

Article

The Modal Logic of Aristotelian Diagrams

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Abstract: In this paper, we introduce and study AD-logic, i.e., a system of (hybrid) modal logic that can be used to reason about Aristotelian diagrams. The language of AD-logic, \mathcal{L}_{AD} , is interpreted on a kind of birelational Kripke frames, which we call “AD-frames”. We establish a sound and strongly complete axiomatization for AD-logic, and prove that there exists a bijection between finite Aristotelian diagrams (up to Aristotelian isomorphism) and finite AD-frames (up to modal isomorphism). We then show how AD-logic can express several major insights about Aristotelian diagrams; for example, for every well-known Aristotelian family A , we exhibit a formula $\chi_A \in \mathcal{L}_{AD}$ and show that an Aristotelian diagram D belongs to the family A iff χ_A is validated by D (when the latter is viewed as an AD-frame). Finally, we show that AD-logic itself gives rise to new and interesting Aristotelian diagrams, and we reflect on their profoundly peculiar status.

Keywords: square of opposition; Aristotelian diagram; modal logic; hybrid logic; logical geometry

MSC: 03B45; 03A05; 03G05



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1. Introduction

The interaction between modal logic and Aristotelian diagrams has a rich and well-documented history. The oldest kind of Aristotelian diagram is the square of opposition for syllogistics, which dates back to the second century CE [1]. However, from the twelfth century onwards, philosophers started to make use of these diagrams to explicate their theorizing on modalities as well [2–5]. Furthermore, historical scholarship has shown that Aristotelian diagrams for modal logic can be reconstructed from the works of many earlier authors, such as Theophrastus [6,7], Chrysippus [8,9], and Avicenna [10,11]. Today, Aristotelian diagrams not only appear in well-known textbooks on modal logic [12,13], but they are also used in applications of modal logic to a variety of philosophical and logical topics, such as paraconsistency [14,15], logic-sensitivity [16–18], and theories of truth [19,20]. It should be emphasized that all of these applications manifest a wide diversity of Aristotelian diagrams: next to the “ordinary” squares of oppositions, we also find several types of hexagons, octagons, and even more complex diagrams [21].

In all of this work, Aristotelian diagrams are used to visualize the logical relations holding among a number of modal statements. In this paper, however, we take a completely different approach, as we will be using a modal language to describe Aristotelian diagrams. This boils down to “reversing the directionality” of the interaction between modal logic and Aristotelian diagrams. Whereas the aforementioned work was mainly concerned with *Aristotelian diagrams for modal logic*, in this paper, we rather study the *modal logic of Aristotelian diagrams*.

More concretely, we start from the basic observation that Aristotelian diagrams are graphs, i.e., relational structures consisting of vertices (viz., formulas) and edges (viz., Aristotelian relations) between them. Consequently, we can view these diagrams as Kripke frames, which allows us to interpret, i.e., to provide a relational semantics for, a modal language. (For reasons of expressivity, we will actually interpret a language of hybrid

logic, rather than one of “basic” modal logic.) We can then express formulas in this modal language that correspond exactly with properties of Aristotelian diagrams that we are interested in. For example, we will exhibit a modal formula $\chi_{classical}$ that precisely captures the property of being a classical square of opposition: an Aristotelian diagram is a classical square if and only if $\chi_{classical}$ is valid on that diagram (considered as a Kripke frame). Similarly, the modal formulas $\chi_{degenerate}$ and χ_{square} capture the properties of being a degenerate square and being a square of opposition “in general”, respectively. (Precise definitions of classical squares, degenerate squares, and squares in general are provided later in the paper.) We can take this idea of viewing Aristotelian diagrams as Kripke frames one step further, which gives rise to the modal consequence relation \models_{AD} . Many results from logical geometry (i.e., the systematic study of Aristotelian diagrams) can now be formally expressed as AD-valid arguments. Consider, for example, the well-known fact that every square of opposition is either a classical square or a degenerate square: this is formally captured by the argument $\chi_{square} \models_{AD} \chi_{classical} \vee \chi_{degenerate}$. (It should be noted that this approach was partially inspired by the work of Grossi [22,23]: he proposed to view argumentation frameworks as Kripke frames, which then allowed him to study abstract argumentation theory using modal logic).

There are several complementary reasons for studying the modal logic of Aristotelian diagrams from the perspective of logical geometry as well as from that of modal logic itself. First of all, it nicely illustrates the expressive power of modal—and in particular, hybrid—languages to describe relational structures. Secondly, even though this is not the main concern of this paper, we will show how a sound and strongly complete axiomatization of \models_{AD} can be obtained almost immediately from general results about hybrid logic. Thirdly, based on the intuitive idea of “viewing an Aristotelian diagram as a Kripke frame”, we formally define the notion of an AD-frame. We will prove that this definition completely captures the structural properties of Aristotelian diagrams, thus shedding new light on the very nature of these diagrams. Fourthly, and perhaps most importantly, the modal language developed here can be used to formulate generic descriptions of families of Aristotelian diagrams. For example, in order to define the family of all classical squares of opposition, we usually refer to an arbitrary logical system (or in more algebraically oriented parts of logical geometry, to an arbitrary Boolean algebra). Such a definition is indeed *general* (because it works for all logical systems), but it is not *generic* (because it still refers to some logical system—regardless of whether that system was chosen arbitrarily or not). For most purposes, such a general definition is appropriate, but there also exist certain theoretical contexts where we really want to work with generic descriptions of Aristotelian families (that do not refer to any logical system at all) [24]. Using the modal language developed in this paper, we can define the family of classical squares of opposition as consisting of precisely those diagrams which (when viewed as Kripke frames) validate the formula $\chi_{classical}$. Since this characterization does not refer to any logical system at all, the modal formula $\chi_{classical}$ provides a fully generic description of the family of classical squares of opposition.

The paper is organized as follows. Section 2 provides some basic background information on Aristotelian diagrams in order to keep this paper self-contained. Section 3 introduces all the key components of AD-logic, including the modal language \mathcal{L}_{AD} , the AD-frames that this language will be interpreted on, and the consequence relation \models_{AD} . In Section 4, we prove that there exists a bijection between finite Aristotelian diagrams and finite AD-frames (up to isomorphism), which shows that AD-frames completely capture the structural properties of Aristotelian diagrams. Section 5 shows how AD-logic allows us to express several key results from logical geometry (in particular, characterizations of and relations between various families of Aristotelian diagrams). In Section 6, we demonstrate how AD-logic can be used to construct interesting new Aristotelian diagrams, which simultaneously contain certain formulas (when viewed as Aristotelian diagrams) and validate those same formulas (when viewed as AD-frames). Finally, Section 7 wraps things up, and mentions some avenues for further research.

2. The Basic Building Blocks of Logical Geometry

Aristotelian diagrams can be defined in several ways [25]. The most general definition is in terms of Boolean algebras, but for the purposes of this paper, it suffices to focus on Aristotelian diagrams relative to logical systems. We first introduce the Aristotelian relations in Definition 1 and then move on to the diagrams themselves in Definition 2.

Definition 1 (Aristotelian relations). *Let S be a logical system, which is assumed to have Boolean operators and a model-theoretic semantics \models_S . The formulas $\alpha, \beta \in \mathcal{L}_S$ are said to be*

<i>S-contradictory</i>	<i>iff</i>	$\models_S \neg(\alpha \wedge \beta)$	<i>and</i>	$\models_S \alpha \vee \beta,$
<i>S-contrary</i>	<i>iff</i>	$\models_S \neg(\alpha \wedge \beta)$	<i>and</i>	$\not\models_S \alpha \vee \beta,$
<i>S-subcontrary</i>	<i>iff</i>	$\not\models_S \neg(\alpha \wedge \beta)$	<i>and</i>	$\models_S \alpha \vee \beta,$
<i>in S-subalternation</i>	<i>iff</i>	$\models_S \alpha \rightarrow \beta$	<i>and</i>	$\not\models_S \beta \rightarrow \alpha.$

These relations are abbreviated as $CD_S, C_S, SC_S,$ and SA_S .

Definition 2 (Aristotelian diagram). *Let S be a logical system, as shown in Definition 1, and let $\mathcal{F} \subseteq \mathcal{L}_S$ be a non-empty fragment of formulas. An Aristotelian diagram for (\mathcal{F}, S) is a vertex- and edge-labeled graph: its vertices are labeled by the formulas from \mathcal{F} , while its edges are labeled by the Aristotelian relations holding between those formulas (relative to S). Furthermore, the fragment \mathcal{F} is required to satisfy the following conditions:*

1. *for every $\alpha \in \mathcal{F}$, there is a $\beta \in \mathcal{F} \setminus \{\alpha\}$ such that $\models_S \beta \leftrightarrow \neg\alpha,$*
2. *for every $\alpha \in \mathcal{F}$, there is no $\gamma \in \mathcal{F} \setminus \{\alpha\}$ such that $\models_S \alpha \leftrightarrow \gamma,$*
3. *there is no $\delta \in \mathcal{F}$ such that $\models_S \delta$ or $\models_S \neg\delta.$*

Figure 1 shows two basic examples of Aristotelian diagrams, viz., classical squares of opposition for $(\{p \wedge q, p \vee q, \neg p \wedge \neg q, \neg p \vee \neg q\}, \text{CPL})$ and for $(\{\Box p, \Diamond p, \Box \neg p, \Diamond \neg p\}, \text{KD})$, where CPL is the system of classical propositional logic, and KD is the system of modal logic that is interpreted on Kripke frames with a serial accessibility relation [13].

Definition 2’s three conditions on \mathcal{F} are motivated by various theoretical and cognitive considerations [26]. The vast majority of diagrams that are found in the extant literature indeed satisfy these conditions. In recent theoretical work on Aristotelian diagrams [27], these conditions are sometimes dropped. However, for the purposes of this paper, we stick to the more traditional approach, and do include these conditions in our definition. Finally, note that because of conditions 1 and 2, we can always write the elements of a finite Aristotelian diagram for (\mathcal{F}, S) (up to S -equivalence) as $\mathcal{F} = \{\alpha_1, \neg\alpha_1, \alpha_2, \neg\alpha_2, \dots, \alpha_n, \neg\alpha_n\}$, which shows that $|\mathcal{F}| = 2n$ is an even number.



Figure 1. Two classical squares of opposition in (a) CPL and (b) KD.

The notion of Aristotelian isomorphism, which was first introduced in [24,28], is of crucial importance in logical geometry, because it allows us to start studying Aristotelian diagrams as objects of independent interest. More concretely, Aristotelian isomorphisms enable us to make abstraction of the specific formulas and logical system that define a given Aristotelian diagram, and thus, to say that two diagrams are “essentially the same”, despite being built from completely different fragments and logical systems.

Definition 3 (Aristotelian isomorphism). *Consider Aristotelian diagrams D for (\mathcal{F}, S) and D' for (\mathcal{F}', S') . An Aristotelian isomorphism from D to D' is a bijection $f: \mathcal{F} \rightarrow \mathcal{F}'$ such that, for all $\alpha, \beta \in \mathcal{F}$ and all Aristotelian relations $R_S \in \{CD_S, C_S, SC_S, SA_S\}$, it holds that $R_S(\alpha, \beta)$ iff $R_{S'}(f(\alpha), f(\beta)).$*

Upon visual inspection, it is immediately clear that function f , defined by $f(p \wedge q) := \Box p, f(p \vee q) := \Diamond p, f(\neg p \wedge \neg q) := \Box \neg p$ and $f(\neg p \vee \neg q) := \Diamond \neg p$, is an Aristotelian isomorphism between the two Aristotelian diagrams shown in Figure 1. For example, we have $SA_{CPL}(p \wedge q, p \vee q)$ and $SA_{KD}(f(p \wedge q), f(p \vee q))$, i.e., $SA_{KD}(\Box p, \Diamond p)$.

3. AD-Logic: The Modal Logic of Aristotelian Diagrams

In this section, we introduce the modal logic of Aristotelian diagrams, or *AD-logic* for short. We first introduce the language \mathcal{L}_{AD} , and then define its semantics in terms of AD-frames. Finally, we provide a sound and strongly complete axiomatization of AD-logic.

3.1. The Syntax of AD-Logic

We begin by defining the language \mathcal{L}_{AD} of AD-logic (Definition 4). This language is highly expressive: next to the “ordinary” modalities $[CD]$ and $[C]$, we introduce the global modality \Box and, especially, some key ingredients from hybrid logic: nominals, the @-operator, and the \downarrow -binder. (Note that although all details are provided below, we do presuppose that the reader has a basic level of familiarity with hybrid logic [29–32].)

Definition 4 (Language). *Let $\mathcal{I} = \{i_1, i_2, \dots\}$ be a countably infinite set of nominals and $\mathcal{V} = \{x_1, x_2, x_3, \dots\}$ be a countably infinite set of state variables. We use s as a metavariable ranging over $\mathcal{I} \cup \mathcal{V}$. The language \mathcal{L}_{AD} is defined by the following BNF:*

$$\varphi ::= s \mid \neg\varphi \mid \varphi \vee \varphi \mid [CD]\varphi \mid [C]\varphi \mid \Box\varphi \mid @_s\varphi \mid \downarrow x(\varphi)$$

The definitions of free and bound (state) variables are as in first-order logic, with \downarrow being the only binding operator.

We now make a number of comments about this language, ranging from technical to more philosophical in nature. We begin by discussing the primitive and defined operators of \mathcal{L}_{AD} . Given negation and disjunction, the other Boolean connectives are defined as usual. Given the modal operators $[CD]$, $[C]$ and \Box , their duals $\langle CD \rangle$, $\langle C \rangle$ and \Diamond are defined as usual. For our purposes later in this paper, it is useful to define two more operators: $\langle SC \rangle\varphi := \langle CD \rangle \langle C \rangle \langle CD \rangle\varphi$ and $\langle SA \rangle\varphi := \langle C \rangle \langle CD \rangle\varphi$; note that these definitions are directly inspired by Lemmas 2 and 3 of [26]. Finally, given the presence of the global modality in \mathcal{L}_{AD} , the @-operator can actually be defined as $@_s\varphi := \Box(s \rightarrow \varphi)$, or alternatively (using an equivalent formula) as $@_s\varphi := \Diamond(s \wedge \varphi)$ (see [31], p. 284). However, later in this paper, we will point out that we often do not have to rely on the global modality at all, and therefore, we follow the more traditional approach and add @ as a primitive operator to our language.

A somewhat peculiar feature of \mathcal{L}_{AD} is that it only contains nominals and state variables (which are true in a unique world), and there are no “ordinary” atomic propositions (which are typically true in multiple worlds). Using standard terminology from hybrid logic [29–32], this means that all formulas of \mathcal{L}_{AD} are, by definition, *pure*. Technically speaking, there is nothing that prevents us from taking such atomic propositions into account as well. However, since they are not needed for the specific purposes of this paper, and some definitions and results can be formulated more elegantly in their absence, we decided not to add “ordinary” atomic propositions to \mathcal{L}_{AD} .

In order to avoid a potentially dire confusion, one should keep in mind which purposes the language \mathcal{L}_{AD} (and, more generally, the system of AD-logic) is meant to serve and which ones it is not. More concretely, recall that Aristotelian diagrams were defined in Section 2 for pairs (\mathcal{F}, S) , and thus exist relative to some “underlying” logical system S (for example, CPL or KD; see Figure 1). Furthermore, formulas from the language \mathcal{L}_S of this underlying logic occur in the diagram for (\mathcal{F}, S) . By contrast, AD-logic does not serve as such an “underlying” logic of one particular diagram, but is rather meant to describe the behavior of Aristotelian diagrams in general. Similarly, formulas from \mathcal{L}_{AD} do not occur in Aristotelian diagrams, but are rather interpreted on them. As a notational reminder of this important distinction, throughout this paper we will systematically use (i) letters from

the *beginning* of the Greek alphabet (α, β, γ , etc.) as metavariables over the language \mathcal{L}_S of the underlying logic of some Aristotelian diagram (\mathcal{F}, S) and (ii) letters from the *end* of the Greek alphabet (φ, χ, ψ , etc.) as metavariables over the language \mathcal{L}_{AD} of AD-logic.

3.2. The Semantics of AD-Logic

We now introduce the semantics of AD-logic. Ultimately, the idea is that \mathcal{L}_{AD} -formulas can be interpreted on Aristotelian diagrams, but for now, we focus on AD-frames (Definition 5) and AD-models (Definition 6), which are a specific kind of birelational Kripke frames and models. The connection with actual Aristotelian diagrams will be discussed in detail in Section 4.

Definition 5 (AD-frame). *An AD-frame is a tuple $F = \langle W, R^{CD}, R^C \rangle$ such that*

1. W is a non-empty set, the elements of which are called “worlds” or “states”,
2. $R^{CD} \subseteq W \times W$ is a relation such that
 - (a) for all $w, w' \in W$, if $\langle w, w' \rangle \in R^{CD}$, then $\langle w', w \rangle \in R^{CD}$, i.e., R^{CD} is symmetric,
 - (b) for all $w \in W$, $\langle w, w \rangle \notin R^{CD}$, i.e., R^{CD} is irreflexive,
 - (c) for all $w \in W$ there is a $w' \in W$ such that $\langle w, w' \rangle \in R^{CD}$, i.e., R^{CD} is serial,
 - (d) for all $w, w', w'' \in W$, if $\langle w, w' \rangle, \langle w, w'' \rangle \in R^{CD}$, then $w' = w''$, i.e., R^{CD} is a partial function,
3. $R^C \subseteq W \times W$ is a relation such that
 - (a) for all $w, w' \in W$, if $\langle w, w' \rangle \in R^C$ then $\langle w', w \rangle \in R^C$, i.e., R^C is symmetric,
 - (b) for all $w \in W$, $\langle w, w \rangle \notin R^C$, i.e., R^C is irreflexive,
4.
 - (a) $R^{CD} \cap R^C = \emptyset$, i.e., R^{CD} and R^C are mutually exclusive,
 - (b) if $\langle w, w' \rangle, \langle w'', w''' \rangle \in R^{CD}$ and $\langle w, w'' \rangle \in R^C$, then $\langle w, w''' \rangle \notin R^C$ and $\langle w', w''' \rangle \notin R^C$,
 - (c) if $\langle w, w'' \rangle, \langle w', w''' \rangle \in R^C$ and $\langle w'', w''' \rangle \in R^{CD}$, then $\langle w, w' \rangle \in R^C$.

Definition 6 (AD-model). *An AD-model is a tuple $M = \langle F, V \rangle$ such that F is an AD-frame and $V: \mathcal{I} \rightarrow W$ is a valuation function that maps each nominal $s \in \mathcal{I}$ onto a state $V(s) \in W$.*

Conditions 2(c–d) of Definition 5 state that the relation R^{CD} can be viewed as a total function $R^{CD}: W \rightarrow W$, so instead of $R^{CD}(w, w')$, we can also write $R^{CD}(w) = w'$. Next, conditions 2(a–b) entail that this function is an involution without any fixed points, i.e., $R^{CD}(R^{CD}(w)) = w \neq R^{CD}(w)$ for all $w \in W$. Consequently, we can always list the elements of a finite AD-frame as $W = \{w_1, R^{CD}(w_1), w_2, R^{CD}(w_2), \dots, w_n, R^{CD}(w_n)\}$, which shows that $|W| = 2n$ is an even number. Needless to say, the fact that finite AD-frames have an even number of worlds corresponds exactly to the fact that finite Aristotelian diagrams have an even number of formulas (see Section 2). More generally, conditions 2(a)–4(c) are all motivated by the definition of Aristotelian diagrams, as will become more clear in Section 4. Finally, with respect to Definition 6, it should be noted that the valuation V only needs to interpret nominals (since there are no “ordinary” atomic propositions in \mathcal{L}_{AD}), and hence, we can simply take its codomain to be W . (Otherwise, we would have to take the codomain of V to be $\wp(W)$, while imposing the requirement that $|V(s)| = 1$ for nominals $s \in \mathcal{I}$, as is usually done in hybrid logic [29–31]).

We are now in a position to define the semantics of AD-logic. We introduce two more auxiliary notions (which are entirely standard from hybrid logic) in Definitions 7 and 8, and then formulate the semantic clauses for \mathcal{L}_{AD} in Definition 9.

Definition 7 (Assignment). *An assignment g on an AD-model $M = \langle \langle W, R^{CD}, R^C \rangle, V \rangle$ is a function $g: \mathcal{V} \rightarrow W$ that maps each state variable $s \in \mathcal{V}$ onto a state $g(s) \in W$.*

Definition 8 (*x*-variant). Consider an assignment g on an AD-model $M = \langle \langle W, R^{CD}, R^C \rangle, V \rangle$, a state variable $x \in \mathcal{V}$, and a world $w \in W$. The *x*-variant of g that maps x to w , denoted $g_w^x: \mathcal{V} \rightarrow W$, is the assignment defined by $g_w^x(x) = w$ and $g_w^x(s) = g(s)$ for all state variables $s \in \mathcal{V} \setminus \{x\}$.

Definition 9 (Semantic clauses). Let $M = \langle \langle W, R^{CD}, R^C \rangle, V \rangle$ be an AD-model, let g be an assignment on M , and let $[s]^{M,g}$ be $V(s)$ if $s \in \mathcal{I}$ and $g(s)$ if $s \in \mathcal{V}$. Then we define

$$\begin{aligned} M, g, w \models s & \quad \text{iff} \quad [s]^{M,g} = w, \\ M, g, w \models \neg\varphi & \quad \text{iff} \quad \text{not } M, g, w \models \varphi, \\ M, g, w \models \varphi \vee \psi & \quad \text{iff} \quad M, g, w \models \varphi \text{ or } M, g, w \models \psi, \\ M, g, w \models [CD]\varphi & \quad \text{iff} \quad \text{for all } v \text{ such that } \langle w, v \rangle \in R^{CD}: M, g, v \models \varphi, \\ M, g, w \models [C]\varphi & \quad \text{iff} \quad \text{for all } v \text{ such that } \langle w, v \rangle \in R^C: M, g, v \models \varphi, \\ M, g, w \models \Box\varphi & \quad \text{iff} \quad \text{for all } v \in W: M, g, v \models \varphi, \\ M, g, w \models @_s\varphi & \quad \text{iff} \quad M, g, [s]^{M,g} \models \varphi, \\ M, g, w \models \downarrow x(\varphi) & \quad \text{iff} \quad M, g_w^x, w \models \varphi. \end{aligned}$$

It is now entirely standard to define the notions of AD-frame validity (Definition 10), AD-validity (Definition 11) and AD-consequence (Definition 12). Upon unpacking the definitions, it is easy to see that AD-validity is a special case of AD-consequence if we take the premise set to be empty (formally: $\models_{AD} \varphi$ iff $\emptyset \models_{AD} \varphi$). Nevertheless, we choose to define these two notions in two separate definitions, because they will be important later in this paper for distinct reasons.

Definition 10 (Validity on an AD-frame). The formula $\varphi \in \mathcal{L}_{AD}$ is said to be valid on the AD-frame $F = \langle W, R^{CD}, R^C \rangle$, written as $F \models \varphi$, iff for every valuation $V: \mathcal{I} \rightarrow W$ on F , every assignment $g: \mathcal{V} \rightarrow W$ on $\langle F, V \rangle$ and every world $w \in W$, it holds that $\langle F, V \rangle, g, w \models \varphi$.

Definition 11 (AD-validity). The formula $\varphi \in \mathcal{L}_{AD}$ is said to be AD-valid, written as $\models_{AD} \varphi$, iff for every AD-frame F , it holds that $F \models \varphi$.

Definition 12 (AD-consequence). Given formulas $\Psi \cup \{\varphi\} \subseteq \mathcal{L}_{AD}$, we say that φ is an AD-consequence of Ψ , written as $\Psi \models_{AD} \varphi$, iff for every AD-model M , every assignment g on M and every world w in M , it holds that if $M, g, w \models \psi$ for every $\psi \in \Psi$, then $M, g, w \models \varphi$.

3.3. Frame Correspondence and Completeness

Since this paper is strongly semantically driven, we will not spend much time on proof theory. However, we will show how a sound and strongly complete axiomatization of AD-logic can be obtained directly from general results about hybrid logic. We start by stating frame correspondences for all the defining properties of AD-frames. In Lemma 1 below, expressions such as “2(a)” and “4(c)” refer to the conditions mentioned in Definition 5.

Lemma 1. Let $F = \langle W, R^{CD}, R^C \rangle$ be an arbitrary birelational Kripke frame.

- F satisfies condition 2(a) (i.e., symmetry of R^{CD}) iff $F \models @_i \langle CD \rangle j \rightarrow @_j \langle CD \rangle i$,
- F satisfies condition 2(b) (i.e., irreflexivity of R^{CD}) iff $F \models @_i \neg \langle CD \rangle i$,
- F satisfies condition 2(c) (i.e., seriality of R^{CD}) iff $F \models \langle CD \rangle \top$,
- F satisfies condition 2(d) (i.e., partial functionality of R^{CD}) iff $F \models (\langle CD \rangle i \wedge \langle CD \rangle j) \rightarrow @_i j$,
- F satisfies condition 3(a) (i.e., symmetry of R^C) iff $F \models @_i \langle C \rangle j \rightarrow @_j \langle C \rangle i$,
- F satisfies condition 3(b) (i.e., irreflexivity of R^C) iff $F \models @_i \neg \langle C \rangle i$,
- F satisfies condition 4(a) iff $F \models @_i \langle CD \rangle j \rightarrow @_i \neg \langle C \rangle j$,
- F satisfies condition 4(b) iff $F \models (@_i \langle CD \rangle j \wedge @_k \langle CD \rangle \ell \wedge @_i \langle C \rangle k) \rightarrow (\neg @_i \langle C \rangle \ell \wedge \neg @_j \langle C \rangle \ell)$,
- F satisfies condition 4(c) iff $F \models (@_i \langle C \rangle j \wedge @_k \langle C \rangle \ell \wedge @_j \langle CD \rangle \ell) \rightarrow @_i \langle C \rangle k$.

Proof. All items are easy to check; many of them are standard frame correspondence results from hybrid logic. \square

Let Ψ_{AD} be the set consisting of all nine \mathcal{L}_{AD} -formulas that are mentioned in Lemma 1. Together with the formula $\diamond i$ (which fully captures the behavior of the global modality \diamond ; see [31], p. 284), this yields a sound and strongly complete axiomatization of AD-logic.

Lemma 2. *A birelational Kripke frame F is an AD-frame iff for every $\psi \in \Psi_{AD}$, $F \models \psi$.*

Proof. This follows immediately from Lemma 1 and Definition 5. \square

Theorem 1. *Let $\mathbf{K}_{\mathcal{H}(\@, \downarrow)}$ be the standard axiomatization of hybrid logic with $\@$ and \downarrow -binder (see [30], p. 833 and [31], p. 304). Then, $\mathbf{K}_{\mathcal{H}(\@, \downarrow)} + \Psi_{AD} \cup \{\diamond i\}$ is sound and strongly complete for the class of AD-frames.*

Proof. Soundness is straightforward. Strong completeness follows from Theorem 5 of [30], p. 833 (also see Theorem 5 of [31], p. 304), since all formulas in $\Psi_{AD} \cup \{\diamond i\}$ are pure, $\diamond i$ is the sole axiom governing the global modality (see [31], p. 284), and Ψ_{AD} defines the class of AD-frames (see Lemma 2). \square

We finish this section with a more general, methodological reflection. At first sight, it might look like the system of AD-logic is quite ad hoc, being a combination of ingredients that have been haphazardly put together with the sole purpose of capturing the intuitive idea of “viewing Aristotelian diagrams as Kripke frames”. However, we hope that throughout this section, it has become clear that AD-logic is firmly rooted in the vast tradition of modal logic research. This includes, most importantly, the notions of nominals, the $\@$ -operator and the \downarrow -binder from hybrid logic [29–32], but also the study of global modalities [33,34] and functional accessibility relations [35,36]. Furthermore, in relational semantics for relevance logic, one of the accessibility relations is an involutive function (usually called the “Routley star” [37–39]), just like R^{CD} is an involutive function in AD-frames.

4. The Relation between AD-Logic and Aristotelian Diagrams

At this point, there might seem to be a subtle mismatch between the informal motivation for this paper, as laid out in Section 1, and the technical development of AD-logic in Section 3. After all, the formulas of \mathcal{L}_{AD} are interpreted on AD-frames and AD-models (see Definitions 9–12), whereas the fundamental intuition was that we wanted to interpret such formulas on the Aristotelian diagrams *themselves*. The distinction is a subtle one, as AD-frames and Aristotelian diagrams are clearly related to each other; for example, the notations R^{CD} and R^C for the relations in an AD-frame are highly suggestive, and the same holds for the parallel observations regarding even numbers of worlds/formulas in finite AD-frames/Aristotelian diagrams. In this section, we will close this gap altogether. We first introduce the notion of an AD-frame being “based on” a given Aristotelian diagram, and then show that there exists a bijection between finite Aristotelian diagrams (up to Aristotelian isomorphism) and finite AD-frames (up to modal isomorphism). (The restriction to finite diagrams/AD-frames is needed for technical reasons; however, this restriction is not very significant, since all Aristotelian diagrams and AD-frames that we will be working with later in the paper are finite anyway).

4.1. AD-Frames Based on Aristotelian Diagrams

We start by introducing the idea of an AD-frame being *based on* an Aristotelian diagram. We first formulate this in terms of arbitrary birelational Kripke frames, and then prove that all such frames are indeed AD-frames (see Lemma 3).

Definition 13 (Birelational Kripke frame based on an Aristotelian diagram). *Consider an Aristotelian diagram D for (\mathcal{F}, S) . A birelational Kripke frame $F = \langle W, R^{CD}, R^C \rangle$ is said to be based on D iff there exists a bijection $h: W \rightarrow \mathcal{F}$ such that for all $w, w' \in W$:*

1. $\langle w, w' \rangle \in R^{CD}$ iff $CD_S(h(w), h(w'))$,

$$2. \quad \langle w, w' \rangle \in R^C \text{ iff } C_S(h(w), h(w')).$$

Note that all components of the Kripke frame F and the Aristotelian diagram D play key roles in Definition 13: the sets W and \mathcal{F} are the domain and codomain of the bijection h , while conditions 1 and 2 specify a correspondence between the relations R^{CD} and R^C of the Kripke frame F and the actual Aristotelian relations that hold among the formulas of D , relative to the logical system S (see Definition 1). Additionally, note that because of its bijective nature and the presence of conditions 1 and 2, the function h from Definition 13 looks very much like an Aristotelian isomorphism (see Definition 3); the only difference is that h connects an AD-frame with an Aristotelian diagram, whereas an Aristotelian isomorphism is defined between two Aristotelian diagrams. (One point of difference is that h is concerned with just *two* relations, R^{CD} and R^C , whereas an Aristotelian isomorphism is concerned with all *four* Aristotelian relations. However, this difference is very minor, since SC_S and SA_S can be defined in terms of CD_S and C_S ; see Lemmas 2 and 3 of [26]).

We now prove that if a birelational Kripke frame is based on an Aristotelian diagram, then it is an AD-frame. This is exactly as it should be, and justifies our terminological choice for the word “AD-frame”.

Lemma 3. *Let $F = \langle W, R^{CD}, R^C \rangle$ be a birelational Kripke frame that is based on an Aristotelian diagram D for (\mathcal{F}, S) . Then F is an AD-frame.*

Proof. Since F is based on D , there exists a bijection $h: W \rightarrow \mathcal{F}$ that satisfies the two conditions mentioned in Definition 13. In order to show that F is an AD-frame, it suffices to check that it satisfies all the conditions of Definition 5. All these conditions follow, via the bijection h , from Definitions 1 and 2. For example, R^C is symmetric because the Aristotelian relation of S -contrariety is symmetric; in detail: for all $w, w' \in W$, we have $\langle w, w' \rangle \in R^C$ iff $C_S(h(w), h(w'))$ iff $\models_S \neg(h(w) \wedge h(w'))$ and $\not\models_S h(w) \vee h(w')$ iff $\models_S \neg(h(w') \wedge h(w))$ and $\not\models_S h(w') \vee h(w)$ iff $C_S(h(w'), h(w))$ iff $\langle w', w \rangle \in R^C$.

For a more elaborate example, we check that condition 4(c) of Definition 5 holds. Consider arbitrary worlds $w, w', w'', w''' \in W$ and suppose that $\langle w, w'' \rangle, \langle w', w''' \rangle \in R^C$ and $\langle w'', w''' \rangle \in R^{CD}$. It follows that $C_S(h(w), h(w''))$ and $C_S(h(w'), h(w'''))$ and $CD_S(h(w''), h(w'''))$. By the definition of CD_S and C_S , it follows that $\models_S h(w) \rightarrow \neg h(w'')$ and $\models_S \neg h(w'') \rightarrow h(w''')$ and $\models_S h(w''') \rightarrow \neg h(w')$, and hence, by transitivity, $\models_S h(w) \rightarrow \neg h(w')$, i.e., $\models_S \neg(h(w) \wedge h(w'))$ (†). Furthermore, it follows from $C_S(h(w), h(w''))$ that $\not\models_S h(w) \vee h(w'')$ and, hence, there exists an S -model M such that $M \not\models h(w)$ and $M \not\models h(w'')$, i.e., $M \models \neg h(w'')$. From $CD(h(w''), h(w'''))$ and $C(h(w'), h(w'''))$, it follows that $\models_S \neg h(w'') \rightarrow h(w''')$ and $\models_S h(w''') \rightarrow \neg h(w')$ and, hence, by transitivity, $\models_S \neg h(w'') \rightarrow \neg h(w')$. Since $M \models \neg h(w'')$, it follows that $M \models \neg h(w')$, i.e., $M \not\models h(w')$. Since $M \not\models h(w)$ and $M \not\models h(w')$, it follows that $M \not\models h(w) \vee h(w')$ and, hence, $\not\models_S h(w) \vee h(w')$ (‡). From (†) and (‡), it follows that $C_S(h(w), h(w'))$ and, hence, $\langle w, w' \rangle \in R^C$, as required. \square

We are now ready to entirely close the gap between Aristotelian diagrams and AD-frames. In particular, validity on an Aristotelian diagram D can straightforwardly be defined in terms of validity on all AD-frames that are based on D (see Definition 10).

Definition 14 (Validity on an Aristotelian diagram). *The formula $\varphi \in \mathcal{L}_{AD}$ is said to be valid on an Aristotelian diagram D , written as $\models_D \varphi$, iff for every AD-frame F that is based on D , it holds that $F \models \varphi$.*

This notion of “diagrammatic validity” will play a crucial role in Sections 5 and 6 later in the paper. However, in the remainder of the current section, we will first investigate the relationship between Aristotelian diagrams and AD-frames in more detail.

4.2. From Aristotelian Diagrams to AD-Frames

We have just introduced the notion of a birelational Kripke frame being *based* on a given Aristotelian diagram, and shown that such frames are always AD-frames. However, there are two questions that have not been answered thus far. First of all, given an Aristotelian diagram D , we can ask whether there always *exists* an AD-frame based on D . Secondly, we can ask to what extent the AD-frames based on a given Aristotelian diagram are *unique*. In order to provide an affirmative answer to the first question, we define the canonical frame F_D for a given Aristotelian diagram D .

Definition 15. Consider an Aristotelian diagram D for (\mathcal{F}, S) . The canonical frame for D is the birelational Kripke frame $F_D = \langle W, R^{CD}, R^C \rangle$, where $W := \mathcal{F}$, $R^{CD} := CD_S \cap (\mathcal{F} \times \mathcal{F})$ and $R^C := C_S \cap (\mathcal{F} \times \mathcal{F})$.

Lemma 4. Consider an Aristotelian diagram D for (\mathcal{F}, S) . Then the canonical frame F_D is based on D , and is thus an AD-frame.

Proof. To see that F_D is based on D , it suffices to note that the identity function $id_{\mathcal{F}}$ has all the required properties to fulfill the role of h in Definition 13. By Lemma 3, it immediately follows that F_D is an AD-frame. \square

Since an Aristotelian diagram D for (\mathcal{F}, S) yields an AD-frame F_D “via” the identity function $id_{\mathcal{F}}$, there is a potentially confusing situation that needs our careful consideration. After all, given a formula $\alpha \in \mathcal{F}$ and a formula φ of AD-logic, it now makes sense to ask whether (for any valuation V on F_D and any assignment g on $\langle F_D, V \rangle$) we have $\langle F_D, V \rangle, g, \alpha \models \varphi$, or, using a common abbreviation, whether $\alpha \models \varphi$. However, this latter expression does *not* mean that we are dealing with a valid argument with a single premise α and conclusion φ , but rather, that the formula $\varphi \in \mathcal{L}_{AD}$ is true at the “formula-viewed-as-a-world” $\alpha \in \mathcal{L}_S$ (recall our notational convention regarding metavariables for \mathcal{L}_S and \mathcal{L}_{AD}). For example, given the classical square of opposition for $(\{\Box p, \Diamond p, \Box \neg p, \Diamond \neg p\}, \text{KD})$ that was already shown in Figure 1b, we can say that $\Box p \models \langle C \rangle \langle CD \rangle \top$, since $C_{\text{KD}}(\Box p, \Box \neg p)$ and $CD_{\text{KD}}(\Box \neg p, \Diamond p)$ and $\Diamond p \models \top$. Furthermore, note that $\Box p \models \langle C \rangle \langle CD \rangle \top$ does *not* mean that the formula $\Box p$ is contrary to the formula $\langle CD \rangle \top$: this would involve a category mistake, since $\Box p$ belongs to the basic modal language \mathcal{L}_{\Box} , whereas $\langle CD \rangle \top$ belongs to \mathcal{L}_{AD} . (Formally speaking, there is no overarching logic that $\Box p$ and $\langle CD \rangle \top$ both belong to, and with respect to which they could be said to be contrary to each other.)

We now turn to the second question about AD-frames based on a given Aristotelian diagram. It is easy to see that two distinct AD-frames can be based on the same Aristotelian diagram. For example, the three frames (i) $\langle \{w, w'\}, \{ \langle w, w' \rangle, \langle w', w \rangle \}, \emptyset \rangle$, (ii) $\langle \{w, v\}, \{ \langle w, v \rangle, \langle v, w \rangle \}, \emptyset \rangle$ and (iii) $\langle \{p, \neg p\}, \{ \langle p, \neg p \rangle, \langle \neg p, p \rangle \}, \emptyset \rangle$ are all nominally distinct from each other, yet all three are based on the diagram for $(\{p, \neg p\}, \text{CPL})$ (recall that CPL is short for classical propositional logic). We thus cannot hope for there to be a *unique* AD-frame based on a given Aristotelian diagram; however, as the example above already suggests, we can show that AD-frames based on a given Aristotelian diagram are unique *up to modal isomorphism*. In order to formulate this correctly, we first recall the notion of modal isomorphism (which is entirely standard from modal logic; see [40], p. 58) in Definition 16. Furthermore, we will actually prove a slightly stronger result: if D_1 and D_2 are two Aristotelian diagrams that are Aristotelian isomorphic to each other, and F_1 and F_2 are AD-frames that are based on resp. D_1 and D_2 , then F_1 and F_2 are modally isomorphic to each other.

Definition 16. Consider AD-frames $F_1 = \langle W_1, R_1^{CD}, R_1^C \rangle$ and $F_2 = \langle W_2, R_2^{CD}, R_2^C \rangle$. A modal isomorphism between these two frames is a bijection $g : W_1 \rightarrow W_2$ such that for all $w, w' \in W$, it holds that $\langle w, w' \rangle \in R_1^{CD}$ iff $\langle g(w), g(w') \rangle \in R_2^{CD}$ and $\langle w, w' \rangle \in R_1^C$ iff $\langle g(w), g(w') \rangle \in R_2^C$.

Lemma 5. Consider Aristotelian diagrams D_1 for (\mathcal{F}_1, S_1) and D_2 for (\mathcal{F}_2, S_2) , and consider an AD-frame $F_1 = \langle W_1, R_1^{CD}, R_1^C \rangle$ based on D_1 and an AD-frame $F_2 = \langle W_2, R_2^{CD}, R_2^C \rangle$ based on D_2 . If D_1 and D_2 are Aristotelian isomorphic, then F_1 and F_2 are modally isomorphic.

Proof. For $i = 1, 2$, since F_i is based on D_i , there exists a bijection $h_i: W_i \rightarrow \mathcal{F}_i$ that satisfies the requirements of Definition 13. Furthermore, since D_1 and D_2 are Aristotelian isomorphic to each other, there exists a bijection $f: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ that satisfies the requirements of Definition 3. We now show that $h_2^{-1} \circ f \circ h_1: W_1 \rightarrow W_2$ is a modal isomorphism.

First of all, note that since $h_2: W_2 \rightarrow \mathcal{F}_2$ is a bijection, the function $h_2^{-1}: \mathcal{F}_2 \rightarrow W_2$ exists and is a bijection as well. Furthermore, instead of saying that for all $w, w' \in W_2$, $\langle w, w' \rangle \in R_2^{CD}$ iff $CD_{S_2}(h_2(w), h_2(w'))$ (see Definition 13), we can equivalently say that for all $\alpha, \alpha' \in \mathcal{F}_2$, $\langle h_2^{-1}(\alpha), h_2^{-1}(\alpha') \rangle \in R_2^{CD}$ iff $CD_{S_2}(\alpha, \alpha')$. (Exactly the same remark applies to R_2^C and C_{S_2} .) The function $h_2^{-1} \circ f \circ h_1: W_1 \rightarrow W_2$ is a composition of three bijections and is itself thus also a bijection. Furthermore, for all $w, w' \in W_1$, we have the following chain of equivalences:

$$\begin{aligned} & \langle (h_2^{-1} \circ f \circ h_1)(w), (h_2^{-1} \circ f \circ h_1)(w') \rangle \in R_2^{CD} \\ \text{iff} & \langle h_2^{-1}(f(h_1(w))), h_2^{-1}(f(h_1(w')))) \rangle \in R_2^{CD} && \text{(def. of functional composition)} \\ \text{iff} & CD_{S_2}(f(h_1(w)), f(h_1(w')))) && \text{(by the remark about } h_2^{-1} \text{ above)} \\ \text{iff} & CD_{S_1}(h_1(w), h_1(w')) && \text{(since } f \text{ is an Arist. isomorphism)} \\ \text{iff} & \langle w, w' \rangle \in R_1^{CD} && \text{(by the properties of } h_1). \end{aligned}$$

In exactly the same way, we can show that $\langle (h_2^{-1} \circ f \circ h_1)(w), (h_2^{-1} \circ f \circ h_1)(w') \rangle \in R_2^C$ iff $\langle w, w' \rangle \in R_1^C$. This shows that $h_2^{-1} \circ f \circ h_1: W_1 \rightarrow W_2$ is a modal isomorphism. \square

We are now in a position to define a function λ from finite Aristotelian diagrams (up to Aristotelian isomorphism) to finite AD-frames (up to modal isomorphism). We introduce some auxiliary notions in Definition 17, and the actual function λ in Definition 18. It is quite easy to show that λ is injective, which we do in Lemma 6.

Definition 17. For each Aristotelian diagram D and AD-frame F , we define

- $[D]_{AI} := \{D' \mid D' \text{ is an Aristotelian diagram that is Aristotelian isomorphic to } D\}$,
- $[F]_{MI} := \{F' \mid F' \text{ is an AD-frame that is modally isomorphic to } F\}$.

Furthermore, we define

- $\mathbf{D} := \{[D]_{AI} \mid D \text{ is a finite Aristotelian diagram}\}$,
- $\mathbf{F} := \{[F]_{MI} \mid F \text{ is a finite AD-frame}\}$.

Definition 18. The function $\lambda: \mathbf{D} \rightarrow \mathbf{F}$ is defined by putting $\lambda([D]_{AI}) := [F_D]_{MI}$, for every $[D]_{AI} \in \mathbf{D}$. Note that this is a well-defined function because of Lemmas 4 and 5: if D_1 and D_2 are Aristotelian isomorphic, then F_{D_1} and F_{D_2} are modally isomorphic.

Lemma 6. The function λ is injective.

Proof. Consider arbitrary Aristotelian diagrams D_1 for (\mathcal{F}_1, S_1) and D_2 for (\mathcal{F}_2, S_2) and suppose that $[F_{D_1}]_{MI} = \lambda([D_1]_{AI}) = \lambda([D_2]_{AI}) = [F_{D_2}]_{MI}$, i.e., F_{D_1} and F_{D_2} are modally isomorphic. We show that $[D_1]_{AI} = [D_2]_{AI}$, i.e., D_1 and D_2 are Aristotelian isomorphic. Since F_{D_1} and F_{D_2} are modally isomorphic, there exists a modal isomorphism $f: \mathcal{F}_1 \rightarrow \mathcal{F}_2$. For all $\alpha, \beta \in \mathcal{F}_1$, we have

$$\begin{aligned} CD_{S_1}(\alpha, \beta) & \text{ iff } \langle \alpha, \beta \rangle \in R_1^{CD} && \text{(by Definition 15)} \\ & \text{ iff } \langle f(\alpha), f(\beta) \rangle \in R_2^{CD} && \text{(by Definition 16)} \\ & \text{ iff } CD_{S_2}(f(\alpha), f(\beta)) && \text{(by Definition 15)}. \end{aligned}$$

Similarly, we can show that $C_{S_1}(\alpha, \beta)$ iff $C_{S_2}(f(\alpha), f(\beta))$. Furthermore, since subcontrariety and subalternation can be defined in terms of contradiction and contrariety (recall Lemmas 2 and 3

from [26]), we also get $SC_{S_1}(\alpha, \beta)$ iff $SC_{S_2}(f(\alpha), f(\beta))$ and $SA_{S_1}(\alpha, \beta)$ iff $SA_{S_2}(f(\alpha), f(\beta))$. This shows that f is an Aristotelian isomorphism between D_1 and D_2 , as desired. \square

Note that the proof of Lemma 6 does not rely on the fact that \mathbf{D} and \mathbf{F} contain only finite Aristotelian diagrams and AD-frames. This stands in sharp contrast with the proof of Lemma 8 below, which deals with the surjectivity of λ .

4.3. From AD-Frames to Aristotelian Diagrams

In this subsection, we finish our investigation of the relation between AD-frames and Aristotelian diagrams, by showing that the function λ defined above is not only injective, but also surjective. This proof will be a bit more involved, so we first introduce the auxiliary notion of (maximal) F -consistency in Definition 19 and prove Lemma 7. The latter is a Lindenbaum-type lemma, which states that every F -consistent set can be extended to a maximally F -consistent set. Finally, it bears emphasizing that the proof of Lemma 8 is closely related to the algorithm described in Section 4 of [24].

Definition 19. Let $F = \langle W, R^{CD}, R^C \rangle$ be an AD-frame. A set $X \subseteq W$ is said to be F -consistent iff there are no $w, v \in X$ such that $\langle w, v \rangle \in R^{CD}$ or $\langle w, v \rangle \in R^C$. Furthermore, X is said to be maximally F -consistent iff (i) X is F -consistent and (ii) for all $w \in W \setminus X$, the set $X \cup \{w\}$ is not F -consistent. Finally, $\Pi(F)$ is defined to be the set of all maximally F -consistent subsets of W .

Lemma 7. Let $F = \langle W, R^{CD}, R^C \rangle$ be an AD-frame. For every F -consistent set $X \subseteq W$, there exists a maximally F -consistent set $A \in \Pi(F)$ such that $X \subseteq A$.

Proof. Let \mathcal{P} be the collection of all sets $S \subseteq W$ such that $X \subseteq S$ and S is F -consistent. Note that \mathcal{P} is non-empty (since $X \in \mathcal{P}$) and that \mathcal{P} is partially ordered by \subseteq . We now prove that every non-empty chain in the poset \mathcal{P} has an upper bound in \mathcal{P} . Consider an arbitrary chain \mathcal{C} in \mathcal{P} . Clearly, $\bigcup \mathcal{C}$ is an upper bound for \mathcal{C} , so it suffices to show that $\bigcup \mathcal{C} \in \mathcal{P}$, i.e., (i) $X \subseteq \bigcup \mathcal{C}$ and (ii) $\bigcup \mathcal{C}$ is F -consistent. Regarding (i), since \mathcal{C} is non-empty, there exists an $S \in \mathcal{C} \subseteq \mathcal{P}$, and hence $X \subseteq S \subseteq \bigcup \mathcal{C}$. Regarding (ii), suppose, toward a reductio, that $\bigcup \mathcal{C}$ is not F -consistent. Hence, there exist $x, y \in \bigcup \mathcal{C}$ such that $\langle x, y \rangle \in R^{CD}$ or $\langle x, y \rangle \in R^C$. Since $x, y \in \bigcup \mathcal{C}$, there exist $C_x, C_y \in \mathcal{C}$ such that $x \in C_x$ and $y \in C_y$. Since \mathcal{C} is a chain, we have $C_x \subseteq C_y$ or $C_y \subseteq C_x$. Consider the case $C_x \subseteq C_y$ (the case $C_y \subseteq C_x$ is analogous). It then follows that $x, y \in C_y$, and thus C_y is not F -consistent, which contradicts $C_y \in \mathcal{C} \subseteq \mathcal{P}$.

We have now checked that every non-empty chain in the poset \mathcal{P} has an upper bound in \mathcal{P} . By Zorn’s lemma, it follows that \mathcal{P} has at least one maximal element, A . Since $A \in \mathcal{P}$, we have, by definition of \mathcal{P} , that $X \subseteq A$ and that A is F -consistent. To show that $A \in \Pi(F)$, it thus suffices to check that A is maximal. Toward a reductio, assume that there exists some $w \in W \setminus A$ such that $A \cup \{w\}$ is F -consistent. Since $X \subseteq A \subseteq A \cup \{w\}$ and $A \cup \{w\}$ is F -consistent, we have $A \cup \{w\} \in \mathcal{P}$. However, since $A \subsetneq A \cup \{w\}$, this contradicts the fact that A is a maximal element of \mathcal{P} . \square

Lemma 8. The function $\lambda: \mathbf{D} \rightarrow \mathbf{F}$ is surjective.

Proof. Consider an arbitrary finite AD-frame $F = \langle W, R^{CD}, R^C \rangle$. We will show that there exists an Aristotelian diagram D such that $[F]_{MI} = \lambda([D]_{AI}) = [F_D]_{MI}$, i.e., such that F is modally isomorphic to F_D . In order to specify D , we have to specify a logical system S and a fragment $\mathcal{F} \subseteq \mathcal{L}_S$. For our logical system, we take classical propositional logic, CPL, while for \mathcal{F} , we take the image of W under the function f , which is defined below.

First of all, note that $|\Pi(F)| \leq 2^{|W|}$, so since F is finite, it follows that $\Pi(F)$ is finite as well. Now fix an ordering on the elements of $\Pi(F)$: $A_1, A_2, \dots, A_{|\Pi(F)|}$. Secondly, we consider a partition Π^{CPL} of CPL consisting of $|\Pi(F)|$ formulas: $\Pi^{CPL} := \{\alpha_1, \alpha_2, \dots, \alpha_{|\Pi(F)|}\}$. The set Π^{CPL} is called a “partition of CPL”, because its elements are mutually exclusive (i.e., $\models_{CPL} \neg(\alpha_i \wedge \alpha_j)$ for distinct $\alpha_i, \alpha_j \in \Pi^{CPL}$) and jointly exhaustive (i.e., $\models_{CPL} \bigvee \Pi^{CPL}$). It is easy to see that such a partition Π^{CPL} always exists; see below for a concrete example

and see the proof of Theorem 4 in [28] for a general construction method. For each $w \in W$, we define

$$f(w) := \bigvee \{ \alpha_i \in \Pi^{\text{CPL}} \mid w \in A_i \}.$$

Now let $f[W] := \{ f(w) \mid w \in W \}$, and consider the Aristotelian diagram D for $(f[W], \text{CPL})$. (This can also be formulated using the well-known representation format of bitstrings [24,28,41]: define $\beta(w) \in \{0, 1\}^{|\Pi(F)|}$ to be the bitstring that has a 1 in its i^{th} bit position iff $w \in A_i$.) We will show that f is a modal isomorphism from F to F_D .

We first check that f is bijective. Surjectivity follows immediately from the definition of f . For injectivity, consider arbitrary $w, v \in W$ and suppose that $w \neq v$. We will show that $\bigvee \{ \alpha_i \in \Pi^{\text{CPL}} \mid w \in A_i \} = f(w) \neq f(v) = \bigvee \{ \alpha_i \in \Pi^{\text{CPL}} \mid v \in A_i \}$, by showing that there exists an $A_i \in \Pi(F)$ such that $w \in A_i$ iff $v \notin A_i$. By the seriality of R^{CD} , there exist $w', v' \in W$ such that $\langle w, w' \rangle, \langle v, v' \rangle \in R^{CD}$. Now consider the following case distinction:

- The set $\{w, v'\} \subseteq W$ is F -consistent.
By Lemma 7, there exists some $A_i \in \Pi(F)$ such that $\{w, v'\} \subseteq A_i$. We have $w \in A_i$ but $v \notin A_i$ (because A_i is F -consistent, $v' \in A_i$ and $\langle v, v' \rangle \in R^{CD}$), as desired.
- The set $\{w', v\} \subseteq W$ is F -consistent.
Analogously to the previous case, we can now show that there exists some $A_i \in \Pi(F)$ such that $w \notin A_i$ but $v \in A_i$, as desired.
- Neither $\{w, v'\}$ nor $\{w', v\}$ is F -consistent.
We will show that this case cannot occur. Since $\{w, v'\}$ is not F -consistent, we have $\langle w, v' \rangle \in R^{CD}$ or $\langle w, v' \rangle \in R^C$. If $\langle w, v' \rangle \in R^{CD}$, then it would follow by the symmetry and functionality of R^{CD} that $w = v$, which contradicts our assumption that $w \neq v$. Hence, $\langle w, v' \rangle \in R^C$. Similarly, since $\{w', v\}$ is not F -consistent, it follows that $\langle w', v \rangle \in R^C$. However, from $\langle w, w' \rangle, \langle v, v' \rangle \in R^{CD}$ and $\langle w, v' \rangle \in R^C$, it follows by condition 4(b) of Definition 5 that $\langle w', v \rangle \notin R^C$.

We now check that for all $w, v \in W$, we have $\langle w, v \rangle \in R^{CD}$ iff $CD_{\text{CPL}}(f(w), f(v))$. For the left to right direction, suppose that $\langle w, v \rangle \in R^{CD}$. By the definition of CD_{CPL} , it suffices to prove the following two claims:

- $\models_{\text{CPL}} \neg(f(w) \wedge f(v))$.
Consider an arbitrary CPL-model M and suppose, toward a contradiction, that $M \models f(w) \wedge f(v)$. Since $M \models f(w) = \bigvee \{ \alpha_i \in \Pi^{\text{CPL}} \mid w \in A_i \}$, there exists $1 \leq i \leq |\Pi(F)|$ such that $M \models \alpha_i$ and $w \in A_i$. Similarly, since $M \models f(v) = \bigvee \{ \alpha_j \in \Pi^{\text{CPL}} \mid v \in A_j \}$, there exists $1 \leq j \leq |\Pi(F)|$ such that $M \models \alpha_j$ and $v \in A_j$. Since α_i and α_j are members of a partition of CPL, it follows from $M \models \alpha_i$ and $M \models \alpha_j$ that $i = j$. Hence, we have $w \in A_i$ and $v \in A_j = A_i$, which, together with $\langle w, v \rangle \in R^{CD}$, contradicts the F -consistency of A_i .
- $\models_{\text{CPL}} f(w) \vee f(v)$.
We have to show that $\models_{\text{CPL}} \bigvee \{ \alpha_i \in \Pi^{\text{CPL}} \mid w \in A_i \} \vee \bigvee \{ \alpha_i \in \Pi^{\text{CPL}} \mid v \in A_i \}$, i.e., that $\models_{\text{CPL}} \bigvee \{ \alpha_i \in \Pi^{\text{CPL}} \mid w \in A_i \text{ or } v \in A_i \}$. Since $\models_{\text{CPL}} \bigvee \Pi^{\text{CPL}}$, it suffices to show that for each $A_i \in \Pi(F)$, we have $w \in A_i$ or $v \in A_i$.
Consider some $A_i \in \Pi(F)$ and suppose that $w \notin A_i$. We will show that $v \in A_i$. Since A_i is maximal, it follows from $w \notin A_i$ that $A_i \cup \{w\}$ is not F -consistent, i.e., there exist $x, y \in A_i \cup \{w\}$ such that $\langle x, y \rangle \in R^{CD}$ or $\langle x, y \rangle \in R^C$. Now, if x and y both come from A_i , then A_i itself would be F -inconsistent. Furthermore, if x and y both come from $\{w\}$, then $x = w = y$, which contradicts the irreflexivity of R^{CD} and R^C . So without loss of generality, we assume that $x \in A_i$ and $y = w$, and thus we have $\langle x, w \rangle \in R^{CD}$ or $\langle x, w \rangle \in R^C$.
In the first case, i.e., $\langle x, w \rangle \in R^{CD}$, it follows by the symmetry and functionality of R^{CD} that $x = v$. Since $x \in A_i$, we thus have $v \in A_i$, as desired. We now turn to the second case: $\langle x, w \rangle \in R^C$. Toward a contradiction, assume $v \notin A_i$. Since A_i is maximal, it follows that $A_i \cup \{v\}$ is not F -consistent, i.e., there exist $y, z \in A_i \cup \{v\}$ such that $\langle y, z \rangle \in R^{CD}$ or $\langle y, z \rangle \in R^C$. Reasoning as before, we again assume, without loss of generality, that $y \in A_i$ and $z = v$, and thus we have $\langle y, v \rangle \in R^{CD}$ or $\langle y, v \rangle \in R^C$.

In the first subcase, i.e., $\langle y, v \rangle \in R^{CD}$, it follows by the symmetry and functionality of R^{CD} that $y = w$. Since $y \in A_i$, we thus have $w \in A_i$, which contradicts our assumption that $w \notin A_i$. In the second subcase, i.e., $\langle y, v \rangle \in R^C$, it follows by condition 4(c) from Definition 5 that $\langle x, y \rangle \in R^C$. However, since $x, y \in A_i$, this contradicts the F -consistency of A_i .

For the right to left direction, suppose that $CD_{CPL}(f(w), f(v))$ and, toward a contradiction, that $\langle w, v \rangle \notin R^{CD}$. We make the following case distinction:

- $\langle w, v \rangle \notin R^C$.
Then the set $\{w, v\} \subseteq W$ is F -consistent, so by Lemma 7, there exists some $A^* \in \Pi(F)$ such that $\{w, v\} \subseteq A^*$. It follows from $CD_{CPL}(f(w), f(v))$ that $\bigvee \{\alpha_i \in \Pi^{CPL} \mid w \in A_i\} = f(w) \models_{CPL} \neg f(v) = \neg \bigvee \{\alpha_i \in \Pi^{CPL} \mid v \in A_i\}$. Hence, for all $A_i \in \Pi(F)$, if $w \in A_i$ then $v \notin A_i$. This is in contradiction with the fact that $w, v \in A^*$.
- $\langle w, v \rangle \in R^C$.
By the seriality of R^{CD} , there exist $w', v' \in W$ such that $\langle w, w' \rangle, \langle v, v' \rangle \in R^{CD}$. Now, if $\langle w', v' \rangle \in R^{CD}$, then by the symmetry and functionality of R^{CD} it would follow that $w = v'$ and $v = w'$, so $\langle w', v' \rangle \in R^{CD}$ would mean that $\langle v, w \rangle \in R^{CD}$, and thus (by the symmetry of R^{CD}) that $\langle w, v \rangle \in R^{CD}$, which contradicts $R^{CD} \cap R^C = \emptyset$. This shows that $\langle w', v' \rangle \notin R^{CD}$. Furthermore, it cannot happen that $\langle w', v' \rangle \in R^C$ either, since together with $\langle w, v \rangle \in R^C$ and the symmetry of R^C , that would contradict condition 4(b) of Definition 5. Since $\langle w', v' \rangle \notin R^{CD}$ and $\langle w', v' \rangle \notin R^C$, it follows that the set $\{w', v'\} \subseteq W$ is F -consistent, and hence, by Lemma 7, there exists some $A^* \in \Pi(F)$ such that $\{w', v'\} \subseteq A^*$. Note that $w \notin A^*$, since together with $w' \in A^*$ and $\langle w, w' \rangle \in R^{CD}$, this would violate the F -consistency of A^* . Analogously, we show that $v \notin A^*$. It follows from $CD_{CPL}(f(w), f(v))$ that $\models_{CPL} f(w) \vee f(v)$, i.e., $\models_{CPL} \bigvee \{\alpha_i \in \Pi^{CPL} \mid w \in A_i\} \vee \bigvee \{\alpha_i \in \Pi^{CPL} \mid v \in A_i\}$. Hence, for all $A_i \in \Pi(F)$, it holds that $w \in A_i$ or $v \in A_i$. This contradicts the fact that $w, v \notin A^*$.

In an analogous fashion, we can check that for all $w, v \in W$, we have $\langle w, v \rangle \in R^C$ iff $C_{CPL}(f(w), f(v))$. To summarize: f is a modal isomorphism from $F = \langle W, R^{CD}, R^C \rangle$ to the AD-frame F_D that is based on the Aristotelian diagram D for $(f[W], CPL)$. \square

In order to better grasp the proof of Lemma 8, it may be useful to work through a concrete example. Consider the AD-frame $F = \langle W, R^{CD}, R^C \rangle$ shown in Figure 2a. An easy calculation yields $\Pi(F) = \{A_1 := \{w, u\}, A_2 := \{u, s\}, A_3 := \{v, s\}\}$. Since $|\Pi(F)| = 3$, we need to consider a partition Π^{CPL} that consists of three formulas, for example $\Pi^{CPL} := \{\alpha_1 := p, \alpha_2 := \neg p \wedge q, \alpha_3 := \neg p \wedge \neg q\}$. Since w only belongs to A_1 , we define $f(w) := \alpha_1 = p$. Since v only belongs to A_3 , we define $f(v) := \alpha_3 = \neg p \wedge \neg q$. Since u belongs to A_1 and A_2 , we define $f(u) := \alpha_1 \vee \alpha_2 = p \vee (\neg p \wedge q) \equiv_{CPL} p \vee q$. Finally, since s belongs to A_2 and A_3 , we define $f(s) := \alpha_2 \vee \alpha_3 = (\neg p \wedge q) \vee (\neg p \wedge \neg q) \equiv_{CPL} \neg p$. In this way, we obtain the Aristotelian diagram D for $(f[W], CPL)$ that is shown in Figure 2b. Upon visual inspection of the left- and right-hand sides of Figure 2, it should be immediately clear that F is modally isomorphic to F_D .



Figure 2. (a) The AD-frame F and (b) an Aristotelian diagram D such that F is modally isomorphic to F_D .

We can now put everything together in Theorem 2. The importance of this theorem, and in particular, of Lemma 8, should not be underestimated. These results tell us that Definition 5 completely captures the structural properties of Aristotelian diagrams. More concretely, the various conditions stated in Definition 5 are sufficient to exhaustively describe the behavior of Aristotelian relations (consider how each of these properties played a role in

the proof of Lemma 8!). Until now, there was a long list of various properties of Aristotelian relations (e.g., we know that subalternation is transitive), but it was unknown whether this list was exhaustive. In other words, it was conceptually possible that some complex property of the Aristotelian relations might exist that we did not yet know about, and that does *not* follow from the properties that are already known. Theorem 2 tells us definitively that no such properties exist.

Theorem 2. *The function $\lambda: \mathbf{D} \rightarrow \mathbf{F}$ is bijective.*

Proof. This follows immediately from Lemmas 6 and 8. \square

To conclude this section, it is worth mentioning that the proof of Theorem 2 immediately yields another interesting result as well: every Aristotelian diagram for some finite fragment $\mathcal{F} \subseteq \mathcal{L}_S$ and some logical system S is Aristotelian isomorphic to a diagram for a finite fragment $\mathcal{F}_{CPL} \subseteq \mathcal{L}_{CPL}$ and the logic CPL (Lemma 9). This means that when we are studying Aristotelian diagrams, we can essentially restrict ourselves to diagrams for (fragments from) *classical propositional logic*.

Lemma 9. *Consider an Aristotelian diagram D_1 for (\mathcal{F}, S) , where \mathcal{F} is finite. There exist a finite fragment $\mathcal{F}_{CPL} \subseteq \mathcal{L}_{CPL}$ and a diagram D_2 for (\mathcal{F}_{CPL}, CPL) such that D_1 is Aristotelian isomorphic to D_2 .*

Proof. Consider the canonical frame F_{D_1} that is based on D_1 . In the proof of Lemma 8, we constructed a finite fragment $\mathcal{F}_{CPL} \subseteq \mathcal{L}_{CPL}$ (namely, $\mathcal{F}_{CPL} := f[\mathcal{F}]$) and an Aristotelian diagram D_2 for (\mathcal{F}_{CPL}, CPL) such that F_{D_1} is modally isomorphic to F_{D_2} . By the proof of Lemma 6, it follows that the Aristotelian diagrams D_1 and D_2 themselves are Aristotelian isomorphic. \square

5. Characterizing Aristotelian Families in AD-Logic

In this section, we really start putting AD-logic to use, in order to capture some important insights from logical geometry. In particular, one of the main ongoing research efforts in logical geometry is to develop a systematic typology of *Aristotelian families* (i.e., maximal classes of Aristotelian diagrams that are closed under Aristotelian isomorphism) [24,28,42]. We will now prove frame correspondence results for several of the most important families of Aristotelian diagrams: for each such family A , we exhibit a formula $\chi_A \in \mathcal{L}_{AD}$, and show that $\models_D \chi_A$ iff D belongs to the Aristotelian family A , for all diagrams D . These frame correspondences do not refer to any underlying logical system, and thus provide fully generic descriptions of the Aristotelian families [24] (recall the distinction between *general* and *generic* characterizations of Aristotelian families from Section 1).

5.1. Expressing Frame Cardinalities

We start by mentioning an auxiliary result. Because our language \mathcal{L}_{AD} is so expressive, it can be used to count the number of worlds in a (finite) AD-frame. We write “)” ^{n} to denote a sequence of n closing brackets.

Lemma 10. *Consider an AD-frame $F = \langle W, R^{CD}, R^C \rangle$. Then we have:*

$$\begin{aligned} |W| = 2 & \text{ iff } F \models \downarrow x_1 (\diamond (\neg x_1 \wedge \downarrow x_2 (\square (x_1 \vee x_2))))), \\ |W| = 4 & \text{ iff } F \models \downarrow x_1 (\diamond (\neg x_1 \wedge \\ & \downarrow x_2 (\diamond (\neg x_1 \wedge \neg x_2 \wedge \\ & \downarrow x_3 (\diamond (\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \\ & \downarrow x_4 (\square (x_1 \vee x_2 \vee x_3 \vee x_4)))))), \end{aligned}$$

and in general, for any even number n :

$$\begin{aligned}
 |W| = n \quad \text{iff} \quad F \models & \downarrow x_1(\Diamond(\neg x_1 \wedge \\
 & \downarrow x_2(\Diamond(\neg x_1 \wedge \neg x_2 \wedge \\
 & \downarrow x_3(\Diamond(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \\
 & \vdots \\
 & \downarrow x_{n-1}(\Diamond(\neg x_1 \wedge \dots \wedge \neg x_{n-1} \wedge \\
 & \downarrow x_n(\Box(x_1 \vee \dots \vee x_n))^{2n}.
 \end{aligned}$$

The aforementioned \mathcal{L}_{AD} -formula is henceforth abbreviated as χ_n , so $|W| = n$ iff $F \models \chi_n$.

Proof. This is an easy exercise in frame correspondence theory for hybrid logic. The structure of the formula χ_n is clear: each variable x_i gets bound to a world $w_i \in W$; for all $1 \leq i < j \leq n$ we have a formula of the form $\neg x_i \wedge \dots \wedge \downarrow x_j(\varphi)$ inside χ_n , which shows that $w_i \neq w_j$; finally, the subformula $\Box(x_1 \vee \dots \vee x_n)$ of χ_n means that w_1, \dots, w_n are all the worlds in W , and thus $|W| = n$. \square

Note that the formula χ_n contains only the binder and the global modality, and hence, there is nothing specifically “(Aristotelian) diagrammatical” about χ_n (in particular, it does not contain $\langle CD \rangle$ or $\langle C \rangle$). This is exactly as it should be, since χ_n merely describes the cardinality of the AD-frame F , which is (nearly) independent of F ’s status as (based on) an Aristotelian diagram. (But do note that we define χ_n only for even numbers n , since finite Aristotelian diagrams/AD-frames have, by definition, an *even* number of formulas/worlds; see Definitions 2 and 5).

5.2. Characterizing the Aristotelian Family of Pairs of Contradictories

Within the typology of Aristotelian diagrams, there exists a unique smallest type of Aristotelian diagram, viz., the *pair of contradictories* (PCD) (see Figure 3). For a diagram for (\mathcal{F}, S) to be a PCD, it suffices to require that \mathcal{F} contains exactly two formulas; from the general Definition 2 of an Aristotelian diagram, it then follows that these two formulas are S-contradictory to each other (hence the term “pair of contradictories”). Consequently, it is also straightforward to characterize PCDs in AD-logic.



Figure 3. A pair of contradictories (PCD).

Definition 20. An Aristotelian diagram for (\mathcal{F}, S) is a pair of contradictories iff $|\mathcal{F}| = 2$.

Theorem 3. Define $\chi_{PCD} := \chi_2$. An Aristotelian diagram D for (\mathcal{F}, S) is a PCD iff $\models_D \chi_{PCD}$.

Proof. For the left to right direction, suppose that D is a PCD and consider an arbitrary AD-frame $F = \langle W, R^{CD}, R^C \rangle$ that is based on D . By Definitions 13 and 20, it follows that $|W| = 2$, so by Lemma 10, we get $F \models \chi_2$, i.e., $F \models \chi_{PCD}$.

For the right to left direction, suppose that $\models_D \chi_{PCD}$, and consider the canonical AD-frame F_D . By Lemma 4, it follows that $F_D \models \chi_{PCD}$, i.e., $F_D \models \chi_2$, so by Lemma 10, we find that $|\mathcal{F}| = 2$, which means that D is a PCD. \square

Let us have another look at Definition 20 and Theorem 3. At first sight, it might look like the right-hand side of Definition 20 does not refer to any underlying logical system S (and is thus fully generic in nature). However, recall that by Definition 2, it follows from the requirement that $|\mathcal{F}| = 2$ that there are $\alpha, \beta \in \mathcal{F}$ such that $CD_S(\alpha, \beta)$, which *does* involve a reference to S . Definition 20 is thus general but not generic in nature, because its right-hand side still (implicitly) refers to some underlying logical system S (even though that system can be chosen arbitrarily). By contrast, the right-hand side of Theorem 3 does not refer to any underlying logical system S whatsoever, and thus offers a fully generic characterization of the Aristotelian family of PCDs. (This contrast between official definitions and \mathcal{L}_{AD} -characterizations will become even clearer when we are dealing with Aristotelian families

of squares, hexagons, and octagons, because their official definitions refer much more explicitly to the underlying logical system S).

5.3. Characterizing the Two Aristotelian Families of Squares of Opposition

Within the typology of Aristotelian diagrams, there exist exactly two families of squares of opposition: the *classical* square and the *degenerate* square; see Figure 4 [24,28,42]. The classical square is by far the oldest and most well-known kind of Aristotelian diagram. In comparison to a classical square, a degenerate square has lost all Aristotelian relations, except for the two relations of contradiction (which are required to be present by the very definition of Aristotelian diagram). Both families of squares are straightforwardly characterized in AD-logic.

Definition 21. Consider an Aristotelian diagram D for (\mathcal{F}, S) . Then:

1. D is a square of opposition iff $|\mathcal{F}| = 4$,
2. D is a classical square of opposition iff $|\mathcal{F}| = 4$ and there exist $\alpha, \beta \in \mathcal{F}$ such that $C_S(\alpha, \beta)$,
3. D is a degenerate square of opposition iff $|\mathcal{F}| = 4$ and there do not exist $\alpha, \beta \in \mathcal{F}$ such that $C_S(\alpha, \beta)$.



Figure 4. (a) A classical square of opposition and (b) a degenerate square of opposition.

Theorem 4. Define the \mathcal{L}_{AD} -formulas $\chi_{square} := \chi_4$, $\chi_{classical} := \chi_4 \wedge \diamond\langle C \rangle \top$ and $\chi_{degenerate} := \chi_4 \wedge \neg\diamond\langle C \rangle \top$. Then for any Aristotelian diagram D for (\mathcal{F}, S) , we have:

- | | | |
|---|-----|-------------------------------|
| 1. D is a square of opposition | iff | $\models_D \chi_{square}$ |
| 2. D is a classical square of opposition | iff | $\models_D \chi_{classical}$ |
| 3. D is a degenerate square of opposition | iff | $\models_D \chi_{degenerate}$ |

Proof. Analogous to the proof of Theorem 3. For purposes of illustration, we prove the second item. For the left to right direction, suppose that D is a classical square of opposition, and consider an arbitrary AD-frame $F = \langle W, R^{CD}, R^C \rangle$ that is based on D . By Definitions 13 and 21, it follows that $|W| = 4$ and that there exist $w, v \in W$ such that $\langle w, v \rangle \in R^C$. From $|W| = 4$, it follows by Lemma 10 that $F \models \chi_4$. From $\langle w, v \rangle \in R^C$, it follows that $F \models \diamond\langle C \rangle \top$. Hence, $F \models \chi_4 \wedge \diamond\langle C \rangle \top$, i.e., $F \models \chi_{classical}$.

For the right to left direction, suppose that $\models_D \chi_{classical}$ and consider the canonical AD-frame F_D . By Lemma 4, it follows that $F_D \models \chi_{classical}$, i.e., $F_D \models \chi_4 \wedge \diamond\langle C \rangle \top$. From $F_D \models \chi_4$, it follows by Lemma 10 that $|\mathcal{F}| = 4$. From $F_D \models \diamond\langle C \rangle \top$, it follows that there exist $\alpha, \beta \in \mathcal{F}$ such that $C_S(\alpha, \beta)$. By Definition 21, this means that D is a classical square of opposition. \square

Just like with the PCDs, let us have another look at Definition 21 and Theorem 4. Note that items 2 and 3 of Definition 21 are general but not generic in nature, because their right-hand sides still refer to some underlying logical system S (even though that system can be chosen arbitrarily). By contrast, the right-hand sides of items 2 and 3 of Theorem 4 do not refer to any underlying logical system S whatsoever, and thus offer fully generic characterizations of the Aristotelian families of classical and degenerate squares. Entirely analogous remarks apply to the characterizations of Aristotelian families of hexagons and octagons in the remainder of this section (compare Definitions 22 and 23 with Theorems 5 and 6, respectively).

Next to characterizing the classical and degenerate squares, we can also easily express in AD-logic that these two types of diagrams jointly constitute an exhaustive classification

of the squares of opposition. To illustrate the versatility of AD-logic, we prove the two parts of Lemma 11 in different fashions: the first item is proved directly “within” AD-logic, while the second is proved by reducing it to the corresponding result in logical geometry. (If we had only considered the proof’s difficulty and length, it would have been more straightforward to prove both items directly within AD-logic).

Lemma 11. *The following hold:*

1. $\chi_{square} \equiv_{AD} \chi_{classical} \vee \chi_{degenerate}$, i.e.,
 $\chi_{square} \models_{AD} \chi_{classical} \vee \chi_{degenerate}$ and $\chi_{classical} \vee \chi_{degenerate} \models_{AD} \chi_{square}$,
2. $\models_{AD} \neg(\chi_{classical} \wedge \chi_{degenerate})$.

Proof. For item 1, propositional reasoning yields $\chi_{classical} \vee \chi_{degenerate} = (\chi_4 \wedge \diamond\langle C \rangle \top) \vee (\chi_4 \wedge \neg\diamond\langle C \rangle \top) \equiv_{AD} \chi_4 \wedge (\diamond\langle C \rangle \top \vee \neg\diamond\langle C \rangle \top) \equiv_{AD} \chi_4 \wedge \top \equiv_{AD} \chi_4 = \chi_{square}$.

For item 2, consider an arbitrary AD-frame F and suppose, toward a contradiction, that $F \models \chi_{classical} \wedge \chi_{degenerate}$. It follows that $F \models \chi_4$, so by Lemma 10, F is finite, and by Lemma 8, there exists an Aristotelian diagram D such that F is modally isomorphic to F_D . Since modal isomorphisms preserve frame validity, it follows from $F \models \chi_{classical}$ that $F_D \models \chi_{classical}$, and, by Lemma 5, that $F' \models \chi_{classical}$ for every AD-frame F' that is based on D . By Definition 14, this means that $\models_D \chi_{classical}$, so by Theorem 4, it follows that D is a classical square of opposition. In exactly the same way, it follows from $F \models \chi_{degenerate}$ that D is a degenerate square of opposition. However, this is impossible, since by Definition 21, a diagram cannot simultaneously be a classical and a degenerate square of opposition. \square

Note that the formula $\chi_{classical}$ contains the global modality, \diamond , both in its left conjunct χ_4 and in its right conjunct $\diamond\langle C \rangle \top$. However, if one prefers to work without the global modality (e.g., because one adheres to the credo that modal logic should be *local*; see [40], p. ix) and is prepared to put the cardinality condition in the metalanguage instead of in \mathcal{L}_{AD} , then we can characterize classical squares of opposition without the global modality altogether. In particular, we have to replace $\diamond\langle C \rangle \top$ with the disjunction $\langle C \rangle \top \vee \langle SC \rangle \top$. (Entirely analogous remarks apply to $\chi_{degenerate}$).

Lemma 12. *Consider an Aristotelian diagram D for (\mathcal{F}, S) and suppose that $|\mathcal{F}| = 4$. Then D is a classical square of opposition iff $\models_D \langle C \rangle \top \vee \langle SC \rangle \top$.*

Proof. For the left to right direction, suppose that D is a classical square of opposition, and consider an arbitrary AD-frame $F = (W, R^{CD}, R^C)$ that is based on D . By Definitions 13 and 21, it follows that $|W| = 4$ and that there exist $w, v \in W$ such that $\langle w, v \rangle \in R^C$ (and thus also $\langle v, w \rangle \in R^C$, by the symmetry of R^C). By the seriality and functionality of R^{CD} , there exist unique $w', v' \in W$ such that $\langle w, w' \rangle, \langle v, v' \rangle \in R^{CD}$. By the irreflexivity of R^{CD} and R^C , it follows that $x \neq y$ for all $x, y \in \{w, v, w', v'\}$, so since $|W| = 4$, it follows that $W = \{w, v, w', v'\}$. Consider an arbitrary valuation V on F and assignment g on $\langle F, V \rangle$. Since $\langle w, v \rangle, \langle v, w \rangle \in R^C$, we have $\langle F, V \rangle, g, x \models \langle C \rangle \top$ for $x \in \{w, v\}$, and $\langle F, V \rangle, g, x \models \langle SC \rangle \top$ for $x \in \{w', v'\}$. Since $W = \{w, v, w', v'\}$, it follows that $F \models \langle C \rangle \top \vee \langle SC \rangle \top$, as desired.

For the right to left direction, suppose that $\models_D \langle C \rangle \top \vee \langle SC \rangle \top$, and consider the canonical AD-frame F_D . By Lemma 4, it follows that $F_D \models \langle C \rangle \top \vee \langle SC \rangle \top$. Consider an arbitrary valuation V on F_D and assignment g on $\langle F_D, V \rangle$, and a world $\alpha \in \mathcal{F}$. We thus have $\langle F_D, V \rangle, g, \alpha \models \langle C \rangle \top \vee \langle SC \rangle \top$. On the one hand, if $\langle F_D, V \rangle, g, \alpha \models \langle C \rangle \top$, then there exists $\beta \in \mathcal{F}$ such that $C_S(\alpha, \beta)$. On the other hand, if $\langle F_D, V \rangle, g, \alpha \models \langle SC \rangle \top$, then there exists $\beta \in \mathcal{F}$ such that $SC_S(\alpha, \beta)$. By Definitions 1 and 2, there exist $\gamma, \delta \in \mathcal{F}$ such that $\gamma \equiv_S \neg\alpha$, $\delta \equiv_S \neg\beta$ and $C_S(\gamma, \delta)$. In both cases, we find, by Definition 21, that D is a classical square of opposition. \square

5.4. Characterizing the Five Aristotelian Families of the Hexagons of Opposition

One can show that there are exactly five families of hexagons of opposition: the Jacoby–Sesmat–Blanché (JSB) hexagon, the Sherwood–Czeżowski (SC) hexagon, and the unconnected- n (U- n) hexagon for $n \in \{4, 8, 12\}$ (see Figures 5 and 6) [42]. The JSB hexagons are named after Jacoby [43], Sesmat [44], and Blanché [45], and the SC hexagons after William of Sherwood [46] and Czeżowski [47]. The U4, U8, and U12 hexagons are less well-known, and are named after the number of pairs of unconnected formulas that they contain. (Recall that two formulas are said to be *unconnected* iff they do not stand in any Aristotelian relation to each other.) Once again, all families of hexagons are straightforwardly characterized in AD-logic.

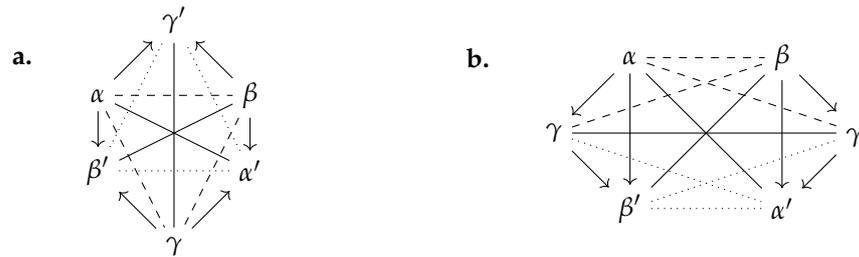


Figure 5. (a) A JSB hexagon and (b) an SC hexagon.

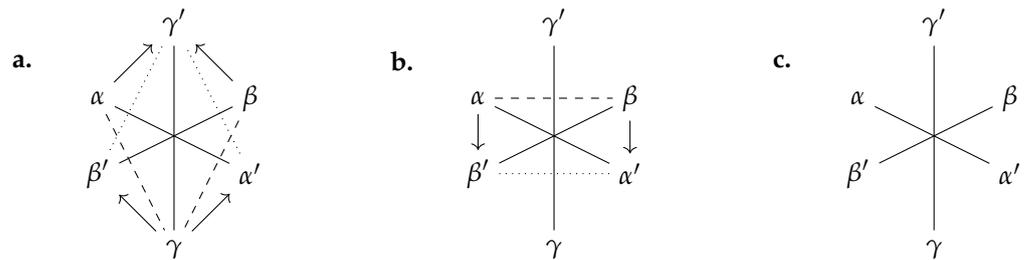


Figure 6. (a) A U4 hexagon, (b) a U8 hexagon, and (c) a U12 hexagon.

Definition 22. Consider an Aristotelian diagram D for (\mathcal{F}, S) . Then:

1. D is a hexagon of opposition iff $|\mathcal{F}| = 6$,
2. D is a JSB hexagon iff $|\mathcal{F}| = 6$ and there exist $\alpha, \beta, \gamma \in \mathcal{F}$ such that $C_S(\alpha, \beta)$, $C_S(\alpha, \gamma)$ and $C_S(\beta, \gamma)$,
3. D is an SC hexagon iff $|\mathcal{F}| = 6$ and there exist $\alpha, \beta, \gamma, \gamma' \in \mathcal{F}$ such that $C_S(\alpha, \beta)$, $C_S(\alpha, \gamma')$, $C_S(\beta, \gamma)$ and $CD_S(\gamma, \gamma')$,
4. D is a U4 hexagon iff $|\mathcal{F}| = 6$ and there exist $\alpha, \beta, \gamma \in \mathcal{F}$ such that $C_S(\alpha, \gamma)$ and $C_S(\beta, \gamma)$, and for all $\delta, \varepsilon \in \mathcal{F}$, if $C_S(\delta, \varepsilon)$, then $\{\delta, \varepsilon\} \in \{\{\alpha, \gamma\}, \{\beta, \gamma\}\}$,
5. D is a U8 hexagon iff $|\mathcal{F}| = 6$ and there exist $\alpha, \beta \in \mathcal{F}$ such that $C_S(\alpha, \beta)$, and for all $\delta, \varepsilon \in \mathcal{F}$, if $C_S(\delta, \varepsilon)$, then $\{\delta, \varepsilon\} = \{\alpha, \beta\}$,
6. D is a U12 hexagon iff $|\mathcal{F}| = 6$ and there do not exist $\alpha, \beta \in \mathcal{F}$ such that $C_S(\alpha, \beta)$.

Theorem 5. Define the following \mathcal{L}_{AD} -formulas:

1. $\chi_{hexagon} := \chi_6$,
2. $\chi_{JSB} := \chi_6 \wedge \diamond \downarrow x(\langle C \rangle \langle C \rangle \langle C \rangle x)$,
3. $\chi_{SC} := \chi_6 \wedge \diamond \downarrow x(\langle C \rangle \downarrow y([C](x \wedge \langle C \rangle (\neg y \wedge \downarrow z(\langle C \rangle ((CD)y \wedge [C]z))))))$,
4. $\chi_{U4} := \chi_6 \wedge \diamond \downarrow x(\langle C \rangle \downarrow y([C](x \wedge \langle C \rangle (\neg y \wedge [C]x))))$,
5. $\chi_{U8} := \chi_6 \wedge \diamond \downarrow x(\langle C \rangle \downarrow y([C](x \wedge [C]y)))$,
6. $\chi_{U12} := \chi_6 \wedge \neg \diamond \langle C \rangle \top$.

Then for any Aristotelian diagram D for (\mathcal{F}, S) , we have:

1. D is a hexagon of opposition iff $\models_D \chi_{hexagon}$,
2. D is a JSB hexagon iff $\models_D \chi_{JSB}$,
3. D is an SC hexagon iff $\models_D \chi_{SC}$,
4. D is a UI4 hexagon iff $\models_D \chi_{UI4}$,
5. D is a UI8 hexagon iff $\models_D \chi_{UI8}$,
6. D is a UI12 hexagon iff $\models_D \chi_{UI12}$.

Proof. This is analogous to the proofs of Theorems 3 and 4. For purposes of illustration, we prove the third item. For the left to right direction, suppose that D is an SC hexagon, and consider an arbitrary AD-frame $F = \langle W, R^{CD}, R^C \rangle$ that is based on D . By Definitions 13 and 22, it follows that $|W| = 6$ and that there exist $w, v, u, u' \in W$ such that $\langle w, v \rangle, \langle w, u' \rangle, \langle v, u \rangle \in R^C$ and $\langle u, u' \rangle \in R^{CD}$. If we let the state variables x, y and z get bound to the worlds w, u' , and v , respectively, then it is easy to check that $F \models \diamond \downarrow x(\langle C \rangle \downarrow y([C](x \wedge \langle C \rangle (-y \wedge \downarrow z(\langle C \rangle (\langle CD \rangle y \wedge [C]z))))))$. (Note that the world u does not get bound to any state variable; u is the world (i) that is R^C -reachable from (the world bound to) z , and (ii) at which $\langle CD \rangle y \wedge [C]z$ holds.) Furthermore, from $|W| = 6$ it follows by Lemma 10 that $F \models \chi_6$. In sum, we find that $F \models \chi_{SC}$.

For the right to left direction, suppose that $\models_D \chi_{SC}$, and consider the canonical AD-frame F_D . By Lemma 4, it follows that $F_D \models \chi_{SC}$, i.e., $F_D \models \chi_6 \wedge \diamond \downarrow x(\langle C \rangle \downarrow y([C](x \wedge \langle C \rangle (-y \wedge \downarrow z(\langle C \rangle (\langle CD \rangle y \wedge [C]z))))))$. From $F_D \models \chi_6$, it follows by Lemma 10 that $|F| = 6$. From $F_D \models \diamond \downarrow x(\langle C \rangle \downarrow y([C](x \wedge \langle C \rangle (-y \wedge \downarrow z(\langle C \rangle (\langle CD \rangle y \wedge [C]z))))))$, it follows that there exist $\alpha, \beta, \gamma, \gamma' \in \mathcal{F}$ such that $C_S(\alpha, \beta), C_S(\alpha, \gamma'), C_S(\beta, \gamma)$, and $CD_S(\gamma, \gamma')$. (Namely, α, γ' and β are the worlds that get bound to the state variables x, y , and z , respectively; γ does not get bound to any state variable, but is the world (i) that is C_S -reachable from (the world bound to) z and (ii) at which $\langle CD \rangle y \wedge [C]z$ holds.) By Definition 22, this means that D is an SC hexagon. \square

Just as before, for some of these characterizing formulas, we can get rid of the global modality, \diamond , if we are prepared to put the cardinality condition in the metalanguage instead of in \mathcal{L}_{AD} . For example, in χ_{JSB} , we have to replace $\diamond \downarrow x(\langle C \rangle \langle C \rangle \langle C \rangle x)$ with the disjunctive formula $\downarrow x(\langle C \rangle \langle C \rangle \langle C \rangle x \vee \langle SC \rangle \langle SC \rangle \langle SC \rangle x)$.

Lemma 13. Consider an Aristotelian diagram D for (\mathcal{F}, S) and suppose that $|\mathcal{F}| = 6$. Then D is a JSB hexagon iff $\models_D \downarrow x(\langle C \rangle \langle C \rangle \langle C \rangle x \vee \langle SC \rangle \langle SC \rangle \langle SC \rangle x)$.

Proof. This is analogous to the proof of Lemma 12. \square

5.5. Characterizing Some Aristotelian Families of Octagons of Opposition

In the project of setting up a systematic typology of Aristotelian diagrams, it has been shown that there are exactly 18 families of octagons of opposition [42]. In this paper, we will not deal with all these families, but rather focus on some of the most well-known ones: the Moretti–Pellissier octagons [48,49], the Lenzen octagons [50–52], the Buridan octagons [53–55], the Beziau octagons [56,57], and the Keynes–Johnson octagons [58,59] (see Figures 7 and 8). Once again, these families of octagons (and also the other ones, which we do not deal with in this paper) can straightforwardly be characterized in AD-logic.

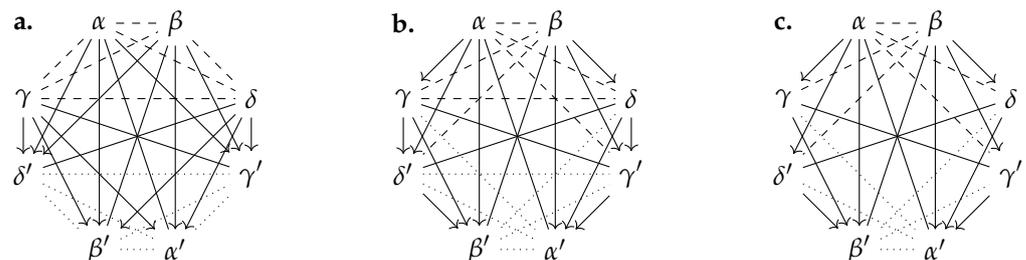


Figure 7. (a) A Moretti–Pellissier octagon, (b) a Lenzen octagon, and (c) a Buridan octagon.

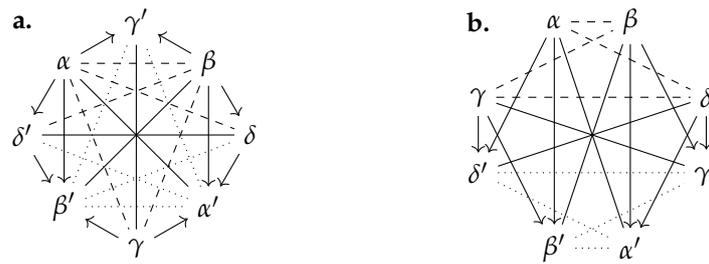


Figure 8. (a) A Beziau octagon and (b) a Keynes–Johnson octagon.

Definition 23. Consider an Aristotelian diagram D for (\mathcal{F}, S) . Then:

1. D is an octagon of opposition iff $|\mathcal{F}| = 8$,
2. D is a Moretti–Pellissier octagon iff $|\mathcal{F}| = 8$ and there exist $\alpha, \beta, \gamma, \delta \in \mathcal{F}$ such that $C_S(\alpha, \beta), C_S(\alpha, \gamma), C_S(\alpha, \delta), C_S(\beta, \gamma), C_S(\beta, \delta)$ and $C_S(\gamma, \delta)$,
3. D is a Lenzen octagon iff $|\mathcal{F}| = 8$ and there exist $\alpha, \beta, \gamma, \gamma', \delta, \delta' \in \mathcal{F}$ such that $C_S(\alpha, \beta), C_S(\alpha, \gamma'), C_S(\alpha, \delta), C_S(\beta, \gamma), C_S(\beta, \delta'), C_S(\gamma, \delta), CD_S(\gamma, \gamma')$, and $CD_S(\delta, \delta')$,
4. D is a Buridan octagon iff $|\mathcal{F}| = 8$ and there exist $\alpha, \beta, \gamma, \gamma', \delta, \delta' \in \mathcal{F}$ such that $C_S(\alpha, \beta), C_S(\alpha, \gamma'), C_S(\alpha, \delta), C_S(\beta, \gamma), C_S(\beta, \delta'), CD_S(\gamma, \gamma')$ and $CD_S(\delta, \delta')$, and for all $\epsilon, \zeta \in \mathcal{F}$, if $C_S(\epsilon, \zeta)$ then $\{\epsilon, \zeta\} \in \{\{\alpha, \beta\}, \{\alpha, \gamma'\}, \{\alpha, \delta\}, \{\beta, \gamma\}, \{\beta, \delta'\}\}$,
5. D is a Beziau octagon iff $|\mathcal{F}| = 8$ and there exist $\alpha, \beta, \gamma, \delta, \delta' \in \mathcal{F}$ such that $C_S(\alpha, \beta), C_S(\alpha, \gamma), C_S(\alpha, \delta), C_S(\beta, \gamma), C_S(\beta, \delta')$ and $CD_S(\delta, \delta')$, and for all $\epsilon, \zeta \in \mathcal{F}$, if $C_S(\epsilon, \zeta)$ then $\{\epsilon, \zeta\} \in \{\{\alpha, \beta\}, \{\alpha, \gamma\}, \{\alpha, \delta\}, \{\beta, \gamma\}, \{\beta, \delta'\}\}$,
6. D is a Keynes–Johnson octagon iff $|\mathcal{F}| = 8$ and there exist $\alpha, \beta, \gamma, \delta \in \mathcal{F}$ such that $C_S(\alpha, \beta), C_S(\alpha, \delta), C_S(\beta, \gamma)$ and $C_S(\gamma, \delta)$, and for all $\epsilon, \zeta \in \mathcal{F}$, if $C_S(\epsilon, \zeta)$, then $\{\epsilon, \zeta\} \in \{\{\alpha, \beta\}, \{\alpha, \delta\}, \{\beta, \gamma\}, \{\gamma, \delta\}\}$.

Theorem 6. Define the following \mathcal{L}_{AD} -formulas:

1. $\chi_{\text{Octagon}} := \chi_8$,
2. $\chi_{\text{MorPel}} := \chi_8 \wedge \diamond \downarrow x(\langle C \rangle \downarrow y(\langle C \rangle (\neg x \wedge \langle C \rangle x \wedge \langle C \rangle (\neg x \wedge \neg y \wedge \langle C \rangle x \wedge \langle C \rangle y))))$,
3. $\chi_{\text{Lenzen}} := \chi_8 \wedge \diamond \downarrow x(\langle C \rangle [C]x \wedge \langle C \rangle \downarrow y(\langle C \rangle [C]y \wedge \langle C \rangle (\neg x \wedge [C]\neg x \wedge \langle C \rangle (\neg x \wedge \neg y \wedge [C]\neg y \wedge \langle C \rangle x))))$,
4. $\chi_{\text{Buridan}} := \chi_8 \wedge \diamond \downarrow x(\langle C \rangle \downarrow y([C](x \wedge \langle C \rangle (\neg y \wedge \downarrow z([C](x \wedge \langle C \rangle \downarrow x_2(\langle C \rangle ((CD)z \wedge [C](x_2 \wedge \langle C \rangle ((CD)y \wedge [C]x_2))))))))$,
5. $\chi_{\text{Beziau}} := \chi_8 \wedge \diamond \downarrow x(\langle C \rangle \downarrow y([C](x \wedge \langle C \rangle \downarrow z(\langle C \rangle \langle C \rangle x \wedge \langle C \rangle ((CD)y \wedge [C]z))))$,
6. $\chi_{\text{KeyJo}} := \chi_8 \wedge \diamond \downarrow x(\langle C \rangle \downarrow y(\langle C \rangle (\neg x \wedge [C]\neg x \wedge \langle C \rangle (\neg x \wedge \neg y \wedge [C]\neg y \wedge \langle C \rangle x))))$.

Then for any Aristotelian diagram D for (\mathcal{F}, S) , we have:

- | | | |
|--|-----|-------------------------------------|
| 1. D is an octagon of opposition | iff | $\models_D \chi_{\text{Octagon}}$, |
| 2. D is a Moretti–Pellissier octagon | iff | $\models_D \chi_{\text{MorPel}}$, |
| 3. D is a Lenzen octagon | iff | $\models_D \chi_{\text{Lenzen}}$, |
| 4. D is a Buridan octagon | iff | $\models_D \chi_{\text{Buridan}}$, |
| 5. D is a Beziau octagon | iff | $\models_D \chi_{\text{Beziau}}$, |
| 6. D is a Keynes–Johnson octagon | iff | $\models_D \chi_{\text{KeyJo}}$. |

Proof. This is analogous to the proofs of Theorems 3–5. For purposes of illustration, we prove the fifth item. For the left to right direction, suppose that D is a Beziau octagon, and consider an arbitrary AD-frame $F = \langle W, R^{CD}, R^C \rangle$ that is based on D . By Definitions 13 and 23, it follows that $|W| = 8$ and that there exist $w, v, u, s, s' \in W$ such that $\langle w, v \rangle, \langle w, u \rangle, \langle w, s \rangle, \langle v, u \rangle, \langle v, s' \rangle \in R^C$ and $\langle s, s' \rangle \in R^{CD}$, and for all $r, t \in W$, if $\langle r, t \rangle \in R^C$, then $\{r, t\} \in \{\{w, v\}, \{w, u\}, \{w, s\}, \{v, u\}, \{v, s'\}\}$. If we let the state variables x, y and z get bound to the worlds w, s and v , respectively, then it is easy to check that $F \models \diamond \downarrow x(\langle C \rangle \downarrow y([C](x \wedge \langle C \rangle \downarrow z(\langle C \rangle \langle C \rangle x \wedge \langle C \rangle ((CD)y \wedge [C]z))))$. (Note that the worlds u and s' do not get bound to any state variables; these two worlds are R^C -reachable from (the world bound to) z . At u , it holds that $\langle C \rangle x$, and at s' , it holds that $\langle CD \rangle y \wedge [C]z$.)

Furthermore, from $|W| = 8$, it follows by Lemma 10 that $F \models \chi_8$. In sum, we find that $F \models \chi_{\text{Beziau}}$.

For the right to left direction, suppose that $\models_D \chi_{\text{Beziau}}$, and consider the canonical AD-frame F_D . By Lemma 4, it follows that $F_D \models \chi_{\text{Beziau}}$, i.e., $F_D \models \chi_8 \wedge \diamond \downarrow x(\langle C \rangle \downarrow y([C](x \wedge \langle C \rangle \downarrow z(\langle C \rangle \langle C \rangle x \wedge \langle C \rangle (\langle CD \rangle y \wedge [C]z))))))$. From $F_D \models \chi_8$, it follows by Lemma 10 that $|\mathcal{F}| = 8$. From $F_D \models \diamond \downarrow x(\langle C \rangle \downarrow y([C](x \wedge \langle C \rangle \downarrow z(\langle C \rangle \langle C \rangle x \wedge \langle C \rangle (\langle CD \rangle y \wedge [C]z))))))$, it follows that there exist $\alpha, \beta, \gamma, \delta, \delta' \in \mathcal{F}$ such that $C_S(\alpha, \beta)$, $C_S(\alpha, \gamma)$, $C_S(\alpha, \delta)$, $C_S(\beta, \gamma)$, $C_S(\beta, \delta')$ and $CD_S(\delta, \delta')$, and that there are no other pairs of S-contrary formulas in \mathcal{F} , apart from the aforementioned ones. (Namely, α, δ and β are the worlds that get bound to the state variables x, y , and z , respectively. The worlds γ and δ' do not get bound to any state variables: these two worlds are C_S -reachable from (the world bound to) z . At γ , it holds that $\langle C \rangle x$, and at δ' , it holds that $\langle CD \rangle y \wedge [C]z$.) By Definition 23, this means that D is a Beziau hexagon. \square

6. Aristotelian Diagrams for AD-Logic

In Section 2, we introduced the notion of an Aristotelian diagram for a fragment of formulas $\mathcal{F} \subseteq \mathcal{L}_S$, relative to some logical system S. For example, we can take S to be some system of modal logic, and thus study *Aristotelian diagrams for modal logic*; for example, recall the classical square of opposition for KD in Figure 1b. Next, in Sections 3–5, we presented the system of AD-logic. This gives rise to a modal consequence relation \models_{AD} , and can thus be viewed as the *modal logic of Aristotelian diagrams*.

Already in the opening paragraphs of this paper, we drew a sharp distinction between these two approaches. More concretely, we distinguished the system of AD-logic from the underlying logic S of some Aristotelian diagram; recall the notational convention introduced at the end of Section 3.1: $\alpha, \beta, \gamma \dots$ as metavariables over \mathcal{L}_S versus $\varphi, \chi, \psi \dots$ as metavariables over \mathcal{L}_{AD} . This distinction was, and still remains, important in order to avoid significant confusions. However, on a strictly technical level, there is no reason why we cannot take the underlying logic S of our Aristotelian diagrams to be the system of AD-logic itself. In light of the results of Section 5, this might even be a very fruitful idea. In this way, we thus end up studying Aristotelian diagrams for AD-logic, or, putting together the two aforementioned italicized expressions: *Aristotelian diagrams for the modal logic of Aristotelian diagrams*.

In this section, we briefly illustrate the viability and fruitfulness of this idea by means of a concrete example. Recall the formulas $\chi_{\text{square}}, \chi_{\text{classical}}, \chi_{\text{degenerate}} \in \mathcal{L}_{AD}$ that were introduced and studied in Section 5. Since the language \mathcal{L}_{AD} is closed under negation, we also have $\neg \chi_{\text{square}}, \neg \chi_{\text{classical}}, \neg \chi_{\text{degenerate}} \in \mathcal{L}_{AD}$. Hence, we can define the following fragment of \mathcal{L}_{AD} -formulas:

$$\mathcal{F}_{AD} := \{\chi_{\text{square}}, \chi_{\text{classical}}, \chi_{\text{degenerate}}, \neg \chi_{\text{square}}, \neg \chi_{\text{classical}}, \neg \chi_{\text{degenerate}}\} \subseteq \mathcal{L}_{AD}.$$

We clearly have all the ingredients at our disposal for defining an Aristotelian diagram (see Definition 2). First of all, note that AD-logic is a logical system with Boolean operators and a model-theoretic semantics \models_{AD} . Secondly, the fragment $\mathcal{F}_{AD} \subseteq \mathcal{L}_{AD}$ is non-empty, and moreover:

Lemma 14. *The fragment \mathcal{F}_{AD} satisfies the following properties:*

1. for every $\chi \in \mathcal{F}_{AD}$, there is a $\chi' \in \mathcal{F} \setminus \{\chi\}$ such that $\models_{AD} \chi' \leftrightarrow \neg \chi$,
2. for every $\chi \in \mathcal{F}_{AD}$, there is no $\chi' \in \mathcal{F} \setminus \{\chi\}$ such that $\models_{AD} \chi' \leftrightarrow \chi$,
3. there is no $\chi \in \mathcal{F}_{AD}$ such that $\models_{AD} \chi$ or $\models_{AD} \neg \chi$.

Proof. Item 1 follows trivially from the definition of \mathcal{F}_{AD} . For item 2, we show that χ_{square} and $\chi_{\text{classical}}$ are not AD-equivalent to each other (all other cases are analogous). Consider some degenerate square of opposition D . By Definition 21, D is a square of opposition but not a classical square. By Theorem 4, it follows that $\models_D \chi_{\text{square}}$ and $\not\models_D \chi_{\text{classical}}$. Hence, there exists an AD-frame F that is based on D such that $F \not\models \chi_{\text{classical}}$ and $F \models \chi_{\text{square}}$.

For item 3, we show that $\chi_{classical}$ is AD-contingent, i.e., $\not\models_{AD} \chi_{classical}$ and $\not\models_{AD} \neg\chi_{classical}$ (all other cases are analogous). Consider some degenerate square D and some classical square D' . By Definition 21 and Theorem 4, it follows that $\not\models_D \chi_{classical}$ and $\models_{D'} \chi_{classical}$. From $\not\models_D \chi_{classical}$, it follows that there exists an AD-frame F that is based on D such that $F \not\models \chi_{classical}$ (and thus $\not\models_{AD} \chi_{classical}$). From $\models_{D'} \chi_{classical}$, it follows that $F_{D'} \models \chi_{classical}$ (and thus $\not\models_{AD} \neg\chi_{classical}$). \square

It remains to be determined what the Aristotelian diagram for $(\mathcal{F}_{AD}, \models_{AD})$ looks like exactly, i.e., which Aristotelian family it belongs to. This is straightforward to check:

Lemma 15. *The following hold:*

1. $\models_{AD} \neg(\chi_{classical} \wedge \chi_{degenerate})$,
2. $\not\models_{AD} \chi_{classical} \vee \chi_{degenerate}$,
3. $\models_{AD} \neg(\chi_{classical} \wedge \neg\chi_{square})$,
4. $\not\models_{AD} \chi_{classical} \vee \neg\chi_{square}$,
5. $\models_{AD} \neg(\chi_{degenerate} \wedge \neg\chi_{square})$,
6. $\not\models_{AD} \chi_{degenerate} \vee \neg\chi_{square}$.

Proof. These are all easy exercises in reasoning about AD-frames. Note that item 1 has already been proved above, as item 2 of Lemma 11. \square

Items 1–2 of Lemma 15 state that $\chi_{classical}$ and $\chi_{degenerate}$ are contrary to each other in AD-logic (see Definition 1), which we abbreviate as $C_{AD}(\chi_{classical}, \chi_{degenerate})$. Completely analogously, items 3–4 state that $C_{AD}(\chi_{classical}, \neg\chi_{square})$, and items 5–6 state that $C_{AD}(\chi_{degenerate}, \neg\chi_{square})$. Since $|\mathcal{F}_{AD}| = 6$, it follows by Definition 22 that the Aristotelian diagram for $(\mathcal{F}_{AD}, \models_{AD})$ is a JSB hexagon, as shown in Figure 9.

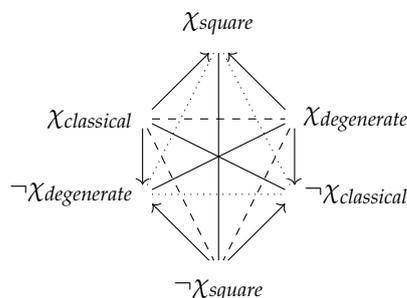


Figure 9. A JSB hexagon for $(\mathcal{F}_{AD}, \models_{AD})$.

We finish this section by making three comments about this JSB hexagon for AD-logic, in increasing order of conceptual importance. First of all, recall that it is common in logical geometry to distinguish two subtypes of JSB hexagons, viz., strong and weak [28]. In general, given a JSB hexagon H for (\mathcal{F}, S) with $C_S(\alpha, \beta)$, $C_S(\alpha, \gamma)$ and $C_S(\beta, \gamma)$, we say that H is a *strong* JSB hexagon iff $\models_S \alpha \vee \beta \vee \gamma$ and that H is a *weak* JSB hexagon iff $\not\models_S \alpha \vee \beta \vee \gamma$. With this definition in mind, it is easy to check that the hexagon for $(\mathcal{F}_{AD}, \models_{AD})$, as shown in Figure 9, is a *strong* JSB hexagon, since it follows from Lemma 11 that $\models_{AD} \chi_{classical} \vee \chi_{degenerate} \vee \neg\chi_{square}$.

Secondly, next to the standard research on Aristotelian diagrams for (formulas coming from) the *object language* \mathcal{L}_S of some logical system S , there is also a flourishing tradition of studying Aristotelian diagrams for *metalogical* notions [25,60–65]. For example, given a logic S , we can say that the Aristotelian relations of S -contrariety and S -subcontrariety are themselves contrary to each other (since two \mathcal{L}_S -formulas cannot simultaneously be S -contraries and S -subcontraries, while they can be neither S -contraries nor S -subcontraries, e.g., when they are S -contradictories). From this perspective, the JSB hexagon for $(\mathcal{F}_{AD}, \models_{AD})$, as shown in Figure 9, is simultaneously an object-level and a meta-level Aristotelian diagram. On the one hand, this hexagon is situated at the object level, because it is concerned

with formulas such as $\chi_{classical}$ and $\chi_{degenerate}$, which come from the object language \mathcal{L}_{AD} of AD-logic. On the other hand, this hexagon is, at the same time, situated at the meta-level as well, because of the correspondence results established in Section 5. For example, $C_{AD}(\chi_{classical}, \chi_{degenerate})$ means that the Aristotelian families of classical and degenerate squares are contrary to each other (since an Aristotelian diagram cannot simultaneously be a classical and a degenerate square, while it can be neither a classical nor a degenerate square, e.g., when it is a PCD).

Finally, and perhaps most subtly, note that because of their specific setup, Aristotelian diagrams for AD-logic can simultaneously contain and validate a given formula. Concretely, let us write D to refer to the JSB hexagon in Figure 9, and consider the formula $\neg\chi_{square}$. On the one hand, we have $\neg\chi_{square} \in \mathcal{F}_{AD}$ and, thus, the formula $\neg\chi_{square}$ occurs inside D when D is viewed as an Aristotelian diagram for $(\mathcal{F}_{AD}, \models_{AD})$. On the other hand, D is a JSB hexagon, and thus not a square, which means that $\models_D \neg\chi_{square}$, i.e., the formula $\neg\chi_{square}$ is validated by D , when D is viewed as an AD-frame. It bears emphasizing that there is nothing paradoxical or inconsistent about this situation. Nevertheless, we do acknowledge that this can quickly get very confusing, and that is precisely the reason why we postponed discussing Aristotelian diagrams for AD-logic until this section, at the very end of the paper.

7. Conclusions

In this paper, we developed the system of AD-logic, i.e., the (hybrid) modal logic of Aristotelian diagrams. We established a sound and strongly complete axiomatization for AD-logic, and proved that there exists a bijection between finite Aristotelian diagrams (up to Aristotelian isomorphism) and finite AD-frames (up to modal isomorphism). This means that the properties of AD-frames are sufficient to derive all structural properties of Aristotelian diagrams. We then showed how AD-logic can capture several major results from logical geometry; for example, for every well-known Aristotelian family A , we exhibited a formula $\chi_A \in \mathcal{L}_{AD}$ and showed that an Aristotelian diagram D belongs to the family A iff χ_A is validated by D (when the latter is viewed as an AD-frame). These correspondence results do not refer to any underlying logical system, and thus offer fully generic characterizations of the Aristotelian families. Finally, we showed that AD-logic gives rise to interesting new Aristotelian diagrams, and we reflected on their profoundly peculiar status.

There are many avenues for future research. For example, continuing along the lines of Section 6, we can investigate further Aristotelian diagrams for AD-logic. In the context of Section 5, we can continue to formalize significant portions of logical geometry in AD-logic. For example, we can define characterizing formulas $\chi_A \in \mathcal{L}_{AD}$ for less well-known Aristotelian families A . More fundamentally, in logical geometry, we not only consider Aristotelian (families of) diagrams in isolation of each other, but also study how they are related to each other. For example, there is work on the notions of *subdiagrams* (e.g., in each JSB hexagon, three classical squares of opposition can be embedded as subdiagrams) and *complementarity* (e.g., a JSB hexagon is complementary to a Buridan octagon, relative to an Aristotelian rhombic dodecahedron) [66–68]. Is AD-logic sufficiently expressive to formalize such relational claims as well?

Finally, we also need to consider what *cannot* be expressed in AD-logic. In particular, in the development of a systematic typology of Aristotelian diagrams, it has become clear that many (though not all) Aristotelian families can be divided into *Boolean subfamilies*; for example, there are two Boolean subfamilies of JSB hexagons (usually called “strong” and “weak”; we briefly encountered these in Section 6), there are three Boolean subfamilies of Buridan octagons, etc. These Boolean subfamilies *cannot* be expressed in AD-logic; for example, there do not exist any formulas $\chi_{strongJSB}, \chi_{weakJSB} \in \mathcal{L}_{AD}$ such that D is a strong JSB hexagon iff $\models_D \chi_{strongJSB}$ and D is a weak JSB hexagon iff $\models_D \chi_{weakJSB}$. At first sight, this might appear to be a disadvantage or a limitation of AD-logic. However, one could also argue that this is exactly what we want, in order to keep AD-logic fully

generic, i.e., independent of any underlying logic S . For example, whether a JSB hexagon for (\mathcal{F}, S) is strong or weak fundamentally depends on the underlying logic S ; from the perspective of AD-logic, we do not (want to) have “access” to the entire logic S , but only to the Aristotelian relations CD_S and C_S that it gives rise to (via the abstract relations R^{CD} and R^C in the AD-frames). It is clear that more thorough reflection is needed regarding this subtle issue.

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