem.

The fifth succession of statements establishes results for nondifferentiable functionals using the Clarke gradient and critical points for this type of map and other specific notions until their insertion for the characterization of weak solutions for Dirichlet problems with the p-Laplacian and the p-pseudo-Laplacian, respectively, in $W_0^{1,p}(\Omega)$. Applications in thermal transfer, for Dirichlet problems derived from previously presented problems of the movement of a glacier, nonlinear elastic membrane, the pseudo torsion problem or nonlinear elastic membrane with the p-Laplacian and p-pseudo-Laplacian will be presented in Part two.

The sixth series of results, using properties of the Clarke subderivative, conditions of the Palais-Smale type and Ekeland principle, are results for Neumann or mixed problems. They are involved in the solution of corresponding problems for the p-Laplacian and the p-pseudo-Laplacian. The novelty resides in applications to solutions for the velocity of solid friction, the study of glacier flow, injection molding, thermal transfer and the pseudo-torsion problem.

The last sequence of assertions starts from the Ghoussoub-Maurey linear principle, which is used in order to solve some minimization problems. Generalizations of the minimization problem for the Laplacian given by Brezis and Nirenberg have been obtained in conjunction with the characterization of weak solutions of Dirichlet problems for the p-Laplacian and for the p-pseudo-Laplacian.

We are particularly interested in these applications, and this work is a necessary study for our future developments since our final goal is to obtain a mathematical model for a specific process involving transfer phenomena for the targeted environmental engineering application. The role of the reactor (in nanofabrication) in nano-liquid-liquid dispersed systems, in which micro- or nano-droplets play this role, and the determinant parameters are related to surface phenomena as a result of special intermolecular forces at the interface. In this context, some innovative mathematical modeling methods have to be proposed and tested in order to properly simulate the physical-chemical interactions and processes specific to nanofabrication. However, one may stress that there are no models available that can be applied for describing the diffusion phenomena involved in the micro-emulsification of dispersed systems in connection with surface properties at the interface in self-organized systems, and this will be the subject of future research.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Diaz, J.I.; Hernandez, J.; Tello, L. On the multiplicity of equilibrium solutions to a nonlinear diffusion equation on a manifold arising in climatology. *J. Math. Anal. Appl.* **1997**, *216*, 593–613. [CrossRef]
2. Diaz, J.I.; Thelin, F. On a nonlinear parabolic problem arising in some models related to turbulent flows. *SIAM J. Math. Anal.* **1994**, *25*, 1085–1111. [CrossRef]
3. Glowinski, R.; Rappaz, J. Approximation of a nonlinear elliptic problem arising in a non-Newtonian fluid model in glaciology. *Modél. Math. Anal. Numér.* **2003**, *37*, 175–186. [CrossRef]
4. King, J.R.; McCue, S.W. Quadrature domains and p-Laplacian growth. *Complex Anal. Oper. Theory* **2009**, *3*, 453–469. [CrossRef]
5. Aronsson, G.; Janfalk, U. On Helle-Shaw flow of power-law fluids. *Eur. J. Appl. Math.* **1992**, *3*, 343–366. [CrossRef]
6. Schowalter, R.E.; Walkington, N.J. Diffusion of fluid in a fissured medium with microstructure. *SIAM J. Math. Anal.* **1991**, *22*, 1702–1722. [CrossRef]
7. Péllissier, M.C.; Reynaud, M.L. Étude d'un modèle mathématique d'écoulement de glacier, R.C. *Acad. Sci. Paris Sér. I Math.* **1974**, *279*, 531–534.
8. Bhattacharya, T.; Dibenedetto, E.; Manfredi, J. Limits as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems. *Rend. Sem. Math. Univ. Pol. Torino Fasc. Spec.* **1989**, *47*, 15–68.
9. Kawohl, B. A family of torsional creep problems. *J. Reine Angew. Math.* **1990**, *410*, 1–22.
10. Benci, V.; Fortunato, D.; Pisani, L. Solitons like solutions of a Lorenz invariant equation in dimension 3. *Rev. Math. Phys.* **1998**, *10*, 315–344. [CrossRef]
11. Meghea, I. Minimax theorems in β-differentiability and theorems of surjectivity and of Fredholm alternative type for operators of the form $J_\varphi - \lambda S$. Ph.D. Thesis, University of Bucharest, Bucharest, Romania, 1999.
12. Meghea, I. Two solutions for a problem of partial differential equations. *UPB Sci. Bull. Ser. A* **2010**, *72*, 41–58.
13. Meghea, I. Some results of Fredholm alternative type for operators of the form $\lambda J_\varphi - S$ with applications. *UPB Sci. Bull. Ser. A* **2010**, *72*, 21–32.
14. Meghea, I. Weak solutions for p-pseudo-Laplacian Δ_p^s using a perturbed variational principle and via surjectivity results. *BSG Proc.* **2010**, *17*, 140–150.
15. Meghea, I. Weak solutions for p-Laplacian and for p-pseudo-Laplacian using surjectivity theorems. *BSG Proc.* **2011**, *18*, 67–76.
16. Meghea, I. On some perturbed variational principles: Connexions and applications. *Rev. Roum. Math. Pure Appl.* **2009**, *54*, 493–511.
17. Meghea, I. Variational approaches to characterize weak solutions for some problems of mathematical physics equations. *Abstr. Appl. Anal.* **2016**, *2016*, 2071926. [CrossRef]
18. Meghea, I.; Stamin, C.S. On a problem of mathematical physics equations. *Bull. Transilv. Univ. Bras. Ser. III Math. Inform. Phys.* **2018**, *11*, 169–180.
19. Meghea, I. Application of a Variant of Mountain Pass Theorem in Modeling Real Phenomena. *Mathematics* **2022**, *10*, 3476. [CrossRef]
20. Meghea, I. Applications of a perturbed linear variational principle via p-Laplacian. *UPB Sci. Bull. Ser. A* **2022**, *84*, 141–152.
21. Meghea, I. Applications for a generalization of two fundamental variational principles. *UPB Sci. Bull. Ser. A* **2020**, *82*, 57–68.
22. Meghea, I. *Ekeland Variational Principles with Generalizations and Variants*; Old City Publishing: Philadelphia, PA, USA; Éditions des Archives Contemporaines: Paris, France, 2009.
23. Burrage, K.; Hale, N.; Kay, D. An efficient FEM scheme for fractional-in-space reaction-diffusion equations. *SIAM J. Sci. Comput.* **2012**, *34*, A2145–A2172. [CrossRef]
24. Dang, J.; Hu, Q.; Xia, S.; Zhang, H. Exponential growth of solution for a reaction-diffusion equation with memory and multiple nonlinearities. *Res. Appl. Math.* **2017**, *1*, 101258. [CrossRef] [PubMed]
25. Fayolle, P.A.; Belyaev, A.G. p-Laplace diffusion for distance function estimation, optimal transport approximation, and image enhancement. *Comput. Aided Geom. Des.* **2018**, *67*, 1–20. [CrossRef]
26. Mukherjee, T.; Sreenadh, K. On Dirichlet problem for fractional p-Laplacian wirh singular non-linearity. *Adv. Nonlinear Anal.* **2019**, *8*, 52–72. [CrossRef]
27. Zhang, L.; Wang, F.; Ru, Y. Existence of nontrivial solutions for fractional differential equations with p-Laplacian. *J. Funct. Spaces* **2019**, *2019*, 3486410. [CrossRef]
28. Benedikt, J.; Girg, P.; Kotrla, L.; Takáč, P. Origin of the p-Laplacian and A. Missbach. *Electron. J. Differ. Equ.* **2018**, *2018*, 16.
29. Lafleche, L.; Salem, S. p-Laplacian Keller-Segel Equation: Fair Competition and Diffusion Dominated Cases. 2018. Available online: https://hal.archives-ouvertes.fr/hal-01883785 (accessed on 1 February 2022).
30. Cellina, A. The regularity of solutions of some variational problems, including the p-Laplace equation for $3 \le p < 4$. *AIMS* **2018**, *38*, 4071–4085.

31. Khan, H.; Li, Y.; Sun, H.; Khan, A. Esistence of solution and Hyers-Ulam stability for a coupled system of fractional differential equations with p-Laplacian operator. *J. Nonlinear Sci. Appl.* **2017**, *10*, 5219–5229. [CrossRef]
32. Xu, X. Existence theorems for a crystal surface model involving the p-Laplace operator. *SIAM J. Math. Anal.* **2017**, *50*, 1–21. [CrossRef]
33. Akagi, G.; Matsuura, K. Nonlinear diffusion equations driven by the $p(\cdot)$-Laplacian. *Nonlinear Differ. Equ. Appl.* **2013**, *20*, 37–64. [CrossRef]
34. Gulsen, T.; Yilmaz, E. Inverse nodal problem for p-Laplacian diffusion equation with polynomoally dependent spectral parameter. *Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat.* **2016**, *65*, 23–36.
35. Lee, Y.S.; Chung, S.Y. Extiction and positivity of solutions of the p-Laplacian evolution equation on networks. *J. Math. Anal. Appl.* **2012**, *386*, 581–592. [CrossRef]
36. Rasouli, S.H. An ecological model with the p-Laplacian and diffusion. *Int. J. Biomath.* **2016**, *9*, 1650008. [CrossRef]
37. Yang, Y.; Deng, J. Qualitative properties of a p-Laplacian population model with delay. *Adv. Differ. Equ.* **2017**, *2017*, 13. [CrossRef]
38. Elmoataz, A.; Toutain, M.; Tenbrinck, D. On the p-Laplacian and ∞-Laplacian on graphs with applications in image and data processing. *SIAM J. Imaging Sci.* **2015**, *8*, 2412–2451. [CrossRef]
39. Gupta, S.; Kumar, D.; Singh, J. Analytical solutions of convection-diffusion problems by combining Laplace transform method and homotopy perturbation method. *Alex. Eng. J.* **2015**, *54*, 645–651. [CrossRef]
40. Liero, M.; Koprucki, T.; Fischer, A.; Scholz, R.; Glitzki, A. p-Laplace thermistor modeling of electrothermal feedback in organic semiconductors devices. *Z. Angew. Math. Phys.* **2015**, *66*, 2957–2977. [CrossRef]
41. Silva, M.A.J. On a viscoelastic plate equation with history setting and perturbation of p-Laplacian type. *IMA J. Appl. Math. Adv. Access* **2012**, *78*, 1130–1146.
42. Fučik, S.; Nečas, J.; Souček, J. *Spectral Analysis of Nonlinear Operators*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1973.
43. Meghea, C.; Meghea, I. *Treatise on Differential Calculus and Integral Calculus for Mathematicians, Physicists, Chemists and Engineers in Ten Volumes*; Old City Publishing: Philadelphia, PA, USA; Éditions des Archives Contemporaines: Paris, France, 2015.
44. Dinca, G.; Jebelean, P. Some existence results for a class of nonlinear equations involving a duality mapping. *Nonlinear Anal.* **2001**, *46*, 347–363. [CrossRef]
45. Lions, J.L. *Quelques Méthodes des Résolution des Problèmes Aux Limites Non Linéaires*; Dunod, Gauthier-Villard: Paris, France, 1969.
46. Vainberg, M.M. *Variational Methods for the Study of Nonlinear Operators*; Holden Day Inc.: San Francisco, CA, USA, 1964.
47. Glowinski, R.; Marocco, A. Sur l'approximation, par éléments finis d'ordre un, et la résolution, par pénalisation—Dualité, d'une classe de problèmes de Dirichlet non linéaires. *Rev. Française D'autom. Inform. Rech. Opérationnelle* **1975**, *9*, 41–76. [CrossRef]
48. Mazur, S. Über schwache Konvergenz in den Räumen (L^p). *Stud. Math.* **1933**, *4*, 128–133. [CrossRef]
49. Stein, E. *Singular Integrals and Differentiability Properties of Functions*; Princeton University Press: Princeton, NJ, USA, 1970.
50. Dinca, G.; Jebelean, P.; Mawhin, J. *Variational and Topological Methods for Dirichlet Problems with p-Laplacian*; Catholic University of Louvain: Louvain-la-Neuve, Belgium, 1999.
51. Adams, R.A. *Sobolev Spaces*; Academic Press: New York, NY, USA, 1975.
52. Jebelean, P. Classical and Generalized Solutions for p-Laplacian; Ph.D. Thesis, University of Bucharest, Bucharest, Romania, 1994.
53. Figueiredo, G. *Topics in Nonlinear Functional Analysis*; The University of Maryland: College Park, MD, USA, 1967.
54. Ambrosetti, A.; Prodi, G. *Analisi Nonlineare*; Editrice Tecnico Scientifica: Pisa, Italy, 1973.
55. Lliboutry, L. *Traité de Glaciologie*; Masson & Cie: Paris, France, 1965; Book II.
56. Pélissier, M.C. *Sur Quelques Problèmes non Linéaires en Glaciologie*; Publications Mathèmatiques d'Orsay, no. 110, U.E.R. Mathèmatique; Université Paris IX: Paris, France, 1975.
57. Lindquist, P. Stability for the solutions of div $(|\nabla u|^{p-2}\nabla u) = f$ with varying p. *J. Math. Anal. Appl.* **1987**, *127*, 93–102. [CrossRef]
58. Cuccu, F.; Emamizadeh, B.; Porru, G. Nonlinear elastic membranes involving the p-Laplacian operator. *Electron. J. Differ. Equ.* **2006**, *2006*, 1–10.
59. Cuccu, F.; Emamizadeh, B.; Porru, G. Optimization or the best eigenvalue in problems involving the p-Laplacian. *Proc. Am. Math. Soc.* **2009**, *137*, 1677–1687. [CrossRef]
60. Belloni, M.; Kawohl, B. The pseudo-p-Laplace eigenvalue problem and viscosity solutions as $p \to \infty$. *ESAIM Control Optim. Calc. Var.* **2004**, *10*, 28–52. [CrossRef]
61. Nečas, J. Sur l'alternative de Fredholm pour les opérateurs nonlinéaires avec applications aux problèmes aux limites. *Ann. Sc. Norm. Sup. Pisa* **1969**, *23*, 331–345.
62. Brezis, H. *Analyse Fonctionnelle. Théorie et Applications*; Masson: Paris, France; Milan, Italy; Barcelone, Spain; Bonn, Germany, 1992.
63. Ekeland, I. *On the variational principle. Cahiers de Mathematique de la Décision*; Université Paris: Paris, France, 1972.
64. Ekeland, I. On the variational principle. *J. Math. Anal. Appl.* **1974**, *47*, 324–353. [CrossRef]
65. Ghoussoub, N. *Duality and Perturbation Methods in Critical Point Theory*; Cambridge University Press: Cambridge, UK, 1993.
66. Costa, D.G.; Gonçalves, J.V. Critical point theory for nondifferentiable functionals and applications. *J. Math. Anal. Appl.* **1990**, *153*, 470–485. [CrossRef]
67. Chang, K.C. Variational methods for non-differentiable functionals and their applications to partial differential equations. *J. Math. Anal. Appl.* **1981**, *80*, 102–129. [CrossRef]

68. Lanchon-Ducauquois, H.; Tulita, C.; Meuris, C. *Modélisation du Transfert Thermique Dans l'He II*; Congrès Français du Thermique: Lyon, France, 2000.
69. Clarke, F.H. *Optimization and Non-Smooth Analysis*; Canadian Mathematical Society: Otawa, ON, Canada, 1983.
70. Aronsson, G. On p-hrmonic functions, convex duality and an asymptotic formula for injection mould filing. *Eur. J. Appl. Math.* **1996**, *7*, 417–437. [CrossRef]
71. Brezis, H.; Nirenberg, L. A minimization problem with critical exponent and non-zero data. *Symmetry Nat. Sc. Norm. Sup. Pisa* **1989**, *1*, 129–140.
72. Lee, C.; Folgar, F.; Tucker, C.L. Simulation of compression molding for fiber-reinforced thermosetting polymers. *Trans. ASME* **1984**, *106*, 114–125. [CrossRef]
73. Bergwall, A. A geometric evolution problem. *Q. Appl. Math.* **2002**, *60*, 37–73. [CrossRef]
74. Janfalk, U. On a minimization problem for vector fields in L^1. *Bull. Lond. Math. Soc.* **1996**, *28*, 165–176. [CrossRef]