axioms

MDPI

Article

Characteristic Function of Maxmax Defensive-Equilibrium Representation for TU-Games with Strategies

Chenwei Liu [1], Shuwen Xiang [2,1,*] and Yanlong Yang [1]

[1] School of Mathematics and Statistics, Guizhou University, Guiyang 550025, China; liuchenwei15@163.com (C.L.); yylong1980@163.com (Y.Y.)
[2] College of Mathematical and Information Science, Guiyang University, Guiyang 550005, China
* Correspondence: shwxiang@vip.163.com

Abstract: In this paper, we propose a characteristic function of the maxmax defensive-equilibrium representation that maps every TU-game with strategies to a TU-game. This characteristic function is given by a two-step procedure in which each of any two complementary coalitions successively selects the equilibrium in a way that maximizes its utility. We then investigate the properties of this characteristic function and present the relations of the cores under three characteristic functions. Finally, as applications of our findings, we provide a firm production advertising game, a supply chain network game, a cost game with strategies, and a Cournot game.

Keywords: cooperative games; characteristic functions; TU-game with strategies; core

MSC: 91A06; 91A12; 91A18

1. Introduction

After von Neumann and Morgenstern [1] introduced the TU-games associated with strategies, the model of the TU-game associated with strategies has been widely used to analyze cooperation in multi-agent decision-making problems. For a class of games that generate coalition values on strategy profiles similar to biform games, existing literature has attempted to reduce the initial model to a strategic game to study its values and solutions. For instance, Ui [2] focused on reducing a TU-game with action choices to a strategic game where the payoff of each strategy profile is determined by the Shapley value [3] of the corresponding TU-game. In the same line, Brandenburger and Stuart [4] proposed the biform game analysis where the value of each coalition depends on the strategies of all players, in the sense that they reduce a biform game to a strategic game, where the payoff of each strategy profile is determined by a particular element in the core [5] of the corresponding TU-game. Following Brandenburger and Stuart [4], the model of biform games has been widely used in multi-agent decision problems. For instance, Ryall et al. [6] developed a biform game that allows the analysis of the dynamics of value appropriation when the topology of a relational network restricts the options available to actors. Feess et al. [7] applied the Shapley value to calculate the revenue of each firm and subtract the investment cost to obtain the payoff of each firm; based on these payoffs, they give the integration among firms using Nash equilibrium, González et al. [8] built novel three-player biform coalitional games to analyze community energy projects in Chile and Scotland, where the payoff of each strategy profile is determined using the core with the confidence index.

In the above studies of the class of games that generate coalition values on strategy profiles, the Nash equilibria were mainly used as the final solution, and the player cooperation occurred only on each strategy profile and was not shown on the set of strategy profiles. Fiestras-Janeiro et al. [9] referred to this class of games as TU-game with strategies. They introduced the maxmin procedure to reflect strategic moves between complementary

check for updates

Citation: Liu, C.; Xiang, S.; Yang, Y. Characteristic Function of Maxmax Defensive-Equilibrium Representation for TU-Games with Strategies. *Axioms* **2023**, *12*, 521. https://doi.org/10.3390/axioms12060521

Academic Editor: Conrado Miguel Manuel García

Received: 13 March 2023
Revised: 22 May 2023
Accepted: 22 May 2023
Published: 25 May 2023

coalitions, where each player in a coalition chooses one of the strategies in a set. Then, a TU-game depends on the chosen strategy profile associated with the coalition. Following the research route of Fiestras-Janeiro et al. [9], Liu et al. [10] presented the study under the minimax representation for the TU-games in characteristic function.

In this paper, we introduce a characteristic function of the maxmax defensive-equilibrium representation into a TU-game with strategies and transform this game into a cooperative game in characteristic function. We present the Shapley value and core as the solution to this cooperative game and investigate the properties of the core. This cooperative game is a new model in which players play defensive strategies in a coalition to obtain the characteristic function value.

Given a strategic game with TU, von Neumann and Morgenstern formulated a cooperative game with the characteristic function of minimax representation which asserts that a coalition's value is the maximum sum of utilities that the members of the coalition can guarantee themselves against the best offensive threat by the complementary coalition [1]. The characteristic function of defensive-equilibrium representation is derived by assuming that complementary coalitions would play an essentially defensive pair of equilibrium strategies against each other [11], and the rational-threats representation [12] is derived by assuming that a coalition S maximizes the difference between its value minus the value of the complementary coalition $-S$. In the TU-games with strategies, Fiestras-Janeiro et al. [9] and Liu et al. [10] respectively proposed the maxmin procedure and minimax representation to obtain the characteristic function values and transform the TU-game with strategies into a TU-game.

The most critical aspect of transforming a strategic form game into a cooperative game is how to determine the characteristic function. In the methods of the maxmin procedure [9] and minimax representation [10], the strategy choices of the players in the complementary coalition $-S$ must be against the coalition S (i.e., minimize the earnings of coalition S). Especially in the 2-person game, the rivalry between these two procedures is shown between the two players, which is significantly higher than the noncooperative behavior in the Nash equilibrium, which corresponds to a solution in which each player maximizes their interests but not confrontation. The characteristic function of defensive-equilibrium representation corresponds to the idea of Nash equilibrium, which shows the alliance behavior of the players in coalition S and the noncooperative behavior of the players in complementary coalition $-S$. Therefore, we will introduce a defensive-equilibrium approach to study the TU-game with strategies.

However, it is difficult to determine the characteristic function values by applying the existing defensive equilibrium. There are two main reasons for this difficulty: First, in the case where the strategy set is a mixed strategy set or a bounded closed convex set, the existence condition for a Nash equilibrium is strict, requiring the utility function to be concave (or quasiconcave) with respect to the multivariate (as shown in literature [13]). Second, even if the set of strategies is finite, due to the multiplicity of Nash equilibria, it is still difficult to obtain a defensive equilibrium. These problems are summarized as how to filter the Nash equilibria, and the methods are mainly refinement and selection. For example, trembling-hand perfect equilibrium [14] and essential equilibrium [15,16] are refining methods; focal equilibrium [17] and the selection of risk dominance and payoff dominance [18] are selecting methods. However, in most cases, it is not possible to refine or select a Nash equilibrium set into a single point set. In this paper, we will try to select Nash equilibria from the defensive rationality of coalitions and complementary coalitions.

To select Nash equilibria, we design a two-step procedure. First, the complementary coalition $-S$ maximizes its utility on the Nash equilibrium set $E(S,-S)$ (i.e., the Nash equilibrium set between coalitions S and $-S$), and the Nash equilibrium set corresponding to this maximized utility is denoted as $E_{-S}(S,-S)$. Second, the coalition S maximizes its utility on set $E_{-S}(S,-S)$, then this maximum utility is its characteristic function value $\omega(X,V)(S)$. Let us say that the function ω corresponding to $\omega(X,V)(S)$ is the characteristic function of the maxmax defensive-equilibrium representation. The characteristic function ω

transforms the TU-game with strategies (X, V) into a TU-game (N, V, ω). The values of all coalitions under characteristic function ω are not lower than the values under the maxmin procedure and minimax representation. Moreover, this two-step procedure provides a reference for selecting Nash equilibria in strategic games.

Models of TU-games with strategies have a wide range of practical applications, such as the aforementioned applications under the biform game analysis. Fiestras-Janeiro et al. [9] considered a set of agents who have to divide a certain amount of money; they can negotiate directly or, on the contrary, they can previously take some (costly) actions that will modify their negotiation power. Any situation of this type can be modeled as a TU-game with strategies. A reasonable recommendation for the players involved in one such process is that they negotiate directly, avoiding the costly actions but taking into account their capacities for changing the negotiation power. This is their main idea regarding TU-games with strategies: to associate each TU-game with strategies to a new TU-game that appropriately reflects the bargaining coalitional power of the involved players. In this paper, we present applications of the TU-games with strategies to a firm production advertising game, a supply chain network game, a cost game with strategies, and a Cournot game. In these applications, the bargaining power of any coalition is reflected through the characteristic function of the maxmax defensive-equilibrium representation.

The remainder of the paper is organized as follows. In Section 2, we recall the TU-games with strategies, introduce a characteristic function to build a cooperative game model, and examine the properties of the characteristic function. In Section 3, we present the Shapley value and core as the solution to the cooperative game and investigate the properties of the core as applications of these relations, the corresponding examples are presented. In Section 4, we discuss the existence and properties of the characteristic function. The paper is concluded in Section 5.

2. Cooperative Game with a Characteristic Function

Let $N = \{1, \cdots, n\}$ be a finite set of players and 2^N be the set of subsets (i.e., coalitions) of N. Denote by $|S|$ the number of players in a nonempty coalition $S \in 2^N$. X_i is a finite pure strategy set of player $i \in N$, $X = \prod_{i \in N} X_i$ is the set of finite pure strategy profiles of all players. For each nonempty coalition $S \in 2^N$, let $X_S = \prod_{i \in S} X_i$ be the set of strategies of S, $x_S = \{x_i : i \in S\} \in X_S$ is a strategy of S. For each $i \in N$ and $S \subseteq N$, denote $-S = N \backslash S$, then $x = (x_S, x_{-S}) \in (X_S, X_{-S}) = X$.

The *coalition function* V [4,9,10] is a map from X to the set of maps from 2^N to the reals. For each $x \in X$, the value of coalition $S \in 2^N$ is given by the map $V(x) : 2^N \to \mathbb{R}$, that is, $V(x)(S)$ is the value created by coalition S on x, with $V(x)(\emptyset) = 0$ for any $x \in X$.

A *TU-game with strategies* [9] involving player set N is a pair (X, V). Denote by $SG(N)$ the set of TU-games with strategies involving player set N and by SG the set of all TU-games with strategies involving a finite set of players. A *TU-game* is a function from 2^N to $\mathbb{R}$ such that the value on the empty set is equal to 0. Denote by $G(N)$ the set of TU-games involving player set N and by G the set of all TU-games involving a finite set of players. A procedure to transform a TU-game with strategies into a TU-game is a map $\phi : SG \to G$ that associates a TU-game $\phi(X, V) \in G(N)$ with every TU-game with strategies $(X, V) \in SG(N)$.

We give the following definition by referring to Myerson [11].

Definition 1. *For each TU-game with strategies $(X, V) \in SG(N)$, we say that μ is a characteristic function of defensive-equilibrium representation if, for every pair of complementary coalitions $S, -S \in 2^N$, there exist strategies $x_S^* \in X_S$ and $x_{-S}^* \in X_{-S}$ such that*

$$
\begin{aligned}
&x_S^* \in \arg\max_{x_S \in X_S} V(x_S, x_{-S}^*)(S),\\
&x_{-S}^* \in \arg\max_{x_{-S} \in X_{-S}} V(x_S^*, x_{-S})(-S),\\
&\mu(X, V)(S) = V(x_S^*, x_{-S}^*)(S), \text{ and } \mu(X, V)(-S) = V(x_S^*, x_{-S}^*)(-S).
\end{aligned}
$$

In particular, N and $\emptyset$ are a pair of complementary coalitions, so $\mu(X,V)(N) = \max_{x \in X} V(x)(N)$.

Obviously, if we consider coalitions S and $-S$ as two single players, then the strategy profile (x_S^*, x_{-S}^*) can be interpreted as a Nash equilibrium between S and $-S$.

For each pair of complementary coalitions $S, -S \in 2^N$, denote by $E(S,-S)$ the set of Nash equilibria between S and $-S$. Particularly, $E(N,\emptyset) = \arg\max_{x \in X} V(x)(N) \neq \emptyset$.

Assumption 1. *For each $(X,V) \in SG(N)$, assume that $E(S,-S) \neq \emptyset$ for all $S, -S \in 2^N$.*

Due to the multiplicity of the Nash equilibria, $E(S,-S)$ is generally not a single-point set. As mentioned in the introduction, in most cases, existing methods of refinement and selection for Nash equilibria frequently fail to produce a unique Nash equilibrium, and hence, it is difficult to confirm the characteristic function value $\mu(X,V)(S)$. Following the idea of the defensive equilibrium, we now select a Nash equilibrium from $E(S,-S)$ by taking the maximum values of complementary coalitions $-S$ and S, respectively, to determine the characteristic function value for coalition S.

Firstly, the coalition $-S$ maximizes its value in $E(S,-S)$, and the set of Nash equilibria corresponding to this maximum value is denoted by

$$E_{-S}(S,-S) = \arg\max_{x \in E(S,-S)} V(x)(-S).$$

Secondly, the coalition S selects the Nash equilibria in $E_{-S}(S,-S)$ to maximize its value; the set of Nash equilibria corresponding to this maximum value is denoted by

$$E_S^*(S,-S) = \arg\max_{x \in E_{-S}(S,-S)} V(x)(S).$$

In particular, if $S = N$, then $E_N^*(N,\emptyset) = E_\emptyset(N,\emptyset) = E(N,\emptyset) = \arg\max_{x \in X} V(x)(N)$.

Definition 2. *The characteristic function of the maxmax defensive-equilibrium representation is the map $\omega : SG \to G$ given, for all N and $(X,V) \in SG(N)$, by*

$$\omega(X,V)(S) = V(y_S^*, y_{-S}^*)(S), (y_S^*, y_{-S}^*) \in E_S^*(S,-S),$$

for all $S \in 2^N$. Specially, for the grand coalition N, $\omega(X,V)(N) = \max_{x \in X} V(x)(N)$.

The characteristic function ω is derived by assuming that complementary coalitions S and $-S$ would play a maximized essentially defensive pair of equilibrium strategies against each other.

For each TU-game with strategies $(X,V) \in SG(N)$, an n-person cooperative game in characteristic function is denoted by (N,V,ω), where V is the coalition function and ω is the characteristic function of the maxmax defensive-equilibrium representation.

For each $S \in 2^N$, the characteristic function of the *maxmin representation* (maxmin procedure) [9] and the characteristic function of *minimax representation* [10] are

$$\psi_1(X,V) = \max_{x_S \in X_S} \min_{x_{-S} \in X_{-S}} V(x_S, x_{-S})(S)$$

and

$$\psi_2(X,V) = \min_{x_{-S} \in X_{-S}} \max_{x_S \in X_S} V(x_S, x_{-S})(S),$$

respectively. Both ψ_1 and ψ_2 implicitly present relations involving coalitions S and $-S$ against each other, which seems to deviate from the idea of cooperation even more than from noncooperation, while the characteristic function ω more appropriately expresses the idea of noncooperation and defense.

Example 1. *Given a TU-game with strategies (X, V), with $N = \{1,2\}$, $A_1 = \{U, D\}$, $A_2 = \{L, R\}$. Its coalition function values are given in Table 1.*

In Table 1, it is easy to see that there are two Nash equilibria between complementary coalitions $\{1\}$ and $\{2\}$, i.e., $E(\{1\},\{2\}) = \{(U, L),(D, R)\}$. Then

$$E_{\{2\}}(\{1\},\{2\}) = E^*_{\{1\}}(\{1\},\{2\}) = \{(D, R)\},$$

it yields $\omega(X, V)(\{1\}) = 4$. Similarly,

$$E_{\{1\}}(\{1\},\{2\}) = E^*_{\{2\}}(\{1\},\{2\}) = \{(U, L)\},$$

thus, $\omega(X, V)(\{2\}) = 5$. Particularly, $\omega(X, V)(\{1,2\}) = 18$.

In addition, in Table 1, the characteristic function μ cannot be confirmed, and in Table 2, $\psi_1(X, V)\{1\} = \psi_1(X, V)\{2\} = 0$ may be unreasonable, because it is seen that the individual rationality requirement of player 2 should be higher than that of player 1 when observed from the whole game pattern.

Table 1. Coalition function values of Example 1.

S	$\{1\}$	$\{2\}$	$\{1,2\}$
$V(U, L)$	6	5	18
$V(U, R)$	0	0	1
$V(D, L)$	0	0	1
$V(D, R)$	4	7	15

Table 2. Characteristic function values of Example 1.

S	$\{1\}$	$\{2\}$	$\{1,2\}$
$\omega(X, V)$	4	5	18
$\psi_1(X, V)$	0	0	18
$\psi_2(X, V)$	4	5	18

Let the general representation of the characteristic function based on the coalition function V be $\mathcal{V}$. We provide the following properties by referring to Carpente et al. [19] and Fiestras-Janeiro et al. [9].

Coalition objectivity. For each $(X, V) \in SG(N)$, if a coalition $S \in 2^N$ is such that $V(x)(S) = c$ for all $x \in X$, then $\mathcal{V}(X, V)(S) = c$.

Let $(X, V) \in SG(N)$. A strategy $x_S \in X_S$ of coalition $S \in 2^N$ is *weakly dominated* in S if there exists a strategy $x'_S \in X_S, x'_S \neq x_S$, such that $V(x'_S, x_{-S})(S) \geq V(x_S, x_{-S})(S)$ for all $x_{-S} \in X_{-S}$. Moreover, (X^{-x_S}, V) denotes the TU-game with strategies that is obtained from (X, V) by deleting strategy x_S.

Irrelevance of weakly dominated strategies. For each $(X, V) \in SG(N)$, if strategy $x_S \in X_S$ is weakly dominated in S, then $\mathcal{V}(X, V)(S) = \mathcal{V}(X^{-x_S}, V)(S)$.

Let $(X, V) \in SG(N)$ and $S \in 2^N$. A strategy $x_{-S} \in X_{-S}$ of coalition $-S$ is a *weakly dominated threat* to coalition S if there exists a strategy $x'_{-S} \in X_{-S}, x'_{-S} \neq x_{-S}$, such that $V(x_S, x'_{-S})(S) \leq V(x_S, x_{-S})(S)$ for all $x_S \in X_S$. Furthermore, $(X^{-x_{-S}}, V)$ denotes the TU-game with strategies that is obtained from (X, V) by deleting strategy x_{-S}.

Irrelevance of weakly dominated threats. For each $(X, V) \in SG(N)$ and $S \in 2^N$, if strategy $x_{-S} \in X_{-S}$ is a weakly dominated threat to coalition S, then $\mathcal{V}(X, V)(S) = \mathcal{V}(X^{-x_{-S}}, V)(S)$.

Let $(X, V) \in SG(N)$ and $\emptyset \neq S \subseteq N$. Denote by N^S the set $\{[S]\} \cup N \backslash S$, i.e., the set of $|N| - |S| + 1$ players in which the coalition S is considered as a single player, and let

(X^S, V^S) be the TU-game with strategies that are obtained from (X, V) by considering the coalition S as a single player.

Merge invariance. Let $(X, V) \in SG(N)$ and $\emptyset \neq S \subseteq N$. Then, for each $T \subseteq N \backslash S$, $\mathcal{V}(X, V)(T) = \mathcal{V}(X^S, V^S)(T)$ and $\mathcal{V}(X, V)(T \cup S) = \mathcal{V}(X^S, V^S)(T \cup \{[S]\})$.

Irrelevance of complementary weakly dominated strategies. Let $(X, V) \in SG(N)$ and $S \subseteq N$, if strategy $x_{-S} \in X_{-S}$ is weakly dominated in $-S$, then $\mathcal{V}(X, V)(S) = \mathcal{V}(X^{-x_{-S}}, V)(S)$.

Irrelevance of complementary weakly dominated threats and strategies. Let $(X, V) \in SG(N)$ and $S \subseteq N$, if strategy $x_{-S} \in X_{-S}$ is a weakly dominated threat to coalition S, at the same time, it is weakly dominated in $-S$, then $\mathcal{V}(X, V)(S) = \mathcal{V}(X^{-x_{-S}}, V)(S)$.

Property 1. *For each $(X, V) \in SG(N)$, the characteristic function of the maxmax defensive-equilibrium representation ω satisfies the coalition objectivity, the irrelevance of weakly dominated strategies, the irrelevance of complementary weakly dominated strategies, and merge invariance.*

Proof. Let $(X, V) \in SG(N)$ and $S \in 2^N$ be such that $V(x)(S) = c$, for all $x \in X$. It is clear that $\omega(X, V) = c$, thus, ω satisfies the coalition objectivity.

If strategy $y_S \in X_S$ is weakly dominated in S, then there exists $y'_S \in X_S$ such that

$$V(y'_S, x_{-S})(S) \geq V(y_S, x_{-S})(S)$$

for all $x_{-S} \in X_{-S}$. Take $y^*_{-S} \in X_{-S}$ with $(y^*_S, y^*_{-S}) \in E^*_S(S, -S)$, then

$$V(y^*_S, y^*_{-S})(S) \geq V(y'_S, y^*_{-S})(S) \geq V(y_S, y^*_{-S})(S).$$

Thus,

$$\omega(X, V)(S) = \omega(X^{-y_S}, V)(S).$$

This shows that ω satisfies the irrelevance of weakly dominated strategies.

To check that ω satisfies the irrelevance of complementary weakly dominated strategies, notice that if strategy $y_{-S} \in X_{-S}$ is weakly dominated in $-S$, then there exists $y'_{-S} \in X_{-S}$ such that

$$V(x_S, y'_{-S})(-S) \geq V(x_S, y_{-S})(-S)$$

for all $x_S \in X_S$. Take $y^*_S \in X_S$ with $(y^*_S, y^*_{-S}) \in E_{-S}(S, -S)$, then

$$V(y^*_S, y^*_{-S})(-S) \geq V(y^*_S, y'_{-S})(-S) \geq V(y^*_S, y_{-S})(-S).$$

Then y_{-S} is not implemented by coalition $-S$, thus, when the coalition S selects $E^*_S(S, -S)$ in $E_{-S}(S, -S)$, it is not related to y_{-S}. Therefore,

$$\omega(X, V)(S) = \omega(X^{-y_{-S}}, V)(S).$$

From the definition of ω, it is clear that ω satisfies the merge invariance. The proof is completed. □

Clearly, the irrelevance of complementary weakly dominated threats and strategies is a special form of the irrelevance of complementary weakly dominated strategies. By their definitions, we obtain the following corollary of Property 1.

Corollary 1. *For each $(X, V) \in SG(N)$, the characteristic function of the maxmax defensive-equilibrium representation ω satisfies the coalition objectivity, the irrelevance of weakly dominated strategies, the irrelevance of complementary weakly dominated threats and strategies, and merge invariance.*

3. Solutions of Cooperative Games and Their Relations

Denote the cooperative game corresponding to $\mathcal{V}$ as $(N, V, \mathcal{V})$. The utility allocation $u = (u_1, \cdots, u_n) \in \mathbb{R}^n$ is said to be *individual rational* if $u_i \geq \mathcal{V}(X, V)(\{i\})$ for all $i \in N$; it is said to be *collective rational* if $\sum_{i \in N} u_i = \mathcal{V}(X,V)(N)$; and it is said to be *coalition rational* if $\sum_{i \in S} u_i \geq \mathcal{V}(X,V)(x)(S)$ for all $S \in 2^N$.

The *Shapley value* $\varphi(\mathcal{V}) \in \mathbb{R}^n$ of cooperative game $(N, V, \mathcal{V})$ is given by

$$\varphi_i(\mathcal{V}) = \sum_{S \subseteq -i} \frac{s!(n-s-1)!}{n!}[\mathcal{V}(S \cup \{i\}) - \mathcal{V}(S)]$$

for all $i \in N$.

For any nonempty coalition $S \in 2^N$, denoted by $1^S \in \mathbb{R}^n$, the characteristic vector of S, its $i-$th coordinate is

$$(1^S)_i = \begin{cases} 1, & \text{if } i \in S, \\ 0, & \text{otherwise.} \end{cases}$$

A map $\lambda : 2^N \backslash \{\emptyset\} \to \mathbb{R}_+$ is called a *balanced map* if $\sum_{S \in 2^N \backslash \{\emptyset\}} \lambda(S)1^S = 1^N$, and a cooperative game $(N, V, \mathcal{V})$ is said to be *balanced* if $\sum_{S \in 2^N \backslash \{\emptyset\}} \lambda(S)\mathcal{V}(X,V)(S) \leq \mathcal{V}(X,V)(N)$ for each balanced map λ.

We say that $(N, V, \mathcal{V})$ is balanced if and only if its *core* $C(\mathcal{V}) \neq \emptyset$ is

$$C(\mathcal{V}) = \{u = (u_1, \cdots, u_n) \in \mathbb{R}^n : \sum_{i \in N} u_i = \mathcal{V}(X,V)(N); \sum_{i \in S} u_i \geq \mathcal{V}(X,V)(S), \forall S \in 2^N, S \neq N\}.$$

In Example 1,

$$C(\omega) = C(\psi_2) = \{(u_1, u_2) \in \mathbb{R}^2 : 4 \leq u_1 \leq 13, 5 \leq u_1 \leq 14\},$$
$$C(\psi_1) = \{(u_1, u_2) \in \mathbb{R}^2 : 0 \leq u_1 \leq 18, 0 \leq u_1 \leq 18\}.$$

Example 2 ([20]). *Consider the case that a business owner approaches three advertising firms, each of which has its cable channel to produce and broadcast an advertising program. Each firm independently decides whether to accept the job offer. The business owner is willing to pay \$17 to each firm for her job. In Table 3, the three advertising firms are players 1, 2, and 3, each one with a pure strategy set $\{0,1\}$, where 0 or 1 indicates that one is turning down the job or accepting the job. The three strategies of each strategy profile are, in turn, owned by players 1, 2, and 3. Let $D(S)$ denote the total cost of creating advertisement artworks for firms in coalition S. Assuming that all the firms take their assigned jobs, the complete cost schedule for advertisement(s) collaboration costs are as follows.*

Table 3. Coalition function values for creating advertisements art works.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$V(0,0,0)$	0	0	0	0	0	0	0
$V(0,1,0)$	0	−1	0	−1	0	−1	−1
$V(1,0,0)$	2	0	0	2	2	0	2
$V(1,1,0)$	2	−1	0	4	2	−1	4
$V(0,0,1)$	0	0	−3	0	−3	−3	−3
$V(0,1,1)$	0	−1	−3	−1	−3	2	2
$V(1,0,1)$	2	0	−3	2	3	−3	3
$V(1,1,1)$	2	−1	−3	4	3	2	4.4

$$D(\{1\}) = 15, D(\{2\}) = 18, D(\{3\}) = 20, D(\{1,2\}) = 30,$$
$$D(\{1,3\}) = 31, D(\{2,3\}) = 32, D(\{1,2,3\}) = 46.6.$$

For each $x \in X$, the value $V(x)(S)$ of coalition S is equal to the income paid by the business to S minus the cost of S for creating advertisement artworks. For instance,

$$V((0,1,1))(\{1\}) = 0 \times 17 - 0 = 0,$$
$$V((0,1,1))(\{2\}) = 1 \times 17 - 18 = -1,$$
$$V((0,1,1))(\{2,3\}) = 2 \times 17 - 32 = 2.$$

By Table 3, it is clear that the coalition function values are generated on strategy profiles except $(0,0,0)$, but rather the utilities of all players.

In Brandenburger and Stuart's [4] biform game analysis for the TU-game with strategies (X,V), the players play the cooperative game on each strategy profile to determine their utilities. In our analysis, players have a cooperative willingness to form coalitions to choose strategies. Coincidentally, this behavior is facilitated by the fact that the utilities of all coalitions are generated on each profile of strategies. In Table 3, when players 1 and 2 form a coalition $\{1,2\}$ with the rationality of maximizing the coalition value, they observe all strategy profiles and then select a strategy $(1,1)$ to maximize its value. In this case, the complementary coalition $\{3\}$ selects strategy 0 to maximize its value. Thus, the unique Nash equilibrium $(1,1,0)$ of coalitions $\{1,2\}$ and $\{3\}$ is obtained. Therefore,

$$\omega(X,V)(\{1,2\}) = 4, \omega(X,V)(\{3\}) = 0.$$

Similarly,

$$\omega(X,V)(\{1\}) = 2, \omega(X,V)(\{2\}) = 0, \omega(X,V)(\{1,3\}) = 3,$$
$$\omega(X,V)(\{2,3\}) = 2, \omega(X,V)(\{1,2,3\}) = 4.4.$$

According to these characteristic values, we get

$$C(\omega) = \{(u_1,u_2,u_3) \in \mathbb{R}^3 : 2 \le u_1 \le 2.4, 0 \le u_2 \le 1.4, 0 \le u_3 \le 0.4; \sum_{i=1}^{3} u_i = 4.4\}.$$

It is easy to check that $\mu = \omega = \psi_1 = \psi_2$; therefore, the cores under these characteristic functions are the same, and the Shapley values are also the same, i.e.,

$$\varphi_1(\mu) = \varphi_1(\omega) = \varphi_1(\psi_1) = \varphi_1(\psi_2) = 2/6 \times 2 + 1/6 \times 4 + 1/6 \times 3 + 2/6 \times (4.4 - 2) = 79/30,$$
$$\varphi_2(\mu) = \varphi_2(\omega) = \varphi_2(\psi_1) = \varphi_2(\psi_2) = 34/30, \varphi_3(\mu) = \varphi_3(\omega) = \varphi_3(\psi_1) = \varphi_3(\psi_2) = 19/30.$$

In Example 2, the definition of coalition function determines that μ, ω, ψ_1, and ψ_2 are identical. However, in general, they are not identical; see for examples below.

Similar to reference [10], relevant properties of the Shapley value $\varphi(\mathcal{V})$ can be obtained. In this paper, we mainly give the properties of the core $C(\mathcal{V})$.

It is easy to obtain that if any allocation $u = (u_1, \cdots, u_n) \in \mathbb{R}^n$ satisfies individual rationality and collective rationality, then $u_i \in [\mathcal{V}(X,V)(\{i\}), \mathcal{V}(X,V)(N) - \sum_{j \neq i} \mathcal{V}(X,V)(\{j\})]$ (abbreviated as $u_i \in u_{\mathcal{V}}[i,-i]$) for all $i \in N$. It can be seen that the smaller $u_{\mathcal{V}}[i,-i]$ for all $i \in N$, the smaller the deviation of the cooperative solution.

Property 2. *For each $(X,V) \in SG(N)$, let $\mathcal{V}_1$ and $\mathcal{V}_2$ be any two characteristic functions for (X,V). If $\mathcal{V}_1(X,V)(N) = \mathcal{V}_2(X,V)(N)$ and $\mathcal{V}_1(X,V)(S) \le \mathcal{V}_2(X,V)(S)$ for all $S \in 2^N$. Then, for every $i \in N$,*

$$u_{\mathcal{V}_2}[i,-i] \subseteq u_{\mathcal{V}_1}[i,-i].$$

Proof. Since $\mathcal{V}_1(X,V)(S) \le \mathcal{V}_2(X,V)(S)$ for all $S \in 2^N$, then

$$\mathcal{V}_1(X,V)(\{i\}) \le \mathcal{V}_2(X,V)(\{i\})$$

for all $i \in N$, and by $\mathcal{V}_1(X,V)(N) = \mathcal{V}_2(X,V)(N)$, it follows that

$$\mathcal{V}_2(X,V)(N) - \sum_{j \neq i} \mathcal{V}_2(X,V)(\{j\}) \leq \mathcal{V}_1(X,V)(N) - \sum_{j \neq i} \mathcal{V}_1(X,V)(\{j\})$$

for all $i \in N$. Therefore, for every $i \in N$,

$$u_{\mathcal{V}_2}[i,-i] \subseteq u_{\mathcal{V}_1}[i,-i].$$

The proof is completed. □

Property 3. *Let $(X,V) \in SG(N)$. Then, for every $i \in N$,*

$$u_\omega[i,-i] \subseteq u_{\psi_2}[i,-i] \subseteq u_{\psi_1}[i,-i].$$

Proof. By the definition of maps ω and ψ_2, we obtain that for every $x^* = (x^*_S, x^*_{-S}) \in E^*_S(S,-S)$,

$$\begin{aligned}\omega(X,V)(S) = V(x^*_S, x^*_{-S})(S) &= \max_{x_S \in X_S} V(x_S, x^*_{-S})(S) \\ &\geq \min_{x_{-S} \in X_{-S}} \max_{x_S \in X_S} V(x_S, x_{-S})(S) = \psi_2(X,V)(S)\end{aligned}$$

for all $S \subseteq N$.

Since for any $x = (x_S, x_{-S}) \in X$,

$$\min_{x_{-S} \in X_{-S}} V(x_S, x_{-S})(S) \leq V(x_S, x_{-S})(S) \leq \max_{x_S \in X_S} V(x_S, x_{-S})(S)$$

for all $S \subseteq N$, then, for any $x_{-S} \in X_{-S}$,

$$\max_{x_S \in X_S} \min_{x_{-S} \in X_{-S}} V(x_S, x_{-S})(S) \leq \max_{x_S \in X_S} V(x_S, x_{-S})(S)$$

for all $S \subseteq N$, it follows that

$$\max_{x_S \in X_S} \min_{x_{-S} \in X_{-S}} V(x_S, x_{-S})(S) \leq \min_{x_{-S} \in X_{-S}} \max_{x_S \in X_S} V(x_S, x_{-S})(S)$$

for all $S \subseteq N$, i.e., $\psi_1(X,V)(S) \leq \psi_2(X,V)(S)$ for all $S \subseteq N$.

Thus,

$$\psi_1(X,V)(S) \leq \psi_2(X,V)(S) \leq \omega(X,V)(S)$$

for all $S \subseteq N$. Therefore, for every $i \in N$,

$$u_\omega[i,-i] \subseteq u_{\psi_2}[i,-i] \subseteq u_{\psi_1}[i,-i]$$

by Property 2. The proof is completed. □

Property 3 shows that the allocation range of cooperative solutions under ω is smaller compared to ψ_1 and ψ_2, and, therefore, the deviation of solution coordinates is smaller.

Theorem 1. *Let $(X,V) \in SG(N)$. Then,*

$$C(\omega) \subseteq C(\psi_2) \subseteq C(\psi_1).$$

Proof. First, we prove that $C(\omega) \subseteq C(\psi_2)$. Assume that allocation $u = (u_1, \cdots, u_n) \in C(\omega)$, but $u = (u_1, \cdots, u_n) \notin C(\psi_2)$. Then, there exists $S \subseteq N$ with $S \neq \emptyset$ such that

$$\sum_{i \in S} u_i < \min_{x_{-S} \in X_{-S}} \max_{x_S \in X_S} V(x_S, x_{-S})(S),$$

by Property 3,

$$\sum_{i\in S} u_i < \min_{x_{-S}\in X_{-S}} \max_{x_S\in X_S} V(x_S, x_{-S})(S) \leq \omega(X,V)(S),$$

this is a contradiction since $u = (u_1, \cdots, u_n) \in C(\omega)$. Therefore, $C(\omega) \subseteq C(\psi_2)$.

Next, we prove that $C(\psi_2) \subseteq C(\psi_1)$. Assume that allocation $u = (u_1, \cdots, u_n) \in C(\psi_2)$, but $u = (u_1, \cdots, u_n) \notin C(\psi_1)$. Then, there exists $S \subseteq N$ with $S \neq \emptyset$ such that

$$\sum_{i\in S} u_i < \max_{x_S\in X_S} \min_{x_{-S}\in X_{-S}} V(x_S, x_{-S})(S),$$

by Property 3,

$$\sum_{i\in S} u_i < \max_{x_S\in X_S} \min_{x_{-S}\in X_{-S}} V(x_S, x_{-S})(S) \leq \min_{x_{-S}\in X_{-S}} \max_{x_S\in X_S} V(x_S, x_{-S})(S),$$

which contradicts that $u = (u_1, \cdots, u_n) \in C(\psi_2)$. Therefore, $C(\psi_2) \subseteq C(\psi_1)$. The proof is completed. □

Example 3 ([21])**.** *In a supply chain network game, there are three members: the manufacturer (player 1), the seller (player 2), and the user (player 3); the manufacturer produces and sells the product, the seller sells the product, and the user buys the product. The manufacturer's strategy set is $\{a_1, a_2\}$, with a_1 and a_2 denoting discount and no-discount strategies, respectively, the seller's strategy set is $\{b_1, b_2\}$, with b_1 and b_2 denoting advertising and no-advertising strategies, respectively, and the user's strategy set is $\{c_1, c_2\}$, with c_1 denoting purchase from the manufacturer and c_2 denoting purchase from the seller. Members of the supply chain can make decentralized decisions (working alone) and centralized decisions (forming coalitions). The coalition function of this game is shown in Table 4. The value created by the entire supply chain, the grand coalition, is related to the purchase volume of users. To maximize the value created by the grand coalition and the benefit of each member, we consider centralized decision-making among the members and allocate the benefits through cooperative games.*

Table 4. Coalition function values for the supply chain network game.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$V(a_1,b_1,c_1)$	1	2	3	3	5	5	12
$V(a_1,b_2,c_1)$	0	2	2	3	5	4	8
$V(a_2,b_1,c_1)$	1	1	3	3	4	5	10
$V(a_2,b_2,c_1)$	0	1	3	3	4	4	7
$V(a_1,b_1,c_2)$	1	2	3	3	3	4	8
$V(a_1,b_2,c_2)$	1	2	2	3	3	5	9
$V(a_2,b_1,c_2)$	1	2	3	3	5	4	7
$V(a_2,b_2,c_2)$	1	2	3	3	5	5	6

In Table 4, due to the multiplicity of Nash equilibria, the characteristic function values corresponding to characteristic function μ cannot be confirmed. The characteristic function values corresponding to characteristic functions ω, ψ_1, and ψ_2 are shown in Table 5, thus,

$$u_\omega[i,-i] \subseteq u_{\psi_2}[i,-i] = u_{\psi_1}[i,-i]$$

for $i = 1,2,3$, and

$$C(\omega) = \{(u_1,u_2,u_3) \in \mathbb{R}^3 : 1 \leq u_1 \leq 7, 2 \leq u_2 \leq 7, 3 \leq u_3 \leq 9; \sum_{i=1}^{3} u_i = 12\} \subseteq$$
$$C(\psi_1) = C(\psi_2) = \{(u_1,u_2,u_3) \in \mathbb{R}^3 : 0 \leq u_1 \leq 7, 1 \leq u_2 \leq 7, 2 \leq u_3 \leq 9; \sum_{i=1}^{3} u_i = 12\}.$$

Table 5. Characteristic function values for the supply chain network game.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\omega(X,V)$	1	2	3	3	5	5	12
$\psi_1(X,V)$	0	1	2	3	5	5	12
$\psi_2(X,V)$	0	1	2	3	5	5	12

In addition,

$$\begin{aligned}&\varphi_1(\omega)=\varphi_1(\psi_1)=\varphi_1(\psi_2)=19/6,\\&\varphi_2(\omega)=\varphi_2(\psi_1)=\varphi_2(\psi_2)=22/6,\\&\varphi_3(\omega)=\varphi_3(\psi_1)=\varphi_3(\psi_2)=31/6.\end{aligned}$$

Let $(X,V)\in SG(N)$. For every $x\in X$, we define a TU-game V^x as

$$\begin{aligned}&V^x(S)=V(x_S,y^*_{-S})(S),\forall S\in 2^N,S\neq N,\\&V^x(N)=\omega(X,V)(N),\end{aligned}$$

where y^*_{-S} is given by (y^*_S,y^*_{-S}) with $(y^*_S,y^*_{-S})\in E^*_S(S,-S)$. The core of game V^x is defined by

$$C(V^x)=\{u=(u_1,\cdots,u_n)\in\mathbb{R}^n:\sum_{i\in N}u_i=\omega(X,V)(N);\sum_{i\in S}u_i\geq V^x(S),\forall S\in 2^N,S\neq N\}.$$

Theorem 2. *Let $(X,V)\in SG(N)$. Then,*

$$\bigcap_{x\in X}C(V^x)=C(\omega).$$

Proof. Let any $u\in\bigcap_{x\in X}C(V^x)$, then, for every $x\in X$,

$$\begin{aligned}&\sum_{i\in S}u_i\geq V^x(S)=V(x_S,x^*_{-S})(S),\forall S\in 2^N,S\neq N,\\&\sum_{i\in N}u_i=V^x(N)=\omega(X,V)(N).\end{aligned}$$

Thus, for every $(x^*_S,x^*_{-S})\in E^*_S(S,-S)$,

$$\begin{aligned}&\sum_{i\in S}u_i\geq\max_{x\in X}V^x(S)=V(x^*_S,x^*_{-S})(S)=\omega(X,V)(S),\forall S\in 2^N,S\neq N,\\&\sum_{i\in N}u_i=\omega(X,V)(N),\forall x\in X.\end{aligned}$$

Therefore, we obtain that $u\in C(\omega)$, i.e., $\bigcap_{x\in X}C(V^x)\subseteq C(\omega)$.

Second, we check that $C(\omega)\subseteq\bigcap_{x\in X}C(V^x)$. Let any $u\in C(\omega)$ and $(x^*_S,x^*_{-S})\in E^*_S(S,-S)$, then,

$$\begin{aligned}&\sum_{i\in S}u_i\geq\omega(X,V)(S)=V(x^*_S,x^*_{-S})(S),\forall S\in 2^N,S\neq N,\\&\sum_{i\in N}u_i=\omega(X,V)(N)=V^x(N),\forall x\in X.\end{aligned}$$

Then, for every $x\in X$,

$$\begin{aligned}&\sum_{i\in S}u_i\geq V(x^*_S,x^*_{-S})(S)\geq V(x_S,x^*_{-S})=V^x(S),\forall S\in 2^N,S\neq N,\\&\sum_{i\in N}u_i=V^x(N).\end{aligned}$$

Therefore, $u\in\bigcap_{x\in X}C(V^x)$, i.e., $C(\omega)\subseteq\bigcap_{x\in X}C(V^x)$. The proof is completed. □

Example 4. *Given a TU-game with strategies (X, V), with $N = \{1,2,3\}$, $A_1 = \{U, D\}$, $A_2 = \{L, R\}$, $A_3 = \{F\}$. The coalition function values are given in Table 6.*

Table 6. Coalition function values of Example 4.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$V(U,L,F)$	2	2	0	5	4	4	5
$V(U,R,F)$	3	0	0	3	3	3	11
$V(D,L,F)$	0	3	0	3	3	3	11
$V(D,R,F)$	2	2	0	8	3	3	12

The corresponding characteristic function values of this game are shown in Table 7; thus, the core of the cooperative game (N, V, ω) is

$$C(\omega) = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : 2 \leq u_1 \leq 8, 2 \leq u_2 \leq 8, 0 \leq u_3 \leq 4; \sum_{i=1}^{3} u_i = 12\}.$$

Table 7. Characteristic function values of Example 4.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\mu(X,V) = \omega(X,V)$	2	2	0	8	4	4	12
$\psi_1(X,V)$	2	2	0	8	3	3	12
$\psi_2(X,V)$	2	2	0	8	3	3	12

The corresponding TU-games are shown in Table 8, thus, the cores of these TU-games are

$$C(V^{(U,L,F)}) = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : 2 \leq u_1 \leq 8, 2 \leq u_2 \leq 8, 0 \leq u_3 \leq 7; \sum_{i=1}^{3} u_i = 12\}.$$

$$C(V^{(U,R,F)}) = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : 2 \leq u_1 \leq 9, 0 \leq u_2 \leq 8, 0 \leq u_3 \leq 9; \sum_{i=1}^{3} u_i = 12\}.$$

$$C(V^{(D,L,F)}) = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : 0 \leq u_1 \leq 8, 2 \leq u_2 \leq 9, 0 \leq u_3 \leq 9; \sum_{i=1}^{3} u_i = 12\}.$$

$$C(V^{(D,R,F)}) = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : 0 \leq u_1 \leq 9, 0 \leq u_2 \leq 9, 0 \leq u_3 \leq 4; \sum_{i=1}^{3} u_i = 12\}.$$

Table 8. TU-games V^x in Example 4.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$V^{(U,L,F)}$	2	2	0	5	4	4	12
$V^{(U,R,F)}$	2	0	0	3	4	3	12
$V^{(D,L,F)}$	0	2	0	3	3	4	12
$V^{(D,R,F)}$	0	0	0	8	3	3	12

It is easy to get

$$C(V^{(U,L,F)}) \cap C(V^{(U,R,F)}) \cap C(V^{(D,L,F)}) \cap C(V^{(D,R,F)}) = C(\mu)$$
$$= \{(u_1, u_2, u_3) \in \mathbb{R}^3 : 2 \leq u_1 \leq 8, 2 \leq u_2 \leq 8, 0 \leq u_3 \leq 4; \sum_{i=1}^{3} u_i = 12\}.$$

By Table 7, the characteristic function values under ω are no less than those under ψ_1 and ψ_2, for all $S \in 2^N$. In addition,

$$\varphi_1(\omega) = \varphi_1(\mu) = 5, \varphi_2(\omega) = \varphi_2(\mu) = 5, \varphi_3(\omega) = \varphi_3(\mu) = 2,$$
$$\varphi_1(\psi_1) = \varphi_2(\psi_1) = 31/6, \varphi_3(\psi_1) = 10/6,$$
$$\varphi_1(\psi_2) = \varphi_2(\psi_2) = 31/6, \varphi_3(\psi_2) = 10/6.$$

Cooperative games about cost are an important aspect of game theory. To study such cost games, we must correspondingly adjust some definitions and properties of TU-games with strategies. Thus, in this case,

$$E_{-S}(S, -S) = \arg\min_{x \in E(S,-S)} V(x)(-S),$$
$$E_S^*(S, -S) = \arg\min_{x \in E_{-S}(S,-S)} V(x)(S).$$

For every $S \in 2^N$,

$$\omega(X, V)(S) = V(y_S^*, y_{-S}^*)(S) with (y_S^*, y_{-S}^*) \in E_S^*(S, -S)$$

is the characteristic function of *minmin defensive-equilibrium representation* for the game $(X, V) \in SG(N)$. In particular, $\omega(X, V)(N) = \min_{x \in X} V(x)(N)$ is the minimum cost created by the grand coalition N. Correspondingly, the *core* of the cooperative game (N, V, ω) is

$$C(\omega) = \{u = (u_1, \cdots, u_n) \in \mathbb{R}^n : \sum_{i \in N} u_i = \omega(X, V)(N); \sum_{i \in S} u_i \leq \omega(X, V)(S), \forall S \in 2^N, S \neq N\}.$$

In addition,

$$\psi_1(X, V) = \min_{x_S \in X_S} \max_{x_{-S} \in X_{-S}} V(x_S, x_{-S})(S),$$
$$\psi_2(X, V) = \max_{x_{-S} \in X_{-S}} \min_{x_S \in X_S} V(x_S, x_{-S})(S),$$

the cores under the ψ_1 and ψ_2 are still represented as $C(\psi_1)$ and $C(\psi_2)$.

Example 5. *Given a TU-cost game with strategies (X, V) [9], with $N = \{1, 2, 3\}$, $A_1 = \{U, D\}$, $A_2 = \{L, R\}$, $A_3 = \{F\}$. The coalition function values and characteristic function values are shown in Table 9 and Table 10, respectively.*

Table 9. Coalition function values for the cost game with strategies.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$V(U,L,F)$	1	8	2	8	2	8	8
$V(U,R,F)$	2	9	5	9	5	9	9
$V(D,L,F)$	5	10	7	10	7	10	10
$V(D,R,F)$	6	7	9	7	9	9	9

Table 10. Characteristic function values for the cost game with strategies.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\mu(X,V) = \omega(X,V)$	1	8	9	7	2	8	8
$\psi_1(X,V)$	2	9	9	7	5	9	8
$\psi_2(X,V)$	2	8	9	7	5	9	8

Cooperative game (N, V, ω) is balanced since it has a nonempty core, i.e.,

$$C(\omega) = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : 1 \geq u_1 \geq 0, 8 \geq u_2 \geq 6, 9 \geq u_3 \geq 1; \sum_{i=1}^{3} u_i = 8\}.$$

The cores under ψ_1 and ψ_2 are

$$C(\psi_1) = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : 2 \geq u_1 \geq -1, 9 \geq u_2 \geq 3, 9 \geq u_3 \geq 1; \sum_{i=1}^{3} u_i = 8\},$$
$$C(\psi_2) = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : 2 \geq u_1 \geq -1, 8 \geq u_2 \geq 3, 9 \geq u_3 \geq 1; \sum_{i=1}^{3} u_i = 8\}.$$

There exist negative cost values of player 1 in cores $C(\psi_1)$ and $C(\psi_2)$, which may lead players 2 and 3 to oppose the allocation of these two cores. Moreover,

$$\varphi_1(\omega) = \varphi_1(\mu) = -1, \varphi_1(\psi_1) = -4/6, \varphi_1(\psi_2) = -3/6,$$

this makes it highly likely that players 2 and 3 also oppose the allocation under the Shapley values $\varphi(\omega)$, $\varphi(\mu)$, $\varphi(\psi_1)$, and $\varphi(\psi_2)$.

Let V be a coalition function. For any coalitions $S_1, S_2 \in 2^N$ with $S_1 \cap S_2 = \emptyset$.

(1) We say that V is *superadditive* on $x \in X$ if $V(x)(S_1 \cup S_2) \geq V(x)(S_1) + V(x)(S_2)$ and V is *superadditive* on X if it is superadditive for any $x \in X$;
(2) We say that V is *subadditive* on $x \in X$ if $V(x)(S_1 \cup S_2) \leq V(x)(S_1) + V(x)(S_2)$ and V is *subadditive* on X if it is subadditive for any $x \in X$; and
(3) We say that V is *additive* on $x \in X$ if $V(x)(S_1 \cup S_2) = V(x)(S_1) + V(x)(S_2)$ and V is *additive* on X if it is additive for any $x \in X$.

If V is additive on X, then the TU-game with strategies (X, V) becomes an n-person strategic game; correspondingly, the cooperative game $(N, V, \mathcal{V})$ becomes the cooperative game $(N, \mathcal{V})$. In this case of the additivity of V, characteristic functions ω, ψ_1, and ψ_2 are still applicable for the cooperative game $(N, \mathcal{V})$.

Example 6 ([22])**.** *Consider a Cournot game involving firms 1, 2, and 3, all of which produce the same product. The strategy set of firm 1 is $\{0, 1\}$, where strategy 1 means that 1 unit of the product is produced per day and strategy 0 means that no product is produced, the strategy sets of firms 2 and 3 are both $\{2, 3\}$, where strategies 2 or 3 means that 2 or 3 units of the product are produced per day. The market price per unit of product is*

$$p = 8 - (q_1 + q_2 + q_3),$$

where q_i is the output of the firm i, determined by its strategies, for $i = 1, 2, 3$. The daily revenue of firm i is pq_i, for $i = 1, 2, 3$. It is easy to obtain the coalition function of this game, as shown in Table 11, where three strategies of each strategy profile are, in turn, owned by firms 1, 2, and 3.

Table 11. Coalition function for the Cournot game.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$V(0,2,2)$	0	8	8	8	8	16	16
$V(0,3,2)$	0	9	6	9	6	15	15
$V(1,2,2)$	3	6	6	9	9	12	15
$V(1,3,2)$	2	6	4	8	6	10	12
$V(0,2,3)$	0	6	9	6	9	15	15
$V(0,3,3)$	0	6	6	6	6	12	12
$V(1,2,3)$	2	4	6	6	8	10	12
$V(1,3,3)$	1	3	3	4	4	6	7

From Table 11, the coalition function V is additive on X. The characteristic function value under μ cannot be confirmed due to the multiplicity of Nash equilibrium of complementary coalitions. The characteristic functions ω, ψ_1, and ψ_2 are shown in Table 12. Since $\psi_1(X,V)(S) \le \psi_2(X,V)(S) \le \omega(X,V)(S)$ for all $S \subseteq N$, then, for $i = 1,2,3$,

$$\begin{aligned}&[\omega(X,V)(\{i\}), 16 - \sum_{j\neq i}\omega(X,V)(\{j\})]\\ \subseteq&[\mathcal{V}_2(X,V)(\{i\}), 16 - \sum_{j\neq i}\mathcal{V}_2(X,V)(\{j\})]\\ \subseteq&[\mathcal{V}_1(X,V)(\{i\}), 16 - \sum_{j\neq i}\mathcal{V}_1(X,V)(\{j\})],\end{aligned}$$

i.e.,

$$u_\omega[i,-i] \subseteq u_{\mathcal{V}_2}[i,-i] \subseteq u_{\mathcal{V}_1}[i,-i],$$

which satisfies Property 3. The cores and their relation is

$$\begin{aligned}&C(\omega) = \{(u_1,u_2,u_3) \in \mathbb{R}^3 : 3 \le u_1 \le 4, 6 \le u_2 \le 10, 6 \le u_3 \le 10; \sum_{i=1}^{3} u_i = 16\}\\ \subseteq&C(\psi_2) = \{(u_1,u_2,u_3) \in \mathbb{R}^3 : 1 \le u_1 \le 4, 4 \le u_2 \le 10, 4 \le u_3 \le 10; \sum_{i=1}^{3} u_i = 16\}\\ \subseteq&C(\psi_1) = \{(u_1,u_2,u_3) \in \mathbb{R}^3 : 1 \le u_1 \le 4, 3 \le u_2 \le 10, 3 \le u_3 \le 10; \sum_{i=1}^{3} u_i = 16\},\end{aligned}$$

which satisfies Theorem 1. Moreover, the Shapley values under ω, ψ_1, and ψ_2 are

$$\begin{aligned}&\varphi_1(\omega) = 14/6, \varphi_2(\omega) = \varphi_3(\omega) = 41/6,\\ &\varphi_1(\psi_1) = 16/6, \varphi_2(\psi_1) = \varphi_3(\psi_1) = 40/6,\\ &\varphi_1(\psi_2) = 14/6, \varphi_2(\psi_2) = \varphi_3(\psi_2) = 41/6.\end{aligned}$$

Table 12. Characteristic functions for the Cournot game.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\omega(X,V)$	3	6	6	6	6	12	16
$\psi_1(X,V)$	1	3	3	6	6	12	16
$\psi_2(X,V)$	1	4	4	6	6	12	16

4. Discussion

Clearly, in the case of finitely pure strategies, the Nash equilibrium between complementary coalitions may not exist. We now present a simple result on the existence of the Nash equilibrium. For each $(X,V) \in SG(N)$, suppose X_i is a mixed strategy set corresponding to the pure strategy set of player $i \in N$.

Assumption 2. *Let $(X,V) \in SG(N)$. Referring to the extension of Nash [23] to the payoff functions, we assume that coalition function $V(x_S, x_{-S})(S)$ is linear on X_S for all $x_S \in X_S$ and all nonempty coalition $S \in 2^N$.*

Theorem 3. *If $(X,V) \in SG(N)$ satisfies Assumption 2, then for every pair of complementary coalitions S and $-S$, there exist strategies $x_S^* \in X_S$ and $x_{-S}^* \in X_{-S}$ such that*

$$\begin{aligned}&x_S^* \in \arg\max_{x_S \in X_S} V(x_S, x_{-S}^*)(S),\\ &x_{-S}^* \in \arg\max_{x_{-S} \in X_{-S}} V(x_S^*, x_{-S})(-S),\\ &\mu(X,V)(S) = V(x_S^*, x_{-S}^*)(S), \text{ and } \mu(-S) = V(x_S^*, x_{-S}^*)(-S).\end{aligned}$$

In particular, $\mu(X,V)(N) = \max_{x \in X} V(x)(N)$.

Proof. For every pair of complementary coalitions $S, -S \in 2^N$, we consider coalitions S and $-S$ as two single players. By Theorem 1 of Nash [23] and Assumption 2, there exist coalition strategies $x^*_S \in X_S$ and $x^*_{-S} \in X_{-S}$ such that

$$\begin{aligned}
&x^*_S \in \arg\max_{x_S \in X_S} V(x_S, x^*_{-S})(S),\\
&x^*_{-S} \in \arg\max_{x_{-S} \in X_{-S}} V(x^*_S, x_{-S})(-S),\\
&\mu(X,V)(S) = V(x^*_S, x^*_{-S})(S), \text{ and } \mu(-S) = V(x^*_S, x^*_{-S})(-S).
\end{aligned}$$

Since X is a bounded closed convex set, and $V(x)(N)$ is a continuous function on X by Assumption 2, there exists $x^* \in X$ such that $V(x^*)(N)$ is the maximum value on X. Therefore, $\mu(X,V)(N) = \max_{x \in X} V(x)(N) = V(x^*)(N)$. □

In the case of Theorem 3, the infinity of the mixed strategies makes it extremely difficult to refine Nash equilibria. Therefore, it is also difficult to obtain characteristic functions ω, ψ_1, and ψ_2.

In this paper, we select the Nash equilibrium by taking the maxmax behavior between complementary coalitions S and $-S$, which is consistent with the noncooperative behavior implied by Nash equilibrium. If the complementary coalitions S and $-S$ choose the Nash equilibrium from the adversarial point of view, then the maxmin procedure or minimax representation can be applied.

Notice that our notion of the maxmax defensive-equilibrium representation is different from the maxmax procedure

$$\max_{x_S \in X_S} \max_{x_{-S} \in X_{-S}} V(x_S, x_{-S})(S).$$

Under the maxmax procedure, the core is likely to be an empty set, and the Shapley value may not be applicable as it may result in unreasonable allocation.

5. Conclusions

Similar to Fiestras-Janeiro et al. [9], studying the TU-game with strategies from the cooperative direction, we transform the TU-game with strategies into a cooperative game in characteristic function and investigate the properties of the core. We derive three main results.

First, to solve the problem that the characteristic function of defensive-equilibrium representation [11] cannot be confirmed due to the multiplicity of Nash equilibria, we establish a characteristic function of the maxmax defensive-equilibrium representation ω, which guarantees that each coalition S gets a characteristic function value. Unlike the fierce rivalry of ψ_1 and ψ_2, ω reflects the maximum defensive selection strategy of the complementary coalitions S and $-S$. Meanwhile, we provide the characteristic function and core for the cost game with strategies and give an example where ω is more reasonable than ψ_1 and ψ_2. Second, we characterize the properties of the general characteristic function based on the coalition function and check that ω satisfies four properties. Third, under the characteristic functions ω, ψ_1, and ψ_2, we present the range of of the allocation coordinate u_i, study the relation among cores, and get that the core under ω is the minimum allocation set. We also show the relation among the cores on all strategy profiles and the core under ω.

Regarding the application of the characteristic function ω, we summarize it as three points: First, the highest individual rationality under ω is obtained by comparing ω, ψ_1, and ψ_2. This means that the allocation solution under ω is narrowed, which is beneficial to obtain a cooperative solution with a smaller deviation in practical problems. Second, the n-person strategic form game with TU is a special form of the TU-game with strategies under the additivity of the coalition function. Therefore, the characteristic function ω can be used to obtain the cooperative solution of the n-person strategic form game with TU. Third, the method of selecting Nash equilibria given by ω can provide an interesting reference for selecting Nash equilibria in n-person strategic form games.

This paper leads us to consider some important questions for future research. One of the problems is to study other characteristic functions based on the Nash equilibria of complementary coalitions. Another issue is to investigate the relation between the cooperative solution under characteristic function ω and the biform game Nash equilibrium solutions. Still, another issue is to further enrich the application of the cooperative solution under characteristic function ω in practical problems such as supply chains.

Author Contributions: Conceptualization, C.L. and S.X.; methodology, C.L. and S.X.; software, C.L. and S.X.; validation, C.L., S.X. and Y.Y.; writing—original draft preparation, C.L.; writing review and editing, C.L. and S.X.; visualization, C.L., S.X. and Y.Y.; supervision, C.L., S.X. and Y.Y.; project administration, S.X. and Y.Y.; funding acquisition, S.X. and Y.Y. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by National Natural Science Foundation of China (Grant Nos. [71961003], [12061020]), Qian Ke He LH (Grant No. [2017] 7223), and Talent Introduction Foundation of Guizhou University (Grant No. [(2019) 49]).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The data sets used and/or analyzed during the current study are available from the corresponding author upon reasonable request.

Acknowledgments: The authors are grateful to the referees for their careful reading of the manuscript and valuable comments. The authors thank the help from the editor too.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. von Neumann, J.; Morgenstern, O. *Theory of Games and Economic Behavior*; Princeton University Press: Princeton, NJ, USA, 2004. [CrossRef]
2. Ui, T. A Shapley value representation of potential games. *Game Econ. Behav.* **2000**, *31*, 121–135. [CrossRef]
3. Shapley, L. A value for n-person games. In *Contributions to the Theory of Game*, 2nd ed.; Princeton University Press: Princeton, NJ, USA, 1953. [CrossRef]
4. Brandenburger, A.; Stuart, H. Biform games. *Manag. Sci.* **2007**, *53*, 537–549. [CrossRef]
5. Gillies, D. Some Theorems on n-Person Games. Ph.D. Thesis, Princeton University, Princeton, NJ, USA, 1953.
6. Ryall, M.D.; Soren, O. Brokers and Competitive Advantage. *Manag. Sci.* **2007**, *53*, 566–583. [CrossRef]
7. Feess, E.; Thun, J.H. Surplus division and investment incentives in supply chains: A biform-game analysis. *Eur. J. Oper. Res.* **2014**, *234*, 763–773. [CrossRef]
8. González, F.F.; van der Weijde, A.H.; Sauma, E. The promotion of community energy projects in Chile and Scotland: An economic approach using biform games. *Energy Econ.* **2020**, *86*, 104677. [CrossRef]
9. Fiestras-Janeiro, M.; Garca-Jurado, I.; Meca, A.; Mosquera, M. On benefits of cooperation under strategic power. *Ann. Oper. Res.* **2020**, *288*, 285–306. [CrossRef]
10. Liu, C.W.; Xiang, S.W.; Yang, Y.L.; Luo, E.Q. Cooperative games based on coalition functions in biform games. *Axioms* **2023**, *12*, 296. [CrossRef]
11. Myerson, R. *Game Theory: Analysis of Conflict*; Harvard University Press: Cambridge, MA, USA, 1991. [CrossRef]
12. Harsanyi, J. A simplified bargaining model for the n-person cooperative game. *Int. Econ. Rev.* **1963**, *4*, 194–220. [CrossRef]
13. Yu, J. Essential equilibria of n-person noncooperative games. *J. Math. Econ.* **1999**, *31*, 361–372. [CrossRef]
14. Selten, R. Reexamination of the perfectness concept for equilibrium points in extensive games. *Int. J. Game Theory* **1975**, *4*, 25–55. [CrossRef]
15. Wu, W.T.; Jiang, J.H. Essential equilibrium points of n-person non-cooperative games. *Sci. Sin.* **1962**, *11*, 1307–1322.
16. Yu, J.; Xiang, S.W. On essential components of the set of Nash equilibrium points. *Nonlinear Anal.-Theor.* **1999**, *38*, 259–264. [CrossRef]
17. Schelling, T.C. *The Strategy of Conflict*; Harvard University Press: Cambridge, MA, USA, 1960.
18. Harsanyi, J.C.; Selten, R. *A General Theory of Equilibrium Selection in Games*; MIT Press: Cambridge, MA, USA, 1988. Available online: https://www.jstor.org/stable/40751255 (accessed on 6 February 2023).

19. Carpente, L.; Casas-Méndez, B.; García-Jurado, I.; van den Nouweland, A. Values for strategic games in which players cooperate. *Int. J. Game Theory* **2005**, *33*, 397–419. [CrossRef]
20. Summerfield, N.; Dror, M. Biform game: Reflection as a stochastic programming problem. *Int. J. Prod. Econ.* **2013**, *142*, 124–129. [CrossRef]
21. Nan, J.X.; Wang, P.P.; Li, D.F. A solution method for Shapley–based equilibrium strategies of biform games. *Chin. J. Manag. Sci.* **2021**, *29*, 202–210. [CrossRef]
22. Shi, X.Q. *Introduction to Cooperative Game Theory*; Peking University Press: Beijing, China, 2012. Available online: https://book.douban.com/subject/11640191/ (accessed on 18 February 2023).
23. Nash, J. Non-cooperative games. *Ann. Math.* **1951**, *54*, 286–295. Available online: https://www.jstor.org/stable/1969529 (accessed on 6 February 2022). [CrossRef]