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Remarks on "Perov Fixed-Point Results on F-Contraction Mappings Equipped with Binary Relation"

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Abstract: Since 1964, when I.A. Perov introduced the so-called generalized metric space where $d(x,y)$ is an element of the vector space $\mathbb{R}^m$, many researchers have considered various contractive conditions in this type of space. In this paper, we generalize, extend and unify some of those established results. We are primarily concerned with examining the existence of a fixed point of some mapping from X to itself, but if (x,y) belongs to some relation R on the set X, then the binary relation R and some F contraction defined on the space cone $\mathbb{R}^m$ are combined. We start our consideration with the recently announced results and give them strict, critical remarks. In addition, we improve several announced results by weakening some of the given conditions.

Keywords: binary relation; vector-valued metric space; fixed point; F contraction; Perov's type mapping

MSC: 47H10; 54H25

Citation: Mitrović, S.; Fabiano, N.; Radojević, S.; Radenović, S. Remarks on "Perov Fixed-Point Results on F-Contraction Mappings Equipped with Binary Relation". *Axioms* **2023**, *12*, 518. https://doi.org/10.3390/axioms12060518

Academic Editor: Hsien-Chung Wu

Received: 24 April 2023
Revised: 16 May 2023
Accepted: 23 May 2023
Published: 25 May 2023

1. Introduction and Preliminaries

In 1906, the French mathematician Maurice Fréchet, for the first time in the history of mathematics, introduced the abstract measurement of the distance between two points in an arbitrary non-empty set X. In order for that distance to correspond to the ordinary distance that a person understands intuitively, known from Euclidean geometry as M, Fréchet introduced it axiomatically in the following way.

Let X be a given non-empty set and d be a mapping (function) defined on the Cartesian product $X^2 = X \times X$, with values in the set $[0, +\infty)$ of nonnegative real numbers that satisfies the following axioms for x, y and z from X:

(d1) $d(x,y) = 0$ if and only if $x = y$;
(d2) $d(x,y) = d(y,x)$;
(d3) $d(x,z) \leq d(x,y) + d(y,z)$.

Then, the pair (X,d) is called a metric space, and the mapping d is a distance or metric on a non-empty set X.

Since S. Banach discovered his famous theorem in 1922 ([1]) about the uniqueness of the fixed point of every contraction Γ defined in the complete metric space (X,d), numerous mathematicians tried to generalize his result. These generalizations basically went in the following two directions:

1. Some of the three above possible metric space axioms are broken.
2. The right side of the Banach condition

$$d(\Gamma x, \Gamma y) \leq k \cdot d(x, y),\ k \in [0, 1)$$

was replaced by new expressions such as $ad(x, y) + bd(x, \Gamma x) + cd(y, \Gamma y)$, where a, b and c are nonnegative real numbers such that $a + b + c < 1$, or with $\varphi(d(x, y))$, where $\varphi : [0, +\infty) \to [0, +\infty)$ and has some additional property.

In case 1, several so-called general metric spaces have been considered, such as: *partial metric spaces*, *metric-like spaces*, *b-metric spaces*, *partial b-metric spaces* and *b-metric-like spaces* (a total of six new different types of spaces, including metric spaces). For more details, see ([2–6]).

In the second direction of generalization, new contractive conditions were obtained, as is well known in the literature: Kannan, Chatterjea, Zamfirescu, Hardy–Rogers, Ćirić, Boyd–Wong, Meir–Keeler and others. For details, see [7]. In all six spaces mentioned above, the mapping d is from $X^2 = X \times X$ to $[0, +\infty)$, where X is different from the empty set.

In 1933, the Serbian mathematician Đ. Kurepa, instead of $[0, +\infty)$, considered the vector space V with the cone P and defined the so-called cone metric spaces. Thus, he considered the mapping d from $X^2 = X \times X$ to the cone P of the real vector space V (i.e., $d(x, y)$ is in the new situation a vector and not a nonnegative real number). For more details, see [8–14].

After Kurepa's introduction of cone metric spaces in 1933, in 1964, Perov defined one of their special types (see [13]). Specifically, instead of a vector space V with a cone P, he used the special case $V = \mathbb{R}^n$ with a cone $P = \{(x_1, x_2, \ldots, x_n) : x_i \geq 0$ for all $i = 1, 2, \ldots, n\}$. Then, $d(x, y) = (x_1, x_2, \ldots, x_n) \in P$. To move on to work with matrices, Perov used $d(x, y) = (x_1, x_2, \ldots, x_n)^T$ (i.e., an $n \times 1$ matrix). From [9], we know that the generalized metric (d i.e., the valued cone metric d) can be represented by a column matrix of a pseudo-metric $p_i, i = 1, 2, \ldots, n$.

Therefore, $d(x, y) = (p_1(x, y), p_2(x, y), \ldots, p_n(x, y))^T$, and at least one of the pseudo-metric p_i is a real metric.

Let us now write the contractive condition that Perov stated as a generalization of the Banach condition in his famous result from 1922. Let M be the given square matrix of the order n and Γ be the mapping from X to X, where (X, d) is the given generalized metric proctor (i.e., valued metric space). If there is a square matrix M that converges to 0 such that $d(\Gamma x, \Gamma y) \leq M \cdot d(x, y)$ for every x, y in X, then Γ has a fixed point in X. Here, $d(x, y)$ and $d(\Gamma x, \Gamma y)$ are columns (i.e., an $n \times 1$ matrix). If $d(x, y)$ is a row (i.e., an ordered n-tuple), then the previous condition has the notation $d(\Gamma x, \Gamma y) \leq d(x, y) \cdot M^T$.

Otherwise, throughout this manuscript, representation is as in [6], where $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{R}_{\geq 0}$, $\mathbb{R}_t$ and $\mathbb{R}_{t>k}$ are the sets of all natural numbers, all integers, real numbers, nonnegative real numbers, real matrices of the order $t \times 1$ and real matrices of the order $t \times 1$ with entries greater than k, respectively. The order being used is the coordinate-wise ordering for $\mathbb{R}$.

We can freely say that the set, relation, function and operation are the four basic pillars of mathematics. We take the set as the basic concept. The operation is a special function from $X \times X$ to X, while the function is a special binary relation of X. The binary relation R in the set X means any subset of $X \times X$. All of this is well known from the previous school course study. Here, we will state the basic terms from binary relations on a given non-empty set X if there is also a mapping T from X to itself.

Now, we will state the basic properties of an arbitrary relation R considered on a non-empty set X. Additionally, if X is provided with some metric or vector metric, and if Γ is a mapping from X to itself, then we will state the properties that we will need to prove the result in the rest of the paper.

Let X be a non-empty set. Then, the Cartesian product on X is defined as follows:

$$X^2 = X \times X = \{(a, b) : a, b \in X\}.$$

All subsets of X^2 are known as the binary relations on X.

Let R be any subset of X^2. Then, notice that for each pair $a, b \in X$, there are two possibilities: either $(a, b) \in R$ or $(a, b) \notin R$. In the first case, we mean that a relates to b under R. For $(a, b) \notin R$, we mean that a does not relate to b under R. For all other matters relating to the binary relations, see [6] (Definitions 3–10, as well as Propositions 2 and 3, Lemma 1 and Example 3) and [4] (Definitions 2.7–2.10, 2.13, 2.14 and 2.16–2.18, Propositions 2.11 and 2.12 and Example 2.15).

The following two lemmas are quite useful, and they can be used in proofs showing that the introduced Picard sequence $x_n = \Gamma x_{n-1}$ is a Cauchy one, where Γ is the mapping of the metric space (X, d) into itself and x_0 is a given point (see [8] and the references therein for the first and second properties):

Lemma 1. *Let (X, d) be a metric space and $\{x_n\}$ be a Picard sequence in it. If*

$$d(x_{n+1}, x_n) < d(x_n, x_{n-1})$$

for all $n \in \mathbb{N}$ then $x_n \neq x_m$ whenever $n \neq m$.

Lemma 2. *Let x_n be a sequence in a metric space (X, d) such that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$. If x_n is not a Cauchy sequence in (X, d), then as a result, there exist two sequences $\{n(k)\}$ and $\{m(k)\}$ of natural numbers such that $n(k) > m(k) > k$ and the sequences*

$$\left\{d\left(x_{n(k)}, x_{m_k}\right)\right\}, \left\{d\left(x_{n(k)+1}, x_{m(k)}\right)\right\}, \left\{d\left(x_{n(k)}, x_{m(k)-1}\right)\right\},$$
$$\left\{d\left(x_{n(k)+1}, x_{m(k)-1}\right)\right\}, \left\{d\left(x_{n(k)+1}, x_{m(k)+1}\right)\right\}, \ldots$$

tend toward ε as $k \to +\infty$ for some $\varepsilon > 0$.

With the mentioned contractive conditions of a mapping Γ from X to X, it is assumed that they are present for both x and y from X or for each pair (x, y) from the Cartesian product $X \times X$. One possible type of weakening of the condition of such results could be, among others, that the contractive condition is fulfilled for all pairs (x, y) belonging to some subset of $X \times X$ or, put in terms of a binary relation defined on a set X, if (x, y) belongs to a given binary relation R on a set X. For more details, see [4,6].

Wardowski [15] (see also [16]) introduced the notion of the F contraction and defined the F contraction as follows:

Definition 1. *Let $F : \mathbb{R}_{>0} \to \mathbb{R}$ be a mapping satisfying the following properties:*

F1: *F is strictly increasing; that is, for all $a, b \in \mathbb{R}_{>0}$, we have*

$$a < b \text{ implies } F(a) < F(b).$$

F2: *For each sequence a_n of $\mathbb{R}_{>0}$, we have*

$$\lim_{n\to+\infty} a_n = 0 \text{ if and only if } \lim_{n\to+\infty} F(a_n) = -\infty.$$

F3: *There exists $\lambda \in (0, 1)$ such that $\lim_{r\to 0^+} r^\lambda F(r) = 0$.*

The set of all functions F satisfying F1–F3 is denoted as $\mathcal{F}$.

I. Altun and M. Olgun in [5] used the concept of an F contraction in a vector-valued metric space in the following way:

Definition 2. *Let $F : \mathbb{R}_{t>0} \to \mathbb{R}_t$ be a function which satisfies the following conditions:*

F1: *F is strictly increasing; that is, for all $a = (a_i)_{i=1}^t, b = (b_i)_{i=1}^t \in \mathbb{R}_{t>0}$, where*

$$a < b \text{ implies } F(a) < F(b),$$

F2: *For each sequence $\{a_n\} = \left(a_1^{(n)}, a_2^{(n)}, \ldots, a_t^{(n)}\right)$ of $\mathbb{R}_{t>0}$, we have*

$$\lim_{n\to+\infty} a_i^{(n)} = 0 \text{ if and only if } \lim_{n\to+\infty} b_i^{(n)} = -\infty$$

for every $i = 1, 2, \ldots, t$, where $F\left(a_1^{(n)}, a_2^{(n)}, \ldots, a_t^{(n)}\right) = \left(b_1^{(n)}, b_2^{(n)}, \ldots, b_t^{(n)}\right)$.

F3: *There exists $\lambda \in (0,1)$ such that $\lim_{a_i \to} a_i^{\lambda} b_i = 0$ for all $i = 1, 2, \ldots, t$, where $F[(a_1, a_2, \ldots, a_t)] = (b_1, b_2, \ldots, b_t)$.*

Here, $\mathbb{R}_{t>0}$ is the set of all $t \times 1$ real matrices with positive entries. Then, the set of all functions F satisfying F1–F3 is denoted as $\mathcal{F}^t$.

Remark 1. *We note that each of the coordinates $F_i, i = 1, 2, \ldots, m$ belongs to the set $\mathcal{F}$. The reverse is also true in a way. Each F from $\mathcal{F}$ can be equal to F_i for each $i = 1, 2, \ldots, m$. In other words, $\underbrace{(F, F, \ldots, F)}_{m}$: $(0, +\infty)^m \to (-\infty, +\infty)^m$, where $(F, \ldots, F)((a_1, a_2, \ldots, a_m)) = (Fa_1, \ldots, Fa_m)$ for each $a_1, a_2, \ldots, a_m$ belonging to $(0, +\infty)$.*

For a large number of proofs of results from generalized metric spaces in the sense of I.A. Perov (i.e., vector-valued metric spaces), they can be used for the following recent result:

Proposition 1 ([9], Proposition 2.1)**.** *Let X be a non-empty set and $d : X \times X \to \mathbb{R}^m$ be a mapping such that $d = (p_1, \ldots, p_m)$, where $p_i : X \times X \to [0, +\infty)$ for $i = 1, 2, \ldots, m$. Then, (X, d) is a generalized metric space (i.e., vector-valued metric space) if and only if $p_1, \ldots, p_m$ is a separating family of pseudometrics (i.e., for any $x, y \in X$, if $x \neq y$, then $p_i(x, y) > 0$ for some $i \in \{1, 2, \ldots, m\}$).*

For some applications, see [8].

Now, we present a brief description of the main properties of generalized metric spaces, which are essentially a special type of so-called cone metric space. For more details, see [2,5,8–11,14]:

- A generalized metric space (X, d) is a metrizable space;
- The generalized metric d is continuous in the sense that $d(x_n, y_n) \to d(x, y)$ whenever $d(x_n, x) \to \theta$ and $d(y_n, y) \to \theta$ as $n \to +\infty$;
- The cone $P = \{(x_1, x_2, \ldots, x_m) : x_i \geq 0, i = 1, 2, \ldots, m\}$ in $\mathbb{R}^m$ is normal with the normal constant $K = 1$ under any of equivalent norms on it. In addition, it is solid (i.e., $intP \neq \emptyset$);
- The cone P is regular in the space $\mathbb{R}^m$, meaning that each decreasing (or increasing) sequence in it is convergent.

2. Main Results

In this part of the paper, we will give strict critical remarks on the results published in [6]. Namely, we will first show that Theorems 4 and 5 from [6] are equivalent. To demonstrate this, we will first consider the mapping of F from $(0, +\infty)^m$ to $(-\infty, +\infty.)^m$. We used this in a similar way in a recent work [11,12]. We notice that the mapping F can be specified by its coordinates $F_1, \ldots, F_m$, which are actually mappings from $(0, +\infty)$ to $(-\infty, +\infty)$.

Now, we recall both Definitions 15 and 16 as well as both Theorems 4 and 5 from [6]:

Definition 3 ([6])**.** *Let (X, σ) be a vector-valued metric space and Γ be a self-mapping on X. If there exist $F \in \mathcal{F}^t$ and $\xi = \left(\xi^{(i)}\right)_{i=1}^{t} \in \mathbb{R}_{t>0}$ such that*

$$\xi + F(\sigma(\Gamma p, \Gamma q)) \leq F(\sigma(p, q)), \text{ for all } p, q \in X \text{ with } \sigma(\Gamma p, \Gamma q) > 0,$$

then Γ is called a Perov-type F contraction.

Definition 4 ([6])**.** *Let (X, σ) be a vector-valued metric space equipped with a binary relation R. Then, a self-mapping Γ on X is called a theoretic-order Perov-type F contraction if there exist $\xi = \left(\xi^{(i)}\right)_{i=1}^{t} \in \mathbb{R}_{t>0}$ and $F \in \mathcal{F}^t$such that*

$$\xi + F(\sigma(\Gamma p, \Gamma q)) \leq F(\sigma(p, q)), \text{ for all } (p, q) \in R \text{ with } \sigma(\Gamma p, \Gamma q) > 0.$$

Theorem 1 ([6])**.** *Let (X, d) be a complete metric space equipped with a binary relation R and Γ be a self-mapping. Suppose the following:*

The pair $(R : \Gamma)$ is a compound structure (Definition 17 in [6]).

For all $(p, q) \in R$ with $d(\Gamma p, \Gamma q) > 0$ such that

$$\xi + F(d(\Gamma p, \Gamma q)) \leq F(d(p, q))$$

where $\xi > 0$ and $F \in \mathcal{F}$, then Γ has a fixed point.

Furthermore, if $C_R(p, q) \neq \emptyset$ (Definition 10 in [6]) for all $p, q \in X$, then Γ has a unique fixed point.

Theorem 2 ([6])**.** *Let (X, σ) be a complete vector-valued metric space equipped with a binary relation R and Γ be a theoretic-order Perov-type contraction such that the pair $(R : \Gamma)$ is a compound structure. Then, Γ has a fixed point. Moreover, Γ has a unique fixed point if $C_R(p, q) \neq \emptyset$ for all $p, q \in X$.*

Before giving more significant results in this paper, we first state two very important remarks on the proof of Theorem 4 from [6]. Specifically, the proof that the defined Picard sequence is a Cauchy one as well as the uniqueness of a possible fixed point are clearly wrong. The authors used the fact that a convergent-order sum is equal to zero, which is incorrect (see pages 9 and 10 in [6]). The error in the Cauchyness proof is easy to fix. First of all, it follows from the obtained condition that $\sigma(x_n, x_m) \leq \sum_{j=n}^{m-1} \sigma\left(c_j, c_{j+1}\right) \to 0$ as $n \to +\infty$ because the series $\sum_{j=1}^{+\infty} \frac{1}{j^{\frac{1}{\lambda}}}$ is convergent and is equal to $\zeta(\frac{1}{\lambda})$, which is larger than 1 for $0 < \lambda < 1$. Similarly, the error in the proof of the uniqueness of a fixed point is eliminated.

We also note that the proof of the Cauchyness of the sequence $\{c_n\}$ constructed in Theorem 6 from [6] is wrong. Namely, it is the same as in Theorems 4 and 5. Furthermore, it can be easily verified that for the proof of Theorem 6, it is sufficient to assume that the function F is only strictly increasing.

Now, we will show that Theorem 5 from [6] is actually not new but is equivalent to Theorem 4, which is also from [6]. We have previously found that many fixed point results in the framework of cone metric spaces are equivalent to the corresponding ones in ordinary metric spaces. It is obvious that from Theorem 5 follows Theorem 4. Since every generalized metric d (i.e., the valued metric d) is given by its coordinates p_i, $i = 1, 2, \ldots, n$ and that at least one pseudo-metric p_i is a real metric, from there, we have the corresponding contractive condition in the metric space (X, p_i), and this gives us the result according to Theorem 4 (for more details, see [11]).

We proved that Theorems 4 and 5 from [6] are equivalent but only if the function F satisfies all three properties in both scalar and vector form. Let us also note that when we consider the general metric space, (i.e., the valued metric space), we can only go up to $n = 2$, which is essentially the same as $n > 2$. It is only about technical matters and complicated writing at first glance. For details, see [11]. If we add to the relation R on the set X the property of its transitivity, then it can be proven that Theorems 4 and 5 (in the new formulation, of course) are equivalent if the mapping F satisfies only the property of strict growth.

In the continuation of this work, the function F will be strictly increasing (i.e., for every $a = (a_i)_{i=1}^m, b = (b_i)_{i=1}^m \in \mathbb{R}^m$, whenever $a < b$, then $F(a) < F(b)$). In order to not use properties F2 and F3 as in [6], we will assume that the relation R given on the non-empty set X is transitive.

Let us formulate and prove the following two results:

Theorem 3. *Let (X, d) be a complete metric space equipped with a transitive binary relation R and Γ be a self-mapping. Suppose the following:*

The pair (R, Γ) is a compound structure.

For all $(p, q) \in R$ with $d(\Gamma p, \Gamma q) > 0$ such that

$$\zeta + F(d(\Gamma p, \Gamma q)) \leq F(d(p, q))$$

where $\zeta \in \mathbb{R}_{>0}$ and F is an increasing function $F \notin \mathcal{F}$, then Γ has a fixed point.

Furthermore, if $C_R(p, q) \neq \emptyset$ for all $p, q \in X$, then Γ has a unique fixed point.

Proof. In our approach to the proof of the formulated theorem, the function F participates only through its strict growth. Instead of properties F2 and F3 of the function F, we assumed the transitivity of the relation R. In this way, we had a hybrid correction of Theorem 4 from [6]. The transitivity of relation R and the strict growth of function F allowed us to prove the Cauchyness of the constructed Picard sequence by applying Lemmas 1 and 2 in their proof of Theorem 4. Due to the assumption of a compound structure (R, Γ), one can construct the Picard sequence $\{c_n\}$ as follows:

$$c_0, c_1 = \Gamma c_0, \ldots, c_n = \Gamma c_{n-1} = \Gamma^n c_0, \ldots$$

where c_0 is the starting point that exists. For the obtained sequence, we have that the adjacent members are in the relation R, but due to the assumption of transitivity, every two members of it are in the relation R. This, with the contractive conditions first, gives that the series $d(c_n, c_{n+1})$ is decreasing, and due to the property of the strictly increasing function F, it is obtained that it tends toward zero as $n \to +\infty$. Now, if the sequence $\{c_n\}$ is not a Cauchy one, by using Lemma 2 and taking $p = x_{n(k)}, q = x_{m(k)}$, it follows that

$$\zeta + F\Big(d\Big(\Gamma x_{n(k)}, \Gamma x_{m(k)}\Big)\Big) \leq F\Big(d\Big(x_{n(k)}, x_{m(k)}\Big)\Big).$$

Again, according to the important property of the strictly increasing function F, we obtain

$$\zeta + F(\varepsilon^+) \leq F(\varepsilon^+)$$

which is a contradiction with $\zeta > 0$. That the mapping Γ has a fixed point and that with an additional condition it is unique is shown in [6]. □

Remark 2. *By adding transitivity to a given relation R, the previous theorem is a significant generalization of Theorem 4 from [6]. Namely, under the new assumption in the proof, we do not need properties F2 or F3. Our recent results were published in, for example, Ref. [16], where various explicit examples are shown in Section 2 and new theorems (Theorems 3.1–3.4) are formulated and proven in Section 3 (see also [8,11,12,14,17]). Otherwise, for the other cases of the same or similar questions, one can see, for example, Refs. [18–30] for further generalizations.*

Theorem 4. *Let (X,σ) be a complete vector-valued metric space equipped with a transitive binary relation R, let Γ be a theoretic-order Perov-type F contraction such that the pair (R,Γ) is a compound structure, and let F be an increasing function. Then, Γ has a fixed point.*

Moreover, Γ has a unique fixed point if $C_R(p,q) \neq \emptyset$ for all $p,q \in X$.

Proof. We see that the proof of this theorem is almost identical to the previous one. Namely, if $m = 1$, then we have the previous theorem, and if $m > 1$, then by using J. Jachymski and J. Klima's famous result [9], we have to give the contractive condition, which becomes

$$\zeta_i + F_i\Big(p_i\Big(\Gamma x_{n(k)}, \Gamma x_{m(k)}\Big)\Big) \leq F_i\Big(p_i\Big(x_{n(k)}, x_{m(k)}\Big)\Big),\ i = 1,2,\ldots,m.$$

As also stated by J. Jachymski and J. Klima [9], at least one pseudo-metric, say p_{i_0}, is then a usual metric. Based on the previous theorem applied to the metric space (X, p_{i_0}), we have a result (i.e., a proof of the assertion of the formulated theorem). □

In a recent paper [4], the authors generalized Perov's result from 1964 by assuming that instead of every x and y from X, the pair (x,y) belongs to some nontrivial relation R on the non-empty set X, which is provided with a generalized metric d. We will quote their result in full:

Theorem 5 ([4])**.** *Let (X,σ) be a complete vector-valued metric space endowed with a binary relation R and $\Gamma : X \to X$ be a mapping. Suppose the following:*

(i) There exists $a \in X$ such that $(a, \Gamma a) \in R$;
(ii) R is Γ-closed; that is, for each $a,b \in X$ with $(a,b) \in R$, we have $(\Gamma a, \Gamma b) \in R$;
(iii) Either Γ is continuous or R is $\sigma-$ self-closed;
(iv) There exists a matrix $A \in \mathbb{M}_m(\mathbb{R}_+)$ convergent to zero such that

$$\sigma(\Gamma a, \Gamma b)_{m\times 1} \leq A_{m\times m}\sigma(a,b)_{m\times 1},$$

for all $a,b \in X$ with $(a,b) \in R$.

Then, Γ has a fixed point;

(v) Furthermore, if $C_R(a,b) \neq \emptyset$ for all $a,b \in X$, then Γ has a unique fixed point.

In the next result, we will state a shorter, much simpler and more natural proof of the previous result. In particular, we will use the recent above-mentioned result of J. Jachymski and J. Klima.

Proof. First, we can consider the given matrix as a bounded linear operator in the standard Banach space $\mathbb{R}^m$, which is provided with one of the equivalent norms in it. Then matrix A has the coordinates $(A_1, \ldots A_m)$, and then the contractive condition given in their work which is expressed through coordinates has the form

$$(p_1(\Gamma a, \Gamma b), \ldots, p_m(\Gamma a, \Gamma b))^T \leq (A_1(p_1(a,b)), \ldots, A_m(p_m(a.b)))_{m\times 1},$$

In other words, we have

$$p_i(\Gamma a, \Gamma b) \leq A_i(p_i(a,b)), i = 1,2,\ldots,m$$

where $\sigma = (p_i, p_2, \ldots, p_m)$. Since for some i_0, p_{i_0} is an ordinary metric, the proof further follows according to the already proven result for metric spaces [3]. □

3. Conclusions

The aim of this paper was to point out the inaccuracies and errors in the recently published paper in the *Mathematics* journal. This was carried out because of young researchers who perform a lot of work in this area of mathematical and functional analysis and because of the reputation of the MDPI journal *Mathematics*. In addition, we pointed out that in F contractions, it can only be assumed that the function F is strictly growing, unlike the paper in which the authors assumed the famous properties of F1, F2 and F3.

Author Contributions: Conceptualization, S.M., N.F., S.R. (Slobodan Radojević) and S.R. (Stojan Radenović); methodology, S.M. and N.F.; validation, S.M., N.F. and S.R. (Slobodan Radojević); formal analysis, S.M., N.F., S.R. (Slobodan Radojević) and S.R. (Stojan Radenović); investigation, S.M. and N.F.; writing—original draft, S.M., N.F., S.R. (Slobodan Radojević) and S.R. (Stojan Radenović); supervision, S.M., N.F., S.R. (Slobodan Radojević) and S.R. (Stojan Radenović). All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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