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Pascu-Rønning Type Meromorphic Functions Based on Sălăgean-Erdély–Kober Operator

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Abstract: In the present investigation, we introduce a new class of meromorphic functions defined in the punctured unit disk $\Delta^* := \{\vartheta \in \mathbb{C} : 0 < |\vartheta| < 1\}$ by making use of the Erdély–Kober operator $\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho}$ which unifies well-known classes of the meromorphic uniformly convex function with positive coefficients. Coefficient inequalities, growth and distortion inequalities, in addition to closure properties are acquired. We also set up a few outcomes concerning convolution and the partial sums of meromorphic functions in this new class. We additionally state some new subclasses and its characteristic houses through specializing the parameters that are new and no longer studied in association with the Erdély–Kober operator thus far.

Keywords: meromorphic functions; starlike function; convolution, positive coefficients; coefficient inequalities; integral operator; Erdély–Kober operator

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1. Introduction and Definitions

Let Σ represent the class of functions f being of the forms given by

$$f(\vartheta) = \frac{1}{\vartheta} + \sum_{n=1}^{\infty} a_n \vartheta^n, \tag{1}$$

defined on the punctured unit disk $\Delta^* := \{\vartheta \in \mathbb{C} : 0 < |\vartheta| < 1\}$ with a simple pole at the origin with one residue there. Denote by means of $\Sigma_P \subset \Sigma$ consisting of the functions of the form

$$f(\vartheta) = \frac{1}{\vartheta} + \sum_{n=1}^{\infty} a_n \vartheta^n, a_n \geq 0. \tag{2}$$

The class of meromorphic starlike and meromorphic convex of order $\wp$ $(0 \leq \wp < 1)$ (see Ref. [1]) are defined as below:

$$\Sigma^*_P(\wp) = \left\{ f \in \Sigma_P : -\Re\left(\frac{\vartheta f'(\vartheta)}{f(\vartheta)}\right) > \wp, \quad \vartheta \in \Delta := \Delta^* \cup \{0\} \right\} \tag{3}$$

and

$$\Sigma^K_P(\wp) = \left\{ f \in \Sigma_P : -\Re\left(\frac{(\vartheta f'(\vartheta))'}{f'(\vartheta)}\right) > \wp, \quad \vartheta \in \Delta := \Delta^* \cup \{0\} \right\}$$

respectively. Since, to a certain extent, the work in the meromorphic univalent case has paralleled that of a regular univalent case, it is natural to search for a subclass of Σ_P that has properties analogous to those of class of analytic univalent functions with negative coefficients [2]. Juneja and Reddy [3] introduced the class Σ_P of functions of the form (2) that are meromorphic and univalent in Δ. They showed that the class possesses properties analogous to those of analytic univalent functions with negative coefficients, and also pointed out the subtle differences between the two classes.

For subclasses of analytic and univalent functions given by

$$\mathcal{A} = \left\{ f \in \mathcal{A} : f(\vartheta) = \vartheta + \sum_{n=2}^{\infty} a_n \vartheta^n \quad \vartheta \in \mathbf{U} = \{\vartheta : |\vartheta| < 1\} \right\}.$$

Goodman [4,5], defined two new functions' classes, namely, uniformly starlike and uniformly convex functions. Inspired by this study, in Ref. [6,7], Rønning introduced and studied the following subclasses of $\mathcal{A}$ called $\hbar-$starlike functions of order $-1 < \wp \leq 1$ if

$$\mathcal{S}_p(\wp, \hbar) = \left\{ f \in \mathcal{A} : \Re\left(\frac{\vartheta f'(\vartheta)}{f(\vartheta)} - \wp \right) > \hbar \left| \frac{\vartheta f'(\vartheta)}{f(\vartheta)} - 1 \right|, \ (\hbar \geq 0; \vartheta \in U) \right\} \tag{4}$$

and uniformly $\hbar-$convex functions $-1 < \wp \leq 1$ if if

$$\mathcal{UCV}(\wp, \hbar) = \left\{ f \in \mathcal{A} : \Re\left(\frac{(\vartheta f'(\vartheta))'}{f'(\vartheta)} - \wp \right) > \hbar \left| \frac{\vartheta f''(\vartheta)}{f'(\vartheta)} \right|, \ (\hbar \geq 0; \vartheta \in U) \right\}. \tag{5}$$

Indeed, it follows from (12) and (5) that

$$f \in \mathcal{UCV}(\wp, \hbar) \Leftrightarrow zf' \in \mathcal{S}_p(\wp, \hbar). \tag{6}$$

The interesting geometric properties of these function classes were extensively studied by Kanas et al. in Ref. [8–11], and Murugusundaramoorthy et al. [12] and references cited therein studied and investigated interesting properties for the subclass of Rønning-type $\hbar$-starlike functions associated with certain fractional calculus operators, and discussed its coefficient estimate, characteristic properties, partial sums and neighbourhood results. In this article, we made an attempt to discuss the class of Pascu–Ronning-type meromorphic functions based on the Sălăgean-Erdély–Kober operator (SEK).

Erdély–Kober Operator (EK)

The fractional calculus plan has currently acquired filled-intensity attention by way of the applications of fractional derivative operators (FD) in analytical functions [13–17]. In the literature, many studies on fractional derivative operators and fractional differential equations, involving different operators such as Riemann–Liouville, Hadamard, Caputo, the Erdély–Kober fractional operator (EKF), Weyl–Riesz operators, Caputo operators, and Grünwald–Letnikov operators, were designed and implemented during the past three decades with applications in other fields. The Riemann–Liouville fractional operator (RLF) has been most frequently used and intentional for information (see Refs. [13–18]). Some requirements and features of EKF operators and FD in the study of analytic functions (Geometric Function Theory) can be found at Refs. [13–22].

Now, we recall the Erdély–Kober-type [23] (also see Ref. [24], Ch. 5) integral operator definition, which will be used throughout the paper as below:

Definition 1. *(Erdély–Kober operator-EK): Let for $\zeta > 0, \tau, \kappa \in \mathbb{C}$, be such that $\Re(\kappa - \tau) \geq 0$, an Erdély–Kober-type integral operator by*

$$\mathcal{V}_{\zeta}^{\tau,\kappa} : \Sigma_P \to \Sigma_P$$

is defined for $\Re(\kappa - \tau) > 0$ and $\Re(\tau) > -\zeta$ by

$$\mathcal{V}_{\zeta}^{\tau,\kappa} f(\vartheta) = \frac{\Gamma(\kappa - \zeta)}{\Gamma(\tau - \zeta)} \frac{1}{\Gamma(\kappa - \tau)} \int_0^1 (1-t)^{\kappa-\tau-1} t^{\tau-1} f(\vartheta t^{\zeta}) dt, \zeta > 0. \tag{7}$$

For $\zeta > 0, \Re(\kappa - \tau) \geq 0, \Re(\tau) > -\zeta$ and $f \in \Sigma_P$ of the form (2), we have

$$\begin{aligned} \mathcal{V}_{\zeta}^{\tau,\kappa} f(\vartheta) &= \frac{1}{\vartheta} + \sum_{n=1}^{\infty} \frac{\Gamma(\kappa-\zeta)\Gamma(\tau+n\zeta)}{\Gamma(\tau-\zeta)\Gamma(\kappa+n\zeta)} a_n \vartheta^n \quad (\vartheta \in \Delta) \\ &= \frac{1}{\vartheta} + \sum_{n=2}^{\infty} \Upsilon_{\zeta}^{\tau,\kappa}(n) a_n \vartheta^n \quad (\vartheta \in \Delta), \end{aligned} \tag{8}$$

where

$$\Upsilon_{\zeta}^{\tau,\kappa}(n) = \frac{\Gamma(\kappa-\zeta)\Gamma(\tau+n\zeta)}{\Gamma(\tau-\zeta)\Gamma(\kappa+n\zeta)}. \tag{9}$$

Note that by fixing $\kappa = \tau$, we obtain

$$\mathcal{V}_{\zeta}^{\tau,\tau} f(\vartheta) = f(\vartheta). \tag{10}$$

Due to El-Ashwah (see Ref. [25,26] with $p = 1$) and [23], we recall the following operators:
For $m \in \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \cdots\}; \ell > 0; \varrho > 0$, let

$$\mathcal{D}_{\ell,\varrho}^{m} : \Sigma_P \to \Sigma_P$$

by

$$\mathcal{D}_{\ell,\varrho}^{m} f(\vartheta) = \begin{cases} \frac{\ell}{\varrho} \vartheta^{-1-\frac{\ell}{\varrho}} \int\limits_0^{\vartheta} t^{\frac{\ell}{\varrho}} \mathcal{D}_{\ell,\varrho}^{m+1} f(t) dt; & \varrho \neq 0, \quad m \in \mathbb{Z}^-; \\ \frac{\varrho}{\ell} \vartheta^{-\frac{\ell}{\varrho}} \frac{d}{d\vartheta} \left(\vartheta^{1+\frac{\ell}{\varrho}} \mathcal{D}_{\ell,\varrho}^{m-1} f(\vartheta) \right) & m \in \mathbb{Z}^+; \\ f(\vartheta), & m = 0. \end{cases}$$

Now we define a new linear operator

$$\mathcal{I}_{\varsigma,\varrho}^{\tau,\kappa} : \Sigma_P \to \Sigma_P$$

by

$$\mathcal{I}_{\varsigma,\varrho}^{\tau,\kappa} f(\vartheta) = \mathcal{D}_{\ell,\varrho}^{m} \mathcal{V}_{\zeta}^{\tau,\kappa} f(\vartheta) = \mathcal{V}_{\zeta}^{\tau,\kappa} \mathcal{D}_{\ell,\varrho}^{m} f(\vartheta)$$

named as the Sălăgean-Erdély–Kober operator (SEK) and is given by the following definition.

Definition 2. *(Sălăgean-Erdély–Kober operator -SEK): For $\zeta > 0, \Re(\kappa - \tau) \geq 0, \Re(\tau) > \zeta$; $m \in \mathbb{Z}; \varsigma > 0; \varrho > 0$ and $f \in \Sigma_P$ of the form (2), we have*

$$\begin{aligned} \mathcal{I}_{\varsigma,\varrho}^{\tau,\kappa} f(\vartheta) &= \frac{1}{\vartheta} + \sum_{n=2}^{\infty} \left[1 + \frac{\varrho(n+1)}{\varsigma} \right]^m \frac{\Gamma(\kappa-\zeta)\Gamma(\tau+n\zeta)}{\Gamma(\tau-\zeta)\Gamma(\kappa+n\zeta)} a_n \vartheta^n, \quad (\vartheta \in \Delta) \\ &= \frac{1}{\vartheta} + \sum_{n=2}^{\infty} \Xi_{\varsigma,\varrho}^{\tau,\kappa}(n) a_n \vartheta^n, \quad (\vartheta \in \Delta) \end{aligned} \tag{11}$$

where

$$\Xi_n = \Xi_{\varsigma,\varrho}^{\tau,\kappa}(n) = \left[1 + \frac{\varrho(n+1)}{\varsigma} \right]^m \frac{\Gamma(\kappa-\zeta)\Gamma(\tau+n\zeta)}{\Gamma(\tau-\zeta)\Gamma(\kappa+n\zeta)}. \tag{12}$$

Particularly,

$$\Xi_1 = \Xi_{\varsigma,\varrho}^{\tau,\kappa}(1) = \left[1 + \frac{2\varrho}{\varsigma} \right]^m \frac{\Gamma(\kappa-\zeta)\Gamma(\tau+\zeta)}{\Gamma(\tau-\zeta)\Gamma(\kappa+\zeta)}. \tag{13}$$

Stimulated by means of earlier works on Σ_P by means of Kumar et al. [27] and function theorists (see Ref. [1,27–34]), in this paper, we tried to define a new subclass $\mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu, \hbar, \wp)$ given in Definition 3, by utilising the generalized operator $\mathcal{I}_{\varsigma,\varrho}^{\tau,\kappa}$ unifying well-known classes of meromorphic uniformly convex functions with positive coefficients, and discuss its notable function properties.

Throughout this paper, we shall confine our attention to the case of real-valued parameters τ and κ, and we will consider that $\vartheta \in \Delta$.

Definition 3. *For $0 \le \wp < 1; 0 \le \mu < 1/2$, and $f \in \Sigma_P$ as assumed in (2), we let $f \in \mathcal{M}^{\tau,\kappa}_{\varsigma,\varrho}(\mu, \hbar, \wp)$ if it holds*

$$\begin{aligned} &- \Re\left(\frac{\vartheta(\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho} f(\vartheta))' + \mu\vartheta^2(\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho} f(\vartheta))''}{(1-\mu)\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho} f(\vartheta) + \mu\vartheta(\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho} f(\vartheta))'} \right) \\ &> \hbar \left| \frac{\vartheta(\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho} f(\vartheta))' + \mu\vartheta^2(\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho} f(\vartheta))''}{(1-\mu)\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho} f(\vartheta) + \mu\vartheta(\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho} f(\vartheta))'} + 1 \right| + \wp \end{aligned} \tag{14}$$

where $\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho}$ is given by (11).

Further, shortly, we were able to state this condition through

$$-\Re\left(\frac{\vartheta G'(\vartheta)}{G(\vartheta)}\right) > \hbar\left|\frac{\vartheta G'(\vartheta)}{G(\vartheta)} + 1\right| + \wp, \tag{15}$$

where

$$G(\vartheta) = (1-\mu)F(\vartheta) + \mu\vartheta F'(\vartheta) = \frac{1-2\mu}{\vartheta} + \sum_{n=1}^{\infty}(n\mu - \mu + 1)\Xi_n a_n \vartheta^n, \quad a_n \ge 0. \tag{16}$$

and $F(\vartheta) = \mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho} f(\vartheta)$.

It is of importance to note that, on specializing the parameters $\mu, \hbar$, we can define or deduce Ronning-type meromorphic function classes of Σ_P based on the Sălăgean-Erdély–Kober operator. We pointed these out as examples, and they will also play important roles for investigations. For this reason, those (more) special classes will be taken into consideration as revealing various applications of our basic result which have not been studied so far, associating with the Sălăgean-Erdély–Kober operator.

Example 1. *For $\mu = 0$, we let*

$$\begin{aligned} \mathcal{M}^{\tau,\kappa}_{\varsigma,\varrho}(0, \hbar, \wp) &\equiv \mathcal{PS}^{\tau,\kappa}_{\varsigma,\varrho}(\hbar, \wp) \\ &= \left\{ f \in \Sigma_P : -\Re\left(\frac{\vartheta(\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho} f(\vartheta))'}{\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho} f(\vartheta)}\right) > \hbar\left|\frac{\vartheta(\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho} f(\vartheta))'}{\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho} f(\vartheta)} + 1\right| + \wp \right\} \end{aligned} \tag{17}$$

where $\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho}$ is given by (11).

Example 2. *For $\mu = 0, \hbar = 0$ we let*

$$\mathcal{M}^{\tau,\kappa}_{\varsigma,\varrho}(0, 0, \wp) \equiv \mathcal{MS}^{\tau,\kappa}_{\varsigma,\varrho}(\wp) = \left\{ f \in \Sigma_P : -\Re\left(\frac{\vartheta(\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho} f(\vartheta))'}{\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho} f(\vartheta)}\right) > \wp \right\} \tag{18}$$

where $\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho}$ is given by (11).

Example 3. *For $\mu = 1$, we let*

$$\begin{aligned} \mathcal{M}^{\tau,\kappa}_{\varsigma,\varrho}(1, \hbar, \wp) &\equiv \mathcal{MK}^{\tau,\kappa}_{\varsigma,\varrho}(\hbar, \wp) \\ &= \left\{ f \in \Sigma_P : -\Re\left(1 + \frac{\vartheta(\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho} f(\vartheta))''}{(\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho} f(\vartheta))'}\right) > \hbar\left|\frac{\vartheta(\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho} f(\vartheta))''}{(\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho} f(\vartheta))'}\right| + \wp \right\} \end{aligned} \tag{19}$$

where $\mathcal{I}^{\tau,\kappa}_{\varsigma,\varrho}$ is given by (11).

Example 4. *For $\mu = 1, \hbar = 0$ we let*

$$\mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(1,0,\wp) \equiv \mathcal{K}_{\varsigma,\varrho}^{\tau,\kappa}(\wp) = \left\{ f \in \Sigma_P : -\Re\left(1 + \frac{\vartheta(\mathcal{I}_{\varsigma,\varrho}^{\tau,\kappa} f(\vartheta))''}{(\mathcal{I}_{\varsigma,\varrho}^{\tau,\kappa} f(\vartheta))'}\right) > \wp \right\} \tag{20}$$

where $\mathcal{I}_{\varsigma,\varrho}^{\tau,\kappa}$ is given by (11).

Remark 1. *Suitably fixing the parameter in the operator $\mathcal{I}_{\varsigma,\varrho}^{\tau,\kappa}$, we can deduce the following*

1. *For $m = \alpha; \varrho = 1; \varsigma = \beta$, and $\tau = \kappa$ we obtain the operator $\mathcal{I}_{\beta,1}^{\tau,\tau} f(\vartheta) = \mathcal{P}_{\beta}^{\alpha} f(\vartheta)$ studied by Lashin [35];*
2. *For $\varrho = 1$ and $\tau = \kappa$, we obtain the operator $\mathcal{I}(m,\varsigma)f(\vartheta)$ studied by Cho et al. [36,37];*
3. *$\varrho = 1, \varsigma = 1$ and $\tau = \kappa$ we obtain the operator $\mathcal{I}_{1,1}^{\tau,\tau} f(\vartheta) = \mathcal{I}(m)f(\vartheta)$ studied by Uralegaddi and Somanatha [38];*
4. *For $m = 0$, it gives $\mathcal{I}_{\varsigma,\varrho}^{\tau,\kappa} f(\vartheta) = \mathcal{I}_{\zeta}(\tau,\kappa)f(\vartheta)$, which was studied by El-Ashwah [23].*

In this study, we achieve the coefficient bounds, distortion bounds, in addition to closure results for the class $\mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$. We additionally discussed a few results regarding the integral operator, convolution, and the partial sums of $f \in \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$. It is noteworthy that the SEK operator defined on meromorphic functions in turn includes various operators illustrated in the Remark 1; thus, our study on the characteristic properties of the function class Σ_P unifies the known (or new) results for the meromorphic functions defined in the parabolic region.

2. Coefficient Inequalities

In the light of the conditions created by Dziok et al. [39], we state the following result without providing any proof.

Lemma 1. *Suppose that $\wp \in [0,1)$, $r \in (0,1]$ and $H \in \Sigma_P(\wp)$ is of the form $H(\vartheta) = \frac{1}{\vartheta} + \sum_{n=1}^{\infty} b_n \vartheta^n$, $\quad 0 < |\vartheta| < r$, with $b_n \geq 0$; then*

$$\sum_{n=1}^{\infty} (n + \wp) b_n r^{n+1} \leq 1 - \wp. \tag{21}$$

In our first theorem, we compose our comprehensive result that is a necessary and sufficient condition for $f \in \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$.

Theorem 1. *Let $f \in \Sigma_P$ be given by (2). Then $f \in \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$ if, and only if*

$$\sum_{n=1}^{\infty} [n(1+\hbar) + (\wp + \hbar)](n\mu - \mu + 1)\, \Xi_n a_n \leq (1 - 2\mu)(1 - \wp). \tag{22}$$

Equivalently,

$$\sum_{n=1}^{\infty} [n(1+\hbar) + (\wp + \hbar)] \aleph(n,\mu)\, a_n \leq (1 - \wp),$$

where

$$\aleph(n,\mu) = \frac{(n\mu - \mu + 1)}{(1 - 2\mu)} \Xi_n. \tag{23}$$

Proof. If $f \in \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$, then by (17), we have

$$\begin{aligned} & -\Re\left(\frac{\vartheta(\mathcal{I}_{\varsigma,\varrho}^{\tau,\kappa}f(\vartheta))' + \mu\vartheta^2(\mathcal{I}_{\varsigma,\varrho}^{\tau,\kappa}f(\vartheta))''}{(1-\mu)\mathcal{I}_{\varsigma,\varrho}^{\tau,\kappa}f(\vartheta) + \mu\vartheta(\mathcal{I}_{\varsigma,\varrho}^{\tau,\kappa}f(\vartheta))'}\right) \\ & > \hbar\left|\frac{\vartheta(\mathcal{I}_{\varsigma,\varrho}^{\tau,\kappa}f(\vartheta))' + \mu\vartheta^2(\mathcal{I}_{\varsigma,\varrho}^{\tau,\kappa}f(\vartheta))''}{(1-\mu)\mathcal{I}_{\varsigma,\varrho}^{\tau,\kappa}f(\vartheta) + \mu\vartheta(\mathcal{I}_{\varsigma,\varrho}^{\tau,\kappa}f(\vartheta))'} + 1\right| + \wp \end{aligned} \tag{24}$$

That is,

$$-\Re\left(\frac{(1+\hbar e^{i\theta})\vartheta G'(\vartheta) + \hbar e^{i\theta}G(\vartheta)}{G(\vartheta)}\right) > \wp,$$

where $G(\vartheta)$ is given by (16). When we replace with $G(\vartheta), G'(\vartheta)$ and we allow $\vartheta \to 1^-$, we obtain

$$\left\{\frac{(1-2\mu)(1-\wp) - \sum_{n=1}^{\infty}[n(1+\hbar) + (\wp+\hbar)](n\mu-\mu+1)\Xi_n a_n}{(1-2\mu) - \sum_{n=1}^{\infty} n(n\mu-\mu+1)\Xi_n a_n}\right\} > 0.$$

This shows that (22) holds.

On the other hand, suppose that (22) is true. Since $-\Re(w) > \wp \Leftrightarrow |w+1| < |w-(1-2\wp)|$, it is sufficient to prove that

$$\left|\frac{w+1}{w-(1-2\wp)}\right| < 1 \text{ and } |w-(1-2\wp)| \neq 0 \text{ for } |\vartheta| < r \leq 1, \quad \vartheta \in \Delta.$$

Using (22), and taking $w(\vartheta) = \frac{(1+\hbar e^{i\theta})\vartheta G'(\vartheta)+\hbar e^{i\theta}G(\vartheta)}{G(\vartheta)}$, we obtain

$$\left|\frac{w+1}{w-(1-2\wp)}\right| \leq \frac{\sum_{n=1}^{\infty}(n\mu-\mu+1)[(n+1)(1+\hbar)]\Xi_n a_n}{2(1-\wp)(1-2\mu) - \sum_{n=1}^{\infty}(n\mu-\mu+1)[n(1+\hbar)+(\hbar+2\wp-1)]\Xi_n a_n} \leq 1.$$

Thus, we have $f \in \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$. □

Firstly, throughout this paper, for brevity, we let the notations

$$\Upsilon_n(\mu,\wp,\hbar) := [n(1+\hbar) + (\wp+\hbar)](n\mu-\mu+1) \tag{25}$$

$$\Upsilon_1(\mu,\wp,\hbar) = (1+\wp+2\hbar)$$

unless specified differently.

Theorem 2 (*Coefficient estimate*)**.** *If $f \in \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$, then*

$$a_n \leq \frac{(1-\wp)(1-2\mu)}{\Upsilon_n(\mu,\wp,\hbar)\Xi_n}, \qquad n = 1,2,3,\ldots.$$

The outcome is precise for

$$f_n(\vartheta) = \frac{1}{\vartheta} + \frac{(1-\wp)(1-2\mu)}{\Upsilon_n(\mu,\wp,\hbar)\Xi_n}\vartheta^n, \qquad n = 1,2,3,\ldots.$$

Theorem 3. *Let us say a positive number*

$$\Lambda = \inf_{n\in\mathbb{N}}\{\Upsilon_n(\mu,\alpha,\hbar)\Xi_n\} \tag{26}$$

exists. If $f \in \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$, then

$$\left|\frac{1}{r} - \frac{(1-\wp)(1-2\mu)}{\Lambda}r\right| \leq |f(\vartheta)| \leq \frac{1}{r} + \frac{(1-\wp)(1-2\mu)}{\Lambda}r, \quad (|\vartheta| = r)$$

and

$$\left|\frac{1}{r^2} - \frac{(1-\wp)(1-2\mu)}{\Lambda}\right| \le |f'(\vartheta)| \le \frac{1}{r^2} + \frac{(1-\wp)(1-2\mu)}{\Lambda}, \quad (|\vartheta| = r).$$

If $\Lambda = \Upsilon_1(\mu,\wp,\hbar)\Xi_1 = (1+2\wp+\hbar)\Xi_1$*, then the result is sharp for*

$$f(\vartheta) = \frac{1}{\vartheta} + \frac{(1-\wp)(1-2\mu)}{(1+\wp+2\hbar)\Xi_1}\vartheta \tag{27}$$

where Ξ_1 *is given in* (13).

Proof. Since $f(\vartheta) = \frac{1}{\vartheta} + \sum_{n=1}^{\infty} a_n\vartheta^n$, we have

$$|f(\vartheta)| \le \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n \le \frac{1}{r} + r\sum_{n=1}^{\infty} a_n.$$

Since

$$\sum_{n=1}^{\infty} a_n \le \frac{(1-\wp)(1-2\mu)}{\Lambda}.$$

Using this, we have

$$|f(\vartheta)| \le \frac{1}{r} + \frac{(1-\wp)(1-2\mu)}{\Lambda}r.$$

Similarly,

$$|f(\vartheta)| \ge \frac{1}{r} - \frac{(1-\wp)(1-2\mu)}{\Lambda}r.$$

Correspondingly, we can prove the other inequality $|f'(\vartheta)|$. Since

$$|f(\vartheta)| \le \frac{1}{r^2} + \sum_{n=1}^{\infty} na_n r^{n-1} \le \frac{1}{r^2} + \sum_{n=1}^{\infty} a_n$$

again by using $\sum_{n=1}^{\infty} a_n \le \frac{(1-\wp)(1-2\mu)}{\Lambda}$, we obtain the desired inequality. Similarly,

$$|f(\vartheta)| \ge \frac{1}{r^2} - \frac{(1-\wp)(1-2\mu)}{\Lambda}.$$

The result is sharp for function (27) with $\Lambda = \Upsilon_1(\mu,\wp,\hbar)\Xi_1 = (1+\wp+2\hbar)\Xi_1$. □

3. Radius of Starlikeness

The radius of starlikeness for $f \in \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$ where f, as provided by (2), meets the condition (3) in $|\vartheta| < r$, we say that it is meromorphically starlike of order ρ, $(0 \le \rho < 1)$, in $|\vartheta| < r$.

Theorem 4. *Let* f *given by* (2) *be in the class* $\mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$*. Then, if there exists*

$$r_1(\wp,\mu,\rho) = \inf_{n\ge1}\left[\frac{(1-\rho)\Upsilon_n(\mu,\wp,\hbar)\Xi_n}{(n+\rho)(1-\wp)(1-2\mu)}\right]^{\frac{1}{n+1}} \tag{28}$$

and it is sharp, then f *is meromorphically starlike of order* ρ *in* $|\vartheta| < r \le r_1(\wp,\mu,\rho)$*.*

Proof. Let $f \in \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$ as in (2). If $0 < r \le r_1(\wp,\mu,\rho)$, then by (28)

$$r^{n+1} \le \frac{(1-\rho)\Upsilon_n(\mu,\wp,\hbar)\Xi_n}{(n+\rho)(1-\wp)(1-2\mu)}, \quad \forall\, n \in \mathbb{N}. \tag{29}$$

From (29), we obtain

$$\frac{n+\rho}{1-\rho}r^{n+1} \le \frac{\Upsilon_n(\mu,\wp,\hbar)\Xi_n}{(1-\wp)(1-2\mu)} \quad \forall\, n \in \mathbb{N}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{n+\rho}{1-\rho} a_n r^{n+1} \le \sum_{n=1}^{\infty} \frac{\Upsilon_n(\mu,\wp,\hbar)\Xi_n}{(1-\wp)(1-2\mu)} a_n \le 1. \tag{30}$$

because of (22). Hence, from (30) and (21), f is meromorphically starlike of order ρ in $|\vartheta| < r \le r_1(\wp,\mu,\rho)$. $\square$

Suppose that there exists a number $\widetilde{r}, \widetilde{r} > r_1(\wp,\mu,\rho)$ such that each $f \in \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$ is meromorphically starlike of order ρ in $|\vartheta| < \widetilde{r} \le 1$. The function

$$f(\vartheta) = \frac{1}{\vartheta} + \frac{(1-\wp)(1-2\mu)}{\Upsilon_n(\mu,\wp,\hbar)\Xi_n}\vartheta^n$$

is $\mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$; thus, it should satisfy (21) with $\widetilde{r}$:

$$\sum_{n=1}^{\infty}(n+\rho)a_n \widetilde{r}^{n+1} \le 1-\rho, \tag{31}$$

while the left-hand side of (31) becomes

$$(n+\rho)\frac{(1-\wp)(1-2\mu)}{\Upsilon_n(\mu,\wp,\hbar)\Xi_n}\widetilde{r}^{n+1} > (n+\rho)\frac{(1-\wp)(1-2\mu)}{\Upsilon_n(\mu,\wp,\hbar)\Xi_n}\frac{(1-\rho)\Upsilon_n(\mu,\wp,\hbar)\Xi_n}{(n+\rho)(1-\wp)(1-2\mu)} = 1-\rho$$

which contradicts with (31). Thus, the number $r_1(\wp,\mu,\rho)$ in Theorem 4 cannot be substituted with a greater number. This means that $r_1(\wp,\mu,\rho)$ is the so-called radius of meromorphical starlikeness of order ρ for the class $\mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$.

4. Integral Operators

In this section, we consider integral transforms of functions in the class $\mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$.

Theorem 5. *Let in $f \in \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$ be given by (2). Then the integral operator*

$$F(\vartheta) = \jmath\int_0^1 u^{\jmath} f(u\vartheta)du \qquad (0 < \jmath < \infty) \tag{32}$$

is in $\mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\delta)$, where

$$\delta \le \frac{n^2(1+\hbar) + n[(\wp+\hbar) + (1+\hbar)(1+\jmath\wp)] + (\jmath+1)(\wp+\hbar) + \jmath\hbar(1-\wp)}{n^2(1+\hbar) + n[(\wp+\hbar) + (1+\jmath)(1+\hbar)] + (1+\jmath)(\wp+\hbar) + \jmath(1-\wp)}.$$

The result is sharp for the function $f(\vartheta) = \frac{1}{\vartheta} + \frac{(1-\wp)(1-2\mu)}{(1+\wp+2\hbar)\Xi_1}\vartheta$.

Proof. Let $f \in \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$. Then

$$F(\vartheta) \;=\; \jmath\int_0^1 u^{\jmath} f(u\vartheta)du = \frac{1}{\vartheta} + \sum_{n=1}^{\infty}\frac{\jmath}{\jmath+n+1}a_n\vartheta^n.$$

It is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{\jmath\,\Upsilon_n(\mu,\delta,\hbar)\Xi_n}{(\jmath+n+1)(1-\delta)(1-2\mu)}a_n \le 1. \tag{33}$$

Since $f \in \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$, we have

$$\sum_{n=1}^{\infty} \frac{\Upsilon_n(\mu,\wp,\hbar)\Xi_n}{(1-\wp)(1-2\mu)} a_n \leq 1.$$

Note that (33) is satisfied if

$$\frac{\jmath\,\Upsilon_n(\mu,\delta,\hbar)\Xi_n}{(\jmath+n+1)(1-\delta)} \leq \frac{\Upsilon_n(\mu,\wp,\hbar)\Xi_n}{(1-\wp)}.$$

Solving for δ, we have

$$\delta \leq \frac{n^2(1+\hbar)+n[(\wp+\hbar)+(1+\hbar)(1+\jmath\wp)]+(\jmath+1)(\wp+\hbar)+\jmath\hbar(1-\wp)}{n^2(1+\hbar)+n[(\wp+\hbar)+(1+\jmath)(1+\hbar)]+(1+\jmath)(\wp+\hbar)+\jmath(1-\wp)} = \Phi(n).$$

A simple computation will show that $\Phi(n)$ is increasing and

$$\begin{aligned}
\Phi(1) &= \frac{(1+\hbar)+[(\wp+\hbar)+(1+\hbar)(1+\jmath\wp)]+(\jmath+1)(\wp+\hbar)+\jmath\hbar(1-\wp)}{(1+\hbar)+[(\wp+\hbar)+(1+\jmath)(1+\hbar)]+(1+\jmath)(\wp+\hbar)+\jmath(1-\wp)} \\
&= \frac{1+\hbar(2+\jmath)+\wp(\jmath+1)}{1+\hbar(2+\jmath)+\wp+\jmath} \leq \Phi(n) < 1.
\end{aligned}$$

Using this, the results follow. □

It is easy to see that if $0 \leq \delta \leq \delta_1 < 1$, then $\mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\delta_1) \subset \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\delta)$.

Corollary 1. *For the integral* (32), *we have*

$$F(\mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)) \tag{34}$$

with

$$\delta \leq \frac{1+\hbar(2+\jmath)+\wp(\jmath+1)}{1+\hbar(2+\jmath)+\wp+\jmath} < 1.$$

If we replace the class $\mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\delta)$ *in* (34) *with a smaller class* $\mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\delta_1)$ *such that*

$$\delta_1 > \frac{1+\hbar(2+\jmath)+\wp(\jmath+1)}{1+\hbar(2+\jmath)+\wp+\jmath}, \tag{35}$$

then (34) *becomes false.*

Proof. The inclusion relation (34) follows directly from Theorem 5. For the proof of sharpness (34), notice that for the function $f(\vartheta) = \frac{1}{\vartheta} + \frac{(1-\wp)(1-2\mu)}{(1+\wp+2\hbar)\Xi_1}\vartheta$. satisfies (22) so it is in the class $\mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$; moreover, we have

$$F(z) = \frac{1}{\vartheta} + \frac{\jmath(1-\wp)(1-2\mu)}{(\jmath+2)(1+\wp+2\hbar)\Xi_1}\vartheta.$$

By the condition (22), the above function F is in the class $\mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\delta_1)$ if and only if

$$\frac{\Upsilon_1(\mu,\delta_1,\hbar)\Xi_1}{(1-\delta_1)(1-2\mu)} \frac{\jmath(1-\wp)(1-2\mu)}{(\jmath+2)(1+\wp+2\hbar)\Xi_1} \leq 1$$

or equivalently,

$$\frac{(1+\delta_1+2\hbar)}{(1-\delta_1)} \frac{\jmath(1-\wp)}{(\jmath+2)(1+\wp+2\hbar)} \leq 1.$$

Solving the above inequality with respect to δ_1, we obtain

$$\delta_1 \leq \frac{(1+\wp+2\hbar)+\jmath\wp(1+\hbar)}{(1+\hbar)(\jmath+1)+\wp},$$

which contradicts with (35). □

5. Results Involving Modified Hadamard Products

The convolution or Hadamard product of f_1 and f_2 when $f_i(\vartheta) = \frac{1}{\vartheta} + \sum_{n=1}^{\infty} a_{n,i}\vartheta^n$, $(a_{n,i} \geq 0), i = 1,2$ then

$$(f_1 * f_2)(\vartheta) := \frac{1}{\vartheta} + \sum_{n=1}^{\infty} a_{n,1}a_{n,2}\vartheta^n. \tag{36}$$

Theorem 6. *For functions $f_j(\vartheta)(j = 1,2)$ defined by (36), let $f_1(\vartheta) \in \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$ and $f_2(\vartheta) \in \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\delta)$. Then $f_1 * f_2 \in \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\eta)$ where*

$$\eta = 1 - \frac{(1-\wp)(1-\delta)(3+\hbar)}{(1+2\wp+\hbar)(1+2\delta+\hbar)\aleph(1,\mu) - 2(1-\wp)(1-\delta)} \tag{37}$$

where

$$\aleph(1,\mu) = \frac{\Xi_1}{1-2\mu} = \frac{1}{1-2\mu}\left[1+\frac{2\varrho}{\varsigma}\right]^m \frac{\Gamma(\kappa-\zeta)\Gamma(\tau+\zeta)}{\Gamma(\tau-\zeta)\Gamma(\kappa+\zeta)}. \tag{38}$$

The results are the best possible for

$$\begin{aligned} f_1(\vartheta) &= \frac{1}{\vartheta} + \frac{1-\wp}{(1+\wp+2\hbar)\aleph(1,\mu)}\vartheta, \\ f_2(\vartheta) &= \frac{1}{\vartheta} + \frac{1-\delta}{(1+\delta+2\hbar)\aleph(1,\mu)}\vartheta \end{aligned}$$

where $\aleph(1,\mu)$ as given by (38).

Proof. According to Theorem 1, it suits to show that

$$\sum_{n=1}^{\infty} \frac{[n(1+\hbar)+(\eta+\hbar)]}{1-\eta}\aleph(n,\mu)a_{n,1}a_{n,2} \leq 1$$

where η is defined by (37) under the hypothesis. The Cauchy's–Schwarz inequality and (22) one leads to the conclusion that

$$\sum_{n=1}^{\infty} \aleph(n,\mu)\sqrt{\frac{[n(1+\hbar)+(\wp+\hbar)][n(1+\hbar)+(\delta+\hbar)]}{(1-\wp)(1-\delta)}}\sqrt{a_{n,1}a_{n,2}} \leq 1. \tag{39}$$

Hence, we must identify the greatest η such that

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{[n(1+\hbar)+(\eta+\hbar)]}{(1-\eta)}\aleph(n,\mu)a_{n,1}a_{n,2} \\ \leq\ &\sum_{n=1}^{\infty} \aleph(n,\mu)\sqrt{\frac{[n(1+\hbar)+(\wp+\hbar)][n(1+\hbar)+(\delta+\hbar)]}{(1-\wp)(1-\delta)}}\sqrt{a_{n,1}a_{n,2}} \\ =\ &1. \end{aligned}$$

By asset of (39), it is appropriate to find the largest η such that

$$\frac{1}{\aleph(n,\mu)}\left(\frac{(1-\wp)(1-\delta)}{[n(1+\hbar)+(\wp+\hbar)][n(1+\hbar)+(\delta+\hbar)]}\right)^{\frac{1}{2}}$$
$$\leq \frac{1-\eta}{[n(1+\hbar)+(\eta+\hbar)]}\left(\frac{[n(1+\hbar)+(\wp+\hbar)][n(1+\hbar)+(\delta+\hbar)]}{(1-\wp)(1-\delta)}\right)^{\frac{1}{2}} \text{ for } n \geq 1$$

where $\aleph(n,\mu)$ is given by (23), and since $\aleph(n,\mu)$ is a decreasing function of n $(n \geq 1)$, we have

$$\eta = 1 - \frac{(1-\wp)(1-\delta)(3+\hbar)}{(1+\wp+2\hbar)(1+\delta+2\hbar)\aleph(1,\mu) - 2(1-\wp)(1-\delta)}$$

and $\aleph(1,\mu)$ as given by (38). Thus concludes the proof. □

Theorem 7. *Let $f_j(\vartheta)$ $(j=1,2)$ be defined by (36) and $f_j \in \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$. Then, $(f_1 * f_2)(\vartheta) \in \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\eta)$ where*

$$\eta = 1 - \frac{(1-\wp)^2(3+\hbar)}{(1+\wp+2\hbar)^2\aleph(1,\mu) - (1-\wp)^2}$$

with $\aleph(1,\mu)$ as given in (38).

Proof. By fixing $\delta = \wp$ in Theorem 6, the results follow. □

Theorem 8. *(Inclusion property) Let $f_j(\vartheta)(j=1,2)$ be defined by (36) and $f_j \in \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$. Then, h defined by*

$$h(\vartheta) = \frac{1}{\vartheta} + \sum_{n=1}^{\infty}(a_{n,1}^2 + a_{n,2}^2)\vartheta^n$$

is in the class $\mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\delta)$ where

$$\delta \leq 1 - \frac{4(1-\wp)^2(1+\hbar)}{[1+\wp+2\hbar)]^2\aleph(1,\mu) + 2(1-\wp)^2}, \tag{40}$$

and $\aleph(1,\mu)$, as given in (38).

Proof. In light of Theorem 1, it is adequate to show that

$$\sum_{n=2}^{\infty}\aleph(n,\mu)\frac{[n(1+\hbar)+(\delta+\hbar)]}{(1-\delta)}(a_{n,1}^2 + a_{n,2}^2) \leq 1 \tag{41}$$

where $f_j \in \mathcal{M}_{\varsigma,\varrho}^{\tau,\kappa}(\mu,\hbar,\wp)$ $(j=1,2)$, from (36) and Theorem 1, we find that

$$\sum_{n=1}^{\infty}\left[\aleph(n,\mu)\frac{[n(1+\hbar)+(\wp+\hbar)]}{1-\wp}\right]^2 a_{n,j}^2 \leq \sum_{n=1}^{\infty}\left[\aleph(n,\mu)\frac{[n(1+\hbar)+(\wp+\hbar)]}{1-\wp}a_{n,j}\right]^2 = 1 \tag{42}$$

which would yields

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\aleph(n,\mu)\frac{[n(1+\hbar)+(\wp+\hbar)]}{1-\wp}\right]^2(a_{n,1}^2 + a_{n,2}^2) \leq 1. \tag{43}$$

On comparing (41) and (43), it can be perceived that inequity (40) will be satisfied if

$$\aleph(n,\mu)\frac{[n(1+\hbar)+(\delta+\hbar)]}{1-\delta}(a_{n,1}^2 + a_{n,2}^2) \leq \frac{1}{2}\left[\aleph(n,\mu)\frac{[n(1+\hbar)+(\wp+\hbar)]}{1-\wp}\right]^2(a_{n,1}^2 + a_{n,2}^2).$$

That is, if

$$\delta \leq 1 - \frac{2(1-\wp)^2[(n+1)(1+\hbar)]}{[n(1+\hbar)+(\wp+\hbar)]^2\aleph(n,\mu)+2(1-\wp)^2} \tag{44}$$

where $\aleph(n,\mu)$ is a decreasing function of $n, (n \geq 1)$ and is given by (23), we have (40), which concludes the proof. □

6. Closure Theorems

Let $f_k(\vartheta)$ be expressed by

$$f_k(\vartheta) = \frac{1}{\vartheta} + \sum_{n=1}^{\infty} a_{n,k}\vartheta^n, \quad k = 1,2,\ldots,m. \tag{45}$$

One can easily prove the following closure theorems for the class $\mathcal{M}^{\tau,\kappa}_{\varsigma,\varrho}(\mu,\hbar,\wp)$ on lines similar to the proofs given in [27,30,31]; hence, we state them without proof.

Theorem 9. *Let the function $f_k(\vartheta)$ defined by (45) be in the class $\mathcal{M}^{\tau,\kappa}_{\varsigma,\varrho}(\mu,\hbar,\wp)$ for every $k = 1,2,\ldots,m$. Then the function $f(\vartheta)$ defined by $f(\vartheta) = \frac{1}{\vartheta} + \sum_{n=1}^{\infty} a_n\vartheta^n$, $(a_n \geq 0)$ belongs to the class $\mathcal{M}^{\tau,\kappa}_{\varsigma,\varrho}(\mu,\hbar,\wp)$, where $a_n = \frac{1}{m}\sum_{k=1}^{m} a_{n,k}$ $(n = 1,2,..)$.*

Theorem 10. *Let $f_0(\vartheta) = \frac{1}{\vartheta}$ and $f_n(\vartheta) = \frac{1}{\vartheta} + \frac{(1-\wp)(1-2\mu)}{\Upsilon_n(\mu,\wp,\hbar)\Xi_n}\vartheta^n$ for $n = 1,2,\ldots$. Then, $f(\vartheta) \in \mathcal{M}^{\tau,\kappa}_{\varsigma,\varrho}(\mu,\hbar,\wp)$ if, and only if $f(\vartheta)$ expressed as $f(\vartheta) = \sum_{n=0}^{\infty}\eta_n f_n(\vartheta)$ where $\eta_n \geq 0$ and $\sum_{n=0}^{\infty}\eta_n = 1$.*

Theorem 11. *The class $\mathcal{M}^{\tau,\kappa}_{\varsigma,\varrho}(\mu,\hbar,\wp)$ is closed under a convex linear combination.*

7. Partial Sums

For the real part of the proportions between the normalised starlike or convex functions and their sequences of partial sums, Silverman [40] came up with resolutely sharp lower bounds. One is drawn to search results for meromorphic univalent functions that are similar to those of Silverman as a logical extension. In this section, we will examine the relationship between a function of the type (ref-e1.2) and its series of partial sums, primarily motivated by Silverman [40] and Cho and Owa [41] (also see Ref. [27,42]).

$$f_k(\vartheta) = \frac{1}{\vartheta} + \sum_{n=1}^{k} a_n\vartheta^n \tag{46}$$

when the coefficients are appropriately small to fulfil the condition comparable to

$$\sum_{n=1}^{\infty} \Upsilon_n(\mu,\wp,\hbar)\Xi_n a_n \leq (1-\wp)(1-2\mu).$$

More precisely, we will determine sharp lower bounds for $\Re\{f(\vartheta)/f_k(\vartheta)\}$ and $\Re\{f_k(\vartheta)/f(\vartheta)\}$. In this connection, we make use of the well-known results that$\Re\left\{\frac{1+w(\vartheta)}{1-w(\vartheta)}\right\} > 0 \quad (\vartheta \in \Delta)$ if and only if $w(\vartheta) = \sum_{n=1}^{\infty} c_n\vartheta^n$ with $|w(\vartheta)| \leq |\vartheta| = 1$.

Theorem 12. *Let $f \in \mathcal{M}^{\tau,\kappa}_{\varsigma,\varrho}(\mu,\hbar,\wp)$ be given by (2), which satisfies condition (22), and suppose that all of its partial sums (46) do not vanish in Δ. Moreover, suppose that*

$$2 - 2\sum_{n=1}^{k} a_n - \frac{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}}{(1-\wp)(1-2\mu)}\sum_{n=k+1}^{\infty} a_n > 0, for \;\; all \;\; k \in \mathbb{N}. \tag{47}$$

Then,

$$Re\left\{\frac{f(\vartheta)}{f_k(\vartheta)}\right\} \geq \frac{(1+\wp)\Upsilon_{k+1}(\mu,\hbar,\wp)\Xi_{k+1}-(1-\wp)(1-2\mu)}{(1+\wp)\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}} \qquad (\vartheta \in \Delta) \tag{48}$$

where

$$\Upsilon_n(\mu,\hbar,\wp) \geq \begin{cases} (1-\wp)(1-2\mu), & if \quad n=1,2,3,\ldots,k \\ \Upsilon_{k+1}(\mu,\hbar.\wp)\Xi_{k+1}, & if \quad n=k+1,k+2,\ldots. \end{cases} \tag{49}$$

The result (48) is sharp with the function given by

$$f(\vartheta) = \frac{1}{\vartheta} + \frac{(1-\wp)(1-2\mu)}{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}}\vartheta^{k+1}. \tag{50}$$

Proof. Define the function $w(\vartheta)$ by

$$\begin{aligned} \frac{1+w(\vartheta)}{1-w(\vartheta)} &= \frac{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}}{(1-\wp)(1-2\mu)}\left[\frac{f(\vartheta)}{f_k(\vartheta)} - \frac{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}-(1-\wp)(1-2\mu)}{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}}\right] \\ &= \frac{1+\sum\limits_{n=1}^{k} a_n\vartheta^{n+1} + \frac{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}}{(1-\wp)(1-2\mu)}\sum\limits_{n=k+1}^{\infty} a_n\vartheta^{n+1}}{1+\sum\limits_{n=1}^{k} a_n\vartheta^{n+1}}. \end{aligned} \tag{51}$$

It suffices to show that $|w(\vartheta)| \leq 1$. Now, from (55), we can write

$$w(\vartheta) = \frac{\frac{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}}{(1-\wp)(1-2\mu)}\sum\limits_{n=k+1}^{\infty} a_n\vartheta^{n+1}}{2+2\sum\limits_{n=1}^{k} a_n\vartheta^{n+1} + \frac{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}}{(1-\wp)(1-2\mu)}\sum\limits_{k=n+1}^{\infty} a_n\vartheta^{n+1}}.$$

Hence we obtain

$$|w(\vartheta)| \leq \frac{\frac{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}}{(1-\wp)(1-2\mu)}\sum\limits_{k=n+1}^{\infty} a_n}{2-2\sum\limits_{n=1}^{k} a_n - \frac{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}}{(1-\wp)(1-2\mu)}\sum\limits_{n=k+1}^{\infty} a_n}.$$

Now $|w(\vartheta)| \leq 1$ if

$$\frac{2\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}}{(1-\wp)(1-2\mu)}\sum_{n=k+1}^{\infty} a_n \leq 2-2\sum_{n=1}^{k} a_n.$$

From (22), it is enough to show that

$$\sum_{n=1}^{k}|a_n| + \frac{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}}{(1-\wp)(1-2\mu)}\sum_{n=k+1}^{\infty}|a_n| \leq \sum_{n=1}^{\infty}\frac{\Upsilon_n(\mu,\wp,\hbar)\Xi_n}{(1-\wp)(1-2\mu)}|a_n|$$

which is equivalent to

$$\begin{aligned} &\sum_{n=1}^{k}\frac{\Upsilon_n(\mu,\wp,\hbar)\Xi_n-(1-\wp)(1-2\mu)}{(1-\wp)(1-2\mu)}|a_n| \\ &+\sum_{n=k+1}^{\infty}\frac{\Upsilon_n(\mu,\wp,\hbar)\Xi_n-\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}}{(1-\wp)(1-2\mu)}|a_n| \\ \geq\ & 0 \end{aligned} \tag{52}$$

To perceive that f specified by (50) gives a sharp result, we see that for $\vartheta = re^{i\pi/(k+2)}$

$$\begin{aligned} \frac{f(\vartheta)}{f_k(\vartheta)} &= 1 + \frac{(1-\wp)(1-2\mu)}{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}}\vartheta^n \to 1 - \frac{(1-\wp)(1-2\mu)}{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}} \\ &= \frac{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1} - (1-\wp)(1-2\mu)}{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}} \quad \text{when } r \to 1^- \end{aligned}$$

which shows the bound (48) is the best possible for each $k \in \mathbb{N}$. □

We next determine bounds for $f_k(\vartheta)/f(\vartheta)$.

Theorem 13. *If f of the form (2) holds the condition (22), then*

$$Re\left\{\frac{f_k(\vartheta)}{f(\vartheta)}\right\} \geq \frac{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}}{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1} + (1-\wp)(1-2\mu)} \qquad (\vartheta \in \Delta), \tag{53}$$

where $\Upsilon_n(\mu,\hbar,\wp)$ is given by (49). The result (53) is sharp with f, assumed by (50).

Proof. As in the previous proof,

$$\begin{aligned} \frac{1+w(\vartheta)}{1-w(\vartheta)} &= \frac{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1} + (1-\wp)(1-2\mu)}{(1-\wp)(1-2\mu)} \\ &\quad \times \left[\frac{f_k(\vartheta)}{f(\vartheta)} - \frac{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}}{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1} + (1-\wp)(1-2\mu)}\right] \\ &= \frac{1 + \sum\limits_{n=1}^{k} a_n\vartheta^{n+1} - \frac{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}}{(1-\wp)(1-2\mu)} \sum\limits_{n=k+1}^{\infty} a_n\vartheta^{n+1}}{1 + \sum\limits_{n=1}^{k} a_n\vartheta^{n+1}}. \end{aligned} \tag{54}$$

Simple computation yields,

$$|w(\vartheta)| \leq \frac{\left(\frac{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}+(1-\wp)(1-2\mu)}{(1-\wp)(1-2\mu)}\right) \sum\limits_{n=k+1}^{\infty} a_n}{2 - 2\sum\limits_{n=1}^{k} a_n - \frac{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1}+(1-\wp)(1-2\mu)}{(1-\wp)(1-2\mu)} \sum\limits_{n=k+1}^{\infty} a_n} \leq 1.$$

This last inequality is equivalent to

$$\sum_{n=1}^{k} a_n + \frac{\Upsilon_{k+1}(\mu,\wp,\hbar)\Xi_{k+1} + (1-\wp)(1-2\mu)}{(1-\wp)(1-2\mu)} \sum_{n=k+1}^{\infty} a_n \leq 1,$$

which immediately leads to the assertion of Theorem 13. □

8. Conclusions

The interaction of geometry and analysis is a crucial component in the study of complex function theory. This rapid expansion is strongly related to the relationship between geometric behaviour and analytical structure. In the current study, we have familiarized a new meromorphic function class which is related to the Sălăgean-Erdély–Kober (SEK) operator. We have also discovered sufficient and necessary criteria for this subclass. We further investigated linear combinations, distortion theory, and other features. One can simply express the conclusions mentioned in this article for the function classes provided in Examples 1 to 4 associated with the SEK operator by suitably specialising the parameter (as in Remark 1). It is worthy to note that they are new and have not been considered so far. For additional research, we may look at specific classes of functions that correspond to fixed second coefficients connected to the SEK operator, and also certain majorization results, neighborhood results, and differential subordination for meromorphic functions.

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