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Some New Jensen–Mercer Type Integral Inequalities via Fractional Operators

Bahtiyar Bayraktar [1,†], Péter Kórus [2,*,†] and Juan Eduardo Nápoles Valdés [3,†]

1 Faculty of Education, Bursa Uludag University, Gorukle Campus, 16059 Bursa, Turkey; bbayraktar@uludag.edu.tr
2 Department of Mathematics, Juhász Gyula Faculty of Education, University of Szeged, Hattyas utca 10, H-6725 Szeged, Hungary
3 Facultad de Ciencias Exactas y Naturales y Agrimensura, Universidad Nacional del Nordeste, Ave. Libertad 5450, Corrientes 3400, Argentina; jnapoles@exa.unne.edu.ar
* Correspondence: korus.peter@szte.hu
† These authors contributed equally to this work.

Abstract: In this study, we present new variants of the Hermite–Hadamard inequality via non-conformable fractional integrals. These inequalities are proven for convex functions and differentiable functions whose derivatives in absolute value are generally convex. Our main results are established using the classical Jensen–Mercer inequality and its variants for (h,m)-convex modified functions proven in this paper. In addition to showing that our results support previously known results from the literature, we provide examples of their application.

Keywords: convex functions; (h,m)-convex functions; Jensen–Mercer inequality; Hermite–Hadamard inequality; Hölder inequality, power mean inequality; non-conformable fractional operators

MSC: 26A33; 26A51; 26D15

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1. Introduction

Jensen's inequality is one of the most studied results in the literature. In the last few decades, quite a few researchers have been interested in refining and generalizing this inequality (see, e.g., [1–6]).

Let $0 < x_1 \leq x_2 \leq \ldots \leq x_n$ and let w_k $(1 \leq k \leq n)$ be positive weights associated with these x_k and let their sum demonstrate unity. Then, Jensen's inequality

$$\Phi\left(\sum_{k=1}^{n} w_k x_k\right) \leq \sum_{k=1}^{n} w_k \Phi(x_k) \tag{1}$$

holds (see [7]).

Mercer investigated a generalized form of Jensen's inequality, which is famously known as the Jensen–Mercer inequality (see [8]): if Φ is a convex function on $[\rho,\sigma]$, then

$$\Phi\left(\rho+\sigma-\sum_{k=1}^{n} w_k x_k\right) \leq \Phi(\rho)+\Phi(\sigma)-\sum_{k=1}^{n} w_k \Phi(x_k) \tag{2}$$

is fulfilled for $x_k \in [\rho,\sigma]$, $w_k \in [0,1]$ with $\sum_{k=1}^{n} w_k = 1$. In case of $n=1$, inequality (2) reads as

$$\Phi(\sigma-x+\rho) \leq \Phi(\rho)+\Phi(\sigma)-\Phi(x) \tag{3}$$

for $x \in [\rho,\sigma]$. Extensions of this result can be found in e.g., [9–11].

The well-known refinement of Jensen's inequality, the Hermite–Hadamard inequality

$$\Phi\left(\frac{\rho+\sigma}{2}\right) \leq \frac{1}{\sigma-\rho}\int_{\rho}^{\sigma}\Phi(x)dx \leq \frac{\Phi(\rho)+\Phi(\sigma)}{2} \tag{4}$$

for convex functions, was proved by Hermite in 1883 and independently by Hadamard in 1893; see, e.g., [12]. This inequality has been generalized by many researchers, taking into account various aspects such as general convexity and fractional operators. For Hermite–Hadamard–Mercer type results, see [13–18].

In general, the concept of convex and general convex functions plays a major role in the theory of integral inequalities. So far, many general convex classes have been described in the literature. A summary of many of these classes was given in [19].

Definition 1. *Let $h : [0,1] \to [0,\infty)$, $h \neq 0$ and $\Phi : I = [0,\infty) \to \mathbb{R}$. If inequality*

$$\Phi(\lambda x + m(1-\lambda)y) \leq h(\lambda)\Phi(x) + mh(1-\lambda)\Phi(y) \tag{5}$$

is fulfilled $\forall\lambda \in [0,1]$ and $x, y \in I$, where $m \in [0,1]$, then function Φ is called (h,m)-convex on I.

In [20,21], the following definitions were presented.

Definition 2. *Let $h : [0,1] \to (0,1]$ and $\Phi : I = [0,\infty) \to \mathbb{R}$. If inequality*

$$\Phi(\lambda x + m(1-\lambda)y) \leq h^s(\lambda)\Phi(x) + m(1-h^s(\lambda))\Phi(y) \tag{6}$$

is fulfilled $\forall\lambda \in [0,1]$ and $x, y \in I$, where $m \in [0,1]$, $s \in [-1,1]$, then function Φ is called (h,m)-convex modified of the first type on I and this set of functions will be denoted as $K^{1,s}_{h,m}(I)$.

Definition 3. *Let $h : [0,1] \to (0,1]$ and $\Phi : I = [0,\infty) \to \mathbb{R}$. If inequality*

$$\Phi(\lambda x + m(1-\lambda)y) \leq h^s(\lambda)\Phi(x) + m(1-h(\lambda))^s\Phi(y) \tag{7}$$

is fulfilled $\forall\lambda \in [0,1]$ and $x, y \in I$, where $m \in [0,1]$, $s \in [-1,1]$, then function Φ is called (h,m)-convex modified of the second type on I and this set of functions will be denoted as $K^{2,s}_{h,m}(I)$.

Throughout the paper, for (h,m)-convex modified functions of the first or, of the second type, we assume that $m \in [0,1]$ and $s \in [-1,1]$.

The following results are extended versions of Jensen–Mercer inequality (3).

Theorem 1. *Let $\Phi : I = [\rho,\sigma] \subset \mathbb{R} \to \mathbb{R}$ be an integrable and (h,m)-convex function. Then, the following Mercer's type inequality holds:*

$$\Phi(x_1 + mx_n - x_k) \leq (h(\lambda) + h(1-\lambda))[\Phi(x_1) + m\Phi(x_n)] - \Phi(x_k) \tag{8}$$

for $x_1 \leq mx_n$, $x_k \in [x_1, mx_n] \subseteq I$ and $\lambda \in [0,1]$, such that $x_k = \lambda x_1 + m(1-\lambda)x_n$.

Proof. Putting $x_k = \lambda x_1 + m(1-\lambda)x_n$ and $y_k = (1-\lambda)x_1 + m\lambda x_n$, we have $y_k + x_k = x_1 + mx_n$. Now, using the (h,m)-convexity of Φ, we have

$$\Phi(y_k) \leq h(1-\lambda)\Phi(x_1) + mh(\lambda)\Phi(x_n),$$
$$\Phi(x_k) \leq h(\lambda)\Phi(x_1) + mh(1-\lambda)\Phi(x_n).$$

By adding the corresponding sides of the inequalities, we obtain

$$\Phi(y_k) + \Phi(x_k) \leq (h(\lambda) + h(1-\lambda))[\Phi(x_1) + m\Phi(x_n)].$$

From the above, the desired inequality (8) is easily obtained. □

Corollary 1. *Let $\Phi : I = [\rho, \sigma] \subset \mathbb{R} \to \mathbb{R}$ be an integrable (h, m)-convex function. Then, from* (8)*, we have*

$$\Phi(x_1 + mx_n - x_k) \leq \mathbf{A}_0[\Phi(x_1) + m\Phi(x_n)] - \Phi(x_k) \tag{9}$$

for $x_1 \leq mx_n$, $x_k \in [x_1, mx_n] \subseteq I$ and $\mathbf{A}_0 = \sup_{\lambda \in [0,1]} (h(\lambda) + h(1-\lambda))$.

Remark 1. *For $m = 1$, Corollary 1 leads to a correct version of Lemma 3.1 of [11].*

Theorem 2. *Let $\Phi : I = [\rho, \sigma] \subset \mathbb{R} \to \mathbb{R}$ be an integrable and $\Phi \in K^{1,s}_{h,m}([\rho, \frac{\sigma}{m}])$. Then, the following Mercer's-type inequality holds:*

$$\Phi(x_1 + mx_n - x_k) \leq (h^s(\lambda) + h^s(1-\lambda))\Phi(x_1) + (2 - h^s(\lambda) - h^s(1-\lambda))m\Phi(x_n) - \Phi(x_k) \tag{10}$$

for $x_1 \leq mx_n$, $x_k \in [x_1, mx_n] \subseteq I$ and $\lambda \in [0,1]$ such that $x_k = \lambda x_1 + m(1-\lambda)x_n$.

Proof. The proof is analogous to that of Theorem 1. Taking $x_k = \lambda x_1 + m(1-\lambda)x_n$, $y_k = (1-\lambda)x_1 + m\lambda x_n$ and combining inequalities

$$\begin{aligned}
\Phi(y_k) &\leq h^s(1-\lambda)\Phi(x_1) + m(1 - h^s(1-\lambda))\Phi(x_n), \\
\Phi(x_k) &\leq h^s(\lambda)\Phi(x_1) + m(1 - h^s(\lambda))\Phi(x_n)
\end{aligned}$$

results in inequality (10). □

Corollary 2. *Let $\Phi : I = [\rho, \sigma] \subset \mathbb{R} \to \mathbb{R}$ be an integrable and $\Phi \in K^{1,s}_{h,m}([\rho, \frac{\sigma}{m}])$. Then, from* (10)*, we have*

$$\Phi(x_1 + mx_n - x_k) \leq \mathbf{A}_1[\Phi(x_1) + m\Phi(x_n)] - \Phi(x_k)$$

for $x_1 \leq mx_n$, $x_k \in [x_1, mx_n] \subseteq I$ and

$$\mathbf{A}_1 = \max\left\{ \sup_{\lambda \in [0,1]} (h^s(\lambda) + h^s(1-\lambda)), \sup_{\lambda \in [0,1]} (2 - h^s(\lambda) - h^s(1-\lambda)) \right\}.$$

Theorem 3. *Let $\Phi : I = [\rho, \sigma] \subset \mathbb{R} \to \mathbb{R}$ be an integrable and $\Phi \in K^{2,s}_{h,m}([\rho, \frac{\sigma}{m}])$. Then, the following Mercer's type inequality holds:*

$$\begin{aligned}
\Phi(x_1 + mx_n - x_k) \leq\ & (h^s(\lambda) + h^s(1-\lambda))\Phi(x_1) \\
& + ((1 - h(\lambda))^s + (1 - h(1-\lambda))^s)m\Phi(x_n) - \Phi(x_k)
\end{aligned} \tag{11}$$

for $x_1 \leq mx_n$, $x_k \in [x_1, mx_n] \subseteq I$ and $\lambda \in [0,1]$, such that $x_k = \lambda x_1 + m(1-\lambda)x_n$.

Proof. The proof is analogous to that of Theorem 1. Taking $x_k = \lambda x_1 + m(1-\lambda)x_n$, $y_k = (1-\lambda)x_1 + m\lambda x_n$ and combining inequalities

$$\begin{aligned}
\Phi(y_k) &\leq h^s(1-\lambda)\Phi(x_1) + m(1 - h(1-\lambda))^s\Phi(x_n), \\
\Phi(x_k) &\leq h^s(\lambda)\Phi(x_1) + m(1 - h(\lambda))^s\Phi(x_n)
\end{aligned}$$

yields inequality (11). □

Corollary 3. *Let $\Phi : I = [\rho, \sigma] \subset \mathbb{R} \to \mathbb{R}$ be an integrable and $\Phi \in K^{2,s}_{h,m}([\rho, \frac{\sigma}{m}])$. Then, from Theorem 3, we have*

$$\Phi(x_1 + mx_n - x_k) \leq \mathbf{A}_2[\Phi(x_1) + m\Phi(x_n)] - \Phi(x_k) \tag{12}$$

for $x_1 \le mx_n$, $x_k \in [x_1, mx_n] \subseteq I$ *and*

$$\mathbf{A}_2 = \max\left\{ \sup_{\lambda\in[0,1]} \left(h^s(\lambda) + h^s(1-\lambda)\right), \sup_{\lambda\in[0,1]} \left((1-h(\lambda))^s + (1-h(1-\lambda))^s\right) \right\}.$$

Remark 2. *For $m = s = 1$ and $h(t) = t$, we have $\mathbf{A}_1 = \mathbf{A}_2 = 1$, moreover, Theorems 2 and 3 (or, Corollaries 2 and 3) become the Jensen–Mercer inequality for convex functions (3).*

Remark 3. *Other variants of the Jensen–Mercer inequality* (2)*, for different notions of convexity, can be found in [16,22–25].*

In the remainder of this paper, we aim to give generalizations of Hermite–Hadamard inequality (4) via non-conformable fractional integrals defined by Nápoles et al. in [26].

Definition 4. *Let $\alpha \in \mathbb{R}$ and $0 < \rho < \sigma$. For each function $\Phi \in L[\rho, \sigma]$, we define*

$${}_{N_3}J_u^{\alpha}\Phi(x) = \int_u^x t^{-\alpha}\Phi(t)dt$$

for every $x, u \in [\rho, \sigma]$.

Definition 5. *Let $\alpha \in \mathbb{R}$ and $\rho < \sigma$. For each function $\Phi \in L_\alpha[\rho, \sigma]$, that is the linear space*

$$L_\alpha[\rho,\sigma] = \left\{\Phi : [\rho,\sigma] \to \mathbb{R} : (t-\rho)^{-\alpha}\Phi(t), (\sigma - t)^{-\alpha}\Phi(t) \in L[\rho,\sigma]\right\},$$

let us define the fractional integrals

$${}_{N_3}J_{\rho+}^{\alpha}\Phi(x) = \int_\rho^x (x-t)^{-\alpha}\Phi(t)dt \quad \text{and} \quad {}_{N_3}J_{\sigma-}^{\alpha}\Phi(x) = \int_x^\sigma (t-x)^{-\alpha}\Phi(t)dt \tag{13}$$

for every $x \in [\rho, \sigma]$. Here, for $\alpha = 0$, we have ${}_{N_3}J_{\rho+}^{\alpha}\Phi(x) = {}_{N_3}J_{\sigma-}^{\alpha}\Phi(x) = \int_\rho^\sigma \Phi(t)dt$.

Definition 6. *More details on the fractional integral and the corresponding fractional derivative N_3^α can be read in [26].*

Fractional differential and integral computations have been widely used in many fields of applied sciences. The interested reader can read about the role of fractional calculus in the study of biological models and chemical processes in [27–29].

2. Inequalities for Convex Functions

In this section, we obtain analogues of Hermite–Hadamard inequality (4) for non-conformable fractional operators (13) using Jensen–Mercer inequalities.

Remark 4. *If in* (2)*, we take $n = 2$ and $w_1 = w_2 = \frac{1}{2}$, then we have*

$$\Phi\left(\sigma - \frac{y_1}{2} + \rho - \frac{x_1}{2}\right) \le \Phi(\rho) + \Phi(\sigma) - \frac{\Phi(x_1) + \Phi(y_1)}{2}. \tag{14}$$

Theorem 4. *Let $\Phi : [\rho, \sigma] \to \mathbb{R}$. If $\Phi \in L_\alpha[\rho, \sigma]$ and Φ is convex on $[\rho, \sigma]$, then*

$$\begin{aligned}\Phi\left(\sigma - \frac{y}{2} + \rho - \frac{x}{2}\right) &\le \Phi(\rho) + \Phi(\sigma) - \frac{1-\alpha}{2(y-x)^{1-\alpha}}\left[{}_{N_3}J_{y-}^{\alpha}\Phi(x) + {}_{N_3}J_{x+}^{\alpha}\Phi(y)\right] \\ &\le \Phi(\rho) + \Phi(\sigma) - \Phi\left(\frac{x+y}{2}\right),\end{aligned} \tag{15}$$

where $x, y \in [\rho, \sigma]$ and $\alpha < 1$.

Proof. If in (14), we choose $x_1 = tx + (1-t)y$ and $y_1 = (1-t)x + ty$, and multiply by $t^{-\alpha}$, then we can write the inequality

$$2\Phi\left(\sigma - \frac{y}{2} + \rho - \frac{x}{2}\right)t^{-\alpha} \leq 2t^{-\alpha}[\Phi(\rho) + \Phi(\sigma)] - t^{-\alpha}\Phi(tx + (1-t)y) - t^{-\alpha}\Phi((1-t)x + ty).$$

Now, by integrating the resulting inequality with respect to t on $[0,1]$ and changing the variable, we obtain

$$\begin{aligned}
&\frac{2}{1-\alpha}\Phi\left(\sigma - \frac{y}{2} + \rho - \frac{x}{2}\right) \\
&\leq 2[\Phi(\rho) + \Phi(\sigma)]\int_0^1 t^{-\alpha}dt - \left[\int_0^1 t^{-\alpha}\Phi(tx + (1-t)y)dt + \int_0^1 t^{-\alpha}\Phi((1-t)x + ty)dt\right] \\
&= \frac{2[\Phi(\rho) + \Phi(\sigma)]}{1-\alpha} - \frac{1}{(y-x)^{1-\alpha}}\left[\int_x^y (y-z)^{-\alpha}\Phi(z)dz + \int_x^y (z-x)^{-\alpha}\Phi(z)dz\right] \\
&= \frac{2[\Phi(\rho) + \Phi(\sigma)]}{1-\alpha} - \frac{1}{(y-x)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{y^-}\Phi(x) + {}_{N_3}J^{\alpha}_{x^+}\Phi(y)\right].
\end{aligned}$$

After dividing both sides of the last inequality by $\frac{2}{1-\alpha}$, we get the left inequality in (15).

For the proof of the second inequality of (15), keeping in mind that Φ is convex, one can write

$$\begin{aligned}
\Phi\left(\frac{x+y}{2}\right) &= \Phi\left(\frac{tx + (1-t)y + ty + (1-t)x}{2}\right) \\
&\leq \frac{\Phi(tx + (1-t)y) + \Phi(ty + (1-t)x)}{2}.
\end{aligned}$$

By multiplying both sides of last inequality by $t^{-\alpha}$ and by integrating with respect to t on $[0,1]$ and changing the variables, we obtain

$$\frac{1}{1-\alpha}\Phi\left(\frac{x+y}{2}\right) \leq \frac{1}{2(y-x)^{1-\alpha}}\left[\int_x^y (y-z)^{-\alpha}\Phi(z)dz + \int_x^y (z-x)^{-\alpha}\Phi(z)dz\right].$$

By multiplying the last inequality by $(\alpha - 1)$ and adding $\Phi(\rho) + \Phi(\sigma)$ to both sides, we get the right-hand side of (15):

$$\begin{aligned}
&\Phi(\rho) + \Phi(\sigma) - \Phi\left(\frac{x+y}{2}\right) \\
&\geq \Phi(\rho) + \Phi(\sigma) - \frac{1-\alpha}{2(y-x)^{1-\alpha}}\left[\int_x^y (y-z)^{-\alpha}\Phi(z)dz + \int_x^y (z-x)^{-\alpha}\Phi(z)dz\right] \\
&= \Phi(\rho) + \Phi(\sigma) - \frac{1-\alpha}{2(y-x)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{y^-}\Phi(x) + {}_{N_3}J^{\alpha}_{x^+}\Phi(y)\right].
\end{aligned}$$

Thus, inequality (15) is proved. □

Corollary 4. *For* $\alpha = 0$*, under the assumptions of Theorem 4, we get*

$$\Phi\left(\sigma - \frac{y}{2} + \rho - \frac{x}{2}\right) \leq \Phi(\rho) + \Phi(\sigma) - \frac{1}{y-x}\int_x^y \Phi(t)dt \leq \Phi(\rho) + \Phi(\sigma) - \Phi\left(\frac{x+y}{2}\right)$$

for all $x, y \in [\rho, \sigma]$*. This inequality was obtained by Kian and Moslehian in ([30], Theorem 2.1), and by Ögülmüs and Sarikaya in ([17], Remark 2.2).*

Theorem 5. *Let $\Phi : [\rho, \sigma] \to \mathbb{R}$. If $\Phi \in L_\alpha[\rho, \sigma]$ and Φ is convex on $[\rho, \sigma]$, then we have*

$$\begin{aligned} &\Phi\left(\sigma - \frac{y}{2} + \rho - \frac{x}{2}\right) \\ &\leq \frac{1-\alpha}{2(y-x)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{(\sigma-y+\rho)^+}\Phi(\sigma - x + \rho) + {}_{N_3}J^{\alpha}_{(\sigma-x+\rho)^-}\Phi(\sigma - y + \rho)\right] \\ &\leq \frac{\Phi(\sigma - x + \rho) + \Phi(\sigma - y + \rho)}{2} \leq \Phi(\rho) + \Phi(\sigma) - \frac{\Phi(x) + \Phi(y)}{2}, \end{aligned} \tag{16}$$

where $x, y \in [\rho, \sigma]$ and $\alpha < 1$.

Proof. To prove inequality (16), we use the left-hand side of (14) and choose $x_1 = tx + (1-t)y$, $y_1 = (1-t)x + ty$ to obtain the auxiliary inequality

$$\begin{aligned} &\Phi\left(\sigma - \frac{y_1}{2} + \rho - \frac{x_1}{2}\right) \\ &= \Phi\left(\frac{\sigma - x_1 + \rho + \sigma - y_1 + \rho}{2}\right) \leq \frac{\Phi(\sigma - x_1 + \rho)}{2} + \frac{\Phi(\sigma - y_1 + \rho)}{2} \\ &= \frac{\Phi(\rho + \sigma - tx - (1-t)y)}{2} + \frac{\Phi(\rho + \sigma - ty - (1-t)x)}{2}. \end{aligned}$$

More precisely, we use the equivalent inequality

$$\Phi\left(\sigma - \frac{y}{2} + \rho - \frac{x}{2}\right) \leq \frac{\Phi(\rho + \sigma - tx - (1-t)y)}{2} + \frac{\Phi(\rho + \sigma - (1-t)x - ty)}{2}. \tag{17}$$

Multiplying both sides of (17) by $t^{-\alpha}$, integrating with respect to t on $[0, 1]$ and changing the variables yields

$$\begin{aligned} &\frac{1}{1-\alpha}\Phi\left(\sigma - \frac{y}{2} + \rho - \frac{x}{2}\right) \\ &\leq \frac{1}{2(y-x)^{1-\alpha}}\left[\int_{\sigma-y+\rho}^{\sigma-x+\rho}(z - (\sigma - y + \rho))^{-\alpha}\Phi(z)dz + \int_{\sigma-y+\rho}^{\sigma-x+\rho}((\sigma - x + \rho) - z)^{-\alpha}\Phi(z)dz\right] \\ &= \frac{1}{2(y-x)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{(\sigma-y+\rho)^+}\Phi(\sigma - x + \rho) + {}_{N_3}J^{\alpha}_{(\sigma-x+\rho)^-}\Phi(\sigma - y + \rho)\right]. \end{aligned}$$

It is easy to see that left-hand side of (16) is proved. To prove the remaining part of (16), we need the following inequalities:

$$\begin{aligned} \Phi(\rho + \sigma - (tx + (1-t)y)) &= \Phi(\rho + \sigma + (\rho + \sigma)t - (\rho + \sigma)t - (tx + (1-t)y)) \\ &= \Phi(t(\sigma - x + \rho) + (1-t)(\sigma - y + \rho)) \\ &\leq t\Phi(\sigma - x + \rho) + (1-t)\Phi(\sigma - y + \rho) \end{aligned}$$

and

$$\Phi(\rho + \sigma - (ty + (1-t)x)) \leq t\Phi(\sigma - y + \rho) + (1-t)\Phi(\sigma - x + \rho).$$

By summing the above inequalities, we have

$$\Phi(\rho + \sigma - (tx + (1-t)y)) + \Phi(\rho + \sigma - (ty + (1-t)x)) \leq \Phi(\sigma - x + \rho) + \Phi(\sigma - y + \rho).$$

By multiplying both sides (17) by $t^{-\alpha}$, integrating with respect to t on $[0, 1]$ and changing the variables, we obtain

$$\begin{aligned} &\frac{1}{(y-x)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{(\sigma-y+\rho)^+}\Phi(\sigma - x + \rho) + {}_{N_3}J^{\alpha}_{(\sigma-x+\rho)^-}\Phi(\sigma - y + \rho)\right] \\ &\leq \frac{1}{1-\alpha}[\Phi(\sigma - x + \rho) + \Phi(\sigma - y + \rho)]. \end{aligned}$$

This inequality implies the remaining part of (16) by keeping (3) in mind. The proof is complete. □

Corollary 5. *For $\alpha = 0$, under the assumptions of Theorem 5, we have*

$$\Phi\Big(\sigma - \frac{y}{2} + \rho - \frac{x}{2}\Big) \leq \frac{1}{y-x}\int_{\sigma-y+\rho}^{\sigma-x+\rho} \Phi(t)dt \leq \Phi(\rho) + \Phi(\sigma) - \frac{\Phi(x)+\Phi(y)}{2} \tag{18}$$

for all $x, y \in [\rho, \sigma]$. This inequality was obtained by Kian and Moslehian in ([30], Theorem 2.1), and by Ögülmüs and Sarikaya in ([17], Remark 2.2).

Remark 5. *If in (18), we choose $x = \rho$ and $y = \sigma$, then we get the Hermite–Hadamard inequality (4).*

3. Inequalities for General Convex Functions

By considering (h, m)-convexity modified in the first and the second sense, we give analogues of Hermite–Hadamard inequality (4) for fractional operators (13) using Jensen–Mercer inequalities proven for these classes. Before that, we recall the following identity obtained by Nápoles et al. in [26] (see Lemma 1).

Lemma 1. *Let $\Phi : [\rho, \sigma] \to \mathbb{R}$ be a differentiable function. If $\Phi' \in L_{\alpha-1}[\rho, \sigma]$, then we have*

$$\frac{\Phi(\rho)+\Phi(\sigma)}{2} - \frac{1-\alpha}{2(\sigma-\rho)^{1-\alpha}}\Big[{}_{N_3}J^{\alpha}_{\sigma^-}\Phi(\rho) + {}_{N_3}J^{\alpha}_{\rho^+}\Phi(\sigma)\Big] = \frac{\sigma-\rho}{2}(I_{01} - I_{02}),$$

where $\alpha < 1$ and

$$I_{01} = \int_0^1 t^{1-\alpha}\Phi'((1-t)\rho + t\sigma)dt,\ I_{02} = \int_0^1 (1-t)^{1-\alpha}\Phi'((1-t)\rho + t\sigma)dt.$$

If in Lemma 1, we substitute $\sigma - y + \rho$ in place of ρ and $\sigma - x + \rho$ in place of σ, we get the next equation.

Corollary 6. *Under the assumptions of Lemma 1, we have*

$$\begin{aligned} &\frac{\Phi(\sigma-y+\rho)+\Phi(\sigma-x+\rho)}{2} \\ &- \frac{1-\alpha}{2(y-x)^{1-\alpha}}\Big[{}_{N_3}J^{\alpha}_{(\sigma-x+\rho)^-}\Phi(\sigma-y+\rho) + {}_{N_3}J^{\alpha}_{(\sigma-y+\rho)^+}\Phi(\sigma-x+\rho)\Big] \\ =& \frac{y-x}{2}(I_1 - I_2), \end{aligned} \tag{19}$$

where $x, y \in [\rho, \sigma]$, $\alpha < 1$ and

$$\begin{aligned} I_1 &= \int_0^1 t^{1-\alpha}\Phi'(\sigma - x + \rho t - (1-t)y)dt, \\ I_2 &= \int_0^1 (1-t)^{1-\alpha}\Phi'(\sigma - x + \rho t - (1-t)y)dt. \end{aligned}$$

Theorem 6. *Let $\Phi : [\rho, \frac{\sigma}{m}] \to \mathbb{R}$ be a differentiable function. If $\Phi' \in L_{\alpha-1}[\rho, \sigma]$ and $|\Phi'| \in K^{1,s}_{h,m}([\rho, \frac{\sigma}{m}])$, then the following inequality holds for all $x, y \in [\rho, \sigma]$, $\alpha < 1$:*

$$\begin{aligned}
&\Bigg| \frac{\Phi(\sigma - y + \rho) + \Phi(\sigma - x + \rho)}{2} \\
&- \frac{1-\alpha}{2(y-x)^{1-\alpha}} \Big[{}_{N_3}J^{\alpha}_{(\sigma-x+\rho)^-} \Phi(\sigma - y + \rho) + {}_{N_3}J^{\alpha}_{(\sigma-y+\rho)^+} \Phi(\sigma - x + \rho) \Big] \Bigg| \\
&\leq \frac{y-x}{2} \Bigg\{ \frac{2\mathbf{A}_1 |\Phi'(\rho)| + 2\mathbf{A}_1 m |\Phi'(\frac{\sigma}{m})| - m\big(|\Phi'(\frac{x}{m})| + |\Phi'(\frac{y}{m})|\big)}{2-\alpha} \\
&- \Big[|\Phi'(x)| + |\Phi'(y)| - m\Big(\Big|\Phi'\Big(\frac{x}{m}\Big)\Big| + \Big|\Phi'\Big(\frac{y}{m}\Big)\Big| \Big) \Big] \int_0^1 t^{1-\alpha} h^s(t) dt \Bigg\},
\end{aligned} \tag{20}$$

where $\mathbf{A}_1$ is from Corollary 2.

Proof. From Corollary 6 and modulus properties, we can write

$$\begin{aligned}
&\Bigg| \frac{\Phi(\sigma - y + \rho) + \Phi(\sigma - x + \rho)}{2} \\
&- \frac{1-\alpha}{2(y-x)^{1-\alpha}} \Big[{}_{N_3}J^{\alpha}_{(\sigma-x+\rho)^-} \Phi(\sigma - y + \rho) + {}_{N_3}J^{\alpha}_{(\sigma-y+\rho)^+} \Phi(\sigma - x + \rho) \Big] \Bigg| \\
&= \frac{y-x}{2} |I_1 - I_2| \leq \frac{y-x}{2} (|I_1| + |I_2|).
\end{aligned} \tag{21}$$

Using (h, m)-convexity of the first sense of function $|\Phi'|$ and Corollary 2, for integral I_1, we get

$$\begin{aligned}
|I_1| &\leq \int_0^1 t^{1-\alpha} \big|\Phi'(\rho + \sigma - (xt + (1-t)y))\big| dt \\
&\leq \int_0^1 t^{1-\alpha} \Big[\mathbf{A}_1 |\Phi'(\rho)| + \mathbf{A}_1 m \Big|\Phi'\Big(\frac{\sigma}{m}\Big)\Big| - \Big(h^s(t)|\Phi'(x)| + m(1 - h^s(t)) \Big|\Phi'\Big(\frac{y}{m}\Big)\Big| \Big) \Big] dt \\
&= \frac{\mathbf{A}_1 \big[|\Phi'(\rho)| + m|\Phi'(\frac{\sigma}{m})| \big]}{2-\alpha} - |\Phi'(x)| \int_0^1 t^{1-\alpha} h^s(t) dt - m\Big|\Phi'\Big(\frac{y}{m}\Big)\Big| \int_0^1 t^{1-\alpha} [1 - h^s(t)] dt \\
&= \frac{\mathbf{A}_1 |\Phi'(\rho)| + \mathbf{A}_1 m |\Phi'(\frac{\sigma}{m})| - m|\Phi'(\frac{y}{m})|}{2-\alpha} - \Big[|\Phi'(x)| - m\Big|\Phi'\Big(\frac{y}{m}\Big)\Big| \Big] \int_0^1 t^{1-\alpha} h^s(t) dt.
\end{aligned}$$

One can write for the second integral I_2 similarly

$$\begin{aligned}
|I_2| &\leq \int_0^1 (1-t)^{1-\alpha} \big|\Phi'(\sigma - x + \rho t - (1-t)y)\big| dt = \int_0^1 t^{1-\alpha} \big|\Phi'(\rho + \sigma - (1-t)x - ty)\big| dt \\
&\leq \frac{\mathbf{A}_1 |\Phi'(\rho)| + \mathbf{A}_1 m |\Phi'(\frac{\sigma}{m})| - m|\Phi'(\frac{x}{m})|}{2-\alpha} - \Big[|\Phi'(y)| - m\Big|\Phi'\Big(\frac{x}{m}\Big)\Big| \Big] \int_0^1 t^{1-\alpha} h^s(t) dt.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
|I_1| + |I_2| \leq\ & \frac{2\mathbf{A}_1 \big(|\Phi'(\rho)| + m|\Phi'(\frac{\sigma}{m})| \big) - m\big(|\Phi'(\frac{x}{m})| + |\Phi'(\frac{y}{m})| \big)}{2-\alpha} \\
&- \Big[|\Phi'(x)| + |\Phi'(y)| - m\Big(\Big|\Phi'\Big(\frac{x}{m}\Big)\Big| + \Big|\Phi'\Big(\frac{y}{m}\Big)\Big| \Big) \Big] \int_0^1 t^{1-\alpha} h^s(t) dt.
\end{aligned}$$

By multiplying the last inequality by $\frac{y-x}{2}$ and taking into account (21), we obtain (20). □

Corollary 7. *If in Theorem 6, we choose $x=\rho$ and $y=\sigma$, then we have*

$$\begin{aligned}
&\left|\frac{\Phi(\rho)+\Phi(\sigma)}{2}-\frac{1-\alpha}{2(\sigma-\rho)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{\sigma^-}\Phi(\rho)+{}_{N_3}J^{\alpha}_{\rho^+}\Phi(\sigma)\right]\right| \\
&\leq \frac{\sigma-\rho}{2}\left\{\frac{2\mathbf{A}_1|\Phi'(\rho)|-m\left|\Phi'\left(\frac{\rho}{m}\right)\right|+(2\mathbf{A}_1-1)m\left|\Phi'\left(\frac{\sigma}{m}\right)\right|}{2-\alpha}\right. \\
&\left.-\left[\left|\Phi'(\rho)\right|+\left|\Phi'(\sigma)\right|-m\left(\left|\Phi'\left(\frac{\rho}{m}\right)\right|+\left|\Phi'\left(\frac{\sigma}{m}\right)\right|\right)\right]\int_0^1 t^{1-\alpha}h^s(t)dt\right\}.
\end{aligned}$$

If, in addition, $m=1$, then

$$\begin{aligned}
&\left|\frac{\Phi(\rho)+\Phi(\sigma)}{2}-\frac{1-\alpha}{2(\sigma-\rho)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{\sigma^-}\Phi(\rho)+{}_{N_3}J^{\alpha}_{\rho^+}\Phi(\sigma)\right]\right| \\
&\leq \frac{(\sigma-\rho)(2\mathbf{A}_1-1)(|\Phi'(\rho)|+|\Phi'(\sigma)|)}{2(2-\alpha)}.
\end{aligned} \tag{22}$$

Theorem 7. *Let $\Phi:\left[\rho,\frac{\sigma}{m}\right]\to\mathbb{R}$ be a differentiable function. If $\Phi'\in L_{\alpha-1}[\rho,\sigma]$ and $|\Phi'|\in K^{2,s}_{h,m}([\rho,\frac{\sigma}{m}])$, then the following inequality holds for all $x,y\in[\rho,\sigma]$, $\alpha<1$:*

$$\begin{aligned}
&\left|\frac{\Phi(\sigma-y+\rho)+\Phi(\sigma-x+\rho)}{2}\right. \\
&\left.-\frac{1-\alpha}{2(y-x)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{(\sigma-x+\rho)^-}\Phi(\sigma-y+\rho)+{}_{N_3}J^{\alpha}_{(\sigma-y+\rho)^+}\Phi(\sigma-x+\rho)\right]\right| \\
&\leq \frac{(y-x)\mathbf{A}_2\left(|\Phi'(\rho)|+m\left|\Phi'\left(\frac{\sigma}{m}\right)\right|\right)}{2-\alpha}-\frac{y-x}{2}\left\{\left(\left|\Phi'(x)\right|+\left|\Phi'(y)\right|\right)\int_0^1 t^{1-\alpha}h^s(t)dt\right. \\
&\left.+m\left(\left|\Phi'\left(\frac{y}{m}\right)\right|+\left|\Phi'\left(\frac{x}{m}\right)\right|\right)\int_0^1 t^{1-\alpha}(1-h(t))^s dt\right\},
\end{aligned}$$

where $\mathbf{A}_2$ is from Corollary 3.

Proof. The proof is analogous to that of Theorem 7, but with the use of Corollary 3 instead of Corollary 2. □

Corollary 8. *If in Theorem 7, we choose $x=\rho$, $y=\sigma$ and $m=1$, then we have*

$$\begin{aligned}
&\left|\frac{\Phi(\rho)+\Phi(\sigma)}{2}-\frac{1-\alpha}{2(\sigma-\rho)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{\sigma^-}\Phi(\rho)+{}_{N_3}J^{\alpha}_{\rho^+}\Phi(\sigma)\right]\right| \\
&\leq \frac{\sigma-\rho}{2}\left(\left|\Phi'(\rho)\right|+\left|\Phi'(\sigma)\right|\right)\left\{\frac{2\mathbf{A}_2}{2-\alpha}-\int_0^1 t^{1-\alpha}\left[h^s(t)+(1-h(t))^s\right]dt\right\}.
\end{aligned} \tag{23}$$

Theorem 8. *Let $\Phi:\left[\rho,\frac{\sigma}{m}\right]\to\mathbb{R}$ be a differentiable function. If $\Phi'\in L_{\alpha-1}[\rho,\sigma]$ and $|\Phi'|^q\in K^{1,s}_{h,m}([\rho,\frac{\sigma}{m}])$, then for all $x,y\in[\rho,\sigma]$, $\alpha<1$, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, the following inequality holds:*

$$\left|\frac{\Phi(\sigma-y+\rho)+\Phi(\sigma-x+\rho)}{2} - \frac{1-\alpha}{2(y-x)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{(\sigma-x+\rho)^-}\Phi(\sigma-y+\rho)+{}_{N_3}J^{\alpha}_{(\sigma-y+\rho)^+}\Phi(\sigma-x+\rho)\right]\right| \leq \frac{y-x}{2}\left(\frac{1}{p-\alpha p+1}\right)^{\frac{1}{p}}(\mathbf{B}_1+\mathbf{C}_1), \tag{24}$$

where

$$\mathbf{B}_1=\left\{\mathbf{A}_1|\Phi'(\rho)|^q+\mathbf{A}_1 m\left|\Phi'\left(\frac{\sigma}{m}\right)\right|^q - m\left|\Phi'\left(\frac{y}{m}\right)\right|^q - \left[|\Phi'(x)|^q - m\left|\Phi'\left(\frac{y}{m}\right)\right|^q\right]\int_0^1 h^s(t)dt\right\}^{\frac{1}{q}},$$

$$\mathbf{C}_1=\left\{\mathbf{A}_1|\Phi'(\rho)|^q+\mathbf{A}_1 m\left|\Phi'\left(\frac{\sigma}{m}\right)\right|^q - m\left|\Phi'\left(\frac{x}{m}\right)\right|^q - \left[|\Phi'(y)|^q - m\left|\Phi'\left(\frac{x}{m}\right)\right|^q\right]\int_0^1 h^s(t)dt\right\}^{\frac{1}{q}}.$$

Proof. From Lemma 6 and modulus properties, we can write (21). Using the well-known Hölder integral inequality and Corollary 2, since $|\Phi'|^q \in K^{1,s}_{h,m}([\rho, \frac{\sigma}{m}])$, we get

$$\begin{aligned}|I_1| &\leq \int_0^1 t^{1-\alpha}|\Phi'(\rho+\sigma-(xt+(1-t)y))|dt \\ &\leq \left(\int_0^1 t^{(1-\alpha)p}dt\right)^{\frac{1}{p}}\left\{\mathbf{A}_1\int_0^1\left(|\Phi'(\rho)|^q+m\left|\Phi'\left(\frac{\sigma}{m}\right)\right|^q\right)dt\right. \\ &\quad \left. - \int_0^1\left[\left(h^s(t)|\Phi'(x)|^q+m(1-h^s(t))\left|\Phi'\left(\frac{y}{m}\right)\right|^q\right)\right]dt\right\}^{\frac{1}{q}} \\ &= \left(\frac{1}{p-\alpha p+1}\right)^{\frac{1}{p}}\left\{\mathbf{A}_1|\Phi'(\rho)|^q+\mathbf{A}_1 m\left|\Phi'\left(\frac{\sigma}{m}\right)\right|^q - m\left|\Phi'\left(\frac{y}{m}\right)\right|^q\right. \\ &\quad \left. - \left[|\Phi'(x)|^q - m\left|\Phi'\left(\frac{y}{m}\right)\right|^q\right]\int_0^1 h^s(t)dt\right\}^{\frac{1}{q}}.\end{aligned} \tag{25}$$

Since

$$\int_0^1 (1-t)^{1-\alpha}|\Phi'(\sigma-x+\rho t-(1-t)y)|dt = \int_0^1 t^{1-\alpha}|\Phi'(\rho+\sigma-(1-t)x-ty)|dt,$$

we can write similarly for the second integral

$$\begin{aligned}|I_2| &\leq \int_0^1 t^{1-\alpha}|\Phi'(\rho+\sigma-(1-t)x-ty)|dt \\ &\leq \left(\frac{1}{p-\alpha p+1}\right)^{\frac{1}{p}}\left\{\mathbf{A}_1|\Phi'(\rho)|^q+\mathbf{A}_1 m\left|\Phi'\left(\frac{\sigma}{m}\right)\right|^q - m\left|\Phi'\left(\frac{x}{m}\right)\right|^q\right. \\ &\quad \left. - \left[|\Phi'(y)|^q - m\left|\Phi'\left(\frac{x}{m}\right)\right|^q\right]\int_0^1 h^s(t)dt\right\}^{\frac{1}{q}}.\end{aligned} \tag{26}$$

By adding inequalities (25) and (26), we get

$$|I_1|+|I_2| \leq \left(\frac{1}{p-\alpha p+1}\right)^{\frac{1}{p}}(\mathbf{B}_1+\mathbf{C}_1).$$

Multiplying both sides of the last inequality by the expression $\frac{y-x}{2}$ and keeping (21) in mind yields (24). The proof is complete. □

Corollary 9. *If in Theorem 8, we choose $x = \rho$, $y = \sigma$ and $m = 1$, then we have*

$$\left|\frac{\Phi(\rho)+\Phi(\sigma)}{2} - \frac{1-\alpha}{2(\sigma-\rho)^{1-\alpha}}\left[\,{}_{N_3}J^{\alpha}_{\sigma^-}\Phi(\rho) + {}_{N_3}J^{\alpha}_{\rho^+}\Phi(\sigma)\right]\right|$$
$$\leq \frac{\sigma-\rho}{2}\left(\frac{1}{p-\alpha p+1}\right)^{\frac{1}{p}}$$
$$\times\left[\left\{\mathbf{A}_1|\Phi'(\rho)|^q + (\mathbf{A}_1-1)|\Phi'(\sigma)|^q - \left[|\Phi'(\rho)|^q - |\Phi'(\sigma)|^q\right]\int_0^1 h^s(t)dt\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left\{(\mathbf{A}_1-1)|\Phi'(\rho)|^q + \mathbf{A}_1|\Phi'(\sigma)|^q - \left[|\Phi'(\sigma)|^q - |\Phi'(\rho)|^q\right]\int_0^1 h^s(t)dt\right\}^{\frac{1}{q}}\right].$$

Theorem 9. *Let $\Phi : [\rho, \frac{\sigma}{m}] \to \mathbb{R}$ be a differentiable function. If $\Phi' \in L_{\alpha-1}[\rho,\sigma]$ and $|\Phi'|^q \in K^{2,s}_{h,m}([\rho, \frac{\sigma}{m}])$, then for all $x, y \in [\rho,\sigma]$, $\alpha < 1$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, the following inequality holds:*

$$\left|\frac{\Phi(\sigma-y+\rho)+\Phi(\sigma-x+\rho)}{2}\right.$$
$$\left. - \frac{1-\alpha}{2(y-x)^{1-\alpha}}\left[\,{}_{N_3}J^{\alpha}_{(\sigma-x+\rho)^-}\Phi(\sigma-y+\rho) + {}_{N_3}J^{\alpha}_{(\sigma-y+\rho)^+}\Phi(\sigma-x+\rho)\right]\right|$$
$$\leq \frac{y-x}{2}\left(\frac{1}{p-\alpha p+1}\right)^{\frac{1}{p}}(\mathbf{B}_2 + \mathbf{C}_2),$$

where

$$\mathbf{B}_2 = \left\{\mathbf{A}_2|\Phi'(\rho)|^q + \mathbf{A}_2 m\left|\Phi'\left(\frac{\sigma}{m}\right)\right|^q\right.$$
$$\left.-|\Phi'(x)|^q\int_0^1 h^s(t)dt - m\left|\Phi'\left(\frac{y}{m}\right)\right|^q\int_0^1 (1-h(t))^s dt\right\}^{\frac{1}{q}},$$
$$\mathbf{C}_2 = \left\{\mathbf{A}_2|\Phi'(\rho)|^q + \mathbf{A}_2 m\left|\Phi'\left(\frac{\sigma}{m}\right)\right|^q\right.$$
$$\left.-|\Phi'(y)|^q\int_0^1 h^s(t)dt - m\left|\Phi'\left(\frac{x}{m}\right)\right|^q\int_0^1 (1-h(t))^s dt\right\}^{\frac{1}{q}}.$$

Proof. The proof is analogous to that of Theorem 8, but with the use of Corollary 3 instead of Corollary 2. □

Corollary 10. *If in Theorem 9, we choose $x = \rho$, $y = \sigma$ and $m = 1$, then we have*

$$\begin{aligned}
&\left|\frac{\Phi(\rho)+\Phi(\sigma)}{2} - \frac{1-\alpha}{2(\sigma-\rho)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{\sigma^-}\Phi(\rho) + {}_{N_3}J^{\alpha}_{\rho^+}\Phi(\sigma)\right]\right| \\
&\leq \frac{\sigma-\rho}{2}\left(\frac{1}{p-\alpha p+1}\right)^{\frac{1}{p}} \\
&\quad \times \left[\left\{|\mathbf{A}_2\Phi'(\rho)|^q + \mathbf{A}_2|\Phi'(\sigma)|^q - |\Phi'(\rho)|^q\int_0^1 h^s(t)dt - |\Phi'(\sigma)|^q\int_0^1 (1-h(t))^s dt\right\}^{\frac{1}{q}}\right. \\
&\qquad \left. + \left\{\mathbf{A}_2|\Phi'(\rho)|^q + \mathbf{A}_2|\Phi'(\sigma)|^q - |\Phi'(\sigma)|^q\int_0^1 h^s(t)dt - |\Phi'(\rho)|^q\int_0^1 (1-h(t))^s dt\right\}^{\frac{1}{q}}\right].
\end{aligned}$$

Theorem 10. *Let $\Phi : [\rho, \frac{\sigma}{m}] \to \mathbb{R}$ be a differentiable function. If $\Phi' \in L_{\alpha-1}[\rho,\sigma]$ and $|\Phi'|^q \in K^{1,s}_{h,m}([\rho, \frac{\sigma}{m}])$, then for all $x, y \in [\rho,\sigma]$, $\alpha < 1$, $q \geq 1$, we have*

$$\begin{aligned}
&\left|\frac{\Phi(\sigma-y+\rho)+\Phi(\sigma-x+\rho)}{2}\right. \\
&\left. - \frac{1-\alpha}{2(y-x)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{(\sigma-x+\rho)^-}\Phi(\sigma-y+\rho) + {}_{N_3}J^{\alpha}_{(\sigma-y+\rho)^+}\Phi(\sigma-x+\rho)\right]\right| \\
&\leq \frac{y-x}{2}\left(\frac{1}{2-\alpha}\right)^{1-\frac{1}{q}}(\mathbf{D}_1 + \mathbf{E}_1),
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
\mathbf{D}_1 &= \left\{\frac{\mathbf{A}_1|\Phi'(\rho)|^q + \mathbf{A}_1 m|\Phi'(\frac{\sigma}{m})|^q - m|\Phi'(\frac{y}{m})|^q}{2-\alpha}\right. \\
&\qquad \left. - \left[|\Phi'(x)|^q - m\left|\Phi'\left(\frac{y}{m}\right)\right|^q\right]\int_0^1 t^{1-\alpha}h^s(t)dt\right\}^{\frac{1}{q}}, \\
\mathbf{E}_1 &= \left\{\frac{\mathbf{A}_1|\Phi'(\rho)|^q + \mathbf{A}_1 m|\Phi'(\frac{\sigma}{m})|^q - m|\Phi'(\frac{x}{m})|^q}{2-\alpha}\right. \\
&\qquad \left. - \left[|\Phi'(y)|^q - m\left|\Phi'\left(\frac{x}{m}\right)\right|^q\right]\int_0^1 t^{1-\alpha}h^s(t)dt\right\}^{\frac{1}{q}}.
\end{aligned}$$

Proof. We first write (21). Then, using the well-known power–mean integral inequality and Corollary 2, since $|\Phi'|^q \in K^{1,s}_{h,m}([\rho, \frac{\sigma}{m}])$, for the integral I_1, we obtain

$$\begin{aligned}
|I_1| &\le \int_0^1 t^{1-\alpha}\left|\Phi'(\rho+\sigma-(xt+(1-t)y))\right|dt \\
&\le \left(\int_0^1 t^{1-\alpha}dt\right)^{1-\frac{1}{q}}\left\{\left(\mathbf{A}_1|\Phi'(\rho)|^q+\mathbf{A}_1 m\left|\Phi'\left(\frac{\sigma}{m}\right)\right|^q\right)\int_0^1 t^{1-\alpha}dt\right. \\
&\quad\left.-\int_0^1 t^{1-\alpha}\left[\left(h^s(t)|\Phi'(x)|^q+m(1-h^s(t))\left|\Phi'\left(\frac{y}{m}\right)\right|^q\right)\right]dt\right\}^{\frac{1}{q}} \\
&=\left(\frac{1}{2-\alpha}\right)^{1-\frac{1}{q}}\left\{\frac{\mathbf{A}_1|\Phi'(\rho)|^q+\mathbf{A}_1 m|\Phi'(\frac{\sigma}{m})|^q}{2-\alpha}-|\Phi'(x)|^q\int_0^1 t^{1-\alpha}h^s(t)dt\right. \\
&\quad\left.-m\left|\Phi'\left(\frac{y}{m}\right)\right|^q\int_0^1 t^{1-\alpha}(1-h^s(t))dt\right\}^{\frac{1}{q}} \\
&=\left(\frac{1}{2-\alpha}\right)^{1-\frac{1}{q}}\left\{\frac{\mathbf{A}_1|\Phi'(\rho)|^q+\mathbf{A}_1 m|\Phi'(\frac{\sigma}{m})|^q-m|\Phi'(\frac{y}{m})|^q}{2-\alpha}\right. \\
&\quad\left.-\left[|\Phi'(x)|^q-m\left|\Phi'\left(\frac{y}{m}\right)\right|^q\right]\int_0^1 t^{1-\alpha}h^s(t)dt\right\}^{\frac{1}{q}}.
\end{aligned}\tag{28}$$

One can write for the second integral similarly

$$\begin{aligned}
|I_2| &\le \int_0^1 t^{1-\alpha}\left|\Phi'(\rho+\sigma-(1-t)x-ty)\right|dt \\
&\le\left(\frac{1}{2-\alpha}\right)^{1-\frac{1}{q}}\left\{\frac{\mathbf{A}_1|\Phi'(\rho)|^q+\mathbf{A}_1 m|\Phi'(\frac{\sigma}{m})|^q-m|\Phi'(\frac{x}{m})|^q}{2-\alpha}\right. \\
&\quad\left.-\left[|\Phi'(y)|^q-m\left|\Phi'\left(\frac{x}{m}\right)\right|^q\right]\int_0^1 t^{1-\alpha}h^s(t)dt\right\}^{\frac{1}{q}}.
\end{aligned}\tag{29}$$

By adding inequalities (28) and (29), we obtain

$$|I_1|+|I_2|\le\left(\frac{1}{2-\alpha}\right)^{1-\frac{1}{q}}(\mathbf{D}_1+\mathbf{E}_1).$$

Multiplying both sides of the last inequality by the expression $\frac{y-x}{2}$ and keeping (21) in mind, we get (27). The proof is complete. □

Corollary 11. *If in Theorem 10, we choose $x=\rho$, $y=\sigma$ and $m=1$, then we have*

$$\begin{aligned}
&\left|\frac{\Phi(\rho)+\Phi(\sigma)}{2}-\frac{1-\alpha}{2(\sigma-\rho)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{\sigma^-}\Phi(\rho)+{}_{N_3}J^{\alpha}_{\rho^+}\Phi(\sigma)\right]\right| \\
&\le\frac{\sigma-\rho}{2}\left(\frac{1}{2-\alpha}\right)^{1-\frac{1}{q}} \\
&\quad\times\left[\left\{\frac{\mathbf{A}_1|\Phi'(\rho)|^q+(\mathbf{A}_1-1)|\Phi'(\sigma)|^q}{2-\alpha}-\left[|\Phi'(\rho)|^q-|\Phi'(\sigma)|^q\right]\int_0^1 t^{1-\alpha}h^s(t)dt\right\}^{\frac{1}{q}}\right. \\
&\quad\left.+\left\{\frac{(\mathbf{A}_1-1)|\Phi'(\rho)|^q+\mathbf{A}_1|\Phi'(\sigma)|^q}{2-\alpha}-\left[|\Phi'(\sigma)|^q-|\Phi'(\rho)|^q\right]\int_0^1 t^{1-\alpha}h^s(t)dt\right\}^{\frac{1}{q}}\right].
\end{aligned}$$

If, in addition, we suppose $q=1$, then we get (22).

Theorem 11. *Let $\Phi : \left[\rho, \frac{\sigma}{m}\right] \to \mathbb{R}$ be a differentiable function. If $\Phi' \in L_{\alpha-1}[\rho,\sigma]$ and $|\Phi'|^q \in K^{2,s}_{h,m}([\rho, \frac{\sigma}{m}])$, then for all $x, y \in [\rho,\sigma]$, $\alpha < 1$, $q \geq 1$, we have*

$$\begin{aligned}
&\left| \frac{\Phi(\sigma - y + \rho) + \Phi(\sigma - x + \rho)}{2} \right. \\
&\left. - \frac{1-\alpha}{2(y-x)^{1-\alpha}} \left[{}_{N_3}J^{\alpha}_{(\sigma-x+\rho)^-} \Phi(\sigma - y + \rho) + {}_{N_3}J^{\alpha}_{(\sigma-y+\rho)^+} \Phi(\sigma - x + \rho) \right] \right| \\
&\leq \frac{y-x}{2} \left(\frac{1}{2-\alpha} \right)^{1-\frac{1}{q}} (\mathbf{D}_2 + \mathbf{E}_2),
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{D}_2 = &\left\{ \frac{\mathbf{A}_2|\Phi'(\rho)|^q + \mathbf{A}_2 m|\Phi'(\frac{\sigma}{m})|^q}{2-\alpha} \right. \\
&\left. - \left[\left|\Phi'(x)\right|^q \int_0^1 t^{1-\alpha} h^s(t)dt + m\left|\Phi'\left(\frac{y}{m}\right)\right|^q \int_0^1 t^{1-\alpha}(1-h(t))^s dt \right] \right\}^{\frac{1}{q}}, \\
\mathbf{E}_2 = &\left\{ \frac{\mathbf{A}_2|\Phi'(\rho)|^q + \mathbf{A}_2 m|\Phi'(\frac{\sigma}{m})|^q}{2-\alpha} \right. \\
&\left. - \left[\left|\Phi'(y)\right|^q \int_0^1 t^{1-\alpha} h^s(t)dt + m\left|\Phi'\left(\frac{x}{m}\right)\right|^q \int_0^1 t^{1-\alpha}(1-h(t))^s dt \right] \right\}^{\frac{1}{q}}.
\end{aligned}$$

Proof. The proof is analogous to that of Theorem 10, but with the use of Corollary 3 instead of Corollary 2. □

Corollary 12. *If in Theorem 11, we choose $x = \rho$, $y = \sigma$ and $m = 1$, then we have*

$$\begin{aligned}
&\left| \frac{\Phi(\rho) + \Phi(\sigma)}{2} - \frac{1-\alpha}{2(\sigma-\rho)^{1-\alpha}} \left[{}_{N_3}J^{\alpha}_{\sigma^-} \Phi(\rho) + {}_{N_3}J^{\alpha}_{\rho^+} \Phi(\sigma) \right] \right| \\
&\leq \frac{\sigma-\rho}{2} \left(\frac{1}{2-\alpha} \right)^{1-\frac{1}{q}} \\
&\times \left[\left\{ \frac{\mathbf{A}_2|\Phi'(\rho)|^q + \mathbf{A}_2|\Phi'(\sigma)|^q}{2-\alpha} - |\Phi'(\rho)|^q \int_0^1 t^{1-\alpha} h^s(t)dt \right. \right. \\
&\left. -|\Phi'(\sigma)|^q \int_0^1 t^{1-\alpha}(1-h(t))^s dt \right\}^{\frac{1}{q}} + \left\{ \frac{\mathbf{A}_2|\Phi'(\rho)|^q + \mathbf{A}_2|\Phi'(\sigma)|^q}{2-\alpha} \right. \\
&\left. \left. -|\Phi'(\sigma)|^q \int_0^1 t^{1-\alpha} h^s(t)dt - |\Phi'(\rho)|^q \int_0^1 t^{1-\alpha}(1-h(t))^s dt \right\}^{\frac{1}{q}} \right].
\end{aligned}$$

If, in addition, we suppose $q = 1$, then we get (23).

4. Applications

Throughout the paper, we examined the fractional integral sums

$${}_{N_3}J^{\alpha}_{y^-} \Phi(x) + {}_{N_3}J^{\alpha}_{x^+} \Phi(y) = \int_x^y (t-x)^{-\alpha} \Phi(t)dt + \int_x^y (y-t)^{-\alpha} \Phi(t)dt,$$

for $x, y \in [\rho,\sigma] \subset \mathbb{R}$.

We demonstrate the scope and strength of our results through three examples, two related to trigonometric functions and one to arithmetic means.

First, consider a convex function. Let $\Phi : [\rho, \sigma] = [\pi, 2\pi] \to \mathbb{R}$, $\Phi(t) = \sin t$, which is convex on $[\pi, 2\pi]$, and fix $\alpha = \frac{1}{2}$. Then, according to Theorem 4, we have the inequality

$$\sin\left(\frac{x+y}{2}\right) \leq -\frac{1}{4\sqrt{y-x}}\left[\int_x^y \frac{\sin t}{\sqrt{t-x}}\,dt + \int_x^y \frac{\sin t}{\sqrt{y-t}}\,dt\right] \leq -\sin\left(\frac{x+y}{2}\right)$$

for all $x, y \in [\pi, 2\pi]$.

Second, we consider a non-convex function that has a convex derivative in absolute value. Let $\Phi : [\pi, 2\pi] \to \mathbb{R}$, $\Phi(t) = t - \cos t$, which has a convex derivative $\Phi'(t) = 1 + \sin t$ on $[\pi, 2\pi]$, and fix $\alpha = \frac{1}{2}$. Keeping Remark 2 in mind, applying Corollary 7 or Corollary 8 (with x in place of ρ and y in place of σ) yields

$$\begin{aligned}&\left| x + y - \cos x - \cos y - \frac{1}{2\sqrt{y-x}}\left[\int_x^y \frac{t - \cos t}{\sqrt{t-x}}\,dt + \int_x^y \frac{t-\cos t}{\sqrt{y-t}}\,dt\right]\right| \\ &\leq \frac{2(y-x)(2 + \sin x + \sin y)}{3}\end{aligned}$$

for all $x, y \in [\pi, 2\pi]$.

Finally, consider the convex function $\Phi : [\rho, \sigma] \subset [0, \infty) \to \mathbb{R}$, $\Phi(t) = t^n$ with $n \geq 1$, and fix $\alpha < 1$. Then, according to Theorem 4, we have

$$\begin{aligned}\left[\sigma - \frac{y}{2} + \rho - \frac{x}{2}\right]^n &\leq \rho^n + \sigma^n - \frac{1-\alpha}{2(y-x)^{1-\alpha}}\left[\int_x^y \frac{t^n}{(t-x)^\alpha}dt + \int_x^y \frac{t^n}{(y-t)^\alpha}dt\right] \\ &\leq \rho^n + \sigma^n - \left(\frac{x+y}{2}\right)^n\end{aligned}$$

for $x, y \in [\rho, \sigma]$, from which we obtain an inequality of arithmetic means:

$$\begin{aligned}[2A(\rho, \sigma) - A(x, y)]^n &\leq 2A(\rho^n, \sigma^n) - \frac{1-\alpha}{2(y-x)^{1-\alpha}}\left[\int_x^y \frac{t^n}{(t-x)^\alpha}dt + \int_x^y \frac{t^n}{(y-t)^\alpha}dt\right] \\ &\leq 2A(\rho^n, \sigma^n) - A^n(x, y),\end{aligned}$$

where $A(u, v)$ denotes the arithmetic mean $A(u, v) = \frac{u+v}{2}$.

5. Conclusions

In the present work, we obtained interesting results pertaining to the Jensen–Mercer-type Hermite–Hadamard inequalities via non-conformable integrals, using the classical convex, (h, m)-convex, and (h, m)-convex modified functions. Thus, we presented various relevant fractional inequalities related to convex functions and differentiable functions of general convex derivative in absolute value.

As applications, we gave examples of functions for which our main inequalities can be applied, and we presented the resulting inequalities.

Our results are expected to provide motivation to generate further research on inequalities that includes other notions of convexity, such as new variants of the Hermite–Hadamard–Mercer inequalities obtained in this work. For example, instead of working with the operators of [26], one can consider the following more general fractional integral:

Definition 7 ([31])**.** *Let $\Phi : [0, \infty) \to [0, \infty)$, such that $\Phi \in L[0, \infty)$. Generalized fractional Riemann–Liouville integral of order $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, $\beta \neq -1$, is given as follows:*

$${}^{\beta}J^{\frac{\alpha}{k}}_{\Phi,u}\Phi(x) = \frac{1}{k\Gamma_k(\alpha)}\int_u^x \frac{\Phi(t)dt}{[\Phi(x,t)]^{1-\frac{\alpha}{k}}\Phi(t,\beta)}$$

with $\Phi(t, \beta) > 0$, $\Phi(t, 0) = 1$ and $\Phi(x, t) = \int_t^x \frac{d\theta}{\Phi(\theta,\beta)}$. Obviously $\Phi(x, t) = -\Phi(t, x)$.

By considering the kernel $\Phi(t,\beta) = t^{-\beta}$, we have

$$\Phi(x,t) = \frac{x^{\beta+1} - t^{\beta+1}}{\beta+1} \quad \text{and} \quad [\Phi(x,t)]^{1-\frac{\alpha}{k}} = \left[\frac{x^{\beta+1} - t^{\beta+1}}{\beta+1}\right]^{1-\frac{\alpha}{k}},$$

and we get the (k,β)–Riemann–Liouville fractional integral in Definition 2.1 of [32]. Furthermore, by setting $k = 1$, we obtain the Katugampola fractional integral (see [33]).

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