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Parametric Expansions of an Algebraic Variety near Its Singularities

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Abstract: Presently, there is a method based on Power Geometry that allows one to find asymptotic forms and asymptotic expansions of solutions to different kinds of non-linear equations near their singularities. The method contains three algorithms: (1) Reducing the equation to its normal form, (2) separating truncated equations, and (3) power transformations of coordinates. Here, we describe the method for the simplest case, a single algebraic equation, and apply it to an algebraic variety, as described by an algebraic equation of order 12 in three variables. The variety was considered in study of Einstein's metrics and has several singular points and singular curves. Near some of them, we compute a local parametric expansion of the variety.

Keywords: algebraic variety; singular point; local parametrization; power geometry

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1. Introduction

Here, we propose a new method for the solution of a polynomial equation

$$f(x_1, \dots, x_n) = 0 \tag{1}$$

near its singular point. In this example, we demonstrate computations of the method for a certain polynomial, f and $n = 3$.

This method is used:

I. The *Newton polyhedron* for separation of truncated equations and

II. *Power transformations* for the simplification of these equations.

Below, we provide a short history of both of these objects.

I. The *Newton polyhedron*. For $n = 2$, in approximately 1670, Newton [1] suggested to use one edge of the "Newton open polygon" [2] (Part I, Ch. I, § 2) of a polynomial $f(x, y)$ to find the branches of solutions to the equation $f(x, y) = 0$, in the form

$$y = \sum b_p x^p \tag{2}$$

with rational power exponents p near the origin $x = y = 0$, where the polynomial f has no constant and linear terms. Puiseux [3] was already using all the edges of the Newton open polygon and had given a rigorous substantiation to the solution of the problem by this method. Liouville [4] was using this approach to find the rational solutions $y = y(x)$ to the linear ordinary differential equation

$$a_0(x) d^n y/dx^n + \dots + a_{n-1}(x) dy/dx + a_n(x) = 0,$$

where $a_i(x)$ are polynomials. Briot and Bouquet [5] were using an analog to the Newton open polygon to find solutions $y(x)$ to the nonlinear ordinary differential equation $dy/dx = f(x,y)/g(x,y)$ near the point $x = y = 0$, where polynomials f and g vanish. A survey of other applications of the Newton (open) polygon was made by Chebotarev [6].

Some properties of solutions in the form of expansions (2) were studied in [7]. In ([2] Part I, Chapter I, § 2, Section 2.9), a method was proposed for computing the second type of solutions to equation $f(x,y) = 0$, in the form

$$x = \sum_i b_i \tau^{p_i}, \quad y = \sum_i c_i \tau^{q_i}, \tag{3}$$

where p_i and q_i are integral numbers and τ is a parameter. It must use continued fractions and power transformations.

For $n > 2$, in 1962, the Newton polyhedron was introduced in [8] for an autonomous system of the ordinary differential equation (ODE) and was called a p o l y h e d r o n $M(G)$. It was used in [9,10] to study solutions to the Equation (1), in the form

$$x_n = \varphi(x_1, \ldots, x_{n-1}), \tag{4}$$

where φ is a power series in a rational exponent of its arguments. Moreover, in [10], supports of the series φ in (4) belong to some cones. Such expansions were considered in [2] (Part I, Chapter I, § 3). However, not all solutions to Equation (1) have the form (4). Here, we consider solutions of the form

$$x_i = \sum_p \varphi_{ip}(t_1, \ldots, t_{n-2})\varepsilon^p, \quad i = 1, \ldots, n,$$

where coefficients $\varphi_{ip}(t_1, \ldots, t_{n-2})$ are rational functions of global parameters $T = (t_1, \ldots, t_{n-2})$ and power exponents p of the small parameter ε are integers.

II. The *power transformation* was used by Newton [1] and all his followers in the simplest form of $y = x^\alpha z$. Weierstrass [11] suggested the sequence of transformations $y = xz$ and $x = zy$ were analogous to the σ-process in Algebraic Geometry. Power transformations in the general form of $\log X = \alpha \log Y$ were suggested in [8]. Hironaka [12] proved the resolution of singularities of any algebraic variety by means of a σ-process. However, power transformations make that happen more quickly (see [2] (Part I, Chapter I, § 2, Section 2.10)).

Here, the basic ideas of this method are explained for the simplest case: a single algebraic equation. In Section 2, we provide a generalization of the Implicit Function Theorem. In Sections 3 and 4, we provide some constructions of Power Geometry [13]. In Section 5, we explain a way of the computation of asymptotic parametric expansions of solutions. In Section 6, we demonstrate the computation of an example in detail.

2. The Implicit Function Theorem

Let $X = (x_1, \ldots, x_n)$, $Q = (q_1, \ldots, q_n)$, then

$$X^Q = x_1^{q_1}, \ldots, x_n^{q_n}, \quad \|Q\| = q_1 + \ldots + q_n.$$

Theorem 1. *Let*

$$f(X, \varepsilon, T) = \Sigma a_{Q,r}(T) X^Q \varepsilon^r, \tag{5}$$

where $0 \leq Q \in \mathbb{Z}^n$, $0 \leq r \in \mathbb{Z}$, the sum is finite and $a_{Q,r}(T)$ are some functions of $T = (t_1, \ldots, t_m)$, besides $a_{00}(T) \equiv 0$, $a_{01}(T) \not\equiv 0$. Then, the solution to the equation $f(X, \varepsilon, T) = 0$ has the form

$$\varepsilon = \Sigma b_R(T) X^R, \tag{6}$$

where $0 \leq R \in \mathbb{Z}^n$, $0 < \|R\|$, the coefficients $b_R(T)$ are functions on T that are polynomials from $a_{Q,r}(T)$ with $\|Q\| + r \leq \|R\|$ divided by $a_{01}^{2\|R\|-1}$. The expansion (6) is unique.

This is a generalization of Theorem 1.1 of [13] (Ch. II) on the implicit function and simultaneously a theorem on reducing the algebraic Equation (5) to its normal form (6) when the linear part $a_{01}(T) \not\equiv 0$ is non-degenerate. In it, we must exclude the values of T near the zeros of the function $a_{01}(T)$.

Let $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$ or $\mathbb{C}^n$, and $f(X)$ be a polynomial. A point $X = X^0$, $f(X^0) = 0$ is called *simple* if the vector $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ in it is non-zero. Otherwise, the point $X = X^0$ is called *singular* or *critical*. By shifting $X = X^0 + Y$, we move the point X^0 to the origin $Y = 0$. If at this point the derivative $\partial f/\partial x_n \neq 0$, then near X^0 all solutions to the equation $f(X) = 0$ have the form $y_n = \Sigma b_{q_1,\ldots q_{n-1}} y_1^{q_1} \cdots y_{n-1}^{q_{n-1}}$, that is, they lie in the $(n-1)$-dimensional space.

3. The Newton Polyhedron

Let the point $X^0 = 0$ be singular. Write the polynomial in the form

$$f(X) = \Sigma a_Q X^Q,$$

where $a_Q = \text{const} \in \mathbb{R}$, or $\mathbb{C}$. Let $\mathbf{S}(f) = \{Q : a_Q \neq 0\} \subset \mathbb{R}^n$.

The set $\mathbf{S}$ is called the *support* of the polynomial $f(X)$. Let it consist of points $Q_1, \ldots, Q_k$. The convex hull of the support $\mathbf{S}(f)$ is the set

$$\Gamma(f) = \left\{ Q = \sum_{j=1}^{k} \mu_j Q_j, \quad \mu_j \geq 0, \quad \sum_{j=1}^{k} \mu_j = 1 \right\},$$

which is called the *Newton polyhedron*.

Its boundary $\partial\Gamma(f)$ consists of generalized faces of $\Gamma_j^{(d)}$, where d is its dimension of $0 \leq d \leq n-1$ and j is its number. The numbering is unique for all dimensions d.

Each (generalized) face $\Gamma_j^{(d)}$ corresponds to its:

- *Boundary subset:*
$$\mathbf{S}_j^{(d)} = \mathbf{S} \cap \Gamma_j^{(d)},$$
- *Truncated polynomial:*
$$\hat{f}_j^{(d)}(X) = \Sigma a_Q X^Q \text{ over } Q \in \mathbf{S}_j^{(d)}, \text{ and}$$
- *Normal: cone:*
$$\mathbf{U}_j^{(d)} = \left\{ P : \langle P, Q' \rangle = \langle P, Q'' \rangle > \langle P, Q''' \rangle,\ Q', Q'' \in \mathbf{S}_j^{(d)},\ Q''' \in \mathbf{S} \backslash \mathbf{S}_j^{(d)} \right\},$$

where $P = (p_1, \ldots, p_n) \in \mathbb{R}_*^n$, the space $\mathbb{R}_*^n$ is conjugate (dual) to the space $\mathbb{R}^n$ and $\langle P, Q \rangle = p_1 q_1 + \ldots + p_n q_n$ is the scalar product.

At $X \to 0$, solutions to the full equation $f(X) = 0$ tend to non-trivial solutions of those truncated equations $\hat{f}_j^{(d)}(X) = 0$, whose normal cone $\mathbf{U}_j^{(d)}$ intersects with the negative orthant $P \leq 0$ in $\mathbb{R}_*^n$.

4. Power Transformations

Let $\ln X = (\ln x_1, \ldots, \ln x_n)$. The linear transformation of the logarithms of the coordinates is

$$(\ln y_1, \ldots, \ln y_n) \stackrel{\text{def}}{=} \ln Y = (\ln X)\alpha, \tag{7}$$

where α, a nondegenerate square n-matrix, is called a *power transformation*.

In the power transformation (7), the monomial X^Q transforms into the monomial Y^R, where $R = Q(\alpha^*)^{-1}$, and the asterisk indicates a transposition.

A matrix α is called *unimodular* if all its elements are integers and $\det \alpha = \pm 1$. For an unimodular matrix α, its inverse α^{-1} and transpose α^* are also unimodular.

Theorem 2. *For the face $\Gamma_j^{(d)}$, there exists a power transformation (7) with the unimodular matrix α which reduces the truncated sum $\hat{f}_j^{(d)}(X)$ to the sum from d coordinates, that is, $\hat{f}_j^{(d)}(X) = Y^S \hat{g}_j^{(d)}(Y)$, where $\hat{g}_j^{(d)}(Y) = \hat{g}_j^{(d)}(y_1, \dots, y_d)$ is a polynomial. Here, $S \in \mathbb{Z}^n$. The additional coordinates $y_{d+1}, \dots, y_n$ are local (small).*

The article [14] specifies an algorithm for computing the unimodular matrix α of Theorem 2.

5. Parametric Expansion of Solutions

Let $\Gamma_j^{(d)}$ be a face of the Newton polyhedron $\Gamma(f)$. Let the full equation $f(X) = 0$ be changed into the equation $g(Y) = 0$ after the power transformation of Theorem 2. Thus, $\hat{g}_j^{(d)}(y_1, \dots, y_d) = g(y_1, \dots, y_d, 0, \dots, 0)$.

Let the polynomial $\hat{g}_j^{(d)}$ be the product of several irreducible polynomials

$$\hat{g}_j^{(d)} = \prod_{k=1}^{m} h_k^{l_k}(y_1, \dots, y_d), \tag{8}$$

where $0 < l_k \in \mathbb{Z}$. Let the polynomial h_k be one of them. Three cases are possible:

Case 1. The equation $h_k = 0$ has a polynomial solution $y_d = \varphi(y_1, \dots, y_{d-1})$. Then, in the full polynomial $g(Y)$, let us substitute the coordinates

$$y_d = \varphi + z_d,$$

for the resulting polynomial $h(y_1, \dots, y_{d-1}, z_d, y_{d+1} \dots, y_n)$, and again construct the Newton polyhedron, separate the truncated polynomials, etc. Such calculations were provided in the Introduction to [13].

Case 2. The equation $h_k = 0$ has no polynomial solution, but has a parametrization of solutions

$$y_j = \varphi_j(T), j = 1, \dots, d, \quad T = (t_1, \dots, t_{d-1}).$$

Then, in the full polynomial $g(Y)$, we substitute the coordinates

$$y_j = \varphi_i(T) + \beta_j \varepsilon, \quad j = 1, \dots, d, \tag{9}$$

where $\beta_j = \text{const}$, $\Sigma|\beta_j| \neq 0$, and from the full polynomial $g(Y)$, we obtain the polynomial

$$h = \Sigma a_{Q'',r}(T) Y''^{Q''} \varepsilon^r, \tag{10}$$

where $Y'' = (y_{d+1}, \dots, y_n)$, $0 \leq Q'' = (q_{d+1}, \dots, q_n) \in \mathbb{Z}^{n-d}$, $0 \leq r \in \mathbb{Z}$. Thus, $a_{00}(T) \equiv 0$, $a_{01}(T) = \sum_{j=1}^{d} \beta_j \partial \hat{g}_j^{(d)} / \partial y_j(T)$.

If in the expansion (8) $l_k = 1$, then $a_{01} \not\equiv 0$. By Theorem 1, all solutions to the equation $h = 0$ have the form

$$\varepsilon = \Sigma b_{Q''}(T) Y''^{Q''},$$

, i.e., according to (9), the solutions to the equation $g = 0$ have the form

$$y_j = \varphi_j(T) + \beta_j \Sigma b_{Q''}(T) Y''^{Q''}, \quad j = 1, \dots, d.$$

Such calculations were proposed in [15] and will be shown in the following example.

If in (8) $l_k > 1$, then in (10) $a_{01}(T) \equiv 0$ and for the polynomial (10) from Y'', ε, we construct the Newton polyhedron by supporting $\mathbf{S}(h) = \{Q'', r : a_{Q'',r}(T) \not\equiv 0\}$, separating the truncations, and so on.

Case 3. The equation $h_k = 0$ has neither a polynomial solution nor a parametric one. Then, using Hadamard's polyhedron [15], one can compute a piecewise approximate parametric solution to the equation $h_k = 0$ and look for an approximate parametric expansion.

Similarly, one can study the position of an algebraic manifold in infinity.

A more conventional approach is given in [16].

6. Variety Ω and Its Singularities

In [17–24], the investigation of the three-parametric family of special homogeneous spaces from the viewpoint of the normalized Ricci flow was started. The Ricci flows describe the evolution of Einstein's metrics on a variety. The equations of the normalized Ricci flow are reduced to a system of two differential equations with three parameters: a_1, a_2 and a_3:

$$\begin{aligned} \frac{dx_1}{dt} &= \tilde{f}_1(x_1, x_2, a_1, a_2, a_3), \\ \frac{dx_2}{dt} &= \tilde{f}_2(x_1, x_2, a_1, a_2, a_3), \end{aligned} \tag{11}$$

here, $\tilde{f}_1$ and $\tilde{f}_2$ are certain functions.

The singular point of this system are associated with the invariant Einstein's metrics. At the singular (stationary) point x_1^0, x_2^0, system (11) has two eigenvalues, λ_1 and λ_2. If at least one of them is equal to zero, then the singular (fixed) point x_1^0, x_2^0 is said to be *degenerate*. It was proved in [17–24] that the set Ω of the values of the parameters a_1, a_2, a_3, in which system (11) has at least one degenerate singular point, is described by the equation

$$\begin{aligned} Q(s_1, s_2, s_3) \equiv &(2s_1 + 4s_3 - 1)\Big(64s_1^5 - 64s_1^4 + 8s_1^3 + 240s_1^2 s_3 - 1536 s_1 s_3^2 - \\ &-4096 s_3^3 + 12 s_1^2 - 240 s_1 s_3 + 768 s_3^2 - 6s_1 + 60 s_3 + 1\Big) - \\ &- 8 s_1 s_2 (2s_1 + 4s_3 - 1)(2s_1 - 32 s_3 - 1)(10 s_1 + 32 s_3 - 5) - \\ &- 16 s_1^2 s_2^2 \Big(52 s_1^2 + 640 s_1 s_3 + 1024 s_3^2 - 52 s_1 - 320 s_3 + 13\Big) + \\ &+ 64(2s_1 - 1) s_2^3 (2s_1 - 32 s_3 - 1) + 2048 s_1 (2s_1 - 1) s_2^4 = 0, \end{aligned}$$

where s_1, s_2, s_3 are elementary symmetric polynomials, equal, respectively, to

$$s_1 = a_1 + a_2 + a_3, \quad s_2 = a_1 a_2 + a_1 a_3 + a_2 a_3, \quad s_3 = a_1 a_2 a_3.$$

In [25], for symmetry reasons, the coordinates $\mathbf{a} = (a_1, a_2, a_3)$ were changed to the coordinates $\mathbf{A} = (A_1, A_2, A_3)$ by the linear transformation

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = M \cdot \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, \quad M = \begin{pmatrix} (1+\sqrt{3})/6 & (1-\sqrt{3})/6 & 1/3 \\ (1-\sqrt{3})/6 & (1+\sqrt{3})/6 & 1/3 \\ -1/3 & -1/3 & 1/3 \end{pmatrix}$$

Definition 1. *Let $\varphi(X)$ be some polynomial, $X = (x_1, \ldots, x_n)$. A point $X = X^0$ of the set $\varphi(X) = 0$ is called the **singular point of the** k**-order**, if all partial derivatives of the polynomial $\varphi(X)$ for the $x_1, \ldots, x_n$ turn into zero at this point, up to and including the k-th order derivatives, and at least one partial derivative of order $k+1$ is nonzero.*

In [25], all singular points of the variety Ω in coordinates $\mathbf{A} = (A_1, A_2, A_3)$ were found. The five points of the third order are:

Name	Coordinates $\mathbf{A}$
$P_1^{(3)}$	$(0, 0, 3/4)$
$P_2^{(3)}$	$(0, 0, -3/2)$
$P_3^{(3)}$	$\left(-\frac{1+\sqrt{3}}{2}, \frac{\sqrt{3}-1}{2}, \frac{1}{2}\right)$
$P_4^{(3)}$	$\left(\frac{\sqrt{3}-1}{2}, -\frac{1+\sqrt{3}}{2}, \frac{1}{2}\right)$
$P_5^{(3)}$	$(1, 1, 1/2)$

three points of the second order

Name	Coordinates $\mathbf{A}$
$P_1^{(2)}$	$\left(\frac{1+\sqrt{3}}{4}, \frac{1-\sqrt{3}}{4}, \frac{1}{2}\right)$
$P_2^{(2)}$	$\left(\frac{1-\sqrt{3}}{4}, \frac{1+\sqrt{3}}{4}, \frac{1}{2}\right)$
$P_3^{(2)}$	$(-1/2, -1/2, 1/2)$

and three more algebraic curves of singular points of the first order:

$$\mathcal{F} = \left\{a_1 = a_2, \quad 16a_1^3 + 16a_1^2 a_3 - 4a_1 - 2a_3 + 1 = 0\right\},$$

$$\mathcal{I} = \left\{A_1 + A_2 + 1 = 0, \quad A_3 = \frac{1}{2}\right\},$$

$$\mathcal{K} = \left\{A_1 = -\frac{9}{4} t h(t),\ A_2 = -\frac{9}{4} h(t),\ A_3 = \frac{3}{4},\ h(t) = \frac{t^2+1}{(t+1)(t^2-4t+1)}\right\}.$$

The points $P_3^{(3)}$, $P_4^{(3)}$ and $P_5^{(3)}$ are of the same type; they pass into each other when rotated in the plane A_1, A_2 by an angle $2\pi/3$, just as all points $P_1^{(2)}$, $P_2^{(2)}$, $P_3^{(2)}$. The curves $\mathcal{F}$, $\mathcal{I}$, $\mathcal{K}$ correspond to two more curves of the same type. Therefore, it is sufficient to study the variety Ω in the neighborhood of points $P_1^{(3)}$, $P_2^{(3)}$, $P_5^{(3)}$, $P_3^{(2)}$ and curves $\mathcal{F}$, $\mathcal{I}$ and $\mathcal{K}$. In Sections 7–10, the neighborhoods of points $P_1^{(3)}$, $P_2^{(3)}$, line $\mathcal{I}$ and point $P_3^{(2)}$ are studied, correspondingly. The methods proposed in [15] and described in Sections 2–5 are implemented.

In coordinates $\mathbf{A}$, the variety Ω is described by a very cumbersome polynomial of degree 12 $R(\mathbf{A}) = Q(\mathbf{s}) = 0$ with rational coefficients, because the transformation from $\mathbf{s}$ to $\mathbf{A}$ has rational coefficients.

In the paper [26], three variants of the global parametrization of the variety Ω were proposed. These parametrizations were computed using the parametric description of the discriminant set of a monic cubic polynomial [27] and can be written in radical form [28]. Such a global description of the variety Ω cannot provide an adequate picture of the Ω structure in the vicinity of its singular points.

7. The Structure of the Variety Ω near the Singular Point $P_1^{(3)}$

Near the point $P_1^{(3)}$, let us introduce the local coordinates x_1, x_2, x_3:

$$A_1 = x_1, \quad A_2 = x_2, \quad A_3 = x_3 + 3/4$$

and from the polynomial $R(\mathbf{A})$, we obtain a polynomial of degree 12 $S_1(x_1, x_2, x_3) = R(\mathbf{A}) = Q(s_1, s_2, s_3)$. We calculate its support, the Newton polyhedron Γ_1, and its faces $\Gamma_j^{(2)}$ and their external normals, using the `PolyhedralSets` package of the Maple 2021 computer algebra system [29]. We obtain five faces of $\Gamma_j^{(2)}$. The graph of the polyhedron Γ_1 is shown in Figure 1.

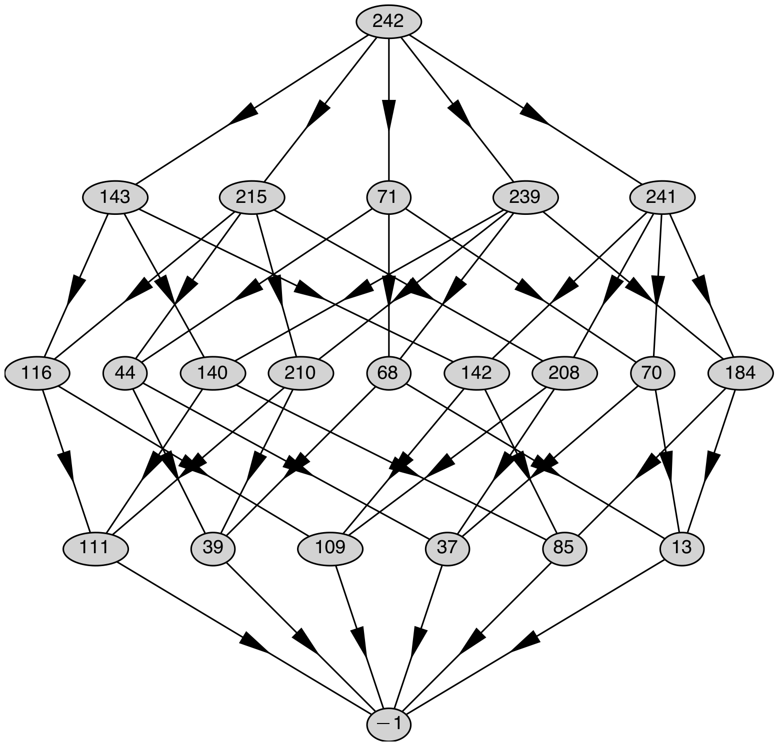

Figure 1. Graph of the polyhedron Γ_1.

Each generalized face of $\Gamma_j^{(d)}$ is presented by its number j in an oval. Numbers j are given by the program automatically. The top line of Figure 1 contains the whole polyhedron Γ_1; the next line contains all the two-dimensional faces $\Gamma_j^{(2)}$.

Below that are the edges of $\Gamma_j^{(1)}$, then the vertices of $\Gamma_j^{(0)}$, and at the bottom, the empty set.

A face $\Gamma_j^{(d)}$ is connected with a face $\Gamma_k^{(d+1)}$ by an arrow, iff $\Gamma_j^{(d)} \subset \Gamma_k^{(d+1)}$.

The external normals to its two-dimensional faces $\Gamma_j^{(2)}$ are

$$N_{71} = (-1,-1,-1/2),\ N_{143} = (1,1,1),\ N_{215} = (-1,0,0),\ N_{239} = (0,-1,0),\ N_{241} = (0,0,-1).$$

The neighborhood of the point $x_1 = x_2 = x_3 = 0$ is approximately described by the truncated equation

$$\begin{aligned} \hat{f}_1 \equiv &-\frac{4096}{81}81x_3^8 + \frac{3}{4}x_1^4 + \frac{3}{4}x_2^4 + \frac{64}{3}x_1^2x_3^4 - \frac{16}{3}x_1^3x_3^2 + \\ &+ \frac{64}{3}x_2^2x_3^4 - \frac{16}{3}x_2^3x_3^2 + \frac{3}{2}x_1^2x_2^2 + 16x_1^2x_2x_3^2 + 16x_1x_2^2x_3^2 = 0, \end{aligned}$$

corresponding to the face of $\Gamma_j^{(1)}$ of number $j = 71$ with the normal $N_{71} = (-2,-2,-1)$, which has all negative coordinates.

According to the article [14], we find the unimodular matrix $\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{pmatrix}$ such that $N\alpha = (0,0,-1)$. Consequently, we have to conduct the power transformation $(\ln y_1, \ln y_2, \ln y_3) = (\ln x_1, \ln x_2, \ln x_3) \cdot \alpha$, i.e., $(\ln x_1, \ln x_2, \ln x_3) = (\ln y_1, \ln y_2, \ln y_3) \cdot \alpha^{-1}$.

Since $\alpha^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$, then

$$x_1 = y_1 y_3^2, \quad x_2 = y_2 y_3^2, \quad x_3 = y_3. \tag{12}$$

Here, $\hat{f}_1(x_1, x_2, x_3) = y_3^8 \cdot F_1(y_1, y_2)$;

$$F_1(y_1, y_2) = -\frac{4096}{81} + \frac{3}{4}y_1^4 + \frac{3}{4}y_2^4 + \frac{64}{3}y_1^2 - \frac{16}{3}y_1^3 + \frac{64}{3}y_2^2 - \frac{16}{3}y_2^3 + \frac{3}{2}y_1^2 y_2^2 + 16 y_1^2 y_2 + 16 y_1 y_2^2. \tag{13}$$

According to the `algcurves` package from the computer algebra system `Maple`, the curve $F_1(y_1, y_2) = 0$ has genus 0, with parametrization

$$\begin{aligned} y_1 = b_1(t) = &- 8(21434756829626557083983t^4 + 141707472789159417720256 0t^3 + \\ &+ 317060381933725804615887 06t^2 + 335726200061958227448792184t + \\ &+ 8333103427347345384379)/\delta, \\ y_2 = b_2(t) = &- 56(3053430900966931440569t^4 + 198407502991736938316080t^3 + \\ &+ 388353320855325331325815 8t^2 + 919355910482049127971584 8t - \\ &- 2622628221833375066586503 23)/\delta, \\ \delta = &9\left(85576987369t^2 + 3099727166140t + 37630556816821\right)^2, \end{aligned} \tag{14}$$

and the plot shown in Figure 2.

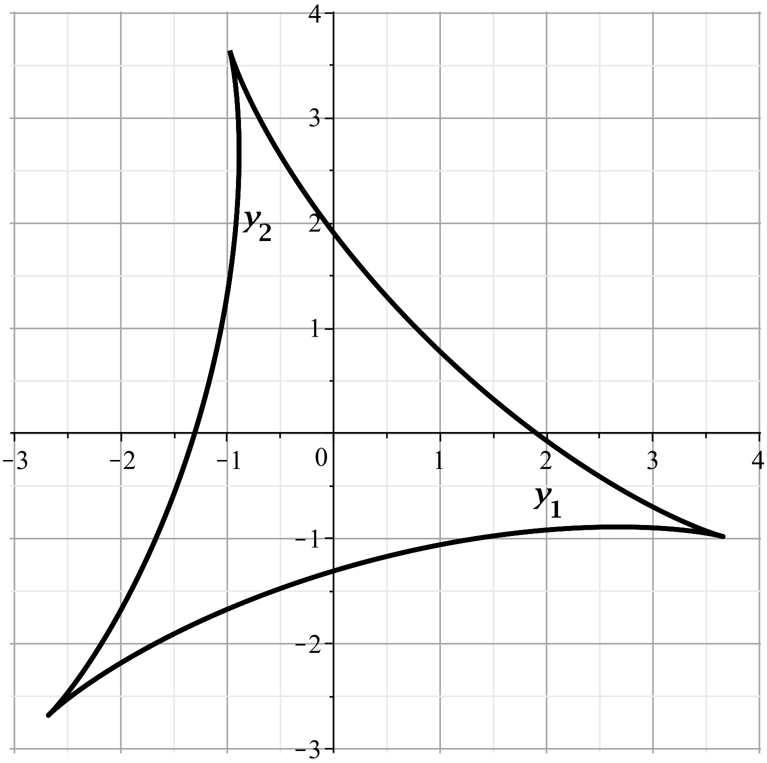

Figure 2. Plot of the curve $F_1(y_1, y_2) = 0$.

This is a curvilinear triangle with vertices

$$(y_1, y_2) = -\frac{8}{3}(1, 1), \quad -\frac{8}{3}\left(-\frac{1+\sqrt{3}}{2}, \frac{\sqrt{3}-1}{2}\right), \quad -\frac{8}{3}\left(\frac{\sqrt{3}-1}{2}, -\frac{\sqrt{3}+1}{2}\right).$$

Now, to describe the structure of the variety Ω near the point $P_1^{(3)}$, we substitute the power transformation (12) into the polynomial $S(x_1, x_2, x_3)$ and obtain the polynomial $T(y_1, y_2, y_3) = S/y_3^8$. It decomposes into the sum

$$T(y_1, y_2, y_3) = \sum_{k=0}^{m} T_k(y_1, y_2) y_3^k$$

with $T_0(y_1, y_2) = F(y_1, y_2)$ and using the command `coeff(f,x[k],m)` in CAS Maple, selecting monomials containing the factor x_k^m; for $k = 3$ and $m = 1$, we obtain

$$\begin{aligned} T_1 \stackrel{\text{def}}{=} G(y_1, y_2) = 8y_1^4 + 16y_1^2 y_2^2 + 8y_2^4 - 1216/27 y_1^3 + 1216/9 y_1^2 y_2 + 1216/9 y_1 y_2^2 - \\ -1216/27 y_2^3 + 3584/27 y_1^2 + 3584/27 y_2^2 - 65536/729. \end{aligned} \tag{15}$$

In the polynomials $T_k(y_1, y_2)$, we conduct the substitution

$$y_1 = b_1(t) + \varepsilon, \quad y_2 = b_2(t) + \varepsilon, \tag{16}$$

and obtain a polynomial $u(\varepsilon, y_3) = T(y_1, y_2, y_3)$ with coefficients depending on t through $b_1(t)$ and $b_2(t)$. In this polynomial

$$u(\varepsilon, y_3) = \sum_{k=0}^{m} T_k(b_1 + \varepsilon, b_2 + \varepsilon) y_3^k = \sum_{p,q \geqslant 0} u_{pq} \varepsilon^p y_3^q,$$

where $u_{00} = F(b_1(t), b_2(t))$ of (13), so $u_{00} \equiv 0$,

$$u_{10} = \frac{\partial F(y_1, y_2)}{\partial y_1} + \frac{\partial F(y_1, y_2)}{\partial y_2} = 3y_1^3 + 128/3 y_1 + 3y_1 y_2^2 + 3y_1^2 y_2 + 64 y_1 y_2 + 3y_2^3 + 128/3 y_2 \stackrel{\text{def}}{=} H(y_1, y_2), \tag{17}$$

and in general

$$u_{pq} = \sum_{p_1+p_2=p \geqslant 1} \frac{1}{p_1! p_2!} \cdot \frac{\partial^p T_q}{\partial y_1^{p_1} \cdot \partial y_2^{p_2}} \tag{18}$$

when $p_1, p_2 \geqslant 0, p \geqslant 1, y_i = b_i(t), i = 1, 2$, according to (14) and in the substitution of (16).

Presently, according to (15) and (17)

$$\begin{aligned} u_{10}(t) = H(b_1(t), b_2(t)) = \\ & - 32768(254517259607t^2 + 8638940893220t + 63662194408079)^3 \times \\ & \times (23525t + 3508186)^4 / (243\gamma^5), \\ u_{01}(t) = G(b_1(t), b_2(t)) = \\ & 5242880(23525t + 3508186)^4 \times \\ & \times (254517259607t^2 + 8638940893220t + 63662194408079)^4 / \\ & /(19683\gamma^6), \\ \gamma = 85576987369t^2 + 3099727166140t + 37630556816821. \end{aligned}$$

The functions $u_{10}(t)$ and $u_{01}(t)$ each have three multiple roots

$$t_1 = -\frac{3508186}{23525}, \quad t_{2,3} = -\frac{4319470446610}{254517259607} \pm \frac{904562081493\sqrt{3}}{254517259607}. \tag{19}$$

The values correspond to the vertices of the curvilinear triangle of Figure 4.

According to Theorem 1 on the implicit function, the equation $u(\varepsilon, y_3) = 0$ has the solution as the power series over y_3

$$\varepsilon = \sum_{k=1}^{\infty} c_k(t) \cdot y_3^k, \tag{20}$$

where $c_k(t)$ are rational functions that are expressed via the coefficients $u_{pq}(t)$, which in turn are expressed via $b_1(t)$ and $b_2(t)$ according to (18). This decomposition is valid for all values of t, except maybe the roots in (19). In particular,

$$c_1(t) = -\frac{u_{01}}{u_{10}} = -\frac{G}{H} = \frac{160(254517259607t^2 + 8638940893220t + 63662194408079)}{81(85576987369t^2 + 3099727166140t + 37630556816821)},$$

where the denominator has no real roots. According to (20), approximate $r \approx c_1(t)y_3$.

Let us return to the initial coordinates, which for small $|y_3|$ on variety Ω are approximated by

$$A_1 = x_1 = (b_1(t) + c_1(t)y_3)y_3^2, \quad A_2 = x_2 = (b_2(t) + c_1(t)y_3)y_3^2. \tag{21}$$

If $y_3 = -1/50$, i.e., $A_3 = 73/100$, the curve (21) is shown in Figure 3.

It is similar to the curve of Figure 11 in [25], with $A_3 = 5/8$ near the origin.

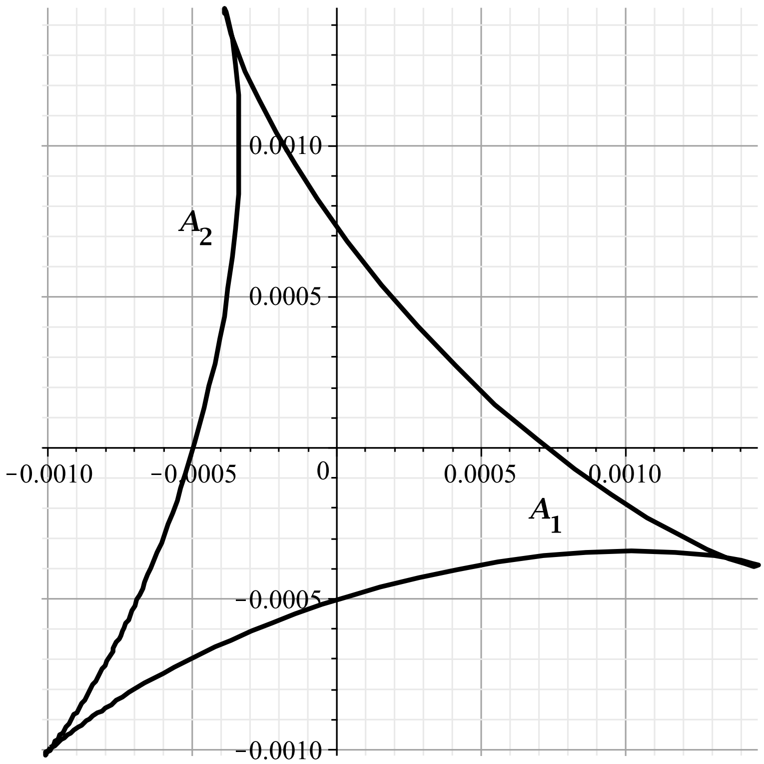

Figure 3. Plot of the curve (21) for $y_3 = -1/50$.

If $y_3 = 1/20$, i.e., $A_3 = 4/5$, the curve (21) is shown in Figure 4 and is similar to the curve of Figure 9 in [25], with $A_3 = 1$ near the origin.

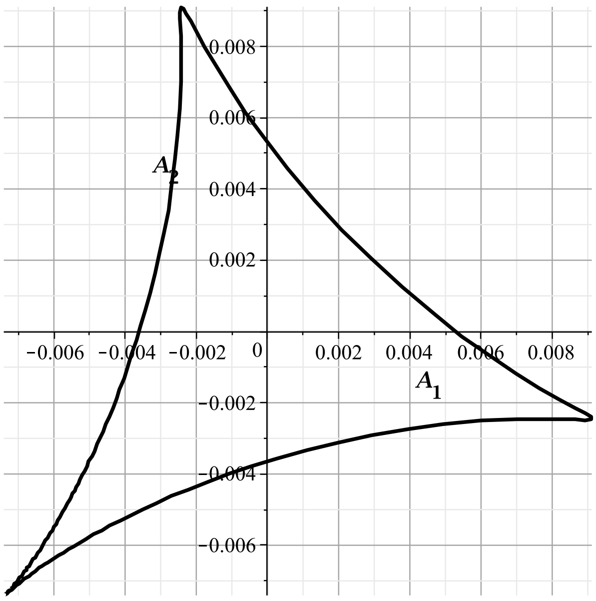

Figure 4. Plot of the curve (21) for $y_3 = 1/20$.

8. The Structure of the Variety Ω near the Singular Point $P_2^{(3)}$

Near the point $P_2^{(3)}$, we introduce the local coordinates x_1, x_2, x_3 :

$$A_1 = x_1,\ A_2 = x_2,\ A_3 = -\frac{3}{2} + x_3 \tag{22}$$

and from the polynomial $R(\mathbf{A})$, we obtain a polynomial of degree 12

$$S_2(x_1, x_2, x_3) = R(\mathbf{A}) = Q(s_1, s_2, s_3).$$

We compute its support, the Newton polyhedron Γ_2, its faces $\Gamma_j^{(2)}$ and their external normals using the package `PolyhedralSets` of the Computer Algebra System (CAS) `Maple 2021` [29]. We obtain five faces of $\Gamma_j^{(2)}$. The graph of the polyhedron Γ_2 is shown in Figure 5.

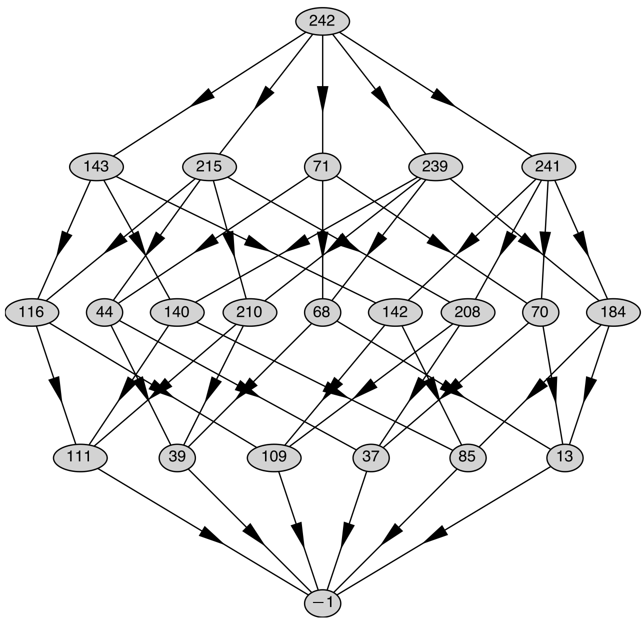

Figure 5. The graph of the polyhedron Γ_2.

The external normals of its two-dimensional faces are $N_{71} = (-1,-1,-1)$, $N_{143} = (1,1,1)$, $N_{215} = (-1,0,0)$, $N_{239} = (0,-1,0)$, $N_{241} = (0,0,-1)$.

The neighborhood of point $x_1 = x_2 = x_3 = 0$ is approximately described by zeros of the truncated polynomial

$$\hat{f}_2 = 192x_1^4 - 768x_1^3x_3 + 384x_1^2x_2^2 + 2304x_1^2x_2x_3 + 1728x_1^2x_3^2 + 2304x_1x_2^2x_3 + 192x_2^4 - 768x_2^3x_3 + 1728x_2^2x_3^2 - 1296x_3^4, \quad (23)$$

corresponding to face 71 with normal $N_{71} = (-1,-1,-1)$, which has all negative coordinates. According to the article [14], we find the unimodular matrix

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

such that

$$N\alpha = (0,0,-1).$$

Hence, we have to perform the power transformation

$$(\ln y_1, \ln y_2, \ln y_3) = (\ln x_1, \ln x_2, \ln x_3) \cdot \alpha,$$

i.e.,

$$(\ln x_1, \ln x_2, \ln x_3) = (\ln y_1, \ln y_2, \ln y_3) \cdot \alpha^{-1}.$$

Since $\alpha^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, then

$$x_1 = y_1y_3, x_2 = y_2y_3, x_3 = y_3. \quad (24)$$

Here

$$\hat{f}_1(x_1,x_2,x_3) = y_3^4 \cdot F_2(y_1,y_2);$$
$$F_2(y_1,y_2) = 192y_1^4 + 384y_2^2y_1^2 + 192y_2^4 - 768y_1^3 + 2304y_1^2y_2 + 2304y_1y_2^2 - 768y_2^3 + 1728y_1^2 + 1728y_2^2 - 1296. \quad (25)$$

According to procedure `genus` from the package `algcurves` of the CAS `Maple`, the curve $F_2(y_1,y_2) = 0$ has genus 0, with parametrization

$$\begin{aligned}
&y_1 = b_1(t) \stackrel{\text{def}}{=} \\
&(-7896743880951563283t^4 + 7295915974692179682 0t^3 + 73345644408971204346t^2 - \\
&- 93447550030150760076 4t22152086631804605310 61)/\delta, \\
&y_2 = b_2(t) \stackrel{\text{def}}{=} \\
&(-8597581719794315283t^4 + 98898571174265195220t^3 + 415981930082178074106t^2 + \\
&917438397740936497924t + 961113999918607711499)/\delta \\
&\delta = 6978188753537418722t^4 + 100485794419088992440t^3 + 590656639737168405916t^2 + \\
&1648137191526734939160t + 1877241261261332663762
\end{aligned} \quad (26)$$

and the graph shown in Figure 6.

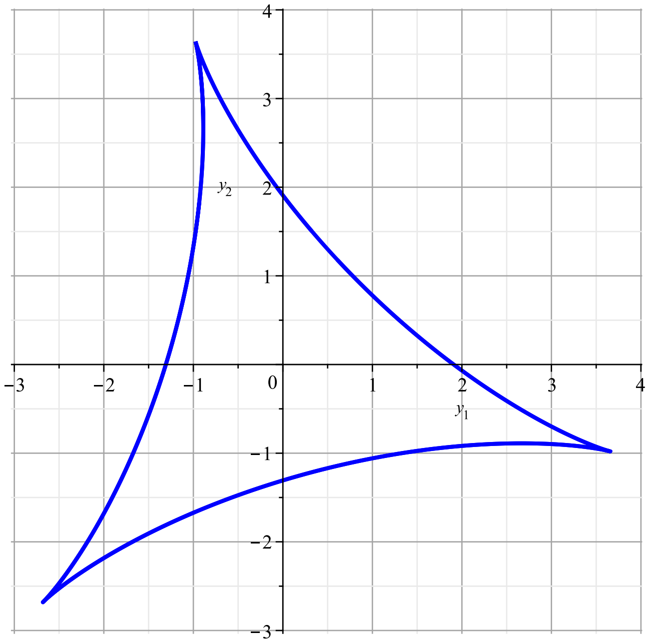

Figure 6. Plot of the curve $F_2(y_1, y_2) = 0$.

This is a curved triangle with vertices

$$(y_1, y_2) = -\frac{3}{2}(1,1),\ -\frac{3}{2}\left(-\frac{1+\sqrt{3}}{2}, \frac{\sqrt{3}-1}{2}\right),\ -\frac{3}{2}\left(\frac{\sqrt{3}-1}{2}, -\frac{1+\sqrt{3}}{2}\right) \tag{27}$$

Now, to describe the structure of the variety Ω near the point $P_2^{(3)}$, we substitute (24) into the polynomial S_2 and obtain the polynomial $T(y_1, y_2, y_3)$. It is divided into the sum

$$T(y_1, y_2, y_3) = y_3^4 \sum_{k=0}^{m} T_k(y_1, y_2) y_3^k,$$

with $T_0(y_1, y_2) = F_2(y_1, y_2)$, and using Maple's command `coeff`, we find

$$\begin{aligned} T_1 \stackrel{\text{def}}{=} G(y_1, y_2) = &-\frac{256}{3}y_1^5 + 256y_1^4y_2 + \frac{512}{3}y_1^3y_2^2 + \frac{512}{3}y_2^3y_1^2 + 256y_1y_2^4 - \frac{256}{3}y_2^5 - 256y_1^4 - \\ &-512y_2^2y_1^2 - 256y_2^4 + \frac{7808}{3}y_1^3 - 7808y_1^2y_2 - 7808y_1y_2^2 + \frac{7808}{3}y_2^3 - 6528y_1^2 + 4608. \end{aligned} \tag{28}$$

In the polynomials $T_k(y_1, y_2)$, we conduct the substitution

$$y_1 = b_1(t) + \varepsilon, y_2 = b_2(t) + \varepsilon. \tag{29}$$

We obtain a polynomial $u(\varepsilon, y_3) = T(y_1, y_2, y_3)/y_3^4$ with coefficients depending on t through $b_1(t)$ and $b_2(t)$. In this polynomial,

$$u(\varepsilon, y_3) = \sum_{k=0}^{m} T_k(b_1 + \varepsilon, b_2 + \varepsilon) y_3^k = \sum_{p,q \geqslant 0} u_{pq} \varepsilon^p y_3^q,$$

where $u_{00} = F_2(b_1(t), b_2(t))$ of (8); thus, $u_{00} = 0$,

$$\begin{aligned} u_{10} = \frac{\partial F_2(y_1, y_2)}{\partial y_1} + \frac{\partial F_2(y_1, y_2)}{\partial y_2} &= 768y_1^3 + 768y_1^2y_2 + 768y_1y_2^2 + \\ &+768y_2^3 + 9216y_1y_2 + 3456y_1 + 3456y_2 \stackrel{\text{def}}{=} H(y_1, y_2) \end{aligned} \tag{30}$$

and in general

$$u_{pq}=\sum_{p_1+p_2=p}\frac{1}{p_1!p_2!}\cdot\frac{\partial^p T_q}{\partial y_1^{p_1}\partial y_2^{p_2}}\text{ at } y_i=b_i(t),\ p_1,p_2\geqslant 0,\ p\geqslant 1, \tag{31}$$

according to the (29) replacement. Presently, according to (28) and (30)

$$\begin{aligned}
u_{10}=&H(b_1(t),b_2(t))=\\
=&-\frac{98304\xi_1^3\xi_2^4}{(1867911769t^2+13448948190t+30636916141)^5},\\
u_{01}=&G(b_1(t),b_2(t))=\\
=&-\frac{65536(42013t+132435)^2(1456633369t^2+4165088670t-21754631523)^2\xi_1^3\xi_2^4}{3(1867911769t^2+13448948190t+30636916141)^9},\\
\xi_1=&5192456907t^2+31062985050t+39519200759,\\
\xi_2=&5070t+57223.
\end{aligned}$$

The functions $u_{10}(t)$ and $u_{01}(t)$ have three multiple roots each

$$t_1=-\frac{57233}{5070},\ t_{2,3}=-\frac{5177164175}{1730818969}\pm\frac{3465328898\sqrt{3}}{5192456907}. \tag{32}$$

In addition, $u_{01}(t)$ has three more multiple roots

$$t_4=-\frac{132435}{42013},\ t_{5,6}=-\frac{2082544335}{1456633369}\pm\frac{3465328898\sqrt{3}}{1456633369}.$$

The values t_1,t_2,t_3 correspond to the vertices (27) of the curved triangle of Figure 6.

By Theorem 1, on the implicit function, the equation $u(\varepsilon,y_3)=0$ has a solution as a power series on y_3

$$\varepsilon=\sum_{k=1}^{\infty}c_k(t)\cdot y_3^k, \tag{33}$$

where $c_k(t)$ are rational functions that are expressed over the coefficients $u_{pq}(t)$, which in turn are expressed over $b_1(t)$ and $b_2(t)$ according to (31). This decomposition holds for all values of t, except, perhaps, the roots of (32). In particular,

$$c_1(t)=-\left(\frac{u_{01}}{u_{10}}\right)=-\frac{G}{H}=-\frac{2(42013t+132435)^2(1456633369t^2+4165088670t-21754631523)^2\xi_1}{9(1867911769t^2+13448948190t+30636916141)^4}$$

where the denominator has no real roots. According to (33), we obtain the approximation of $\varepsilon\approx c_1(t)y_3$.

Let us return to the original coordinates, which for small $|y_3|$ on the variety Ω are approximately equal

$$x_1=(b_1(t)+c_1(t)y_3)y_3,\ x_2=(b_2(t)+c_1(t)y_3)y_3, \tag{34}$$

in this case

$$A_1=x_1,\ A_2=x_2,\ A_3=-\frac{3}{2}+y_3.$$

If $y_3=-1/50$, i.e., $A_3=-76/50$, then the curve (34) is shown in Figure 7. It is similar to the curve of Figure 2 in [25] near the origin, corresponding to $A_3=-2$.

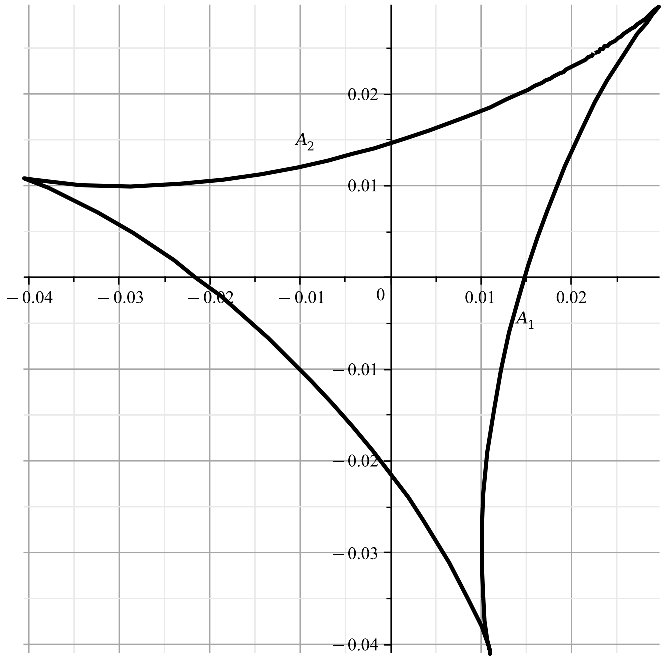

Figure 7. Plot of the curve (34) for $y_3 = -1/50$.

If $y_3 = 1/20$ $(A_3 = -29/20)$, it is shown in Figure 8 and is similar to the curve of Figure 4 in [25] near the origin, corresponding to $A_3 = -5/4$.

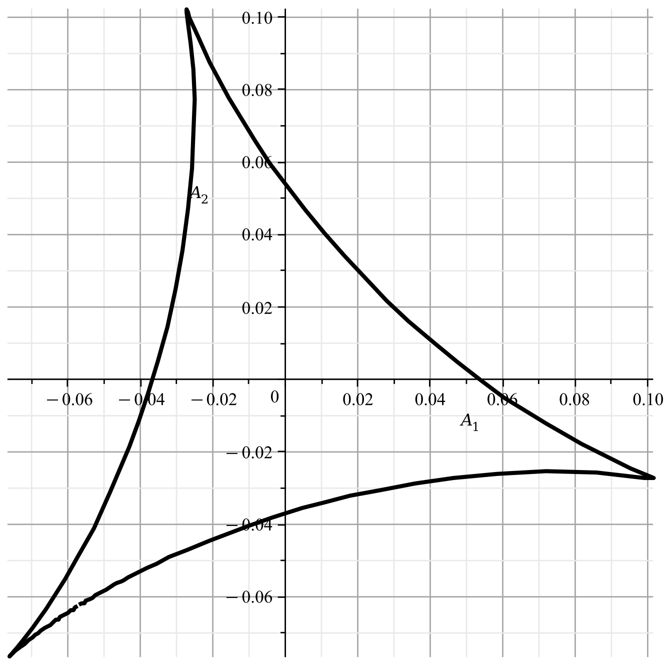

Figure 8. Plot of the curve (34) for $y_3 = 1/20$.

The similarity of these curves confirms the correctness of the found parameterization, which can be refined.

9. The Structure of the Variety Ω Near the Curve $\mathcal{J}$ of Singular Points

On the curve $\mathcal{J}$ and near it, let us introduce the local coordinates x_1, x_2, x_3 :

$$A_1 = x_1 - x_2 - \frac{1}{2},\ A_2 = x_1 + x_2 - \frac{1}{2},\ A_3 = \frac{1}{2} + x_3. \tag{35}$$

On the line $\mathcal{J}$, the coordinates $x_1 = x_3 = 0$ and x_2 are arbitrary.

From the polynomial $R(\mathbf{A})$, we obtain a polynomial of degree 12

$$S_3(x_1, x_2, x_3) = R(\mathbf{A}) = Q(s_1, s_2, s_3), \tag{36}$$

we compute its support, the Newton polyhedron Γ_3, its faces $\Gamma_j^{(2)}$ and their external normals, using the `PolyhedralSets` package of the CAS `Maple` 2021 [29]. We obtain seven faces $\Gamma_j^{(2)}$. The graph of the polyhedron Γ_3 is shown in Figure 9.

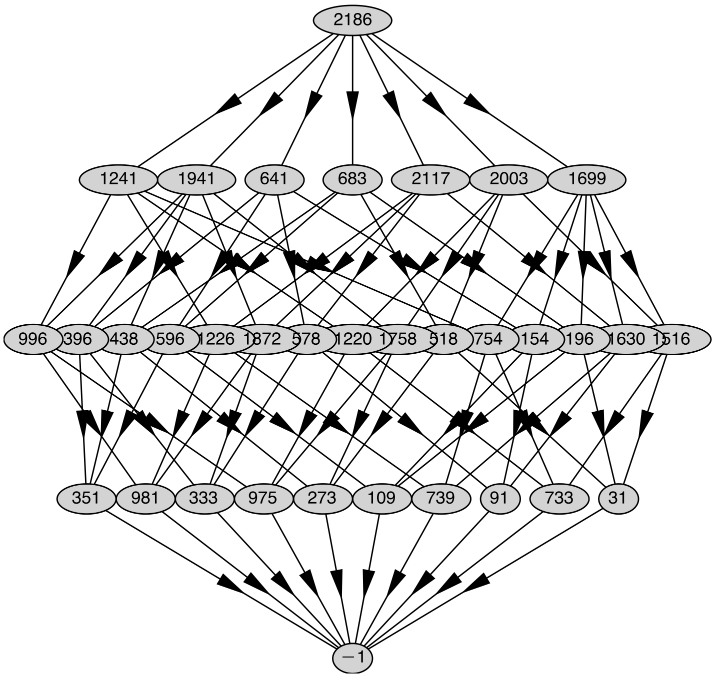

Figure 9. The graph of the polyhedron Γ_3.

The external normals of its two-dimensional faces are $N_{641} = (-1, 0, -1)$, $N_{683} = (-1, -1, -2)$, $N_{1241} = (1, 1, 1)$, $N_{1699} = (0, 0, -1)$, $N_{1941} = (-1, 0, 0)$, $N_{2003} = (0, -1, 0)$, $N_{2117} = (0, 1, 0)$.

The neighborhood of the line $x_1 = x_3 = 0$ is approximately described by the zeros of the truncated polynomial

$$\begin{aligned}\hat{f}_1 = -\frac{1024}{81}x_1^2 x_2^4 - \frac{16384}{729}x_2^8 x_1^2 + \frac{8192}{729}x_2^8 x_3^2 + \frac{8192}{243}x_1^2 x_2^6 + \frac{1664}{81}x_2^4 x_3^2 - \frac{16}{3}x_2^2 x_3^2 - \\ -\frac{6400}{243}x_2^6 x_3^2 + \frac{4096}{243}x_1 x_2^6 x_3 - \frac{8192}{729}x_2^8 x_1 x_3 - \frac{512}{81}x_1 x_2^4 x_3,\end{aligned} \tag{37}$$

corresponding to face 641 with normal $N_{641} = (-1, 0, -1)$, which has two negative coordinates. According to the paper [14], we find the unimodular matrix

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

such that

$$N\alpha = (0, 0, -1).$$

Hence, we have to perform the power transformation

$$(\ln y_1, \ln y_2, \ln y_3) = (\ln x_1, \ln x_2, \ln x_3) \cdot \alpha,$$

i.e.,

$$(\ln x_1, \ln x_2, \ln x_3) = (\ln y_1, \ln y_2, \ln y_3) \cdot \alpha^{-1}.$$

Since $\alpha^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, then

$$x_1 = y_1 y_3, \; x_2 = y_2, \; x_3 = y_3. \tag{38}$$

In this case

$$\hat{f}_1(x_1, x_2, x_3) = y_3^2 \cdot F_3(y_1, y_2); F_3(y_1, y_2) = -\frac{16y_2^2(4y_2^2 - 3)^2(64y_2^2y_1^2 + 32y_1y_2^2 - 32y_2^2 + 27)}{729}.$$

The equation $F_3(y_1, y_2) = 0$ has three solutions:

1. $y_2 = 0$. It corresponds to the point $P_3^{(2)}$, which we will study separately in Section 10.
2. $y_2 = \pm\sqrt{3}/2$. It corresponds to points $P_4^{(3)}$ and $P_5^{(3)}$, which we will study separately.
3. Curve

$$\Phi(y_1, y_2) \stackrel{\text{def}}{=} 64_2^2 y_1^2 + 32y_1y_2^2 - 32y_2^2 + 27 = 0. \tag{39}$$

According to the procedure `genus` from the package `algcurves` program from the CAS Maple, the curve $\Phi(y_1, y_2) = 0$ has a genus 0, parameterization

$$y_1 = b_1(t) \stackrel{\text{def}}{=} \frac{5t^2 + 2t - 1}{19t^2 + 22t + 7}, \quad y_2 = b_2(t) \stackrel{\text{def}}{=} -\frac{19t^2 + 22t + 7}{16t^2 + 24t + 8}, \tag{40}$$

as shown in the graph in Figure 10.

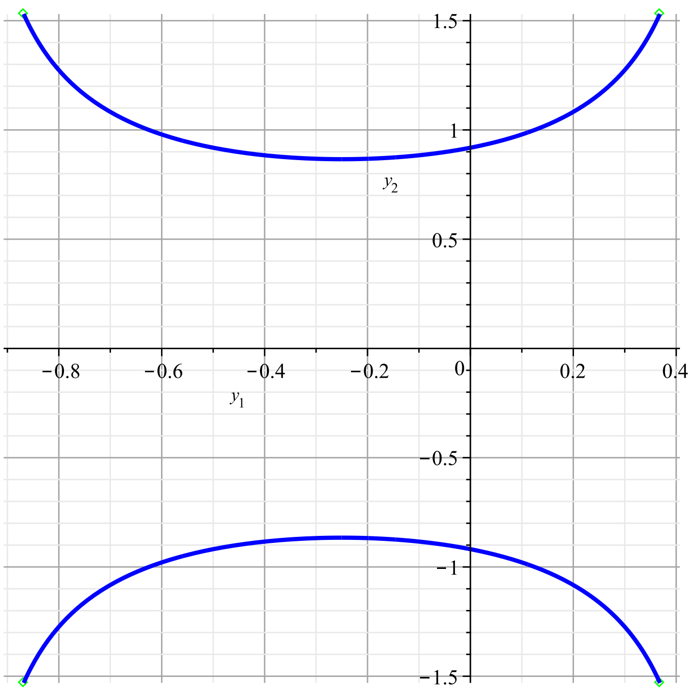

Figure 10. Plot of the curve $\Phi(y_1, y_2) = 0$.

This curve is located in the band $-1 < y_1 < \frac{1}{2}$, it is symmetric relative to the axis $y_2 = 0$ and the vertical $y_1 = -\frac{1}{4}$. When $y_1 = -\dfrac{1}{4}$, on it $y_2 = \pm\dfrac{\sqrt{3}}{2} = \pm 0.8660254$, $t = (-5 \mp 2\sqrt{3})/13$ (i.e., on the curve $t = -0.651084$ and $t = -0.118146$). In this $|y_2| \geqslant \sqrt{3}/2$. At $y_1 = -1$, $t = -1/2$, at $y_1 = 1/2$, $t = -1$, and $y_2 = \pm\infty$.

Presently, to describe the structure of variety Ω near the line $\mathcal{J}$, we substitute (38) into the polynomial $S_3(\mathbf{x})$ and obtain the polynomial $T(y_1, y_2, y_3)$. It splits into the sum

$$T(y_1, y_2, y_3) = y_3^2 \sum_{k=0}^{m} T_k(y_1, y_2) y_3^k$$

with $T_0(y_1, y_2) = F_3(y_1, y_2)$; using the `coeff` command, we obtain

$$\begin{aligned} T_1 \stackrel{\text{def}}{=} G(y_1, y_2) = {} & \frac{16384}{243} y_1^3 y_2^4 - \frac{32768}{243} y_2^6 y_1^3 + \frac{131072}{2187} y_1^3 y_2^8 - \frac{11776}{81} y_1 y_2^4 + \\ & + \frac{8192}{81} y_1 y_2^6 + \frac{1280}{27} y_2^2 y_1^2 + \frac{65536}{729} y_2^8 y_1^2 + \frac{1408}{27} y_1 y_2^2 - \frac{4096}{81} y_1^2 y_2^6 - \\ & - \frac{2048}{27} y_1^2 y_2^4 + 16 + \frac{4096}{243} y_2^6 - - \frac{65536}{2187} y_2^8 + \frac{13312}{243} y_2^4. \end{aligned} \tag{41}$$

In the polynomials $T_k(y_1, y_2)$, we substitute

$$y_1 = b_1(t) + \varepsilon, \quad y_2 = b_2(t). \tag{42}$$

We obtain a polynomial $u(\varepsilon, y_3) = T(y_1, y_2, y_3)/y_3^2$ with coefficients depending on t through $b_1(t)$ and $b_2(t)$. In this polynomial

$$u(\varepsilon, y_3) = \sum_{k=0}^{m} T_k(b_1 + \varepsilon, b_2) y_3^k = \sum_{p,q \geq 0} u_{pq} \varepsilon^p y_3^q,$$

where $u_{00} = F_3(b_1(t), b_2(t))$ from (42), so $u_{00} = 0$,

$$u_{10} = \frac{\partial F(y_1, y_2)}{\partial y_1} = -\frac{512 y_2^4 (4y_2^2 - 3)^2 (4y_1 + 1)}{729} \stackrel{\text{def}}{=} H(y_1, y_2) \tag{43}$$

when $y_i = b_i(t)$, $i = 1, 2$, and in general

$$u_{pq} = \frac{1}{p!} \cdot \frac{\partial^p T_q}{\partial y_1^p}, \text{ when } y_i = b_i(t),\ i = 1, 2, \tag{44}$$

according to (40). Presently, according to (41) and (43)

$$\begin{aligned} u_{10}(t) = {} & H(b_1(t), b_2(t)) = -\frac{(19t^2 + 22t + 7)^3 (13t^2 + 10t + 1)^5}{497664 \zeta^8}, \\ u_{01}(t) = {} & G(b_1(t), b_2(t)) = \frac{\eta}{1728 \zeta^4}, \\ \zeta = {} & (t+1)(2t+1), \\ \eta = {} & 2224717t^8 + 12017960t^7 + 28029436t^6 + 37008760t^5 + \\ & + 30350558t^4 + 15868120t^3 + 5174044t^2 + 963080t + 78397 \end{aligned}$$

By generalized Theorem 1 on the implicit function, the equation $u(\varepsilon, y_3) = 0$ has a solution as a power series over y_3

$$\varepsilon = \sum_{k=1}^{\infty} c_k(t) \cdot y_3^k, \tag{45}$$

where $c_k(t)$ are rational functions that are expressed through the coefficients $u_{pq}(t)$, which in turn are expressed through $b_1(t)$ and $b_2(t)$ according to (44). This expansion is valid for all values of t, except maybe the roots of the function $H(t)$. They correspond to points $y_1 = -1/4$, $y_2 = \pm\sqrt{3}/2$. Therefore, we have to remove them together with their neighborhoods. In particular,

$$c_1(t) = -\left(\frac{u_{01}}{u_{10}}\right) = -\frac{G}{H} = \left(288\eta\zeta^4\right)/\left(\left(19t^2 + 22t + 7\right)\times\right.$$
$$\left.\times\left(6997t^6 + 24846t^5 + 37479t^4 + 30484t^3 + 13971t^2 + 3390t + 337\right)\left(13t^2 + 10t + 1\right)^4\right),$$

where the denominator has two real roots $t_{1,2} = (-5 \mp 2\sqrt{3})/13$. According to (45), approximate $\varepsilon \approx c_1(t)y_3$.

Let us return to the original coordinates, which for small $|y_3|$ on the variety Ω are approximated by

$$x_1 = (b_1(t) + c_1(t)y_3)y_3,\ x_2 = b_2(t),\ x_3 = y_3, \tag{46}$$

in which case

$$A_1 = x_1 - x_2 - \frac{1}{2},\ A_2 = x_1 + x_2 - \frac{1}{2},\ A_3 = \frac{1}{2} + y_3. \tag{47}$$

Figure 11 at $y_3 = 1/20$ (i.e., $A_3 = 11/20$) shows the upper and lower sections of the curve (46) and (47) for $1.4 < |b_2(t)| < 3$. The sections where $|b_2(t)| < 1.4$ are discarded, because they are affected by singularities of the singular points $P_4^{(3)}$ and $P_5^{(3)}$. We observe that these curves are like parallel line segments and almost coincide. In the corresponding $A_3 = 0.505$ in Figure 12 in [25], similar branches merge.

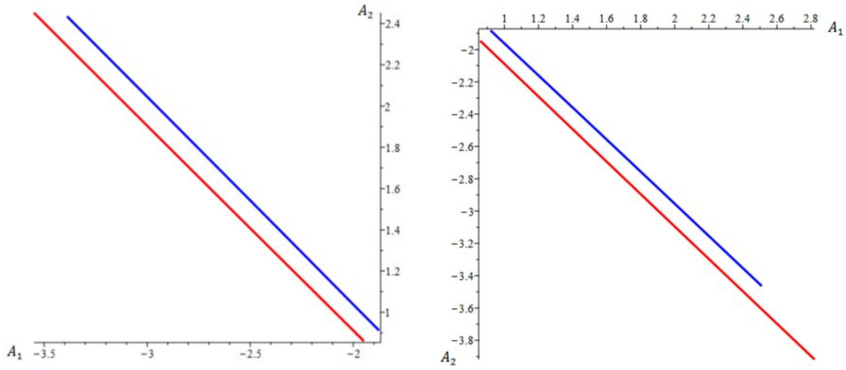

Figure 11. Plots of curves (46), (47) at $y_3 = 1/20$.

Figure 12 shows the upper and lower sections of the curve (46) and (47) at $y_3 = -1/20$ (i.e., $A_3 = 9/20$). Here, the distance between the branches is larger, which corresponds to Figure 8 in [25], with $A_3 = 0.45$, where these branches do not merge.

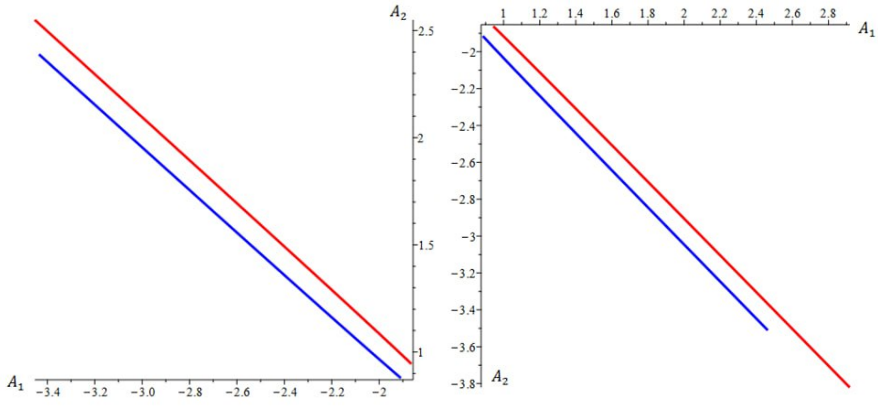

Figure 12. Plots of curves (46), (47) at $y_3 = -1/20$.

10. The Structure of the Variety Ω near the Singular Point $P_3^{(2)}$

In Section 10, we moved from the coordinates $\mathbf{A}$ to the coordinates x_1, x_2, x_3, which are local and near the point $P_3^{(2)}$. For the polynomial $S_3(\mathbf{x})$, we have already calculated the Newton polyhedron Γ_3, its faces and the normals to the faces (Figure 9). There was a normal $N_{683} = (-1, -1, -2)$. It corresponds to a truncated polynomial

$$\hat{f}_2 = -\frac{1024}{729}x_1^6 + 16x_3^3 + \frac{2048}{243}x_1^4x_2^2 + \frac{256}{27}x_1^4x_3 - \frac{1024}{81}x_1^2x_2^4 - \frac{64}{3}x_1^2x_3^2 - \frac{16}{3}x_2^2x_3^2 + \frac{1280}{27}x_1^2x_2^2x_3. \tag{48}$$

Presently, we conduct the power transformation

$$x_1 = y_1y_3, \quad x_2 = y_2y_3, \quad x_3 = y_3^2, \tag{49}$$

and obtain

$$\hat{f}_2 = -\frac{16}{729}y_3^6\left(64y_1^6 - 384y_1^4y_2^2 + 576y_1^2y_2^4 - 432\,y_1^4 - 2160y_1^2\,y_2^2 + 972\,y_1^2 + 243y_2^2 - 729\right).$$

Here,

$$F_4(y_1, y_2) = 64y_1^6 - 384y_1^4y_2^2 + 576y_1^2y_2^4 - 432\,y_1^4 - 2160y_1^2\,y_2^2 + 972\,y_1^2 + 243y_2^2 - 729.$$

The curve $F_4(y_1, y_2) = 0$ has genus 0, with parameterization

$$\begin{aligned}
y_1 =& b_1(t) = -\frac{3\left(t^2 + 229582512\right)\left(t^2 - 52488t - 229582512\right)}{2\beta},\\
y_2 =& b_2(t) = -\frac{9(t + 26244)^3(t - 8748)^3}{4(t^2 + 229582512)\beta},\\
\beta =& t^4 + 104976t^3 - 1377495072t^2 - 24100653779712t + 52708129816230144
\end{aligned} \tag{50}$$

and its graph shown in Figure 13.

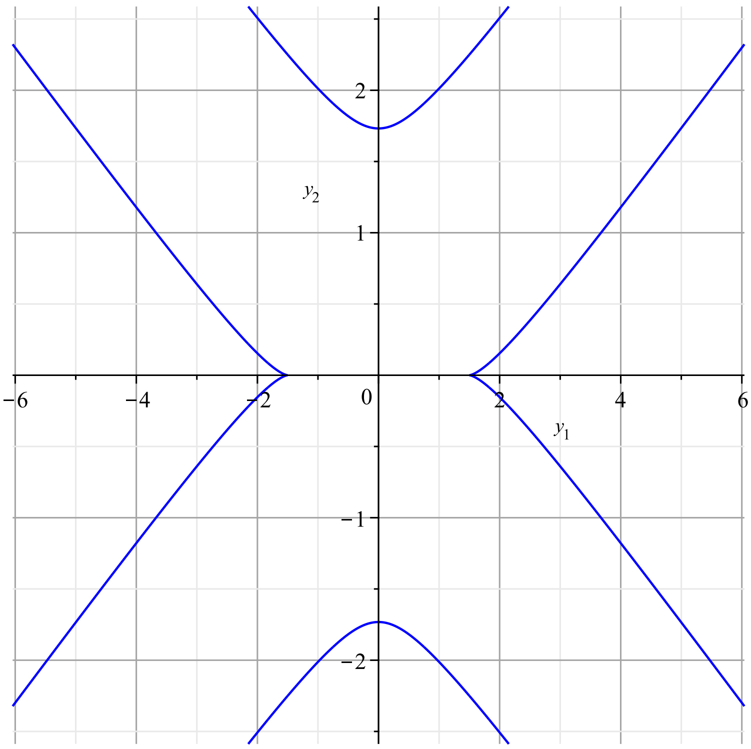

Figure 13. Plot of the curve $F_4(y_1, y_2) = 0$.

Presently, in the full polynomial $S_3(\mathbf{x})$, we conduct the power transformation in (49) to $\mathbf{y}$ and extract from it all terms with y_3 in the seventh degree with the procedure `mtaylor`. We obtain the polynomial

$$\frac{32y_1(512y_1^6 - 3072y_1^4y_2^4 + 4608y_1^2y_2^4 - 3600y_1^4 - 16704y_1^2y_2^2 - 432y_2^4 + 8424y_1^2 + 3564y_2^2 - 6561)y_3^7}{2187}$$

Its division by $\frac{32}{2187}y_1y_3^7$ will provide a polynomial $G(y_1, y_2)$, which we factorize according to the parameterization (50)

$$G(t) = \frac{54(t^4 - 34992t^3 + 1071385056t^2 + 8033551259904t + 52708129816230144)^3(t+26244)^4(t-8748)^4}{(t^2+229582512)^4\beta^3}. \quad (51)$$

In the full polynomial $S_3(x_1, x_2, x_3)$ from Section 9, we replace $\mathbf{x}$ with $\mathbf{y}$ by the power transformation (49) and assume

$$T(\mathbf{y}) = \frac{S_3(x)}{y_3^6} = \sum_{k=0}^{m} y_3^k T_k(y_1, y_2).$$

If we substitute $y_2 = b_1(t)$, $y_3 = b_2(t) + \varepsilon$ in $T(\mathbf{y})$, then the equation $T(\mathbf{y}) = 0$ takes the form

$$u(\varepsilon, y_3) = \sum_{k=0}^{m} T_k(b_1(t), b_2(t) + \varepsilon)y_3^k = \sum_{p,q \geq 0} u_{pq}\varepsilon^p y_3^q = 0, \quad (52)$$

where $u_{00} = -\frac{16}{729}F_4(b_1(t), b_2(t)) = 0$, $u_{10} = -\frac{1024}{729}\frac{\partial F_4}{\partial y_2} = -\frac{1024}{729}H(b_1(t), b_2(t))$,

$$H(t) = -\frac{59049\left(t^4 - 34992t^3 + 1071385056t^2 + 8033551259904t + 52708129816230144\right)^3(t+26244)^3(t-8748)^3}{2\beta^4(t^2+229582512)},$$

and in general

$$u_{pq} = \frac{1}{p!} \cdot \frac{\partial^p T_q}{\partial y_2^p} \text{ when } y_1 = b_1(t), y_2 = b_2(t).$$

According to Theorem 1, the Equation (52) has a solution

$$\varepsilon = \sum_{k=1}^{\infty} c_k(t)y_3^k$$

i.e., according to (49)

$$x_1 = b_1(t)y_3, \quad x_2 = b_2(t)y_3 + \sum_{k=1}^{\infty} c_k(t)y_3^{k+1}, \quad x_3 = y_3^2. \quad (53)$$

According to (35), in the first approximation when y_3 is small, we obtain

$$A_1 = -\frac{1}{2} + b_1(t)y_3 - b_2(t)y_3, A_2 = -\frac{1}{2} + (b_1(t) + b_2(t))y_3, A_3 = \frac{1}{2} + y_3^2. \quad (54)$$

For the real coordinate y_3, coordinate $x_3 \geqslant 0$. Indeed, in Figure 8 of [25], corresponding to $A_3 = 0.45$, i.e., $x_3 = -0.05$, there is no section of the variety Ω near the point $A_1 = -1/2 = A_2$. Assume $y_3 = 0.07$. Then, in the first approximation $x_1 = b_1(t)y_3$, $x_2 = b_2(t)y_3$, $x_3 = 0.005$. Let us draw a curve (54) in coordinates A_1, A_2 at $-1 \leqslant A_1, A_2 \leqslant 0$ (see Figure 14).

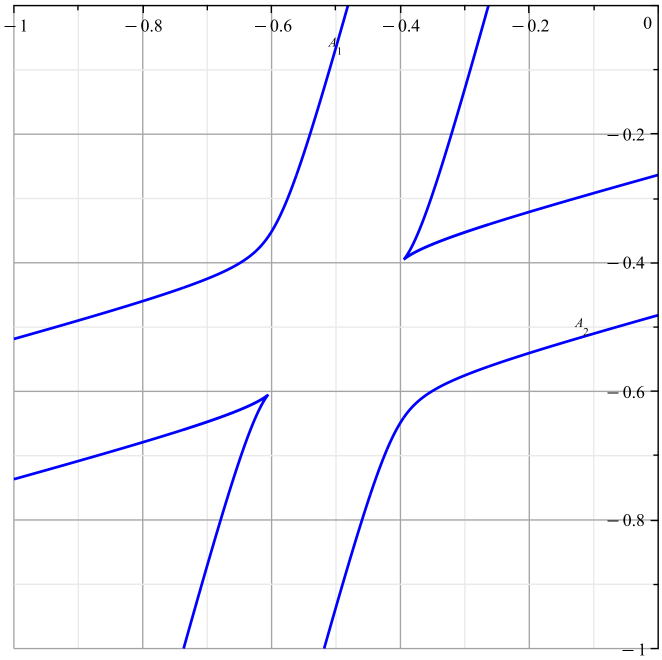

Figure 14. Plot of the curve (54) for $y_3 = 0.07$.

It is similar to Figure 12 of [25] corresponding to $A_3 = 0.505$ in the neighborhood of the point $A_1 = A_2 = -\frac{1}{2}$. This confirms the correctness of the expansion (53).

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