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Inference for Partially Accelerated Life Test from a Bathtub-Shaped Lifetime Distribution with Progressive Censoring

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Abstract: The analysis of the constant-stress partially accelerated life test was considered under progressive Type-II censoring when the lifetime of the products follows a two-parameter bathtub-shaped distribution. The maximum likelihood estimates of the unknown parameters were established, where the expectation–maximization iterative solution is proposed for the estimation. The approximate confidence intervals were also constructed based on asymptotic theory via the Fisher information matrix. For comparison purposes, the bootstrap (i.e., Studentized-t and percentile) confidence intervals of the unknown parameters were also obtained. Finally, simulation studies and a real-life data example are presented to examine the performance of the different results.

Keywords: maximum likelihood estimator; bathtub-shaped distribution; bootstrap confidence intervals; credible intervals; Bayes estimates

MSC: 62F10; 62F15

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1. Introduction

With the development of manufacturing design technology, modern products have the characteristics of high reliability and a long lifespan. Under these usage conditions, life testing becomes increasingly difficult within a reasonable period due to the time and cost limitations. To conduct life testing in an effective way, the accelerated life test (ALT) has been introduced in practice, which provides a low-cost and quick method to obtain the failure information. In the ALT, in order to obtain the failure data, products are tested either under a harsher environment or under more intensive usage than the usual use conditions, and the common stress factors include the temperature, pressure, and voltage, among others. Data collected in such accelerated conditions are then extrapolated through a physically appropriate statistical model to estimate the lifetime distribution under normal use conditions. Generally, there are mainly three types of acceleration tests, namely the constant-stress ALT, the step-stress ALT, and the progressive-stress ALT. All these types of ALTs have been studied by a number of authors [1–5]; for more details about the ALT, one can refer to the monograph by Nelson [6].

Under accelerating life testing, when the acceleration factor has an unknown value, the partially accelerated life test (PALT) is usually conducted as a reasonable alternative to the life test, where the units are investigated under both accelerated and regular use conditions. There are mainly two types of PALTs, the constant-stress PALT and the step-stress PALT. In the constant-stress PALT, the units are divided into two groups; one group is assigned to run under the regular use conditions, while the other group is tested under accelerated conditions. For the step-stress PALT, the units are initially run under the regularuse conditions for a pre-specified period of time, and if a test unit does not fail for the specified time, it is then run under accelerated conditions until failure occurs or the observation is censored. PALTs have been discussed by several authors. For instance, Abdel-Hamid [7] considered the parameter estimation of the Burr Type-XII distribution in a constant-stress PALT for progressively Type-II-censored data. Abdel-Hamid and Al-Hussaini [8] studied the estimation problem in the step-stress PALT when the lifetime of the tested units under the regular use conditions followed a finite mixture of a general class of distributions. Abdel-Ghaly et al. [9] investigated the maximum likelihood estimation (MLE) method to estimate the parameters of the Weibull distribution in the step-stress PALT. A similar problem was also studied by Ismail [10] under the progressive hybrid Type-II censoring scheme. Cheng and Wang [11] discussed the MLE of the Burr XII distribution under the constant-stress PALT when multiple censored data were observed.

In practice, the lifetime distributions with bathtub-shaped hazard functions have attracted the interest of many authors and provide appropriate conceptual models for some electronic and mechanical products, as well as the lifetimes of humans. Chen [12] proposed a two-parameter distribution with a bathtub-shaped or increasing failure rate function. The survival function (SF), cumulative distribution function (CDF), probability density function (PDF), and hazard rate function (HRF) of the Chen distribution are given by

$$\begin{aligned} S_1(t) &= e^{\alpha(1-e^{t^\beta})}, t > 0, \\ F_1(t) &= 1 - e^{\alpha(1-e^{t^\beta})}, t > 0, \\ f_1(t) &= \alpha\beta t^{\beta-1} \exp\left\{\alpha(1-e^{t^\beta}) + t^\beta\right\}, t > 0, \\ h_1(t) &= \alpha\beta t^{\beta-1} e^{t^\beta}, t > 0, \end{aligned} \tag{1}$$

where $\alpha > 0$ is the scale parameter and $\beta > 0$ is the shape parameter, respectively. In practice, the applicability of a model may partially be attributed to the fact that its reliability, hazard rate, and probability density functions all have nice expressions. Due to its flexible structural properties and practical significance, the Chen distribution has found wide applications in life test studies. It has been observed that the Chen distribution has a bathtub-shaped hazard function when $0 < \beta < 1$, and when $\beta \geq 1$, it features an increasing hazard rate function. The case $\alpha = 1$ corresponds to the exponential power distribution. Several authors have discussed the Chen distribution in different cases. For example, based on Type-II-censored samples, Chen [12] constructed exact confidence intervals for the shape parameter and also obtained exact confidence regions for both model parameters. Rastogi and Tripathi [13] discussed the parameter estimation problem of the Chen distribution for hybrid censored data. Wu [14] investigated the MLE method to estimate the parameters of the Chen distribution under progressive Type-II censoring and also derived exact confidence intervals and confidence regions for the related parameters. Ahmed [15] presented the Bayesian approach to estimate the parameters of the Chen distribution for progressively Type-II-censored data. Elshahhat and Rastogi [16] considered the Bayesian life analysis of a generalized Chen's population under progressive censoring.

In life testing, reliability analysis, and other related fields, censoring is a very common phenomenon, and the experiments are often terminated before all units fail due to the cost and time considerations. In such cases, the exact failure times are known for only a portion of the units under study. The most-common censoring schemes are Type-I and Type-II censoring, which, however, can only remove units at the termination point, lacking flexibility in practical life tests. Therefore, progressive censoring is further proposed to conduct the tests, which allows units to be removed at different testing stages. The strategy of progressive censoring is vital to planning duration experiments in the field of reliability and lifetime analysis and includes progressive Type-I and Type-II censoring as its special population cases. For details about the censoring scheme, the reader can refer to the recent review paper of Balakrishnan [17] and the monograph of Balakrishnan and Aggarwala [18], as well as the references therein. In this paper, we considered the constant-stress PALT applied to units whose lifetime under the use conditions was assumed to be the Chen distribution under a progressive Type-II censoring scheme. It is worth mentioning that, although there are many discussions that have focused on the inference of the Chen distribution such as the previously mentioned ones and others (e.g., Zhang and Gui [19], Soliman et al. [20]), the inference for the partially constant-stress accelerated life test of the Chen distribution has not been discussed in the literature. In addition, except for the traditional likelihood-based inferential approach, another aim of this paper was that the expectation–maximization estimation for the Chen model be proposed under the constant-stress PALT.

The rest of this article is organized as follows. Section 2 provides a brief description of the constant-stress PALT and some basic assumptions. Section 3 deals with the estimation problem of the MLEs of the parameters. The corresponding confidence intervals are proposed in Section 4, and this section also discusses two parametric bootstrap confidence intervals. An illustration example and Monte Carlo simulation studies are presented in Section 5 to investigate the performance the proposed results. Finally, some concluding remarks are addressed in Section 6.

2. Model Description and Basic Assumptions

2.1. Model Description

In the constant-stress PALT, assuming that there are n identical test units, n_1 units are randomly selected from n test units for testing in the regular use conditions and the remaining $n_2 = n - n_1$ units are tested under accelerated stress. Progressive Type-II censoring is applied as follows. For $j = 1, 2$, when the first failure, say T_{j1}, has occurred, R_{j1} units are randomly removed from the remaining $n_j - 1$ surviving units. When the second failure, say X_{j2}, has occurred, R_{j2} units from the remaining $n_j - 2 - R_{j1}$ units are randomly removed. The test proceeds at different stress levels until the m_jth failure, say T_{jm_j}, has occurred, at which time all remaining $R_{jm_j} = n_j - m_j - \sum_{k=1}^{m_j-1} R_{jm_j}$ units are withdrawn and the constant-stress PALT is terminated. In our discussion, the values of the censoring numbers $R_{ji}, j = 1, 2, i = 1, 2, \ldots, m_j$ were predetermined with $m_j < n_j$. It is noted that the complete sample and conventional Type-II censoring are special cases of the progressive Type-II censoring scheme.

In our study, the progressively Type-II-censored samples $T_{j1} < T_{j2} < \ldots < T_{jm_j}$, $j = 1, 2$ were from two testing conditions, of which the CDFs and the PDFs are $F_j(t)$ and $f_j(t)$, respectively, with censoring schemes $R_j = (R_{j1}, R_{j2}, \ldots, R_{jm_j})$. Denote $t_{j1} < t_{j2} < \ldots < t_{jm_j}$ as the observed values of $T_{j1}, T_{j2}, \ldots, T_{jm_j}$; the joint PDF based on the two progressively Type-II-censored samples can be expressed as

$$L(\alpha, \beta, \lambda; t) = \prod_{j=1}^{2}\left[C_j \prod_{i=1}^{m_j} f_j(t_{ji})(1 - F_j(t_{ji}))^{R_{ji}}\right], \tag{2}$$

where $t = (t_1, t_2), t_j = (t_{j1}, t_{j2}, \ldots, t_{jm_j}), j = 1, 2$ and $C_j = n_j \prod_{i=1}^{m_j-1}[n_j - \sum_{k=1}^{i}(1 + R_{jk})]$.

2.2. Basic Assumptions

1. The lifetime of a unit tested under the regular use conditions follows a Chen distribution with the CDF and PDF presented in (1).
2. The hazard rate of a unit tested under accelerated conditions is given by $h_2(t) = \lambda h_1(t)$, where λ is an acceleration factor satisfying $\lambda > 1$. Therefore, the HRF, SF, CDF, and PDF under the accelerated conditions are given by

$$\begin{aligned}
&h_2(t) = \alpha\beta\lambda t^{\beta-1}e^{t^\beta}, t > 0,\\
&S_2(t) = \exp\left\{-\int_0^t h_2(u)du\right\} = \exp\left\{\alpha\lambda(1-e^{t^\beta})\right\}, t > 0,\\
&F_2(t) = 1 - e^{\alpha\lambda(1-e^{t^\beta})}, t > 0,\\
&f_2(t) = \alpha\beta\lambda t^{\beta-1}\exp\left\{\alpha\lambda(1-e^{t^\beta}) + t^\beta\right\}, t > 0.
\end{aligned} \tag{3}$$

3. There are $n_j, j = 1,2$ units allocated under the regular use conditions with $j = 1$ and the accelerated conditions with $j = 2$, of which the lifetimes $T_{ij}, i = 1,2,\dots,n_j, j = 1,2$ are mutually independent.

3. Maximum Likelihood Estimation

Based on (1)–(3), the log-likelihood function of $L(\alpha,\beta,\lambda;t)$, say ℓ, can be expressed as

$$\begin{aligned}
\ell \propto & (m_1+m_2)\log(\alpha\beta) + m_2\log(\lambda) + \beta\sum_{j=1}^{2}\sum_{i=1}^{m_j}\log t_{ji} + \sum_{j=1}^{2}\sum_{i=1}^{m_j} t_{ji}^{\beta}\\
&+ \alpha\sum_{j=1}^{2}\sum_{i=1}^{m_j}\lambda^{j-1}(1+R_{ji})(1-e^{t_{ji}^{\beta}}),
\end{aligned} \tag{4}$$

where the notation $\propto$ means "be proportional to" without additive constant terms.

Taking the derivatives of ℓ with respect to α, β, λ and equating them to zero, one can derive the MLEs of the parameters α, β, and λ from the following likelihood equations:

$$\ell_\alpha = 0, \quad \ell_\beta = 0, \quad \text{and} \quad \ell_\lambda = 0, \tag{5}$$

where

$$\begin{aligned}
\ell_\alpha &= \frac{\partial\ell}{\partial\alpha} = \frac{m_1+m_2}{\alpha} + \sum_{j=1}^{2}\sum_{i=1}^{m_j}\lambda^{j-1}(1+R_{ji})(1-e^{t_{ji}^{\beta}}),\\
\ell_\beta &= \frac{\partial\ell}{\partial\beta} = \frac{m_1+m_2}{\beta} + \sum_{j=1}^{2}\sum_{i=1}^{m_j}\log t_{ji} + \sum_{j=1}^{2}\sum_{i=1}^{m_j} t_{ji}^{\beta}\log t_{ji}\\
&\qquad - \alpha\sum_{j=1}^{2}\sum_{i=1}^{m_j}\lambda^{j-1}(1+R_{ji})e^{t_{ji}^{\beta}}t_{ji}^{\beta}\log t_{ji},\\
\ell_\lambda &= \frac{\partial\ell}{\partial\lambda} = \frac{m_2}{\lambda} + \alpha\sum_{i=1}^{m_2}(1+R_{2i})(1-e^{t_{2i}^{\beta}}).
\end{aligned}$$

Based on the likelihood equations $\ell_\alpha = 0$ and $\ell_\lambda = 0$ in (5), one has

$$\alpha = \frac{m_1}{\sum_{i=1}^{m_1}(1+R_{1i})(e^{t_{1i}^{\beta}}-1)}, \quad \lambda = \frac{m_2\sum_{i=1}^{m_1}(1+R_{1i})(e^{t_{1i}^{\beta}}-1)}{m_1\sum_{i=1}^{m_2}(1+R_{2i})(e^{t_{2i}^{\beta}}-1)}, \tag{6}$$

which implies that

$$\frac{m_1+m_2}{\beta} + \sum_{j=1}^{2}\sum_{i=1}^{m_j}(1+t_{ji}^{\beta})\log t_{ji} - \sum_{j=1}^{2}\frac{m_j\sum_{i=1}^{m_j}(1+R_{ji})e^{t_{ji}^{\beta}}t_{ji}^{\beta}\log t_{ji}}{\sum_{i=1}^{m_j}(1+R_{ji})(e^{t_{ji}^{\beta}}-1)} = 0. \tag{7}$$

Therefore, using numerical iterative programs such as the Newton–Raphson algorithm to solve the nonlinear Equation (7), the MLE of β can be calculated. Further, by substituting the MLE of β into (6), the MLEs of α and λ can be obtained. Alternatively, one can also use the expectation–maximization (EM) algorithm to deduce the MLEs of the unknown parameters. The EM algorithm is extensively used to the iterative computation of maximum likelihood estimates and is very useful in a variety of fields such as survival analysis, reliability theory, and other fields. The EM iteration alternates between performing an expectation step or E-step and a maximization step or M-step. The E-step creates a function of the expectation of the log-likelihood of the current estimation evaluation using the parameters, and the M-step calculates the parameters that maximize the expected log-likelihood found in the E-step. For more details, one can refer to the work of Dempster et al. [21], where the authors first introduced the EM algorithm to handle some missing or incomplete data situations, and the monograph by McLachlan and Krishnan [22], as well as the references therein.

In the constant-stress PALT, the progressively Type-II-censored samples can be viewed as an incomplete dataset in each life test stage. Therefore, the EM algorithm will provide a good alternative to the conventional iterative method in the process of numerically computing the MLEs.

For $j = 1, 2$, let $X = (X_{j1}, X_{j2}, \ldots, X_{jm_j})$ with $X_{jk} = (X^1_{jk}, X^2_{jk}, \ldots, X^{R_{jk}}_{jk}), k = 1, 2, \ldots, m_j$, represent the censored data under the normal use and accelerated conditions, respectively. We treated the censored observations as missing data. Thus, the combination of (T, X) forms the complete constant-stress PALT failure dataset, for which the likelihood function can be expressed as

$$L^*(\alpha, \beta, \lambda; t) = \prod_{j=1}^{2}\prod_{i=1}^{m_j} f_j(t_{ji}) \cdot \prod_{j=1}^{2}\prod_{k=1}^{m_j}\prod_{s=1}^{R_{jk}} f_j(x^s_{jk}). \tag{8}$$

The log-likelihood function of $L^*(\alpha, \beta, \lambda; t)$, say ℓ^*, can be expressed as

$$\begin{aligned}
\ell^* \propto &\left[m_1 + m_2 + \sum_{j=1}^{2}\sum_{k=1}^{m_j} R_{jk}\right]\ln\alpha + \left[m_1 + m_2 + \sum_{j=1}^{2}\sum_{k=1}^{m_j} R_{jk}\right]\ln\beta + \left[m_2 + \sum_{k=1}^{m_2} R_{2k}\right]\ln\lambda \\
&+ \beta\left[\sum_{j=1}^{2}\sum_{i=1}^{m_j}\ln t_{ji} + \sum_{j=1}^{2}\sum_{k=1}^{m_j}\sum_{s=1}^{R_{jk}}\ln x^s_{jk}\right] + \sum_{j=1}^{2}\sum_{i=1}^{m_j}[\alpha\lambda^{j-1}(1 - e^{t^{\beta}_{ji}}) + t^{\beta}_{ji}] \\
&+ \sum_{j=1}^{2}\sum_{k=1}^{m_j}\sum_{s=1}^{R_{jk}}[\alpha\lambda^{j-1}(1 - e^{(x^s_{jk})^{\beta}}) + (x^s_{jk})^{\beta}].
\end{aligned}$$

The MLEs of the parameters α, β, and λ for the complete failure sample (T, X) can be derived by taking the derivatives for the log-likelihood function ℓ^* with respect to α, β, and λ and setting them to zero, which can be expressed as follows:

$$\begin{aligned}
\frac{\partial\ell^*}{\partial\alpha} =& \frac{1}{\alpha}\left[m_1 + m_2 + \sum_{j=1}^{2}\sum_{k=1}^{m_j} R_{jk}\right] + \sum_{j=1}^{2}\sum_{i=1}^{m_j}\lambda^{j-1}(1 - e^{t^{\beta}_{ji}}) + \sum_{j=1}^{2}\sum_{k=1}^{m_j}\sum_{s=1}^{R_{jk}}\lambda^{j-1}(1 - e^{(x^s_{jk})^{\beta}} = 0, \\
\frac{\partial\ell^*}{\partial\beta} =& \frac{1}{\beta}\left[m_1 + m_2 + \sum_{j=1}^{2}\sum_{k=1}^{m_j} R_{jk}\right] + \left[\sum_{j=1}^{2}\sum_{i=1}^{m_j}\ln t_{ji} + \sum_{j=1}^{2}\sum_{k=1}^{m_j}\sum_{s=1}^{R_{jk}}\ln x^s_{jk}\right] \\
&+ \sum_{j=1}^{2}\sum_{i=1}^{m_j} t^{\beta}_{ji}(1 - \alpha\lambda^{j-1}e^{t^{\beta}_{ji}})\ln t_{ji} + \sum_{j=1}^{2}\sum_{k=1}^{m_j}\sum_{s=1}^{R_{jk}}(x^s_{jk})^{\beta}(1 - \alpha\lambda^{j-1}e^{(x^s_{jk})^{\beta}})\ln x^s_{jk} = 0, \qquad (9) \\
\frac{\partial\ell^*}{\partial\lambda} =& \frac{1}{\lambda}\left[m_2 + \sum_{k=1}^{m_2} R_{2k}\right] + \alpha\left[\sum_{i=1}^{m_2}(1 - e^{t^{\beta}_{2i}}) + \sum_{k=1}^{m_2}\sum_{s=1}^{R_{2k}}(1 - e^{(x^s_{2k})^{\beta}})\right] = 0.
\end{aligned}$$

Furthermore, one has

$$\begin{aligned}
\alpha \triangleq \alpha(\beta) &= \frac{\mathbb{B}(e^{t_{2i}^{\beta}}, e^{(x_{2k}^{s})^{\beta}})\mathbb{E}(e^{t_{2i}^{\beta}}, e^{(x_{2k}^{s})^{\beta}}) - \mathbb{A}\mathbb{D}}{\mathbb{D}\mathbb{C}(e^{t_{1i}^{\beta}}, e^{(x_{1k}^{s})^{\beta}})}, \\
\lambda \triangleq \lambda(\beta) &= \frac{\mathbb{C}(e^{t_{1i}^{\beta}}, e^{(x_{1k}^{s})^{\beta}})\mathbb{D}^2}{\mathbb{E}(e^{t_{2i}^{\beta}}, e^{(x_{2k}^{s})^{\beta}})[\mathbb{B}(e^{t_{2i}^{\beta}}, e^{(x_{2k}^{s})^{\beta}})\mathbb{E}(e^{t_{2i}^{\beta}}, e^{(x_{2k}^{s})^{\beta}}) - \mathbb{A}\mathbb{D}]},
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
\mathbb{A} &= m_1 + m_2 + \sum_{j=1}^{2}\sum_{k=1}^{m_j} R_{jk}, \\
\mathbb{B}(b_{2i}, b_{2k}^{s}) &= \sum_{i=1}^{m_2}(1 - b_{2i}) + \sum_{k=1}^{m_2}\sum_{s=1}^{R_{2k}}(1 - b_{2k}^{s}), \\
\mathbb{C}(c_{1i}, c_{1k}^{s}) &= \sum_{i=1}^{m_1}(1 - c_{1i}) + \sum_{k=1}^{m_1}\sum_{s=1}^{R_{1k}}(1 - c_{1k}^{s}), \\
\mathbb{D} &= m_2 + \sum_{k=1}^{m_2} R_{2k}, \\
\mathbb{E}(e_{2i}, e_{2k}^{s}) &= \sum_{i=1}^{m_2}(1 - e_{2i}) + \sum_{k=1}^{m_2}\sum_{s=1}^{R_{2k}}(1 - e_{2k}^{s}).
\end{aligned}$$

and $\alpha \triangleq \alpha(\beta)$ represents α equivalent to $\alpha(\beta)$

By substituting (10) to $\frac{\partial \ell^*}{\partial \beta} = 0$ in (9), one has

$$\beta = -\frac{m_1 + m_2 + \sum_{j=1}^{2}\sum_{k=1}^{m_j} R_{jk}}{\left(\begin{array}{c} \sum_{j=1}^{2}\sum_{i=1}^{m_j} \ln t_{ji} + \sum_{j=1}^{2}\sum_{k=1}^{m_j}\sum_{s=1}^{R_{jk}} \ln x_{jk}^{s} \\ +\sum_{j=1}^{2}\sum_{i=1}^{m_j} t_{ji}^{\beta}(1 - \alpha(\beta)\lambda^{j-1}(\beta)e^{t_{ji}^{\beta}}) \ln t_{ji} + \\ \sum_{j=1}^{2}\sum_{k=1}^{m_j}\sum_{s=1}^{R_{jk}} (x_{jk}^{s})^{\beta}(1 - \alpha(\beta)\lambda^{j-1}(\beta)e^{(x_{jk}^{s})^{\beta}}) \ln x_{jk}^{s} \end{array}\right)}. \tag{11}$$

Therefore, it can be observed that the pseudo likelihood equations can be rewritten as a nonlinear function of β.

Given $T = (T_{j1}, T_{j2}, \ldots, T_{jm_j}), j = 1,2,$ for $k = 1,2,\ldots,m_j$, the conditional distribution of $X_{jk}^{s}, s = 1,2,\ldots,R_{jk}$ follows a truncated Chen distribution with scale parameter $\alpha\lambda^{j-1}$ and shape parameter β, of which the density can be expressed as

$$f_{X|T}(x_{jk}|t) = \frac{f_j(x_{jk})}{1 - F_j(t_{jk})} = \frac{\alpha\beta\lambda^{j-1}x_{jk}^{\beta-1}e^{\alpha\lambda^{j-1}(1-e^{x_{jk}^{\beta}}) + x_{jk}^{\beta}}}{e^{\alpha\lambda^{j-1}(1-e^{t_{jk}^{\beta}}) + t_{jk}^{\beta}}}, x_{jk} > t_{jk}.$$

For $j = 1,2$ and $k = 1,2,\ldots,m_j$, one has

$$\begin{aligned}
E_{j1}(\alpha,\beta,\lambda) =& E_j(e^{(X_{jk}^{s})^{\beta}}|X_{jk} > t_{jk}) = \frac{1 + \alpha\lambda^{j-1}e^{t_{jk}^{\beta}}}{\alpha\lambda^{j-1}}, \\
E_{j2}(\alpha,\beta,\lambda) =& E_j(\ln X_{jk}^{s}|X_{jk} > t_{jk}) = \ln t_{jk} + \frac{e^{\alpha\lambda^{jk}e^{t_{jk}^{\beta}}}}{\beta}\int_{t_{jk}^{\beta}}^{\infty} t^{-1}e^{-\alpha\lambda^{j-1}e^{t}}dt, \\
E_{j3}(\alpha,\beta,\lambda) =& E_j((X_{jk}^{s})^{\beta}\ln X_{jk}^{s}|X_{jk} > t_{jk}) = \frac{\alpha\lambda^{j-1}}{\beta S_j(t_{jk})}\int_{t_{jk}^{\beta}}^{\infty} te^{\alpha\lambda^{j-1}(1-e^{t})+t}\ln t dt,
\end{aligned}$$

$$E_{j4}(\alpha,\beta,\lambda) = E_j((X_{jk}^s)^\beta e^{(X_{jk}^s)^\beta} \ln X_{jk}^s | X_{jk} > t_{jk}) = \alpha\beta^{-1}\lambda^{j-1} e^{\alpha\lambda^{j-1} e^{t_{ij}^\beta}} \int_{t_{jk}^\beta}^{\infty} t e^{2t-\alpha\lambda^{j-1}e^t} \ln t dt.$$

Hence, an EM algorithm (Algorithm 1) to calculate the MLEs of the parameters α, β, and λ is proposed as follows.

Algorithm 1 EM iterative algorithm.

Step 1 Let $\alpha^{(0)}, \beta^{(0)}$, and $\lambda^{(0)}$ be the initial guess values of the MLEs for α, β, and λ.

Step 2 In the $(d+1)$th iterative:

- (E-step) Replace any function of X_{jk}^s (say $h_{X_{jk}^s}$) by $E(h(X_{jk}^s)|X_{jk}^s > T_{jk})$, and the likelihood equations are replaced by

$$\beta = -\frac{m_1 + m_2 + \sum_{j=1}^2 \sum_{k=1}^{m_j} R_{jk}}{\left(\begin{array}{c} \sum_{j=1}^2 \sum_{i=1}^{m_j} \ln t_{ji} + \sum_{j=1}^2 \sum_{k=1}^{m_j} \sum_{s=1}^{R_{jk}} E_{j2}(\alpha^{(d)}, \beta^{(d)}, \lambda^{(d)}) \\ + \sum_{j=1}^2 \sum_{i=1}^{m_j} t_{ji}^\beta (1 - [\alpha(\beta)]^{(d)}([\lambda(\beta)]^{(d)})^{j-1} e^{t_{ji}^\beta}) \ln t_{ji} \\ + \sum_{j=1}^2 \sum_{k=1}^{m_j} \sum_{s=1}^{R_{jk}} E_{j3}(\alpha^{(d)}, \beta^{(d)}, \gamma^{(d)}) \\ - \sum_{j=1}^2 \sum_{k=1}^{m_j} \sum_{s=1}^{R_{jk}} E_{j4}(\alpha^{(d)}, \beta^{(d)}, \gamma^{(d)})[\alpha(\beta)]^{(d)}([\lambda(\beta)]^{(d)})^{j-1} \end{array}\right)},$$

where

$$[\alpha(\beta)]^{(d)} = \frac{\mathbb{B}(e^{t_{2i}^\beta}, E_{21}(\alpha^{(d)}, \beta^{(d)}, \lambda^{(d)}))\mathbb{E}(e^{t_{2i}^\beta}, E_{21}(\alpha^{(d)}, \beta^{(d)}, \lambda^{(d)})) - \mathbb{AD}}{\mathbb{DC}(e^{t_{1i}^\beta}, E_{11}(\alpha^{(d)}, \beta^{(d)}, \lambda^{(d)}))},$$

$$[\lambda(\beta)]^{(d)} = \frac{\mathbb{C}(e^{t_{1i}^\beta}, e^{(x_{1k}^s)^{\beta^{(d)}}})\mathbb{D}^2}{\left(\begin{array}{c} \mathbb{E}(e^{t_{2i}^\beta}, E_{21}(\alpha^{(d)}, \beta^{(d)}, \lambda^{(d)})) \\ \cdot[\mathbb{B}(e^{t_{2i}^\beta}, E_{21}(\alpha^{(d)}, \beta^{(d)}, \lambda^{(d)}))\mathbb{E}(e^{t_{2i}^\beta}, E_{21}(\alpha^{(d)}, \beta^{(d)}, \lambda^{(d)})) - \mathbb{AD}] \end{array}\right)}.$$

- (M-step) The estimate of β, namely $\beta^{(k+1)}$, can be derived iteratively by solving the following nonlinear equations:

$$\beta^{(d+1)} = -\frac{m_1 + m_2 + \sum_{j=1}^2 \sum_{k=1}^{m_j} R_{jk}}{\left(\begin{array}{c} \sum_{j=1}^2 \sum_{i=1}^{m_j} \ln t_{ji} + \sum_{j=1}^2 \sum_{k=1}^{m_j} R_{jk} E_{j2}(\alpha^{(d)}, \beta^{(d)}, \lambda^{(d)}) \\ + \sum_{j=1}^2 \sum_{i=1}^{m_j} t_{ji}^{\beta^{(d+1)}} (1 - [\alpha(\beta)]^{(d)}([\lambda(\beta)]^{(d)})^{j-1} e^{t_{ji}^{\beta^{(d+1)}}}) \ln t_{ji} \\ + \sum_{j=1}^2 \sum_{k=1}^{m_j} R_{jk} E_{j3}(\alpha^{(d)}, \beta^{(d)}, \gamma^{(d)}) \\ - \sum_{j=1}^2 \sum_{k=1}^{m_j} R_{jk} E_{j4}(\alpha^{(d)}, \beta^{(d)}, \gamma^{(d)})[\alpha(\beta)]^{(d)}([\lambda(\beta)]^{(d)})^{j-1} \end{array}\right)},$$

and the estimates of α and λ, namely $\alpha^{(d+1)}, \lambda^{(d+1)}$, can be further derived as

$$\alpha^{(d+1)} = \frac{\mathbb{B}(e^{t_{2i}^{\beta^{(d+1)}}}, E_{21}(\alpha^{(d)}, \beta^{(d)}, \gamma^{(d)}))\mathbb{E}(e^{t_{2i}^{\beta^{(d+1)}}}, E_{21}(\alpha^{(d)}, \beta^{(d)}, \gamma^{(d)})) - \mathbb{AD}}{DC(e^{t_{1i}^{\beta^{(d+1)}}}, E_{11}(\alpha^{(d)}, \beta^{(d)}, \gamma^{(d)}))},$$

$$\lambda^{(d+1)} = \frac{\mathbb{C}(e^{t_{1i}^{\beta^{(d+1)}}}, E_{11}(\alpha^{(d)}, \beta^{(d)}, \gamma^{(d)}))\mathbb{D}^2}{\left(\begin{array}{c} \mathbb{E}(e^{t_{2i}^{\beta^{(d+1)}}}, E_{21}(\alpha^{(d)}, \beta^{(d)}, \gamma^{(d)})) \\ \cdot[\mathbb{B}(e^{t_{2i}^{\beta^{(d+1)}}}, E_{21}(\alpha^{(d)}, \beta^{(d)}, \gamma^{(d)}))\mathbb{E}(e^{t_{2i}^{\beta^{(d+1)}}}, E_{21}(\alpha^{(d)}, \beta^{(d)}, \gamma^{(d)})) - \mathbb{AD}] \end{array}\right)}.$$

Step 4 Stop the iteration, and find the EM-based estimates of α, β, and λ when $|\alpha^{(d+1)} - \alpha^{(d)}| < \varepsilon, |\beta^{(d+1)} - \beta^{(d)}| < \varepsilon$ and $|\lambda^{(d+1)} - \lambda^{(d)}| < \varepsilon$ for some given tolerance limit ε, for example $\varepsilon = 0.001$.

Remark 1. *From Wu [14], it is observed that, for samples $T_{j1} < T_{j2} < \ldots < T_{jm_j}$, $j = 1, 2$, one has that*

$$T_j(\beta) = \frac{1}{n_j(m_j - 1)} \sum_{i=1}^{m_j} (R_{ji} + 1) \frac{e^{t_{ji}^{\beta}} - 1}{e^{t_{j1}^{\beta}} - 1} - \frac{1}{m_j - 1}$$

follows the F distribution with $2m_j - 2$ and 2 degrees of freedom and $T_j(\beta)$ is a strictly increasing function of β. Hence, for $0 < \gamma < 1$, one can choose the initial estimate of β from the following $100(1-\gamma)\%$ confidence interval:

$$\Big(\rho(t, F_{(2(n-1),2)}(1 - \frac{\gamma}{2})), \rho(t, F_{(2(n-1),2)}(\frac{\gamma}{2}))\Big),$$

where $F_{(a,b)}(p)$ is the $100p\%$ right-tail percentile of the F distribution with a and b degrees of freedom and $\rho(t, y)$ is the solution of β for equation $T_j(\beta) = y$. Meanwhile, one can also let $T_j(\beta), j = 1, 2$ equal the median of the $F_{(2(n-1),2)}$ distribution and find the root of β, which can also be utilized as the initial estimate for β.

4. Confidence Interval Estimation

In this section, common large-sample-based asymptotic confidence intervals of the model parameters were constructed via the Fisher information. In addition, the bootstrap sampling technique was also used to obtain the bootstrap confidence intervals for comparison.

4.1. Asymptotic Confidence Intervals

Under some mild regularity (see, e.g., Casella and Berger [23]), the asymptotic confidence intervals of α, β, and λ can be derived from the usually asymptotic normality of the maximum likelihood estimation with empirical variances estimated from the inverse of the observed Fisher information matrix.

By direct calculation, the second partial derivatives of the log-likelihood function in (5) can be expressed as

$$\ell_{\alpha\alpha} = \frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{m_1 + m_2}{\alpha^2}, \;\; \ell_{\alpha\beta} = \frac{\partial^2 \ell}{\partial \alpha \partial \beta} = -\sum_{j=1}^{2} \sum_{i=1}^{m_j} \lambda^{j-1} (1 + R_{ji}) e^{t_{ji}^{\beta}} t_{ji}^{\beta} \log t_{ji},$$

$$\ell_{\alpha\lambda} = \frac{\partial^2 \ell}{\partial \alpha \partial \lambda} = \sum_{i=1}^{m_2} (1 + R_{2i})(1 - e^{t_{2i}^{\beta}}),$$

$$\ell_{\beta\beta} = \frac{\partial^2 \ell}{\partial \beta^2} = -\frac{m_1 + m_2}{\beta^2} + \sum_{j=1}^{2} \sum_{i=1}^{m_j} t_{ji}^{\beta} \log^2 t_{ji} - \alpha \sum_{j=1}^{2} \sum_{i=1}^{m_j} \lambda^{j-1} (1 + R_{ji}) e^{t_{ji}^{\beta}} t_{ji}^{\beta} (1 + t_{ji}^{\beta}) \log^2 t_{ji},$$

$$\ell_{\beta\lambda} = \frac{\partial^2 \ell}{\partial \beta \partial \lambda} = -\alpha \sum_{i=1}^{m_2} (1 + R_{2i}) e^{t_{2i}^{\beta}} t_{2i}^{\beta} \log t_{2i}, \;\; \ell_{\lambda\lambda} = \frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{m_2}{\lambda^2}.$$

Thus, the observed Fisher information matrix $\mathrm{I} = \mathrm{I}(\alpha, \beta, \lambda)$ can be expressed as

$$\mathrm{I}(\hat{\alpha}, \hat{\beta}, \hat{\lambda}) = \begin{pmatrix} -\ell_{\alpha\alpha} & -\ell_{\alpha\beta} & -\ell_{\alpha\lambda} \\ -\ell_{\alpha\beta} & -\ell_{\beta\beta} & -\ell_{\beta\lambda} \\ -\ell_{\alpha\lambda} & -\ell_{\beta\lambda} & -\ell_{\lambda\lambda} \end{pmatrix} \Bigg|_{\hat{\alpha}, \hat{\beta}, \hat{\lambda}} = \begin{pmatrix} \mathrm{Var}(\hat{\alpha}) & \mathrm{Cov}(\hat{\alpha}, \hat{\beta}) & \mathrm{Cov}(\hat{\alpha}, \hat{\lambda}) \\ \mathrm{Cov}(\hat{\alpha}, \hat{\beta}) & \mathrm{Var}(\hat{\beta}) & \mathrm{Cov}(\hat{\beta}, \hat{\lambda}) \\ \mathrm{Cov}(\hat{\alpha}, \hat{\lambda}) & \mathrm{Cov}(\hat{\beta}, \hat{\lambda}) & \mathrm{Var}(\hat{\lambda}) \end{pmatrix}^{-1}.$$

Based on the asymptotic theory, the asymptotic distribution of $(\hat{\alpha}, \hat{\beta}, \hat{\lambda})'$ is normally distributed with mean vector $(\alpha, \beta, \lambda)'$ and variance–covariance matrix $\mathrm{I}^{-1}(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$, i.e.,

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\lambda} \end{pmatrix} - \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} \xrightarrow{\mathrm{d}} \mathrm{N}\Big(0, \mathrm{I}^{-1}(\hat{\alpha}, \hat{\beta}, \hat{\lambda})\Big),$$

where $\xrightarrow{d}$ denotes the convergence in the distribution and $\mathrm{I}^{-1}(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ is the inverse of the Fisher information matrix $\mathrm{I}(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$. For $0 < \gamma < 1$, the asymptotic $100(1-\gamma)\%$ confidence intervals of α, β, and λ are given by

$$\left(\hat{\theta}_i - z_{\gamma/2}\sqrt{\mathrm{Var}(\hat{\theta}_i)},\ \hat{\theta}_i + z_{\gamma/2}\sqrt{\mathrm{Var}(\hat{\theta}_i)}\right),\ i = 1,2,3,$$

where $\theta_1 = \alpha, \theta_2 = \beta$ and $\theta_3 = \lambda$, and z_γ is the upper γth quantile of the standard normal distribution.

Sometimes, previous asymptotic confidence intervals may have negative lower bounds. In order to overcome this drawback, the logarithmic transformation and delta methods can be employed to obtain the asymptotic normality distribution of $\ln \hat{\theta}_i$ as $(\ln \hat{\theta}_i - \ln \theta_i) \xrightarrow{d} \mathrm{N}\big(0, \mathrm{Var}(\hat{\theta}_i)/\theta_i^2\big)$, $i = 1,2,3$, which implies that the asymptotic $100(1-\gamma)\%$ confidence interval of $\ln \theta_i$ is

$$(\ln \hat{\theta}_i \pm z_{\lambda/2}\sqrt{\mathrm{Var}(\hat{\theta}_i)}/\hat{\theta}_i) \triangleq (A_{i1}, A_{i2}), i = 1,2,3.$$

Hence, the asymptotic $100(1-\gamma)\%$ confidence interval of $\theta_i, i = 1,2,3$ can be constructed as $(e^{A_{i1}}, e^{A_{i2}})$, respectively.

4.2. Bootstrap Confidence Intervals

The bootstrap method [24] is known to enlarge the sample by simulation so as to overcome the shortage of experimental data. Compared with the exact method, the bootstrap estimation is widely used due to its great properties such as its easy calculation. Generally, one can derive the bootstrap confidence interval by both nonparametric and parametric approaches. Following the suggestion of Kundu et al. [25], the parametric bootstrap confidence intervals of the unknown model parameters were constructed in this part, where the Studentized-t bootstrap confidence interval (SBCI) from Hall [26] and the percentile bootstrap confidence interval (PBCI) suggested by Efron [24] were obtained, respectively.

Under the constant-stress PALT, the following procedure of the progressively Type-II-censored samples is provided to generate the associated bootstrap data (e.g., Balakrishnan and Sandhu [27]):

1. Based on the original progressively Type-II-censored sample $t_{j1}, t_{j2}, \ldots, t_{jm_j}, j = 1,2$ and censoring scheme $R_j = (R_{j1}, R_{j2}, \ldots, R_{jm_j})$, obtain the estimates of α, β, and λ, say $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$.
2. For the given m_j and $n_j, j = 1,2$, generate m_j independent observations $(W_{j1}, W_{j2}, \ldots, W_{jm_j})$ from uniform distribution $U(0,1)$.
3. Set $E_{ji} = 1/(i + \sum_{k=m_j-i+1}^{m_j} R_{jk})$, for $j = 1,2, i = 1,2,\ldots,m_j$.
4. Set $V_{ji} = W_{ji}^{E_{ji}}$, for $j = 1,2, i = 1,2,\ldots,m_j$.
5. Set $U_{ji}^* = 1 - \prod_{k=m_j-i+1}^{m_j} V_{jk}, i = 1,2,\ldots,m_j$. Then, $U_{j1}^*, U_{j2}^*, \ldots, U_{jm_j}^*, j = 1,2$ is the progressively Type-II-censored data from $U(0,1)$.
6. Based on $\hat{\alpha}, \hat{\beta}$, and $\hat{\lambda}$, two random samples $(t_{j1}^*, t_{j2}^*, \ldots, t_{jm_j}^*), j = 1,2$ from CDFs $F_1(t)$ and $F_2(t)$ presented in (1) and (3) are generated as follows:
$$t_{ji}^* = \left[\ln\left(1 - \frac{\ln(1-U_{ji}^*)}{\hat{\alpha}\hat{\lambda}^{j-1}}\right)\right]^{1/\hat{\beta}}.$$
7. As in Step 1, based on $(t_{j1}^*, t_{j2}^*, \ldots, t_{jm_j}^*), j = 1,2$, the bootstrap sample estimates of α, β, λ are computed, namely $\hat{\alpha}^*, \hat{\beta}^*$, and $\hat{\lambda}^*$.
8. Repeat Steps 2–7 N times; we can generate N different bootstrap samples for α, β, and λ, i.e., $\hat{\theta}_k^{*1}, \hat{\theta}_k^{*2}, \ldots, \hat{\theta}_k^{*N}, k = 1,2,3$, where $\theta_1^* = \alpha^*, \theta_2^* = \beta^*$, and $\theta_3^* = \lambda^*$, respectively.

9. Arrange all $\hat{\alpha}^*$, $\hat{\beta}^*$, and $\hat{\lambda}^*$ in an ascending order; one can obtain the bootstrap samples for α, β, and λ as $\hat{\theta}_k^{*[1]}, \hat{\theta}_k^{*[2]}, \ldots, \hat{\theta}_k^{*[N]}, k = 1, 2, 3$.

- Percentile bootstrap confidence intervals:

Let $G(t) = P(\hat{\theta}_k^* \leq t)$ be the CDF of $\hat{\theta}_k^*$ and $\hat{\theta}_{kboot}^* = G^{-1}(t)$ for given t. For $0 < \gamma < 1$, the approximate $100(1-\gamma)\%$ PBCI of θ_k is given by

$$\left(\hat{\theta}_{kboot}^*\left(\frac{\gamma}{2}\right), \hat{\theta}_{kboot}^*\left(1 - \frac{\gamma}{2}\right)\right), k = 1, 2, 3.$$

- Studentized-t bootstrap confidence intervals:

In order to construct the Studentized-t bootstrap confidence intervals for parameters α, β, and λ, we derive the order statistics:

$$\eta_k^{*[1]} < \eta_k^{*[2]} < \cdots < \eta_k^{*[N]}, \quad \text{with} \quad \eta_k^{*[d]} = \frac{\hat{\theta}_k^{*[d]} - \hat{\theta}_k}{\sqrt{\mathrm{Var}(\hat{\theta}_k^{*[d]})}}, k = 1, 2, 3, d = 1, 2, \ldots, N.$$

Denote $\mathbb{G}(t) = P(\eta_k^* \leq t)$ as the CDF of η_k^* and $\eta_{kboot-t} = \hat{\theta}_k + \sqrt{\mathrm{Var}(\hat{\theta}_k)}\mathbb{G}^{-1}(t)$ for given t. Thus, for $0 < \gamma < 1$, the $100(1-\gamma)\%$ SBCI of θ_k can be expressed as

$$\left(\hat{\eta}_{kboot-t}\left(\frac{\gamma}{2}\right), \hat{\eta}_{kboot-t}\left(1 - \frac{\gamma}{2}\right)\right), k = 1, 2, 3,$$

where $\mathrm{Var}(\hat{\theta}_k)$ can be estimated by the asymptotic variance from the observed Fisher information matrix $\mathrm{I}(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$.

5. Numerical Studies

In this section, a real data example and simulation studies are presented for illustrative purposes.

5.1. Data Analysis

The dataset from a light-emitting diode (LED) life test was analyzed, which was also analyzed by Cheng and Wang [11]. Since the original data included 155 samples with censoring level 0.25, for the sake of simplicity, we just extracted the observed failure samples obtained under the use or stress conditions as the set of complete constant-stress PALT data, for which the details are listed in Table 1.

Table 1. Complete LED constant-stress PALT failure data.

Regular Use Condition
0.18, 0.19, 0.19, 0.34, 0.36, 0.40 0.44, 0.44, 0.45, 0.46, 0.47, 0.53, 0.57, 0.57, 0.63, 0.65, 0.70, 0.71, 0.71, 0.75, 0.76, 0.76, 0.79, 0.80, 0.85, 0.98, 1.01, 1.07, 1.12, 1.14, 1.15, 1.17, 1.20, 1.23, 1.24, 1.25, 1.26, 1.32, 1.33, 1.33, 1.39, 1.42, 1.50, 1.55, 1.58, 1.59, 1.62, 1.68, 1.70, 1.79, 2.00, 2.01, 2.04, 2.54, 3.61, 3.76, 4.65, 8.97.
Stress condition
0.13, 0.16, 0.20, 0.20, 0.21, 0.25, 0.26, 0.28, 0.28, 0.30, 0.31, 0.33, 0.35, 0.35, 0.35, 0.39, 0.50, 0.52, 0.58, 0.60, 0.60, 0.62, 0.63, 0.67, 0.71, 0.73, 0.75, 0.75, 0.78, 0.80, 0.80, 0.86, 0.90, 0.91, 0.93, 0.93, 0.94, 0.98, 0.99, 1.01, 1.03, 1.06, 1.06, 1.10, 1.22, 1.22, 1.24, 1.28, 1.39, 1.39, 1.46, 1.48, 1.52, 1.74, 1.95, 2.46, 3.02, 5.16.

For the purposes of comparison, we fit the Chen distribution, as well as the normal, the Weibull, gamma, and log-normal to fit this real dataset. Since the exponential distribution is a special case of the Weibull and gamma distributions, the exponential distribution was also considered as a candidate model as well. In order to select the best-fitting model from the candidate distributions, the goodness-of-fit criteria quantities are presented in Table 2, including the Akaike information criterion (AIC), Bayesian information criterion

(BIC), the second-order Akaike information criterion (AICc), as well as the values of the log-likelihood. Let k be the number of free parameters in the related model and n be the sample size, the related criteria are defined as

$$\text{AIC}{=}2k - 2\ln(\text{likelihood}), \quad \text{BIC}{=}k\ln n - 2\ln(\text{likelihood}), \quad \text{AICc}{=}\text{AIC} + \frac{2k(k+1)}{n-k-1}.$$

The best model is the one with the least values of the AIC, BIC, and AICc. The results presented in Table 2 indicate that the Chen distribution provided the best fit and can be used to analyze these data.

Table 2. Summary of the goodness-of-fit test for the fit distributions.

Results under Data from Regular Use Conditions				
Model	log-likelihood	AIC	BIC	AICc
Chen	−52.9318	109.8636	113.9845	110.0818
Weibull	−71.6092	147.2185	151.3394	147.4367
Gamma	−69.2207	142.4414	146.5623	142.6596
normal	−126.8569	257.7138	261.8347	257.9320
log-normal	−93.9304	191.8608	195.9817	192.0790
exponential	−74.3596	150.7193	152.7797	150.7907
results under data from stress conditions				
Model	log-likelihood	AIC	BIC	AICc
Chen	−41.1271	86.2542	90.3751	86.4724
Weibull	−49.0693	102.1386	106.2595	102.3568
Gamma	−47.4702	98.9404	103.0613	99.1586
normal	−97.2140	198.4279	202.5488	198.6461
log-normal	−93.9592	191.9184	196.0393	192.1366
exponential	−53.4458	108.8916	110.9520	108.9630

Based on the complete data given in Table 1, a group of progressively Type-II-censored constant-stress PALT data was generated, and the details are presented in Table 3. Therefore, different point and interval estimates were obtained, and the results are provided in Table 4 with the 90% significance level for the interval estimates. It can be seen from Table 4, the bootstrap confidence intervals outperformed the ACIs, where the SBCIs of α, β and λ also featured a narrower interval length than those of PBCIs.

Table 3. Progressively Type-II-censored LED constant-stress PALT data.

Regular Use Conditions
$T_1 = (0.18, 0.40, 0.47, 0.65, 0.79, 1.07, 1.20, 1.32, 1.50, 1.68, 2.04, 4.65)$ $R_1 = (4, 4, 4, 6, 4, 4, 4, 4, 4, 4, 3, 1)$
Stress conditions
$T_2 = (0.13, 0.21, 0.30, 0.35, 0.58, 0.63, 0.75, 0.86, 0.94, 1.03, 1.22, 1.39, 1.95, 5.16$ $R_2 = (3, 4, 3, 4, 3, 4, 3, 4, 3, 4, 3, 4, 2, 0)$

Table 4. Estimates under LED constant-stress PALT data.

	MLE	**ACI**	**PBCI**	**SBCI**
α	0.0833	(0.0389,0.2177) 0.1788	(0.0496,0.2224) 0.1728	(0.0244,0.1931) 0.1687
β	0.7286	(0.6137,0.8435) 0.2298	(0.6234,0.8460) 0.2226	(0.5461,0.7459) 0.1998
λ	1.6067	(0.5604,2.6530) 2.0926	(1.1400,2.5212) 1.3812	(1.2593,2.6011) 1.3418

5.2. Simulation Studies

In this part, extensive numerical studies were conducted to investigate the performance of our methods. To generate the progressively Type-II-censored samples under the constant-stress PALT, the algorithm developed by Balakrishnan and Aggarwala [18] was implemented. In addition, in order to compare the performance of different estimates, four quantiles were examined as follows:

(a) The mean-squared errors (MSEs) of the estimates of the parameters α, β, and λ computed as $\frac{1}{M}\sum_{d=1}^{M}(\phi_l - \hat{\phi}_{ld})^2$, repeated M-times, where $\phi_1 = \alpha, \phi_2 = \beta$, and $\phi_3 = \lambda$ denote the selected values of the parameters α, β, and λ and $\hat{\phi}_{ld}, d = 1, 2, \ldots, M$ is the related estimates of $\phi_l, l = 1, 2, 3$.

(b) The average estimates (AVEs) of the MLEs for the parameters α, β, and λ, which are defined as $\hat{\phi}_l = \frac{1}{M}\sum\hat{\phi}_{ld}$.

(b) The coverage probabilities (CPs) of the confidence intervals of the parameters α, β, and λ, which are defined as the probability that the interval estimate contains the true parameters.

(c) The average widths (AWs) of the confidence intervals of the parameters α, β, and λ.

For $j = 1, 2$, three progressive censoring schemes were considered as follows:

Scheme I: $R_{j1} = R_{j2} = \cdots = R_{j(m_j-1)} = 0$ and $R_{jm_j} = n_j - m_j$;

Scheme II: $R_{j1} = n_j - m_j$ and $R_{j2} = \cdots = R_{jm_j} = 0$;

Scheme III: $R_{j1} = R_{j2} = R_{j(m_j-1)} = 1$ and $R_{jm_j} = n_j - 2m_j + 1$.

Under different choices of the parameter values, sample sizes, and censoring schemes, the results of criteria quantities are presented in Tables 5–8 based on 10,000 repetitions, where the numerical computational work was conducted by using the MATLAB 2017a software; the interval results were obtained under a 90% significance level, and these same censoring schemes were used under both the normal use and accelerated stress stages. It is noted that the criteria quantities, the MSEs and AVEs, of the parameters α, β, and λ decreased when the effective sample size m, n increased, which indicated that the MLEs were consistent and worked satisfactorily under the simulated scenarios. In addition, the point estimates obtained from Censoring Scheme II outperformed the results from Censoring Schemes I and III in terms of the MSEs and AVEs in general. Furthermore, it was seen that the different AWs of the various interval estimates decreased when the sample size increased, whereas the CPs increased correspondingly with the same trend. Moreover, it was also observed that the two bootstrap confidence intervals were superior to the likelihood-based ACIs according to the quantities' AWs, whereas the SBCI provided relatively shorter interval widths than those of the PBCI in most cases, which is in agreement with the results of Hall [26]. Similar to the point results and compared to the other two censoring schemes, the interval estimates obtained from Censoring Scheme II also had relatively shorter interval lengths under fixed simulated scenarios.

Table 5. AVEs and MSEs (with brackets) for the parameters with $(\alpha, \beta, \lambda) = (1, 0.5, 2)$ at $n_1 = n_2 = n$ and $m_1 = m_2 = m$.

n	m	CS	α	β	λ
30	10	I	1.2217 (0.3254)	0.6041 (0.2322)	1.8104 (0.2416)
		II	1.2034 (0.3135)	0.5783 (0.2158)	1.8329 (0.2337)
		III	1.2516 (0.3217)	0.5926 (0.2411)	2.2021 (0.2504)
	20	I	1.1510 (0.3126)	0.4124 (0.2258)	2.1765 (0.2300)
		II	1.1379 (0.2989)	0.5473 (0.2122)	2.1341 (0.2213)
		III	1.2037 (0.3097)	0.5721 (0.2194)	1.8647 (0.2321)

Table 5. *Cont.*

n	m	CS	α	β	λ
50	15	I	0.8153 (0.2205)	0.5678 (0.1601)	1.8790 (0.1818)
		II	1.1536 (0.2114)	0.5545 (0.1534)	1.9444 (0.1726)
		III	1.1764 (0.2198)	0.5581 (0.1643)	1.8426 (0.1769)
	25	I	0.8437 (0.2143)	0.5454 (0.1532)	2.1095 (0.1745)
		II	0.8924 (0.2009)	0.4792 (0.1521)	2.0423 (0.1708)
		III	0.9609 (0.2132)	0.5562 (0.1530)	2.1111 (0.1723)
100	35	I	1.1111 (0.1421)	0.5333 (0.1532)	2.0899 (0.1251)
		II	1.0613 (0.1346)	0.5201 (0.1467)	2.0351 (0.1139)
		III	1.1453 (0.1399)	0.5400 (0.1499)	2.0932 (0.1300)
	50	I	1.0712 (0.1314)	0.5204 (0.1406)	2.0532 (0.1126)
		II	1.0221 (0.1289)	0.5172 (0.1375)	2.0241 (0.1089)
		III	0.9105 (0.1357)	0.5198 (0.1437)	1.9613 (0.1231)

Table 6. AWs and CPs (with brackets) for the parameters with $(\alpha, \beta, \lambda = (1, 0.5, 2)$ at $n_1 = n_2 = n$ and $m_1 = m_2 = m$ ($\gamma = 0.1$).

n	m	CS	α			β			λ		
			ACI	SBCI	PBCI	ACI	SBCI	PBCI	ACI	SBCI	PBCI
30	10	I	0.8213	0.5306	0.6189	2.3216	1.5324	1.6041	1.4279	0.9078	0.9111
			(0.8722)	(0.8631)	(0.8624)	(0.8701)	(0.8623)	(0.8712)	(0.8671)	(0.8641)	(0.8650)
		II	0.8011	0.4946	0.5717	2.2075	1.4561	1.5102	1.2436	0.8568	0.8389
			(0.8693)	(0.8556)	(0.8579)	(0.8643)	(0.8656)	(0.8697)	(0.8632)	(0.8637)	(0.8725)
		III	0.9311	0.5000	0.6034	2.2453	1.5656	1.5322	1.2979	0.8769	0.8662
			(0.8731)	(0.8615)	(0.8611)	(0.8679)	(0.8611)	(0.8714)	(0.8574)	(0.8679)	(0.8734)
	20	I	0.7891	0.5008	0.5869	1.9785	1.2562	1.3008	1.3153	0.8359	0.8533
			(0.8760)	(0.8698)	(0.8658)	(0.8697)	(0.8630)	(0.8620)	(0.8602)	(0.8635)	(0.8639)
		II	0.7222	0.4625	0.5634	1.8011	1.1054	1.2121	1.1137	0.7886	0.8216
			(0.8663)	(0.8571)	(0.8567)	(0.8635)	(0.8627)	(0.8654)	(0.8551)	(0.8612)	(0.8654)
		III	0.8010	0.4773	0.5743	1.8884	1.3409	1.2564	1.2532	0.8042	0.8480
			(0.8723)	(0.8673)	(0.8712)	(0.8722)	(0.8543)	(0.8635)	(0.8563)	(0.8627)	(0.8640)
50	15	I	0.8059	0.4672	0.4918	1.5534	1.2074	1.2300	0.9157	0.7993	0.7899
			(0.8924)	(0.8731)	(0.8710)	(0.8822)	(0.8717)	(0.8652)	(0.8791)	(0.8663)	(0.8711)
		II	0.7474	0.4551	0.4635	1.5307	1.1935	1.2115	0.8993	0.7269	0.7425
			(0.8812)	(0.8678)	(0.8689)	(0.8769)	(0.8642)	(0.8657)	(0.8790)	(0.8659)	(0.8661)
		III	0.7713	0.4750	0.5001	1.6089	1.2617	1.3021	0.9334	0.7619	0.7713
			(0.8929)	(0.8754)	(0.8723)	(0.8843)	(0.8754)	(0.8690)	(0.8814)	(0.8660)	(0.8700)
	25	I	0.7451	0.4300	0.4419	1.2230	0.9611	0.9887	0.8656	0.6890	0.6998
			(0.8809)	(0.8754)	(0.8665)	(0.8669)	(0.8713)	(0.8743)	(0.8691)	(0.8658)	(0.8671)
		II	0.7009	0.4214	0.4327	1.2132	0.8794	0.9236	0.8102	0.6572	0.6627
			(0.8807)	(0.8629)	(0.8624)	(0.8614)	(0.8610)	(0.8689)	(0.8678)	(0.8654)	(0.8665)
		III	0.7513	0.4267	0.4678	1.2345	0.8999	0.9542	0.8423	0.7314	0.7320
			(0.8816)	(0.8742)	(0.8741)	(0.8736)	(0.9645)	(0.8720)	(0.8684)	(0.8637)	(0.8702)
100	35	I	0.5327	0.2811	0.2841	0.8678	0.6231	0.6423	0.6217	0.5013	0.5134
			(0.9132)	(0.8956)	(0.9012)	(0.9231)	(0.8942)	(0.8972)	(0.8963)	(0.8956)	(0.9142)
		II	0.5202	0.2679	0.2700	0.7591	0.6124	0.6222	0.5936	0.4897	0.4999
			(0.8965)	(0.8817)	(0.8835)	(0.8922)	(0.8879)	(0.8938)	(0.8856)	(0.8901)	(0.9009)
		III	0.5220	0.2744	0.2786	0.7777	0.6515	0.6231	0.6110	0.5110	0.5183
			(0.9111)	(0.8924)	(0.8971)	(0.9045)	(0.9023)	(0.8939)	(0.8942)	(0.9014)	(0.9125)
	50	I	0.4712	0.1849	0.2013	0.6135	0.5214	0.5327	0.5121	0.4217	0.3868
			(0.9121)	(0.9121)	(0.9178)	(0.9127)	(0.9011)	(0.9126)	(0.8971)	(0.9131)	(0.9063)
		II	0.4657	0.1735	0.1989	0.5670	0.5173	0.5199	0.4785	0.3561	0.3620
			(0.8934)	(0.8903)	(0.8997)	(0.8930)	(0.8954)	(0.8979)	(0.8849)	(0.8943)	(0.8974)
		III	0.4699	0.2352	0.2247	0.5978	0.5336	0.5231	0.4990	0.3754	0.3926
			(0.9023)	(0.9156)	(0.9234)	(0.8965)	(0.9087)	(0.9118)	(0.8937)	(0.9072)	(0.9125)

Table 7. AVE and MSEs (with brackets) for the parameters with $(\alpha, \beta, \lambda) = (0.7, 1, 1.5)$ at $n_1 \neq n_2$ and $m_1 \neq m_2$.

n_1 n_2	m_1 m_2	CS	α	β	λ
30	10	I	0.9755 (0.3257)	1.3221 (0.2214)	1.7987 (0.2032)
40	15	II	0.9537 (0.3216)	1.3024 (0.2185)	1.7611 (0.2011)
		III	0.9701 (0.3309)	1.3176 (0.2210)	1.8432 (0.2058)
	16	I	0.9314 (0.3203)	1.3010 (0.2179)	1.7534 (0.1978)
	18	II	0.8973 (0.3168)	1.2877 (0.2124)	1.7518 (0.1963)
		III	0.9102 (0.3200)	1.3112 (0.2213)	1.7642 (0.1991)
50	15	I	0.8333 (0.2372)	1.3124 (0.1843)	1.7013 (0.1816)
60	20	II	0.8259 (0.2341)	1.2013 (0.1785)	1.6752 (0.1807)
		III	0.8367 (0.2405)	1.2657 (0.1841)	1.6986 (0.1821)
	25	I	0.7910 (0.2316)	1.1211 (0.1742)	1.6923 (0.1709)
	30	II	0.7832 (0.2254)	1.0799 (0.1699)	1.3485 (0.1636)
		III	0.7932 (0.2332)	1.1154 (0.1721)	1.6000 (0.1687)
100	35	I	0.7543 (0.1550)	1.0799 (0.1174)	1.6032 (0.1125)
120	40	II	0.6815 (0.1537)	1.0231 (0.1112)	1.5649 (0.1062)
		III	0.8011 (0.1589)	1.1014 (0.1217)	1.6101 (0.1198)
	45	I	0.7385 (0.1421)	1.0473 (0.1069)	1.5892 (0.1054)
	50	II	0.7112 (0.1378)	1.0117 (0.1052)	1.5437 (0.1012)
		III	0.7361 (0.1436)	1.0762 (0.1110)	1.6045 (0.1103)

Table 8. AWs and CPs (with brackets) for the parameters with $(\alpha, \beta, \lambda = (0.7, 1, 1.5)$ at $n_1 \neq n_2$ and $m_1 \neq m_2$ ($\gamma = 0.1$).

n_1 n_2	m_1 m_2	CS	α ACI	α SBCI	α PBCI	β ACI	β SBCI	β PBCI	λ ACI	λ SBCI	λ PBCI
30	10	I	0.9528	0.7600	0.8179	2.6223	1.1253	1.2638	1.3879	1.1022	1.2135
40	15		(0.8864)	(0.8872)	(0.8912)	(0.8804)	(0.8815)	(0.8872)	(0.8726)	(0.8810)	(0.8821)
		II	0.8491	0.7306	0.7525	2.5762	1.1094	1.2114	1.3564	0.9423	0.9745
			(0.8793)	(0.8765)	(0.8823)	(0.8765)	(0.8724)	(0.8769)	(0.8683)	(0.8659)	(0.8721)
		III	1.0372	0.7962	0.8213	2.5990	1.1102	1.2471	1.4011	0.9987	1.1104
			(0.8910)	(0.8898)	(0.8913)	(0.8796)	(0.8793)	(0.8834)	(0.8795)	(0.8746)	(0.8793)
	16	I	0.9200	0.6983	0.7232	2.1032	0.9345	0.9814	1.1123	0.8971	0.9214
	18		(0.8845)	(0.8849)	(0.8842)	(0.8799)	(0.8829)	(0.8823)	(0.8728)	(0.8753)	(0.8812)
		II	0.8719	0.6854	0.7121	1.9967	0.8972	0.9423	1.1059	0.8865	0.8937
			(0.8762)	(0.8754)	(0.8834)	(0.8734)	(0.8721)	(0.8800)	(0.8667)	(0.8640)	(0.8711)
		III	0.8989	0.7156	0.7358	2.1124	0.9291	0.9878	1.1207	0.8992	0.9148
			(0.8906)	(0.8900)	(0.8873)	(0.8812)	(0.8794)	(0.8907)	(0.8779)	(0.8801)	(0.8734)
50	15	I	0.8661	0.6374	0.6392	1.1231	0.7314	0.7569	0.7980	0.5896	0.6474
60	20		(0.8976)	(0.8823)	(0.8859)	(0.8821)	(0.8797)	(0.8894)	(0.8809)	(0.8809)	(0.8902)
		II	0.8219	0.6011	0.6238	1.0989	0.6111	0.6777	0.7237	0.5619	0.5938
			(0.8810)	(0.8748)	(0.8795)	(0.8769)	(0.8730)	(0.8825)	(0.8710)	(0.8726)	(0.8788)
		III	0.8432	0.6455	0.6299	1.1237	0.7368	0.7495	0.7595	0.6035	0.6371
			(0.8931)	(0.8847)	(0.8810)	(0.8824)	(0.8806)	(0.8876)	(0.8769)	(0.8864)	(0.8869)
	25	I	0.8324	0.6011	0.6213	0.9871	0.6457	0.6534	0.7351	0.5600	0.5990
	30		(0.8865)	(0.8789)	(0.8902)	(0.8812)	(0.8879)	(0.8907)	(0.8735)	(0.8798)	(0.8847)
		II	0.8107	0.5846	0.5997	0.9658	0.5892	0.6004	0.7032	0.5478	0.5612
			(0.8812)	(0.8736)	(0.8769)	(0.8753)	(0.8734)	(0.8798)	(0.8694)	(0.8731)	(0.8762)
		III	0.8485	0.6109	0.6144	1.0216	0.6235	0.6278	0.7444	0.5592	0.6349
			(0.8927)	(0.8824)	(0.8873)	(0.8835)	(0.8812)	(0.8859)	(0.8738)	(0.8765)	(0.8924)

Table 8. *Cont.*

n_1 / n_2	m_1 / m_2	CS	α			β			λ		
			ACI	SBCI	PBCI	ACI	SBCI	PBCI	ACI	SBCI	PBCI
100	35	I	0.5124	0.3283	0.3611	0.7241	0.4653	0.5224	0.5531	0.3459	0.3618
120	40		(0.8910)	(0.8875)	(0.8913)	(0.8967)	(0.8923)	(0.9078)	(0.8952)	(0.8924)	(0.9078)
		II	0.4837	0.3241	0.3548	0.6973	0.4327	0.4419	0.5136	0.3327	0.3412
			(0.8867)	(0.8856)	(0.8879)	(0.8952)	(0.8856)	(0.8978)	(0.8911)	(0.8874)	(0.8942)
		III	0.4965	0.3360	0.3716	0.7536	0.5061	0.5372	0.5429	0.3446	0.3547
			(0.8881)	(0.8907)	(0.8942)	(0.9011)	(0.9108)	(0.9116)	(0.9035)	(0.8979)	(0.9011)
	45	I	0.4611	0.3237	0.3572	0.6893	0.4325	0.4611	0.4803	0.3315	0.3600
	50		(0.8889)	(0.8912)	(0.8931)	(0.9015)	(0.8897)	(0.9174)	(0.8977)	(0.8993)	(0.9057)
		II	0.4352	0.3120	0.3345	0.6479	0.4234	0.4330	0.4760	0.3244	0.3309
			(0.8845)	(0.8849)	(0.8908)	(0.8947)	(0.8861)	(0.8921)	(0.8914)	(0.8868)	(0.8937)
		III	0.4458	0.3296	0.3478	0.6642	0.4496	0.4537	0.4993	0.3278	0.3469
			(0.8876)	(0.8924)	(0.8920)	(0.8969)	(0.8974)	(0.9123)	(0.9123)	(0.8895)	(0.8986)

6. Concluding Remarks

In this paper, statistical inference was discussed for a constant-stress partially accelerated life test. Under progressive Type-II censoring, parameter estimation was conducted, when the lifetime of the units followed the two-parameter bathtub-shaped Chen distribution. The expectation–maximization procedure was implemented for maximum likelihood estimation, and the confidence intervals of the model parameters were derived by using the asymptotic theory and bootstrap techniques. The performance of the point and interval estimates was investigated by simulation studies and a real-life example, and the results indicated that the proposed methods worked satisfactorily. Although the constant-stress partially accelerated life test was considered for the bathtub-shaped Chen distribution in this paper, the results could be extended to other lifetime models with a proper modification. For further works, the testing design and optimal sampling planning for the constant-stress partially accelerated life test seem also interesting and important in practice, which will be discussed in the future.

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