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Approximation Relations on the Posets of Pseudoultrametrics

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Abstract: In this paper we study pseudoultrametrics, which are a natural mixture of ultrametrics and pseudometrics. They satisfy a stronger form of the triangle inequality than usual pseudometrics and naturally arise in problems of classification and recognition. The text focuses on the natural partial order on the set of all pseudoultrametrics on a fixed (not necessarily finite) set. In addition to the "way below" relation induced by a partial order, we introduce its version which we call "weakly way below". It is shown that a pseudoultrametric should satisfy natural conditions closely related to compactness, for the set of all pseudoultrametric weakly way below it to be non-trivial (to consist not only of the zero pseudoultrametric). For non-triviality of the set of all pseudoultrametrics way below a given one, the latter must be compact. On the other hand, each compact pseudoultrametric is the least upper bound of the directed set of all pseudoultrametrics way below it, which are compact as well. Thus it is proved that the set $CPsU(X)$ of all compact pseudoultrametric on a set X is a continuous poset. This shows that compactness is a crucial requirement for efficiency of approximation in methods of classification by means of ultrapseudometrics.

Keywords: pseudoultrametric; approximation; weakly way below; way below; compactness

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1. Introduction

Pseudoultrametrics [1] are a generalization of ultrametrics that relaxes the non-degeneracy requirement (i.e., distinct points are not necessarily separated by a positive distance). A (pseudo)ultrametric on a set can be regarded as a tree-like classification of its elements, in which the numeral value is a measure of (dis-)similarity. This leads to applications to taxonomy and phylogenetic tree construction [2]. Ultrametrics have proved to be useful in the analysis of complex systems such as networks and social structures [3].

Let us consider a simple example of a pseudoultrametric which appears in a classification problem. Suppose that we need to classify a chemical substance, having at our disposal some tests T_1, T_2, T_3, …, which are applied to samples in this order and can give identical results for different chemicals. Let X be the set of all substances we can encounter. Then for all $x, y \in X$ the number

$$d(x,y) = \begin{cases} 0, & \text{if all tests } T_i \text{ give identical results for } x \text{ and } y, \\ \frac{1}{n}, & \text{if } T_n \text{ is the first test with different results for } x \text{ and } y, \end{cases}$$

can be used as the measure of possible (dis-)similarity between x and y: the later the distinction between x and y is revealed, the lower is $d(x,y)$. If $x = y$, then $d(x,y) = 0$, but the converse implication fails (we may simply lack a means to tell x from y- then $x \neq y$ but $d(x,y) = 0$).

The above-defined function d is a pseudoultrametric on X that gradually classifies the substances: X is first partitioned into balls of radii 1, i.e., two chemicals fall into the same ball if they are indiscriminable by the test T_1. Then these classes are subdivided into balls of radii $\frac{1}{2}$, and each of the latter contains substances that produce identical results both under T_1 and T_2, etc.

Roots of the theory of ultrametric spaces are in computer science. Hence it is natural that ultrapseudometric spaces have applications in the study of abstract data types and algorithms. As was pointed out by M. Krötzsch [4], "Domain theory and the theory of metric spaces are the two central utilities in the study of denotational semantics in computer science." Intrinsic relations between ultrametrics and orders were revealed in the latter work. In particular, it was shown that the space of formal balls in a generalized ultrametric space is (under reasonable assumptions) a continuous poset, i.e., a partially ordered set such that each of its elements is the least upper bound of the directed set of the elements approximating it from below.

It turned out [5] that partial orders are closely related to topologies. In particular, a "decent" ordering of a set determines quite natural and useful topologies, e.g., Scott topology, upper/lower topology, Lawson topology, etc. For these topologies to have nice properties, the original order has to satisfy certain requirements, mostly related to approximation relations, and called "continuity" in domain theory.

These requirements are met by surprisingly many natural partial orders, e.g., on the sets of closed subsets of fixed topological spaces [6], on the sets of inclusion hyperspaces [7], on the sets of capacities [8], etc. This has had fruitful implications on topological and algebraic properties of these sets.

Therefore it is natural to apply the apparatus of domain theory to naturally (i.e., pointwise) ordered sets of metrics or metric-like structures. We arrived at the conclusion that the most suitable class for this approach consists of pseudoultrametrics. Categories of ultrametrics were studied [9], but order properties have not been yet investigated. We are going to fill this gap.

The paper describes approximation relations on the set PsU of all pseudoultrametrics on a set X, and on its subsets $CPsU(X)$ and $LCPsU(X)$, that consists of all compact pseudoultrametrics and of all locally compact pseudoultrametrics respectively. The first section, "Preliminaries", contains basic definitions and notation. In the section "Posets of pseudoultrametrics" we describe properties of the mentioned sets with pointwise orders. A significant part of the section consists of counterexamples, which show disadvantages of $PsU(X)$ and $LCPsU(X)$ (e.g., lack of meet continuity). The only "positive" result (which later turns out to be crucial) here is Theorem 1 on meet continuity of $CPsU(X)$.

The section "Approximation from below" contains most of the results of the paper. We introduce an auxiliary relation, which we call "weakly way below". First we show that pseudoultrametric d on X is the least upper bound of the directed set of compact pseudoultrametrics, which implies immediately that no non-compact pseudoultrametric can be weakly way below any pseudoultrametric (Theorem 2). This radically reduces the search. To show that a similar fact is valid "on the other side", namely, no nonzero pseudoultrametric is way below a non-compact pseudoultrametric (Theorem 3), we prove a series of lemmas on the "weakly way below" relation. So we restrict our attention to compact pseudoultrametrics, for which the relations "way below" and "weakly way below" coincide (Theorem 4). Theorem 5 shows how to construct pseudoultrametrics way below a given one recursively. Finally, it is proved (Theorem 6) that the poset $CPsU(X)$ is continuous in the above sense.

2. Preliminaries

Below, "poset" stands for a partially ordered set, i.e., a set with a reflexive antisymmetric transitive binary relation.

Recall that a poset $(D, \leqslant)$ is directed (resp. filtered) if for all $d_1, d_2 \in D$ there is $d \in D$ such that $d_1, d_2 \leqslant d$ (resp. $d_1, d_2 \geqslant d$).

Definition 1. *An element x_0 is said to be way below an element x_1 (or approximates x_1 from below) in a poset $(X, \leqslant)$ (denoted $x_0 \ll x_1$) if for every non-empty directed subset $D \subset X$ such that $x_1 \leqslant \sup D$ there is an element $d \in D$ such that $x_0 \leqslant d$.*

Definition 2. *An element x_0 is said to be way above an element x_1 (or approximates x_1 from above) in a poset $(X, \leqslant)$ (denoted $x_0 \gg x_1$) if for every non-empty filtered subset $D \subset X$ such that $x_1 \geqslant \inf D$ there is an element $d \in D$ such that $x_0 \geqslant d$.*

Obviously $x_0 \ll x_1$ or $x_0 \gg x_1$ imply respectively $x_0 \leqslant x_1$ or $x_0 \geqslant x_1$ (see more in [5]).

A poset is called continuous (resp. dually continuous) if each element is the least upper bound of the directed set of all elements approximating it from below (resp. the greatest lower bound of the filtered set of all elements approximating it from above).

Example 1. *Consider the set $BC(\mathbb{R}^2)$ of all bounded closed non-empty subsets of the plane $\mathbb{R}^2$. It is natural for subsets $A \subseteq B$ to regard A as a lesser element of $BC(\mathbb{R}^2)$ than B and write $A \leqslant B$. Then $BC(\mathbb{R}^2)$ is a poset. It is a routine exercise in metric topology to verify that the following statements are equivalent:*

1. *A is contained in the interior of B.*
2. *For any filtered (with respect to inclusion) family $\{F_i \mid i \in \mathcal{I}\}$ of bounded closed non-empty subsets of plane such that $\bigcap_{i \in \mathcal{I}} F_i \subseteq A$, there is $F_i \subseteq B$.*

Thus B approximates A from above in $BC(\mathbb{R}^2)$ if and only if A is contained in the interior of B, i.e., B is a closed neighborhood of A.

As all closed neighborhoods of each $A \in BC(\mathbb{R}^2)$ form a filtered family with the intersection equal to A, the poset $(BC(\mathbb{R}^2), \subseteq)$ is dually continuous.

This example shows why the term "approximates" is used: it is not possible to get closer to A from outside by F_is without some F_i becoming "trapped" in B. Then B is a "safe" approximation of A: even if the precise position of A can be measured with some measurement errors only, for small enough errors we always are in B.

Now the reader can catch the essence of (dual) continuity of a poset: every element can be "safely" approximated from below (resp. from above). Then any two directed sets that approximate the same element from below are "intertwined" as follows: each element of the first set precedes an element of the second one, and vice versa (analogously for approximations from above). Thus all "safe" approximations are essentially "the same".

Another important point about the latter example is that a subset $A \subset \mathbb{R}^2$ can be regarded as a piece of information about the actual position of an invisible point on a plane. Then it is natural to consider a subset A that is contained in B as a *bigger* portion of information than B because it describes where the point is more specifically. Therefore in computer science, when it comes to information theory, subsets are often ordered by *reverse inclusion*: $A \leqslant B$ if $A \supseteq B$. Then $(BC(\mathbb{R}^2), \supseteq)$ is a continuous poset, and B is way below A if and only if A is contained in the interior of B. See [10] for more sophisticated use of this approach for image recognition.

In this paper we are not interested in the "way above" relation and restrict our attention solely to "way below". We adopt the following definition.

Definition 3. *An element x_0 is called weakly way below an element x_1 in a poset $(X, \leqslant)$ (denoted $x_0 \prec\!\!\!\prec x_1$) if for every non-empty directed subset $D \subset X$ such that $x_1 = \sup D$ there is an element $d \in D$ such that $x_0 \leqslant d$.*

Observe the equality sign that differs from the precedence sign in Definition 1. It will be shown further that "weakly way below" is a strictly weaker property indeed than "way below".

We are going to apply the above apparatus to the set of all pseudometrics on a fixed set, and to its subset that consists of all pseudoultrametrics. Ultrametrics (or non-Archimedean metrics [11]) have been studied since the beginning of XX century, cf. a review in [9]. They have found numerous applications, e.g., in computer science.

Monotone families of (pseudo-)ultrametrics were studied in [1], but approximation relations were out of the scope of the latter paper.

The following notion is a natural mixture of the notions of ultrametric and pseudometric.

Definition 4. *A mapping $d : X \times X \to \mathbb{R}$ that satisfies the conditions:*

- $d(x, y) \geqslant 0$ *for all* $x, y \in X$ *(nonnegativeness);*
- $d(x, x) = 0$ *for all* $x \in X$ *(identity);*
- $d(x, y) = d(y, x)$ *for all* $x, y \in X$ *(symmetry);*
- $d(x, y) \leqslant \max\{d(y, z), d(z, x)\}$ *for all* $x, y, z \in X$ *(triangle inequality),*

is called a pseudoultrametric on the set X*.*

It is just a pseudometric such that the usual triangle inequality $d(x, y) \leqslant d(y, z) + d(z, x)$ holds in a stronger form.

A pair (X, d) of a set X and a pseudoultrametric d on it is called a pseudoultrametric space. For any subset A in (X, d) the (finite or infinite) least upper bound $\operatorname{diam} A = \sup\{d(x, y) \mid x, y \in A\}$ is called the diameter of A with respect to d.

Just as for any (pseudo-)metric, the ball $B_r(x)$ and the closed ball $\bar{B}_r(x)$ for $r > 0$ are defined as follows:

$$B_r(x) = \{y \in X \mid d(x, y) < r\}, \qquad \bar{B}_r(x) = \{y \in X \mid d(x, y) \leqslant r\}.$$

3. Posets of Pseudoultrametrics

Denote PsU as the set $PsU(X)$ of all pseudoultrametrics on a set X. Its subsets $CPsU(X)$ and $LCPsU(X)$ consist of all compact pseudoultrametrics and of all locally compact pseudoultrametrics respectively, i.e., $CPsU(X)$ is the set of all pseudoultrametrics that make X a compact space. Similarly $LCPsU(X)$ denotes the set of all pseudoultrametrics on X such that each point of X is the centre of a compact closed ball (note that we *do not require* the Hausdorff property, and the mentioned functions may not be ultrametrics) [12].

Example 2. *Let X be an arbitrary set. The discrete metric defined with the formula*

$$d(x, y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y, \end{cases} \qquad x, y \in X,$$

is an ultrametric and therefore a pseudoultrametric. Each ball in X if the radius is 1 is a singleton (one-point set), hence is compact. Hence $d \in LCPsU(X)$, but $d \notin CPsU(X)$ for infinite X because any sequence of distinct points in (X, d) has no limit.

Example 3. *Consider a finite partition $A_1, A_2, \ldots, A_n$ of a set X and define a function $\rho : X \times X \to \mathbb{R}$ with the formula*

$$\rho(x, y) = \begin{cases} 0, & x, y \in A_i \text{ for some } i \in \{1, 2, \ldots, n\}, \\ 1, & x \in A_i, y \in A_j \text{ for some } i \neq j,\ i, j \in \{1, 2, \ldots, n\}, \end{cases} \qquad x, y \in X.$$

Clearly ρ is a pseudoultrametric, but it fails to be an ultrametric if at least one A_i contains more than one point. Each ball in (X, ρ) either is equal to X, if the radius is greater than 1, or coincides with one of A_i otherwise. Thus $\rho \in CPsU(X)$.

Remark 1. *Recall that two balls of equal radii in a pseudoultrametric space (X, d) either coincide or have empty intersection. Hence the balls of a fixed radius R form a partition of X, therefore are* *open and closed. This implies that X is complete if and only if for each point $x \in X$ there is $R > 0$ such that for each decreasing sequence of balls*

$$B_R(x) \supset B_{r_1}(x_1) \supset B_{r_2}(x_2) \supset \ldots \text{ with } R > r_1 > r_2 > \ldots \searrow 0$$

the intersection is non-empty.

Likewise X is compact if and only if X is complete and for all $r > 0$ there is only a finite number of distinct balls of radius r in X. The space (X, d) is locally compact if and only if it is complete and for each point $x \in X$ there is $R > 0$ such that for all $0 < r < R$ the ball $B_R(x)$ is the union of a finite number of balls of radius r.

The partial orders on the set $PsU(X)$ of all pseudoultrametrics on X and its subsets $CPsU(X)$ and $LCPsU(X)$ are defined pointwise: a pseudoultrametric d_1 precedes a pseudoultrametric d_2 (written $d_1 \leqslant d_2$ or $d_2 \geqslant d_1$) if $d_1(x,y) \leqslant d_2(x,y)$ holds for all points $x, y \in X$. The trivial pseudometric $d \equiv 0$ is the least element of $PsU(X)$, $CPsU(X)$, and of $LCPsU(X)$. We write $d_1 < d_2$ or $d_2 > d_1$ if $d_1 \leqslant d_2$ and $d_1 \neq d_2$ (this does not mean that $d_1(x,y) < d_2(x,y)$ for all x, y).

Observe that if $d_1 \leqslant d_2$ for $d_1, d_2 \in PsU(X)$, then the identity mapping $\mathbf{1}_X : (X, d_2) \to (X, d_1)$ is continuous. The continuous image of a compact space is compact. Hence if $d_2 \in CPsU(X)$, then $d_1 \in CPsU(X)$, i.e., $CPsU(X) \subset PsU(X)$ is a lower subset:

$$CPsU(X){\downarrow} = \{d' \in PsU(X) \mid d' \leqslant d \text{ for some } d \in CPsU(X)\} \subset CPsU(X).$$

The least upper bound of pseudoultrametrics d_1, d_2 in $PsU(X)$ is the pointwise maximum $d^*(x,y) = \max\{(d_1(x,y), d_2(x,y)\}$ for all $x, y \in X$.

Example 4. *There are compact pseudoultrametrics d_1, d_2 on a countable set X such that $\sup\{d_1, d_2\}$ is not a locally compact pseudoultrametric. Let $Y_+ = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ and $Y = Y_+ \cup \{-1, 0\}$. Define $\rho : Y \times Y \to \mathbb{R}$ by the formula*

$$\rho(u,v) = \begin{cases} 0, & u = v, \\ \max\{|u|, |v|\}, & u \neq v, \end{cases} \qquad u, v \in Y.$$

Consider

$$X = \big\{(0,0)\big\} \cup \bigcup_{n \in \mathbb{N}} \Big\{\frac{1}{n}\Big\} \times \Big(\{0\} \cup \Big\{\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \ldots\Big\}\Big)$$

with the compact ultrametric $d_1((u,v),(u',v')) = \max\{\rho(u,u'), \rho(v,v')\}$ and with the compact pseudoultrametric

$$d_2((u,v),(u',v')) = \begin{cases} 1, & (u,v),(u',v') \text{ are in distinct of the sets } Y_+ \times \{0\} \\ & \text{and } X \setminus (Y_+ \times \{0\}), \\ 0 & \text{otherwise,} \end{cases}$$

for $(u,v),(u',v') \in X$.

Then X with the pseudoultrametric $d^* = \sup\{d_1, d_2\}$ is isometric to the set

$$X_{\pm} = \big\{(0,0)\big\} \cup \bigcup_{n \in \mathbb{N}} \{\frac{1}{n}\} \times \Big\{\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \ldots\Big\} \cup Y_+ \times \{-1\}$$

with the pseudoultrametric $d_{\pm}((u,v),(u',v')) = \max\{\rho(u,u'), \rho(v,v')\}$, which is not locally compact.

Thus neither of the sets $CPsU(X)$ and $LCPsU(X)$ is an upper subsemilattice in the lattice $PsU(X)$.

Remark 2. *If all balls in X with respect to a pseudoultrametric d_1 are open with respect to a (locally) compact pseudoulrametric d_2 (i.e., d_1 is continuous with respect to d_2), then the pseudoultrametric* $\sup\{d_1, d_2\}$ *is obviously (locally) compact as well.*

The formula

$$d_*(x,y) = \inf\big\{\max\{\min\{d_1(t_k,t_{k+1}), d_2(t_k,t_{k+1})\} \mid 0 \leqslant k \leqslant n-1\} \mid n \in \mathbb{N},\\ t_0 = x, \{t_1, ..., t_{n-1}\} \subset X, t_n = y\big\}$$

determines the infimum of d_1, d_2 in the set of all pseudoultrametrics. The identity mappings $\mathbf{1}_X : (X, d_1) \to (X, d_*)$ and $\mathbf{1}_X : (X, d_1) \to (X, d_*)$ are continuous, hence compactness of either of d_1 and d_2 implies compactness of d_*.

Example 5. *There exist locally compact pseudoultrametrics d_1, d_2 on a countable set X such that the pseudoultrametric $d_* = \inf\{d_1, d_2\}$ is not locally compact.*

Put $Y = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\} \cup \{0\}$ and $X = Y \times \mathbb{N}$, and define d_1, d_2 as follows:

$$d_1((x,n),(x',n')) = \begin{cases} 0, & (x,n) = (x',n'), \\ \max\{x,x'\}, & n = n', x \neq x', \\ 1 & \textit{otherwise}, \end{cases}$$

and

$$d_2((x,n),(x',n')) = \begin{cases} 0, & (x,n) = (x',n') \textit{ or } x = x' = 0, \\ 1 & \textit{otherwise}, \end{cases}$$

for all $(x,n), (x',n') \in X$. Then

$$d_*((x,n),(x',n')) = \begin{cases} 0, & (x,n) = (x',n') \textit{ or } x = x' = 0, \\ \max\{x,x'\} & \textit{otherwise}, \end{cases}$$

and a point $(0,1)$ does not have a compact neighborhood in (X, d_*): for all $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon$, hence the ball $B_\varepsilon((0,0))$ contains a sequence $(\frac{1}{k}, n)_{n=1}^\infty$ without convergent subsequences.

Thus $CPsU(X)$ is a lower subsemilattice of the lattice $PsU(X)$ of all pseudoultrametrics on X, but $LCPsU(X)$ is not.

Clearly none of the posets $PsU(X)$, $LCPsU(X)$, and $CPsU(X)$ for $|X| > 1$ has a greatest element, therefore they are not complete upper semilattices. Nevertheless, $PsU(X)$ and $CPsU(X)$ are bounded complete upper semilattices, i.e., if (compact) pseudoultrametrics d_α, $\alpha \in \mathcal{A}$, satisfy $d_\alpha \leqslant d$ for a (compact) pseudoultrametric d, then the pointwise supremum of all d_α is a (compact) pseudoultrametric that is the least upper bound of $\{d_\alpha \mid \alpha \in \mathcal{A}\}$.

Example 6. *There are locally compact pseudoultrametrics ρ, d, and $d_1 \leqslant d_2 \leqslant \dots$ on a countable set X such that $d = \sup\{d_1, d_2, \dots\}$, but $\inf\{\rho, d\} \neq \sup\{\inf\{\rho, d_1\}, \inf\{\rho, d_2\}, \dots\}$.*

Let $X = \{-1, -\frac{1}{2}, -\frac{1}{3}, \ldots\} \cup \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}\}$*, and*

$$\rho(u,v) = \begin{cases} 0, & |u| = |v|, \\ 1, & |u| \neq |v|, \end{cases}$$

$$d(u,v) = \begin{cases} 0, & u = v = 0, \\ 0, & \{u,v\} \subset \{-\frac{1}{2k-1}, -\frac{1}{2k}\} \text{ or} \\ & \{u,v\} \subset \{\frac{1}{2k}, \frac{1}{2k+1}\} \text{ for some } k \in \mathbb{N}, \\ 1 & \text{otherwise}, \end{cases}$$

$$d_n(u,v) = \begin{cases} 0, & |u|, |v| < \frac{1}{2n+1}, \\ 0, & \{u,v\} \subset \{-\frac{1}{2k-1}, -\frac{1}{2k}\} \text{ or} \\ & \{u,v\} \subset \{\frac{1}{2k}, \frac{1}{2k+1}\} \text{ for some } 1 \leqslant k \leqslant n, \\ 1 & \text{otherwise}, \end{cases}$$

for all $u, v \in X$*. It is straightforward to verify that* $\inf\{\rho, d_n\}(-1, 0) = 0$ *for all* $n \in \mathbb{N}$*, but* $\inf\{\rho, d\}(-1, 0) = 1$.

In other words, the posets $PsU(X)$ and $LCPsU(X)$ are not meet continuous.

Theorem 1. *For each pseudoultrametric* ρ *and a directed set* $\{d_\alpha \mid \alpha \in \mathcal{A}\}$ *of pseudoultrametrics such that there is a compact pseudoultrametric* $d = \sup\{d_\alpha \mid \alpha \in \mathcal{A}\}$ *(on the same set* X*), the equality*

$$\inf\{\rho, d\} = \sup\{\inf\{\rho, d_\alpha\} \mid \alpha \in \mathcal{A}\}$$

is valid.

Proof. Denote $\rho' = \inf\{\rho, d\}$ and observe $\inf\{\rho, d_\alpha\} = \inf\{\rho, d, d_\alpha\} = \inf\{\rho', d_\alpha\}$, hence $\rho' = \sup\{\inf\{\rho', d_\alpha\} \mid \alpha \in \mathcal{A}\} = d'$ is the equality to be proved, given $\rho' \leqslant d = \sup\{d_\alpha \mid \alpha \in \mathcal{A}\}$. Clearly ρ' is compact, as well as d' and all $\inf\{\rho', d_\alpha\}$, and the right side is less than or equal to the left side.

Let there be $x, y \in X$ such that $\rho'(x,y) = \varepsilon > \theta > d'(x,y)$. Then $x \in B = B_\varepsilon(x)$, $y \in C = X \setminus B_\varepsilon(c)$ (balls are with respect to ρ'), and $\rho'(u,v) \geqslant \varepsilon$ for all $u \in B, v \in C$.

For all $\alpha \in \mathcal{A}$ we have $\inf\{\rho', d_\alpha\} < \theta$. Each sequence of points "from x to y" has to jump once from B to C. Hence by the formula for $\inf\{\rho', d_\alpha\}$ there should be $u \in B$, $v \in C$ such that $\inf\{\rho'(u,v), d_\alpha(u.v)\} \leqslant \theta$. Taking into account $\rho'(u,v) \geqslant \varepsilon > \theta$, we obtain $d_\alpha(u,v) \leqslant \theta$. Hence the closed set $\{z \in C \mid d_\alpha(z,B) \leqslant \theta\}$ is non-empty for all α. Observe $\{z \in C \mid d_\alpha(z,B) \leqslant \theta\} \subset \{z \in C \mid d_\beta(z,B) \leqslant \theta\}$ if $d_\alpha \geqslant d_\beta$. Therefore the family of compact sets $\{z \in C \mid d_\alpha(z,B) \leqslant \theta\}$ for all $\alpha \in \mathcal{A}$ is filtered. Therefore its intersection is non-empty, and there is $z \in C$ such that $d_\alpha(z,B) \leqslant \theta$ for all $\alpha \in A$. This implies $d(z,B) \leqslant \theta$, which is contradictory to $d(z,B) \geqslant \rho'(z,B) \geqslant \varepsilon > \theta$. Thus $\rho' = d'$, and the proof is complete. $\square$

Therefore the poset $PsU(X)$ is meet continuous, i.e., the equality

$$\inf\{\rho, \sup\{d_\alpha \mid \alpha \in \mathcal{A}\}\} = \sup\{\inf\{\rho, d_\alpha\} \mid \alpha \in \mathcal{A}\}$$

is valid provided all pseudoultrametrics and their suprema here are compact.

We do not discuss existence or properties of least upper bounds of bounded sets in the poset $LCPsU(X)$.

4. Approximation from Below

Let d be a pseudoultrametric on X. Then for all $\varepsilon > 0$ all ε-balls in X with respect to d are open and disjoint. Hence the sets $A = B_\varepsilon(x)$ and $B = X \setminus B_\varepsilon(x) = \bigcup_{y \notin B_\varepsilon(x)} B_\varepsilon(y)$ are open and disjoint as well. Clearly $d(u,v) \geqslant \varepsilon$ for all $u \in A$, $v \in B$. Therefore the formula

$$d_\varepsilon^x(u,v) = \begin{cases} \varepsilon, & \text{exactly one of } u,v \text{ is in } B_\varepsilon(x), \\ 0 & \text{otherwise,} \end{cases} \qquad u,v \in X,$$

defines a compact pseudoultrametric $d_\varepsilon^x \leqslant d$ on X. Moreover,

$$d(u,v) = \sup\{d_\varepsilon^x(u,v) \mid x \in X, \varepsilon > 0\}$$

for all $u,v \in X$. This implies that any pseudoultrametric d on X is the least upper bound of the directed set of all compact pseudoultrametrics of the form

$$\sup\{d_{\varepsilon_1}^{x_1}, d_{\varepsilon_2}^{x_2}, \ldots, d_{\varepsilon_n}^{x_n}\}, \quad n \in \mathbb{N},\ x_1, x_2, \ldots, x_n \in X,\ \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n > 0.$$

This has an immediate consequence on the "way below" relations in the posets $PsU(X)$ and $LCPsU(X)$.

Theorem 2. *Let d, d_0 be pseudoultrametrics on X, $d_0 \notin CPsU(X)$. Then d_0 is not weakly way below d (hence is not way below d) neither in $PsU(X)$ nor in $LCPsU(X)$.*

Recall that a pseudoultrametric d on a set X is compact if and only if:

- it attains its least upper bound $\varepsilon = \operatorname{diam} X = \sup\{d(u,v) \mid u,v \in X\}$;
- X is the disjoint union of finitely many ε-balls $B_\varepsilon(x_1)$, $B_\varepsilon(x_2)$, ..., $B_\varepsilon(x_n)$;
- each ball $B_\varepsilon(x_i)$ is compact with respect to d.

These properties imply that $\operatorname{diam} B_\varepsilon(x_i) < \varepsilon$ for all $i = 1, 2, \ldots, n$.

We consider the "weakly way below" relation in $PsU(X)$ in the five following lemmas.

Lemma 1. *Let for a pseudoultrametric d on X a ball $B_\varepsilon(x)$ exist such that the values $d(u,v)$ for $u,v \in B_\varepsilon(x)$ **do not attain** their least upper bound $\varepsilon_0 = \operatorname{diam} B_\varepsilon(x)$. Then for any pseudoultrametric $d_0 \lll d$ and all $u,v \in B_\varepsilon(x)$ the equality $d_0(u,v) = 0$ holds.*

Proof. Observe that $B_\varepsilon(x)$ is not compact with respect to d. Assume there are $s,t \in B_\varepsilon(x)$ such that $d_0(s,t) = \delta > 0$, then $B_\varepsilon(x)$ is the union of the disjoint (or coinciding) balls $B_\delta^0(y) \cap B_\varepsilon(x)$ with respect to d_0 for all $y \in B_\varepsilon(x)$.

The sets $A = B_\delta^0(s)$ and $B = X \setminus B_\delta^0(s)$ are closed and open in X with respect to d_0, hence $A_0 = A \cap B_\varepsilon(x) \ni s$ and $B_0 = B \cap B_\varepsilon(x) \ni t$ are closed and open in $B_\varepsilon(x)$ with respect to d, and at least one of A_0, B_0, say B_0, has a diameter ε_0.

Consider a construction based on a pseudoultrametric d on a set Y. For arbitrary $x \in Y$ and $\theta > 0$ denote

$$d_{\leqslant\theta}^x(u,v) = \begin{cases} d(u,v), & u,v \in B_\theta(x), \\ 0, & u,v \notin B_\theta(x), \\ d(u,x), & u \in B_\theta(x),\ v \notin B_\theta(x), \\ d(v,x), & u \notin B_\theta(x),\ v \in B_\theta(x), \end{cases} \qquad u,v \in Y.$$

Observe that $d_{\leqslant\theta}^x$ is a pseudoultrametric less than or equal to d.

Choose an increasing sequence

$$0 < \theta_1 < \theta_2 < \ldots < \theta_n < \ldots \nearrow \varepsilon_0,$$

and consider the sequence

$$d^s_{\leqslant\theta_1} \leqslant d^s_{\leqslant\theta_2} \leqslant \ldots \leqslant d^s_{\leqslant\theta_n} \leqslant \ldots$$

of pseudoultrametrics on $B_\varepsilon(x) \subset X$. We extend each of these pseudoultrametrics to X by putting $d^s_{\leqslant\theta_n}(u,v) = d(u,v)$, if either of u,v is not in $B_\varepsilon(x)$. Then it is straightforward to verify that $d^s_{\leqslant\theta_n}(u,v) \nearrow d(u,v)$ as $n \to \infty$ for all $u,v \in X$.

On the other hand, the pseudoutrametric

$$d_\delta^{A,B}(u,v) = \begin{cases} 0, & u,v \in A \text{ or } u,v \in B, \\ \delta & \text{otherwise}, \end{cases} \qquad u,v \in X,$$

satisfies $d_\delta^{A,B} \leqslant d_0$, hence $d_\delta^{A,B} \twoheadleftarrow\!\!\!\!< d$. Therefore $n \in \mathbb{N}$ must exist such that $d^s_{\leqslant\theta_n} \geqslant d_\delta^{A,B}$, which is impossible because there is $y \in B_0$ such that $d(s,y) \geqslant \theta_{n+1} > \theta_n$, hence $d^s_{\leqslant\theta_n}(s,y) = d(s,s) = 0 \ngeqslant d_\delta^{A,B}(s,y) = \delta$.

This contradiction completes the proof that $d_0(u,v) = 0$ for all $u,v \in B_\varepsilon(x)$. $\square$

Lemma 2. *Assume that for a pseudoultrametric d on X a ball $B_\varepsilon(x)$ exists such that the values $d(u,v)$ for $u,v \in B_\varepsilon(x)$* ***attain*** *their least upper bound $\varepsilon_0 = \operatorname{diam} B_\varepsilon(x) > 0$ and there are infinitely many points $x_1, x_2, x_3, \ldots \in B_\varepsilon(x)$ such that $d(x_i, x_j) = \varepsilon_0$ for $i \neq j$. Then for any pseudoultrametric $d_0 \twoheadleftarrow\!\!\!\!< d$ and all $u,v \in B_\varepsilon(x)$ the equality $d_0(u,v) = 0$ holds.*

Proof. By the assumption $B_\varepsilon(x)$ is the disjoint union of infinitely many balls $B_{\varepsilon_0}(y)$, $y \in B_\varepsilon(x)$. Let $d_0 \twoheadleftarrow\!\!\!\!< d$, and $s,t \in B_\varepsilon(x)$ exist such that $d_0(s,t) = \delta > 0$.

Consider again the sets $A = B^0_\delta(s)$ and $B = X \setminus B^0_\delta(s)$ which are closed and open in X with respect to d_0, and the closed and open in $B_\varepsilon(x)$ with respect to d intersections $A_0 = A \cap B_\varepsilon(x) \ni s$ and $B_0 = B \cap B_\varepsilon(x) \ni t$. At least one of A_0 and B_0, say B_0, intersects infinitely many disjoint balls $B_{\varepsilon_0}(x_1)$, $B_{\varepsilon_0}(x_2)$, $B_{\varepsilon_0}(x_3)$, ... We may also assume $s \notin B_{\varepsilon_0}(x_n)$ for all $n = 1,2,3,\ldots$.

For all $n = 1,2,3,\ldots$ denote $C_n = B_{\varepsilon_0}(x_n) \cup B_{\varepsilon_0}(x_{n+1}) \cup B_{\varepsilon_0}(x_{n+2}) \cup \ldots$. Then the sets $C_1 \supset C_2 \supset C_3 \supset \ldots$ are closed and open, and their intersection is empty. Define a pseudoultrametric d_n for any $n \in \mathbb{N}$ with the formula

$$d_n(u,v) = \begin{cases} d(u,v), & u,v \notin C_n, \\ 0, & u,v \in C_n, \\ d(u,s), & u \notin C_n,\ v \in C_n, \\ d(v,s), & u \in C_n,\ v \notin C_n, \end{cases} \qquad u,v \in X,$$

(we glue all points of C_n with s). Clearly $d_n(u,v) \nearrow d(u,v)$ as $n \to \infty$.

Analogously to the previous lemma, we show that neither of d_n is greater than or equal to the pseudoultrametric

$$d_\delta^{A,B}(u,v) = \begin{cases} 0, & u,v \in A \text{ or } u,v \in B, \\ \delta & \text{otherwise}, \end{cases} \qquad u,v \in X,$$

which satisfies $d_\delta^{A,B} \leqslant d_0$, hence $d_\delta^{A,B} \twoheadleftarrow\!\!\!\!< d$ should be valid. By the choice of x_i we have $B_0 \cap C_n \neq \varnothing$, therefore there is $y \in B_0 \cap C_n$. Then $d_n(s,y) = d(s,s) = 0 \ngeqslant d_\delta^{A,B}(s,y) = \delta$, which is a contradiction. $\square$

Slight modifications of the latter arguments yield the conclusion that:

Lemma 3. *If either the values $d(u,v)$ for u,v in the entire X* ***do not attain*** *their least upper bound $\varepsilon_0 = \operatorname{diam} X$, or they* ***attain*** *ε_0 and there are infinitely many points $x_1, x_2, x_3, \ldots$ such that $d(x_i, x_j) = \varepsilon_0$ for all $i \neq j$, then $d_0 \equiv 0$ is a unique pseudoultrametric weakly way below d.*

Lemma 4. *Assume that a pseudoultrametric d on X `attains` its supremum $\varepsilon = \operatorname{diam} X > 0$ and there is a ball $B_\varepsilon(x)$ with the diameter **equal** to ε. Then $d_0 \equiv 0$ is the unique pseudoultrametric weakly way below d.*

Observe that the above condition means that the values $d(u,v)$ for $u, v \in B_\varepsilon(x)$ **do not attain** their least upper bound that is equal to ε.

Proof. Let $d_0 \prec\!\!\prec d$. It is proved already that $d_0(u,v) = 0$ for all $u, v \in B_\varepsilon(x)$. Choose $0 < \theta < \varepsilon$ and arbitrary $y \notin B_\varepsilon(x)$, and define a pseudoultrametric $d^{x,y}_{\leqslant\theta}$ similarly to how $d^x_{\leqslant\theta}$ was defined:

$$d^{x,y}_{\leqslant\theta}(u,v) = \begin{cases} \min\{d(u,v),\theta\}, & u,v \in B_\varepsilon(x), \\ \min\{d(u,v),\theta\}, & u,v \in B_\theta(x) \cup B_\varepsilon(y), \\ \min\{d(v,y),\theta\}, & u \in B_\varepsilon(x) \setminus B_\theta(x), v \in B_\varepsilon(y) \\ \min\{d(u,y),\theta\}, & v \in B_\varepsilon(x) \setminus B_\theta(x), u \in B_\varepsilon(y), \\ d(u,v) & \text{otherwise,} \end{cases} \qquad u,v \in X.$$

Choose an increasing sequence

$$0 < \theta_1 < \theta_2 < \ldots < \theta_n < \ldots \nearrow \varepsilon,$$

and then the sequence

$$d^{x,y}_{\leqslant\theta_1} \leqslant d^{x,y}_{\leqslant\theta_2} \leqslant \ldots \leqslant d^{x,y}_{\leqslant\theta_n} \leqslant \ldots$$

converges to d. On the other hand, for each $n = 1, 2, 3, \ldots$ and $u \in B_\varepsilon(x) \setminus B_{\theta_n}(x)$ we have $d^{x,y}_{\leqslant\theta_n}(u,y) = 0$. As $d_0 \leqslant d^{x,y}_{\leqslant\theta_n}$ for some n, $d_0(u,y) = 0$ for all $u \in B_\varepsilon(x) \setminus B_{\theta_n}(x)$. For any other $z \notin B_\varepsilon(x)$ there is also m such that $d_0(u,z) = 0$ for all $u \in B_\varepsilon(x) \setminus B_{\theta_m}(x)$. Let $k = \max\{n, m\}$ and $u \in B_\varepsilon(x) \setminus B_{\theta_k}(x)$, and take into account $d_0(x,u) = 0$, then $d_0(u,y) = d_0(u,z) = 0$ implies $d_0(x,y) = d_0(x,z) = d_0(y,z) = 0$.

This completes the proof that $d_0 \equiv 0$. □

Lemma 5. *Assume that a pseudoultrametric d on X **attains** its supremum $\varepsilon = \operatorname{diam} X > 0$ and X is the disjoint union of balls $X_1 = B_\varepsilon(x_1)$, $X_2 = B_\varepsilon(x_2), \ldots, X_n = B_\varepsilon(x_n)$ with the diameters **less than** ε.*

Then a pseudoultrametric $d_0 \leqslant d$ is weakly way below d if and only if:

(a) all $d_0(x_i, x_j)$ are less than ε;

(b) for all $i = 1, 2, \ldots, n$ the restriction d_{0i} of d_0 to the ball X_i is way below the restriction d_i of d to X_i.

Proof. *Necessity.* If $d_0(x_i, x_j) = \varepsilon$ for some i, j, then the increasing sequence of pseudoultrametrics $(1 - \frac{1}{k})d$, $k = 1, 2, \ldots$, converges to d, but $(1 - \frac{1}{k})d(x_i, x_j) \leqslant (1 - \frac{1}{k})\varepsilon < \varepsilon = d_0(x_i, x_j)$ for all k, hence $d_0 \not\prec\!\!\prec d$. Thus (a) is necessary.

Consider a directed set $\{\rho_\alpha \mid \alpha \in \mathcal{A}\}$ of pseudoultrametrics on a ball X_i with the least upper bound d_i. Extend each ρ_α to a pseudoultrametric $\bar\rho_\alpha$ on X by the formula

$$\bar\rho_\alpha = \begin{cases} \rho_\alpha(u,v), & u,v \in X_i, \\ d(u,v), & u \notin X_i \text{ or } v \notin X_i, \end{cases} \qquad u,v \in X.$$

Then the set $\{\bar\rho_\alpha \mid \alpha \in \mathcal{A}\}$ is directed and has the least upper bound d, therefore $d_0 \leqslant \bar\rho_\alpha$ for some $\alpha \in \mathcal{A}$. Going back to restrictions on X_i, we obtain $d_{0i} \leqslant \rho_\alpha$, hence $d_{0i} \ll d_i$. Therefore (b) is necessary as well.

Sufficiency. Assume (a) and (b). Let $\{d_\alpha \mid \alpha \in \mathcal{A}\}$ be a directed set of pseudoultrametrics such that the supremum of d_α is equal to d. We show that for all $0 < \theta < \varepsilon$

the inequality $d_\alpha(x_i, x_j) > \theta$ is valid for all $i \neq j$ and some $\alpha \in \mathcal{A}$. Assuming the contrary for some $i \neq j$, we obtain from $d_\alpha \leqslant d$ for all $\alpha \in \mathcal{A}$ that

$$\begin{aligned} d_\alpha(u,v) \leqslant d'(u,v) = \min\{&d(u,v), \\ &\max\{d(u,x_i),\theta,d(x_j,v)\}, \\ &\max\{d(v,x_i),\theta,d(x_j,u)\}\}, \qquad u,v \in X. \end{aligned}$$

Then d' is a pseudoultrametric, $d_\alpha < d'$ (because $d'(x_i, x_j) = \theta < \varepsilon = d(x_i, x_j)$), which contradicts d being the supremum of all d_α.

By the assumption we can choose θ so that $\max\{\operatorname{diam}_0 X_1, \operatorname{diam}_0 X_1, \ldots, \operatorname{diam}_0 X_n\} < \theta < \varepsilon$, $\theta \geqslant \max\{d_0(x_i, x_j) \mid i \neq j\}$, then there is $\alpha_0 \in \mathcal{A}$ such that $d_{\alpha_0}(x_i, x_j) > \theta$ for all $i \neq j$.

Analogously to the previous arguments, it can be shown that the restrictions $d_{\alpha i}$ of all d_α to X_i have the least upper bound d_i in $PsU(X_i)$. Thus $d_i \geqslant d_i$ for all i, hence $d_{\alpha_i} \geqslant d_{0i}$ for some α_i.

Now there is $\beta \in \mathcal{A}$ such that d_β is greater than or equal to all $d_{\alpha_0}, d_{\alpha_1}, \ldots, d_{\alpha_n}$. Clearly $d_\beta(u,v) \geqslant d_{0i}(u,v) = d_0(u,v)$ for all $u, v \in X_i$. If $i \neq j$, $u \in X_i$, $v \in X_j$, then $d_\beta(x_i, x_j) > \theta$, $d_\beta(u, x_i) \leqslant \theta$, $d_\beta(x_j, v) \leqslant \theta$, which implies $d_\beta(u,v) > \theta \geqslant d_0(u,v)$. Thus $d_\beta \geqslant d_0$, which completes the proof that $d_0 \ll d$. □

Remark 3. *It is straightforward to verify that, if in the five latter lemmas the pseudoultrametric d is locally compact, then, by Remark 2, all the auxiliary pseudoultrametrics constructed in the proofs are locally compact as well. Hence these lemmas are valid for the "weakly way below" relation not only in $PsU(X)$, but also in $LCPsU(X)$.*

Now we can obtain a corollary on what is "way below" a non-compact pseudoultrametric.

Theorem 3. *If a pseudoultrametric (a locally compact pseudoultrametric) d is not compact, then $d_0 \equiv 0$ is the unique pseudoultrametric way below d in $PsU(X)$ (resp. in $LCPsU(X)$).*

Proof. By Remark 1 for d either there is a decreasing sequence of balls

$$B_R(x) \supset B_{r_1}(x_1) \supset B_{r_2}(x_2) \supset \ldots \text{ with } R > r_1 > r_2 > \ldots \searrow 0$$

with the empty intersection, or for some $r > 0$ there are distinct balls $B_r(x_1), B_r(x_2), \ldots$.

In the first case we denote $A_1 = B_R(x) \setminus B_{r_1}(x_1)$, $A_2 = B_{r_1}(x_1) \setminus B_{r_2}(x_2)$, $\ldots$. In the second case we simply put $A_1 = B_r(x_1)$, $A_2 = B_r(x_2)$, $\ldots$. In both cases all sets $A_1, A_2, \ldots$, as well as $A_0 = X \setminus (A_1 \cup A_2 \cup \ldots)$, are open and form a partition of X. Hence the pseudoultrametric

$$\rho(u,v) = \begin{cases} 0, & u,v \text{ are in the same } A_i, \\ \max\{i,j\}, & u \in A_i, v \in A_j, i \neq j, \end{cases} \qquad u,v \in X,$$

is continuous with respect to d. Then $d' = \sup\{d, \rho\}$ is a pseudoultrametric, and is locally compact provided so is d. It satisfies conditions of Lemma 3, hence only the zero pseudoultrametric is way below d' in $PsU(X)$ (resp. in $LCPsU(X)$). As $d \leqslant d'$ and $d_0 \ll d$ implies $d_0 \ll d'$, the proof is complete. □

Remark 4. *Together with the previous lemma this implies that the relations "way below" and "weakly way below" for non-compact pseudoultrametrics are different.*

Theorem 4. *The relations "way below" and "weakly way below" on the poset $CPsU(X)$ coincide.*

Proof. We need only to prove $d_0 \lll d \implies d_0 \ll d$ in $CPsU$. Let $\{d_\alpha \mid \alpha \in \mathcal{A}\}$ be a directed set of pseudoultrametrics such that the supremum ρ of d_α is compact and greater than or equal to d. By the virtue of Theorem 1 (i.e., meet continuity) the supremum of $\inf\{d, d_\alpha\}$ is equal to $\inf\{d, \rho\} = d$, therefore there is $\inf\{d, d_\alpha\} \geqslant d_0$, which implies $d_\alpha \geqslant d_0$. □

Theorem 5. *Let $d_0, d_1 \in CPsU(X)$, $d_0 \leqslant d_1$. Then $d_0 \ll d_1$ if and only if the following holds:*

(1) if $d_0(x,y) = d_1(x,y)$ for some $x, y \in X$, then $d_1(x,y) = 0$;
(2) there are $k \in \{0,1,2,\ldots\}$ and $z_1, \ldots, z_k \in X$ such that for all $x \in X$ the equality $d_0(x, z_i)$ is valid for some $1 \leqslant i \leqslant k$.

Proof. *Sufficiency.* Assume (1) and (2). We can assume $d_0(z_i, z_j) > 0$ for all $i \neq j$ (otherwise we can drop either z_i or z_j, etc., until the condition is satisfied). We prove the statement by induction. If $k = 0$ or $k = 1$, then $d_0 \equiv 0$, hence $d_0 \ll d_1$.

If the statement holds for $k \leqslant n$, $n \geqslant 1$, then for $k = n+1$ there is $\varepsilon = \max\{d_0(z_i, z_j) \mid 1 \leqslant i < j \leqslant n+1\} > 0$, therefore X is a finite union of compact balls $B_\varepsilon(z_i)$ with respect to d_0, which are pairwise either disjoint or equal. The restrictions of d_0 and d_1 to each ball clearly satisfy 1) and 2), hence $d_0|_{B_\varepsilon(z_i)} \ll d_1|_{B_\varepsilon(z_i)}$. The conditions of Lemma 5 are satisfied, which implies $d_0 \lll d_1 \implies d_0 \ll d_1$.

Necessity. Let $d_0 \ll d_1$. To show (1), assume that there are $x, y \in X$ such that $d_0(x,y) = d_1(x,y) > 0$. Consider the set $D = \left\{\left(1 - \frac{1}{n}\right)d_1 \mid n \in \mathbb{N}\right\}$. Then $D \subset CPsU(X)$ is directed, and $\sup D = d_1$. For all $n \in \mathbb{N}$ we have $d_0(x,y) = d_2(x,y) > \left(1 - \frac{1}{n}\right)d_1(x,y)$, hence no element of D dominates d_1, which contradicts that $d_1 \ll d_2$. Thus 1) holds.

We show that $d_0 \ll d_1$ implies (2). For $\varepsilon > 0$ let

$$d^{(\varepsilon)}(x,y) = \begin{cases} d_1(x,y), & \text{if } d_1(x,y) \geqslant \varepsilon, \\ 0, & \text{otherwise,} \end{cases}$$

for $x, y \in X$. Obviously $d^{(\varepsilon)}$ is a pseudoultrametric, $d^{(\varepsilon)} \leqslant d_1$, and compactness of d_1 implies that $d^{(\varepsilon)}$ is compact as well. Consider the set $D = \left\{d^{(\varepsilon)} \mid \varepsilon > 0\right\}$. It is directed and $(\sup D)(x,y) = \sup\left\{d^{(\varepsilon)}(x,y) \mid \varepsilon > 0\right\} = d_1(x,y)$ for all $x, y \in X$, hence $\sup D = d_1$. Taking into account $d_0 \ll d_1$, we can choose $\varepsilon > 0$ such that $d_0 \leqslant d^{(\varepsilon)}$. There is a finite partition $B_\varepsilon(z_1)$, $B_\varepsilon(z_2)$, …, $B_\varepsilon(z_m)$ of X into balls with respect to $d^{(\varepsilon)}$, and $d^{(\varepsilon)}$ does not attain values in $(0, \varepsilon)$, hence for each $x \in B_{\varepsilon'}(z_i)$ we have $d_0(x, z_i) \leqslant d^{(\varepsilon)}(x, z_i) = 0$. This completes the proof. □

The latter theorem implies that for all $d \in CPsU(X)$, $\varepsilon > 0$ and $\lambda \in (0,1)$ the pseudoultrametric with the formula

$$d^\varepsilon_\lambda(x,y) = \begin{cases} \lambda d(x,y), & d(x,y) \geqslant \varepsilon, \\ 0, & d(x,y) < \varepsilon, \end{cases}$$

is way below d. The set

$$D = \{d^\varepsilon_\lambda | \lambda \in [0;1), \varepsilon > 0\}$$

is directed, and

$$(\sup D)(x,y) = \lim_{\substack{\varepsilon \to 0 \\ \lambda \to 1}} d^\varepsilon_\lambda(x,y) = d(x,y).$$

Thus the following theorem is obtained:

Theorem 6. *For all $d \in CPsU(X)$ there is a directed set of compact pseudoultrametrics way below d such that* $\sup D = d$.

5. Conclusions and Future Research

We have proved that the poset $CPsU(X)$ is continuous. It lacks directed completeness, hence is not a domain, but each of its subsets of the form $d{\downarrow} = \{\rho \in CPsU(X) \mid \rho \leqslant d\}$

for $d \in CPsU(X)$ is a domain, namely a complete continuous lower semilattice. We will describe its properties in a subsequent paper. This will allow the entire well-elaborated apparatus of domain theory [5] to be applied to classification problems.

Morphisms between $CPsU(X)$ and $CPsU(Y)$, that preserve the structure to different extents, will be described. In particular, tools of category theory will be used. We expect fruitful applications of this theory to denotational semantics in computer science. For example, an information system (maybe a neural net) can be studied such that its input and/or output is a tree-like classification. Then this system behavior is stable (tolerant to inaccuracies) if the respective mapping input $\to$ output is continuous in order sense. This means that, the closer an input information is to the "actual" state of things, the more accurate is the output. The analysis of continuity and comparison of information precision involves "way below" relations.

A "decent" order on a poset determines several classical topologies, namely, lower/upper, Scott, and Lawson topologies [5,13]. The continuity of $CPsU(X)$ (and hence of its lower subsets) implies that these topologies have nice properties (e.g., are Hausdorff and/or compact). These topologies and related metrics are also a topic of ongoing research.

The other conclusion of this work is that approximations of pseudoultrametrics (in the sense of order theory) are efficient only for the compact case. This is not a big surprise because similar limitations appeared earlier in similar circumstances, e.g., for approximations of possibility measures or non-additive measures. In practice compactness corresponds to gradual classification such that the set of classes (clusters etc.) at any stage is finite. In fact this happens in most cases, hence compact pseudoultrametrics are quite sufficient for applications.

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