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Recent Results on Expansive-Type Evolution and Difference Equations: A Survey

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Abstract: In this survey, we review some old and new results initiated with the study of expansive mappings. From a variational perspective, we study the convergence analysis of expansive and almost-expansive curves and sequences governed by an evolution equation of the monotone or non-monotone type. Finally, we propose two well-defined algorithms to remedy the shortcomings concerning the ill-posedness of expansive-type evolution systems.

Keywords: first-order evolution equation; asymptotic behavior; maximal monotone operator; difference equation; periodic solution; expansive-type gradient system

MSC: 47H05; 47H25; 39A12; 37A30; 39A23

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Citation: Djafari Rouhani, B.; Piranfar, M.R. Recent Results on Expansive-Type Evolution and Difference Equations: A Survey. *Axioms* **2023**, *12*, 373. https://doi.org/10.3390/axioms12040373

Academic Editors: Patricia J. Y. Wong and Delfim F. M. Torres

Received: 1 March 2023
Revised: 23 March 2023
Accepted: 6 April 2023
Published: 13 April 2023

1. Introduction

Let H be a real Hilbert space, with an inner product $\langle \cdot, \cdot \rangle$, induced norm $\| \cdot \|$, and identity operator I. The study of the existence and approximation of solutions to nonlinear equations is an important topic and an active field of research in nonlinear analysis. However, nonlinear equations, even with strong restrictive conditions imposed, may not have a solution. An important case is the question raised by L. Nirenberg.

Let $D \subset H$. A self-mapping $T : D \to D$ is said to be expansive (expanding) if

$$\|x - y\| \leq \|Tx - Ty\|, \quad \forall x, y \in D.$$

Nirenberg's question states: "Is any continuous expansive mapping $T : H \to H$ such that $T(H)$ has nonempty interior, surjective?" [1]. This question can be formulated as whether for every continuous expansive mapping T and every $u \in H$, does the equation $T(x) = u$ have a solution? In spite of the strong conditions in Nirenberg's question, one may think that the answer is positive; however, recently, Ives and Preiss [2] answered this question negatively. Indeed, they provided a counterexample in $L^2(0, +\infty)$, which gives a negative answer to Nirenberg's problem even in general separable Hilbert spaces. This question had been already asked for more general spaces, such as Banach spaces, where Morel and Steinlein [3] constructed a counterexample in l^1. In any case, before this negative answer, many attempts to solve this question ended up giving affirmative answers to Nirenberg's question under additional conditions. Among them, we point out [4], where the interior of the range of the expansive mapping is assumed to be unbounded. For more results, see [5–8].

From a variational point of view, one can find a correspondence between expansive mappings and nonexpansive operators. We will get back to this correspondence, but before going further. Let us review briefly some classical results on nonexpansive mappings and their variational analysis. A mapping $T : D \subset H \to H$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in D,$$

where D is a nonempty subset of H. Nonexpansive mappings are generalizations of contractions (with a Lipschitz constant $k < 1$); however, their behaviors can be extremely different. One of the basic problems for nonlinear mappings concerns the following:

$$\text{find } x \in D \text{ such that } \quad T(x) = x.$$

Every solution to the above problem is called a fixed point of T, and the set of all fixed points of T is denoted by $\mathrm{Fix}\,(T)$. If T is nonexpansive, then $\mathrm{Fix}\,(T)$ is closed and convex. The most important properties of contractions are described by the celebrated Banach contraction principle:

Theorem 1 ([9])**.** *Let $D \subset H$, and let $T : D \to D$ be a contraction. Then, (i) T has a unique fixed point, say p, and (ii) for each $x \in D$, $\lim_{n\to+\infty} T^n(x) = p$.*

This theorem does not hold for nonexpansive mappings without any additional conditions. The following theorem, which extends the first part of Banach's contraction principle, was independently proved in 1965 by Browder [10], Kirk [11] and Göhde [12]. We state the theorem here in Hilbert space to stay in the framework of our paper; however, the theorem is proved in more general Banach spaces.

Theorem 2. *Suppose that $T : D \to D$ is a nonexpansive mapping, where D is a nonempty, closed and convex subset of H. Then, T has a fixed point, and the set of all fixed points of T, which may not be a singleton, is closed and convex.*

The second part of Banach's contraction principle does not hold for nonexpansive mappings either. Indeed, according to Banach's contraction principle, all orbits of a contraction T converge to the unique fixed point of T, while orbits of a nonexpansive mapping may not converge at all. Baillon, in 1975, proved that the Cesaro means of the Picard iterates of any nonexpansive mapping T always converge weakly to a fixed point of T, provided that $\mathrm{Fix}\,(T) \neq \emptyset$.

Theorem 3. *Let D be a nonempty, closed, and convex subset of H, and T be a nonexpansive mapping from D into itself. If the set $\mathrm{Fix}\,(T)$ is nonempty, then for each $x \in D$, the Cesaro means*

$$S_n(x) = \frac{1}{n} \sum_{k=1}^{n-1} T^k x,$$

converge weakly to some $y \in \mathrm{Fix}(T)$.

For more details, we refer the reader to [13] and the beautiful books by Goebel and Kirk [14], and by Goebel and Reich [15].

If D is not convex, then $\mathrm{Fix}(T)$ may be empty, and then Baillon's proof is not applicable anymore. To avoid the convexity assumption on D, Djafari Rouhani [16,17] introduced the notions of nonexpansive and almost-nonexpansive sequences and curves.

In this survey, after reviewing some backgrounds on nonexpnasive curves and related notions, we take an expansive-type variational approach to problems of the form

$$\text{find } x \in D \text{ such that } \quad 0 \in A(x),$$

where $A : D \subset H \rightrightarrows H$ is a (possibly multivalued) nonlinear operator.

Section 3, briefly, provides some intuition and backgrounds on the celebrated steepest-descent method and its monotone generalizations. In Section 4, we review some definitions and results on expansive curves. Applying the results in Section 4, Section 5 describes the asymptotic behavior of an expansive-type quasi-autonomous system. In Section 6, we recall discrete versions of the definitions and propositions in Section 4 and apply them to study the asymptotic behavior of an almost-nonexpansive sequence. Section 7 studies the periodic behavior of the expansive sequence described in Section 6. Section 8 is devoted to the study of continuous- and discrete-time non-monotone expansive-type dynamics. As will be seen later, the system considered in Section 5 is "strongly ill-posed". In Section 9, we introduce new well-posed expansive-type systems, which yield weak and strong convergence to zeros of any maximal monotone operator.

Notation 1. *Let u be a curve in H, and $C \subset H$.*

(i) *Convergence in weak and strong topologies are, respectively, denoted by $\to$ and $\rightharpoonup$.*
(ii) *$\overline{\text{conv}}(C)$ denotes the closed convex hull of C.*
(iii) *$\omega_w(u)$ denotes the set of all sequential weak limit points of u.*
(iv) $L(u) = \{q \in H : \lim_{t\to+\infty} \|u(t) - q\| \text{ exists}\}$.
(v) *The weighted average of u is $\sigma_T := \frac{1}{T}\int_0^T u(t)dt$.*

2. Nonexpansive and Almost-Nonexpansive Curves

We recall the following definition from [17]:

Definition 1. *(i) The curve $u(t)$ in H is nonexpansive if for all $r, s, h \geq 0$, we have $\|u(r+h) - u(s+h)\| \leq \|u(r) - u(s)\|$.*

(ii) $u(t)$ is an almost-nonexpansive curve if for all $r, s, h \geq 0$, we have $\|u(r+h) - u(s+h)\|^2 \leq \|u(r) - u(s)\|^2 + \varepsilon(r,s)$, where $\lim_{r,s\to+\infty} \varepsilon(r,s) = 0$.

The following concept introduced in [18] will play an important role:

Definition 2. *Given a bounded curve $u(t)$ in H, the asymptotic center c of $u(t)$ is defined as follows: for every $q \in H$, let $\phi(q) = \limsup_{t\to+\infty} \|u(t) - q\|^2$. Then, ϕ is a continuous and strictly convex function on H, satisfying $\phi(q) \to +\infty$ as $\|q\| \to +\infty$. Therefore, ϕ achieves its minimum on H at a unique point c called the asymptotic center of the curve $u(t)$.*

To the best of our knowledge, Edelstein [18] was the first one who applied the technique of an asymptotic center to fixed-point theory. Combining the notion of nonexpansive curves and the concept of an asymptotic center, Djafari Rouhani proved theorems regarding the asymptotic behavior of nonexpansive and almost-nonexpansive curves without assuming the existence of a fixed point.

Theorem 4 ([17])**.** *Let $u(t)$ be an almost-nonexpansive curve in H. Then, the following are equivalent:*

(i) $L(u) \neq \emptyset$.
(ii) $\liminf_{T\to+\infty} \|\sigma_T\| < +\infty$.
(iii) *σ_T converges weakly to $p \in H$.*

Moreover, under these conditions, we have:

- $\overline{\text{conv}}(\omega_w(u)) \cap L(u) = \{p\}$.
- *p is the asymptotic center of the curve $u(t)$.*

Browder and Petryshyn [19] introduced the notion of asymptotically regular mappings. A mapping $T : D \to D$ is (weakly) asymptotically regular on D if

$$(T^{n+1}x - T^n x \rightharpoonup 0) \quad T^{n+1}x - T^n x \to 0, \quad \forall x \in D.$$

They also showed that if $T : D \to D$ is nonexpansive, then for every $0 < \lambda < 1$, $T_\lambda = \lambda I + (1-\lambda)T$ is asymptotically regular, and $\mathrm{Fix}\,(T_\lambda) = \mathrm{Fix}\,(T)$. Djafari Rouhani extended the notion of asymptotically regular mappings to curves in H:

Definition 3. *(i) The curve $u(t)$ in H is asymptotically regular if for all $h > 0$, $u(t+h) - u(t) \to 0$ as $t \to +\infty$.*

(ii) $u(t)$ is a weakly asymptotically regular curve in H if $u(t+h) - u(t) \rightharpoonup 0$ as $t \to +\infty$.

The following theorem provides sufficient conditions for the weak convergence of asymptotically regular almost-nonexpansive curves:

Theorem 5 ([17])**.** *Let $u(t)$ be a weakly asymptotically regular almost-nonexpansive curve in H. Then, the following are equivalent:*

(i) $L(u(t)) \neq \emptyset$.
(ii) $\liminf_{t\to+\infty} \|u(t)\| < +\infty$.
(iii) *$u(t)$ converges weakly to $p \in H$.*

3. A Steepest-Descent-like Method

For a smooth function $\phi : H \to \mathbb{R}$, the gradient operator $\nabla\phi$ shows the direction of steepest ascent of a particle traveling along the graph of ϕ, hence $-\nabla\phi$ shows the direction of steepest descent. If we consider the curve $u(t)$ as the position of a particle in time t, then the above discussion shows that if the velocity vector $\dot{u}(t)$ equals the value of $-\nabla\phi$ at $u(t)$, then $u(t)$ travels along the steepest-descent direction on the graph of ϕ. In this case, if ϕ has a minimum point, then it may happen that $u(t)$ goes to a minimum point of ϕ. This leads to one of the most celebrated methods in optimization:

Let ϕ be convex with a nonempty set of minimizers. Then, every solution trajectory to the following system

$$\dot{u}(t) = -\nabla\phi(u(t)), \tag{1}$$

converges weakly to a minimizer of ϕ. This method is called the steepest-descent method. A counterexample due to Baillon [20,21] shows that, in general, solutions to the above system may not be strongly convergent in H; see also [22] (Proposition 3.3). Generalizations of this method to nonsmooth and monotone cases were studied by several authors in the 1970s. If $A^{-1}(0)$ is nonempty, Baillon and Brézis [23,24] proved the weak convergence of the mean of solutions to the following system:

$$-\dot{u}(t) \in Au(t), \tag{2}$$

where A is a maximal monotone operator in H and $u(0) = u_0 \in \overline{D(A)}$ is arbitrary. Bruck [25] established the weak convergence of solutions to (2) with an additional condition on A, which is called demipositivity. Motivated by the approach of nonexpansive curves, Djafari Rouhani studied the convergence analysis of a quasi-autonomous version of (2) without assuming $A^{-1}(0)$ to be nonempty.

Theorem 6 ([17])**.** *If u is a weak solution (for the notion of weak and strong solutions, see [26]) of the system*

$$\begin{cases} -\dot{u}(t) \in Au(t) + f(t), \\ u(0) = u_0 \in \overline{D(A)}, \end{cases} \tag{3}$$

on every interval $[0,T]$, and satisfies $\sup_{t>0} \|u(t)\| < +\infty$, and if $f - f_\infty \in L^1((0,+\infty);H)$ for some $f_\infty \in H$, then $\sigma_T = (1/T)\int_0^T u(t)dt$ converges weakly to the asymptotic center of the curve $u(t)$.

The following theorems, respectively, study the weak and strong convergence of trajectories of (3).

Theorem 7 ([17])**.** *If u is a weak solution of the system* (3) *on every interval* $[0, T]$*, and satisfies* $\sup_{t>0} \|u(t)\| < +\infty$ *and for all* $h \geq 0$, $u(t+h) - u(t) \rightharpoonup 0$ *as* $t \to +\infty$*, and if* $f - f_\infty \in L^1((0, +\infty); H)$ *for some* $f_\infty \in H$*, then* $u(t)$ *converges weakly as* $t \to +\infty$ *to the asymptotic center of the curve* $u(t)$*.*

Theorem 8 ([17])**.** *If u is a weak solution of the system* (3) *on every interval* $[0, T]$*, and satisfies* $\lim_{t\to+\infty}\langle u(t), u(t+h)\rangle = \alpha(h)$ *exists uniformly in* $h \geq 0$*, then* $\sigma_T = (l/T)\int_0^T u(t)dt$ *converges strongly as* $T \to +\infty$ *to the asymptotic center of the curve* $u(t)$*.*

4. Expansive Curves and Autonomous Systems

Now, we are in a position to go back to expansive mappings. In general, contrary to nonexpansive mappings, an expansive mapping may not be continuous. As we have seen, the set of fixed points of a nonexpansive mapping may be empty, but it always remains closed and convex. Djafari Rouhani [27] provided examples to show that there are expansive self-mappings of the closed unit ball of H, namely empty, nonconvex, or nonclosed sets of fixed points. The first mean ergodic theorem for expansive mappings was proved by Djafari Rouhani [27]. A continuous time approach to the orbits of an expansive mapping was considered by Djafari Rouhani, and introduced as the notion of expansive curves.

Definition 4. *An expansive curve u in H is a curve satisfying* $\|u(t+h) - u(s+h)\| \geq \|u(t) - u(s)\|$ *for all* $s, t, h \geq 0$*.*

Expansive curves inherit many properties of orbits of expansive mappings, including the lack of convexity and lack of closedness of the set of their fixed points. In any case, the following two sets, which can be defined for any curve, are closed and convex (or empty) sets.

$$F_1(u) = \{q \in H : \|u(t) - q\| \text{ is nonincreasing; }\}$$
$$E_1(u) = \{q \in H : \|u(t) - q\| \text{ is nondecreasing.}\}$$

The following theorem describes the ergodic, weak, and strong convergence of expansive curves in H:

Theorem 9 ([27])**.** *Let u be an expansive curve in H and* $\sigma_T = \frac{1}{T}\int_0^T u(t)dt$ *for* $T > 0$*.*

(i) *If* $\liminf_{T\to+\infty}\|\sigma_T\| < +\infty$ *and* $\|u(t)\| = o(\sqrt{t})$*, then the weak limit q of any weakly convergent subsequence of* σ_T *belongs to* E_1*.*

(ii) *If in addition to (i),* $\liminf_{t\to+\infty}\|u(t)\| < +\infty$*, then u is a bounded curve and* σ_T *converges weakly to the asymptotic center p of* $u(t)$*. Moreover we have* $p = \lim_{t\to+\infty} P_{E_1}u(t)$*.*

(iii) *If in addition to (ii), u is weakly asymptotically regular, then* $u(t)$ *converges weakly to p as* $t \to +\infty$*.*

(iv) *If* $\lim_{t\to+\infty}\|u(t)\|$ *exists, then* σ_T *converges strongly to the asymptotic center p of* $u(t)$*, and moreover in addition to* $p = \lim_{t\to+\infty} P_{E_1}u(t)$*, we have* $p = P_K 0$*, where* $K_t = \overline{\mathrm{conv}}\{u(s); s \geq t\}$ *and* $K = \cap_{t\geq 0}K_t$*.*

Now, let A be a monotone operator in H. If u is weak solution of

$$\begin{cases} \dot{u}(t) \in Au(t), \\ u(0) = u_0, \end{cases} \tag{4}$$

on $[0, T]$ for every $T > 0$, then u is an expansive curve in H [27] (Lemma 5.3); hence, Theorem 9 describes the asymptotic behavior of any weak solution to (4). Unfortunately, the system (4) is "strongly ill-posed". For example, consider the simple linear case of $A = -\Delta$ with Dirishlet boundary conditions, which yields the heat equation with final Cauchy data and is not solvable in general. In Section 9, we try to fix this problem.

5. Almost-Expansive Curves and Quasi-Autonomous Evolution Systems

By introducing an expansive counterpart to the notion of almost-nonexpansive curves, we will be able to study the asymptotic behavior of solutions to (4) for the quasi-autonomous case. Before going further, let us first recall the definition of an almost-expansive curve and a description of its asymptotic behavior from [28].

Definition 5. *The curve u in H is called almost expansive if*

$$\limsup_{s,t\to+\infty}\left[\sup_{h\geq 0}(\|u(s)-u(t)\|^2-\|u(s+h)-u(t+h)\|^2)\right]\leq 0,$$

where for every $\varepsilon>0$, there exists $t_0\geq 0$, such that for all $s,t\geq t_0$, and for all $h\geq 0$, we have

$$\|u(s)-u(t)\|^2\leq\|u(s+h)-u(t+h)\|^2+\varepsilon.$$

We note that if u is bounded, then this definition is equivalent to

$$\limsup_{s,t\to+\infty}\sup_{h\geq 0}(\|u(s)-u(t)\|-\|u(s+h)-u(t+h)\|)\leq 0.$$

The following theorem describes the ergodic, weak, and strong convergence of almost-expansive curves in H.

Theorem 10 ([27])**.** *Let u be an almost expansive curve in H.*

(i) If $\liminf_{T\to+\infty}\|\sigma_T\|<+\infty$ and $\|u(t)\|=o(\sqrt{t})$, then either the weak limit q of any weakly convergent subsequence σ_{T_n} of σ_T belongs to $L(u)$ or $\|u(t)\|\to+\infty$ as $t\to+\infty$.
(ii) If in addition to (i), $\liminf_{t\to+\infty}\|u(t)\|<+\infty$, then u is bounded and σ_T converges weakly as $T\to+\infty$ to the asymptotic center p of u.
(iii) Assuming the conditions in (ii), $u(t)$ converges weakly as $t\to+\infty$ to the asymptotic center p of u if, and only if, u is weakly asymptotically regular.
(iv) If $0\in L(u)$, then σ_T converges strongly as $T\to+\infty$ to the asymptotic center p of u. Moreover, we have $p=P_K0$, where $K_t=\overline{\text{conv}}\{u(s);s\geq t\}$ and $K=\cap_{t\geq 0}K_t$.
(v) If u is asymptotically regular, then $\lim_{t\to+\infty}u(t)=p=P_K0$, where p is the asymptotic center of u and $K_t=\overline{\text{conv}}\{u(s);s\geq t\}$ and $K=\cap_{t\geq 0}K_t$.

The following proposition relates the asymptotic behavior of expansive-type evolution equations to that of almost-expansive curves.

Proposition 1 ([28])**.** *If u is a weak solution of*

$$\begin{cases}\dot{u}(t)+f(t)\in Au(t),\\ u(0)=u_0,\end{cases}\tag{5}$$

on $[0,T]$ for every $T>0$, and if $\sup_{t\geq 0}\|u(t)\|<+\infty$ and

$$\lim_{s,r\to+\infty}\int_s^{+\infty}\|f(\theta+(r-s))-f(\theta)\|d\theta=0,$$

then the curve u is almost expansive in H.

Therefore, similar to the expansive case, one can apply the results on the asymptotic behavior of almost-expansive curves to describe the asymptotic behavior of solutions to (5).

Theorem 11 ([28])**.** *Assume u is a weak solution of (5) on every interval $[0,T]$ and $\sup_{t\geq 0}\|u(t)\|<+\infty$. Assume $f-f_\infty\in L^1((0,+\infty);H)$ for some $f_\infty\in H$. Then, the following hold:*

(i) $\sigma_T\rightharpoonup p$ as $T\to+\infty$, where p is the asymptotic center of u.

(ii) $u(t) \rightharpoonup p$ *as* $t \to +\infty$*, if and only if* u *is weakly asymptotically regular.*
(iii) *If* $\lim_{t\to+\infty} \|u(t)\|$ *exists, then* $\lim_{T\to+\infty} \sigma_T = p = P_K 0$*, where* K *is as defined above.*
(iv) $\lim_{t\to+\infty} u(t) = p = P_K 0$ *if and only if* u *is asymptotically regular.*

6. Expansive-Type Difference Equations

As we have already explained, the dissipative systems of the form (3) have a unique weak solution, whereas for solutions to (4), neither existence nor uniqueness is guaranteed. A similar situation occurs for the backward discretization of (4):

$$u_{n+1} - u_n \in \lambda_n A u_{n+1}.$$

Hence, we consider the following forward discretization:

$$u_{n+1} - u_n \in \lambda_n A u_n, \tag{6}$$

which is always well defined.

Similar to the continuous case, by introducing the notion of almost-expansive sequences and studying their asymptotic behavior under some suitable conditions, we describe the asymptotic behavior of the solution to (6).

Definition 6. *A sequence* u_n *in* H *is said to be almost-expansive if for all* $i, j, k \geq 0$*, we have*

$$\limsup_{i,j\to\infty} \left[\sup_{k\geq 0} (\|u_i - u_j\|^2 - \|u_{i+k} - u_{j+k}\|^2) \right] \leq 0.$$

i.e., $\forall \varepsilon > 0$, $\exists N_0$ *such that* $\forall i, j \geq N_0$, $\forall k \geq 0$, $\|u_i - u_j\|^2 \leq \|u_{i+k} - u_{j+k}\|^2 + \varepsilon$.

We note that if u_n is bounded, then this definition is equivalent to

$$\limsup_{i,j\to\infty} \left[\sup_{k\geq 0} (\|u_i - u_j\| - \|u_{i+k} - u_{j+k}\|) \right] \leq 0.$$

The sequence of averages of u_n is denoted by s_n and defined by $s_n = \frac{1}{n}\sum_{k=0}^{n-1} u_k$. The following theorem provides a discrete version of Theorem 10.

Theorem 12 ([27])**.** *Let* u_n *be an almost expansive sequence in* H*.*

(i) *If* $\liminf_{n\to+\infty} \|s_n\| < +\infty$ *and* $\|u_n\| = o(\sqrt{t})$*, then either the weak limit* q *of any weakly convergent subsequence* s_{n_k} *of* s_n *belongs to* $L(u_n)$ *or* $\|u_n\| \to +\infty$ *as* $n \to +\infty$*.*
(ii) *If in addition to (i),* $\liminf_{n\to+\infty} \|u_n\| < +\infty$*, then* u_n *is bounded and* s_n *converges weakly as* $n \to +\infty$*, to the asymptotic center* p *of* u_n*.*
(iii) *Assuming the conditions in (ii),* u_n *converges weakly as* $n \to +\infty$ *to the asymptotic center* p *of* u_n*, if and only if* u_n *is weakly asymptotically regular.*
(iv) *If* $0 \in L(u_n)$*, then* s_n *converges strongly as* $n \to +\infty$ *to the asymptotic center* p *of* u_n*. Moreover, we have* $p = P_K 0$*, where* $K_n = \overline{\mathrm{conv}}\{u_k; k \geq n\}$ *and* $K = \cap_{n\geq 0} K_n$*.*
(v) *If* u_n *is asymptotically regular, then* $\lim_{n\to+\infty} u_n = p = P_K 0$*, where* p *is the asymptotic center of* u_n*, and* K *is defined above.*

We still need an additional condition for the sequence u_n governed by (6) to be almost expansive.

Proposition 2 ([29])**.** *Let* λ_n *be a nondecreasing sequence of positive numbers, such that*

$$\limsup_{\substack{j\geq i \\ i,j\to+\infty}} \sum_{l=i}^{+\infty} \left(\frac{\lambda_{(j-i)+l}}{\lambda_l} - 1 \right) = 0. \tag{7}$$

If u_n is a bounded solution to (6), then u_n is almost expansive.

Note that the condition (7) in the above proposition is in particular satisfied if $\sup_{n\geq 1} \lambda_n \leq \lambda$ for some $\lambda > 0$, and $\frac{\lambda}{a_n+1} \leq \lambda_n$ for some $a_n \in l^1$. For example, the sequence $\lambda_n = \frac{n^2}{1+n^2}$ satisfies the conditions of the above proposition. Now, we are in a position to apply our results on almost-expansive sequences to describe the asymptotic behavior of the sequence u_n governed by (6).

Theorem 13 ([29])**.** *Assume that λ_n is a nondecreasing sequence satisfying the condition (7), and u_n is a bounded solution to (6). Then, the following hold:*

(i) $s_n \rightharpoonup p$*, as* $n \to +\infty$*, where p is the asymptotic center of u_n.*
(ii) $u_n \rightharpoonup p$*, as* $n \to +\infty$ *if and only if u is weakly asymptotically regular.*
(iii) *If* $\lim_{n\to+\infty} \|u_n\|$ *exists, then* $\lim_{n\to+\infty} s_n = p = P_K 0$*, where K is as defined above.*
(iv) $\lim_{n\to+\infty} u_n = p = P_K 0$ *if and only if u_n is asymptotically regular.*

In the following theorem, by assuming the zero set of A to be nonempty, we can obtain stronger results:

Theorem 14 ([29])**.** *Let u_n be the sequence generated by (6), where $A^{-1}(0) \neq \emptyset$ and $\liminf_{n\to+\infty} \lambda_n \geq \lambda$ for some $\lambda > 0$. If u_n is bounded, then there exists some $p \in A^{-1}(0)$, such that $u_n \rightharpoonup p$ as $n \to +\infty$. Otherwise, $\|u_n\| \to +\infty$ as $n \to +\infty$.*

Note that if the step size λ_n goes to infinity as $n \to +\infty$, then the existence of a bounded solution to (6) implies that $A^{-1}(0) \neq \emptyset$. In fact, let u_n be a bounded solution to (6) and $b_n = \frac{u_{n+1}-u_n}{\lambda_n}$. Clearly, $b_n \in Au_n$ and $b_n \to 0$. Since u_n is bounded, there exist some $q \in H$ and a subsequence u_{n_k}, such that $u_{n_k} \rightharpoonup q$ as $k \to +\infty$. Now, the maximality of A implies that $q \in A^{-1}(0)$.

7. Periodic Solutions in Discrete Time

In this section, we will need the following extended version of expansive mappings

Definition 7. *The mapping $T : D(T) \subset H \to H$ is said to be α-expansive if*

$$\alpha\|x-y\| \leq \|Tx-Ty\|, \quad \forall x,y \in D(T).$$

If $\alpha = 1$, we say that T is expansive.

Clearly, letting $\alpha = 1$, the above definition coincides with the definition of an expansive mapping, and if $T : H \to H$ is α-expansive, then T^{-1} exists and it is $\frac{1}{\alpha}$-Lipschitz continuous. The following theorem provides sufficient conditions for the system (6) to have a periodic solution.

Theorem 15 ([29])**.** *Suppose that A is a single-valued and maximal strongly monotone operator in H. If λ_n is a periodic sequence with period N, then there exists an N-periodic solution to (6).*

The above theorem does not hold for a general maximal monotone operator A; not even for subdifferentials of proper, convex and lower semicontinuous functions, nor for inverse strongly monotone operators. To see this, let $A : \mathbb{R} \to \mathbb{R}$ be the constant function $A \equiv 1$, and $\lambda_n \equiv 1$. Then, (6) reduces to $u_{n+1} = u_n + 1$, which shows that the sequence u_n tends to $+\infty$, as $n \to +\infty$, for all $u_0 \in \mathbb{R}$. Therefore, it does not have a periodic solution. However, assuming (6) has a periodic solution, is it possible that (6) has another solution (by starting from a different initial point) that behaves differently? The following theorem answers this question.

Theorem 16 ([29]). *Assume that A is a single-valued and maximal monotone operator in H, and the sequence λ_n is periodic with period N. If* (6) *has an N-periodic solution w_n, then every bounded solution to* (6) *is also periodic with period N and differs from w_n by an additive constant.*

In general, the existence of periodic solutions does not imply the boundedness of all solutions to (6). For this, let $D = [0,1]$, $A = (I - P_D)$, and $\lambda_n \equiv 1$. Then, (6) reduces to $u_{n+1} = 2u_n - P_D u_n$. If we choose $u_0 = 0$, then $u_n \equiv 0$, which is a periodic solution with period N for all $N \in \mathbb{N}$. However, if we choose $u_0 = 2$, then $u_{n+1} = 2u_n - 1$, which clearly goes to $+\infty$, as $n \to +\infty$.

8. A Gradient System of Expansive Type

In this section, we consider a particular case of non-monotone operators. This case is motivated by the prominent example of a maximal monotone operator that is the subdifferential of a proper, convex, and lower semicontinuous function. A quasiconvex function is an extension of a convex function, which has found many applications in economics [30]. Unlike the convex case, quasiconvex functions do not have a convex epigraph, but have convex sublevel sets. This is stated formally in the following definition:

Definition 8. *(i)* *A function $\phi : H \to (-\infty, +\infty]$ is quasiconvex if*

$$\phi(\lambda x + (1-\lambda)y) \leq \max\{\phi(x), \phi(y)\}, \quad \forall x, y \in H \text{ and } \forall \lambda \in [0,1].$$

(ii) *A function $\phi : H \to (-\infty, +\infty]$ is strongly quasiconvex if there is $\alpha > 0$ such that*

$$\phi(\lambda x + (1-\lambda)y) \leq \max\{\phi(x), \phi(y)\} - \alpha\lambda(1-\lambda)\|x-y\|^2, \quad \forall x, y \in H \text{ and } \forall \lambda \in [0,1].$$

The notion of a subdifferential has been generalized for nonconvex functions by many authors. Nevertheless, in any circumstance, the subdifferential operator of a quasiconvex function is not monotone. However, in the case where the quasiconvex function $\phi : H \to \mathbb{R}$ is Gâteaux differentiable, then the following characterization holds:

$$\phi \text{ is quasiconvex on } H \Leftrightarrow (\forall x, y \in H, \;\; \phi(y) \leq \phi(x) \Rightarrow \langle \nabla\phi(x), x - y \rangle \geq 0).$$

This characterization will be useful in the rest of this section to make up for the lack of monotonicity.

We consider the expansive system governed by the non-monotone operator $\nabla\phi$, where $\phi : H \to \mathbb{R}$ is a differentiable quasiconvex function. Indeed, as in [31], we consider the following differential equation

$$\dot{u}(t) = \nabla\phi(u(t)) + f(t), \quad t \in [0, +\infty), \tag{8}$$

where $\phi : H \to \mathbb{R}$ is a differentiable quasiconvex function, such that $\nabla\phi$ is Lipschitz continuous and $f \in W^{1,1}((0,+\infty); H)$. The Cauchy–Lipschitz theorem implies the existence of a unique solution of the system (8) with an initial condition, where $\nabla\phi$ is Lipschitz continuous. In order to study the asymptotic behavior of solutions to systems of the form (8), the authors in [31] introduced the following set for a function ϕ along a curve u:

$$L_\phi(u) = \{y \in H : \exists T > 0 \text{ s.t. } \phi(y) \leq \phi(u(t)) \quad \forall t \geq T\}.$$

Denoting the set of all global minimizers of ϕ by Argmin ϕ, then Argmin $\phi \subset L_\phi(u)$. The following proposition shows that if u is a solution to (8), then $L_\phi(u) \subset L(u)$.

Proposition 3 ([31])**.** *Let $u(t)$ be a solution to (8). For an arbitrary interval $[a, b]$, where $b \geq a \geq 0$, and each $y \in L_\phi(u)$, we have*

$$\|u(a) - y\| \leq \|u(b) - y\| + \int_a^b \|f(t)\| dt,$$

and therefore $\lim_{t \to +\infty} \|u(t) - y\|$ exists (it may be infinite).

Proposition 4 ([31])**.** *Let $u(t)$ be a solution to (8). If $\liminf_{t \to +\infty} \|u(t)\| < +\infty$, then*

(i) $\lim_{t \to +\infty} \nabla\phi(u(t)) = 0$.
(ii) $\lim_{t \to +\infty} \phi(u(t))$ *exists and is finite.*
(iii) $L_\phi(u) \neq \emptyset$.
(iv) *u is bounded.*

The following theorem describes the asymptotic behavior of solutions to (8).

Theorem 17 ([31])**.** *Let $u(t)$ be a solution to (8). If $\liminf_{t \to +\infty} \|u(t)\| < +\infty$, then there exists some $p \in (\nabla\phi)^{-1}(0)$, such that $u(t) \rightharpoonup p$ as $t \to +\infty$, and if $p \notin \operatorname{Argmin} \phi$, the convergence is strong. If $u(t)$ is unbounded, then $\|u(t)\| \to +\infty$ as $t \to +\infty$.*

Note that the above theorem shows that if $(\nabla\phi)^{-1}(0) = \emptyset$, then for any solution to (8), we have $\lim_{t \to +\infty} \|u(t)\| = +\infty$.

The following two theorems provide sufficient conditions for the strong convergence of solutions to (8).

Theorem 18 ([31])**.** *With either one of the following assumptions, bounded solutions to (8) converge strongly to some point in $(\nabla\phi)^{-1}(0)$:*

(i) *Sublevel sets of ϕ are compact.*
(ii) $\operatorname{int} L_\phi(u) \neq \emptyset$.

Theorem 19 ([31])**.** *Assume that $\phi : H \to \mathbb{R}$ is a strongly quasiconvex function and $u(t)$ is a bounded solution to (8). Then, $\operatorname{Argmin} \phi$ is a singleton and $u(t)$ converges strongly to the unique minimizer of ϕ.*

For a differentiable quasiconvex function $\phi : H \to \mathbb{R}$ whose gradient $\nabla\phi$ is Lipschitz continuous with Lipschitz constant K, as in Section 6, we consider the forward finite-difference discrete version of (8), which yields a well-defined sequence:

$$u_{n+1} - u_n = \lambda_n \nabla\phi(u_n) + f_n, \tag{9}$$

where the sequence f_n belongs to l^1 and $\lambda_n \geq \varepsilon$ for some $\varepsilon > 0$.

In order to study the asymptotic behavior of u_n, we define the following discrete version of $L_\phi(u)$:

$$L_\phi(u_n) = \{y \in H : \exists N > 0 \text{ s.t. } \phi(y) \leq \phi(u_n) \quad \forall n \geq N\}.$$

The following proposition is a discrete version of Proposition 3.

Proposition 5 ([31])**.** *Let u_n be the sequence generated by (9). For each $y \in L_\phi(u_n)$, and $k < m$, we have*

$$\|u_k - y\| \leq \|u_m - y\| + \sum_{n=k}^{m-1} \|f_n\|, \tag{10}$$

and consequently $\lim_{n \to +\infty} \|u_n - y\|$ exists (it may be infinite).

Proposition 6 ([31]). *Let u_n be a solution to (9), such that $\liminf_{n\to+\infty}\|u_n\| < +\infty$. Then, $L_\phi(u_n)$ is nonempty if and only if $\lim_{n\to+\infty}\phi(u_n)$ exists, and in this case u_n is bounded.*

If ϕ is convex, we can omit the Lipschtz continuity condition on $\nabla\phi$ in Proposition 6.

Proposition 7 ([31]). *Assume that u_n is a solution to (9), such that $\liminf_{n\to+\infty}\|u_n\| < +\infty$. If either one of the following conditions is satisfied, then $L_\phi(u_n)$ is nonempty.*

(i) ϕ *is convex and the sequence of step sizes λ_n is bounded above.*
(ii) $\limsup_{n\to+\infty}\lambda_n < \frac{2}{K}$.

In the continuous case, we showed that if $\liminf_{t\to+\infty}\|u(t)\| < +\infty$, then $L_\phi(u) \neq \emptyset$. However, in the discrete case, it remains an open problem whether without any additional assumption that $\liminf_{n\to+\infty}\|u_n\| < +\infty$ implies that $L_\phi(u_n)$ is nonempty.

The following theorems describe the weak and strong convergence of solutions to (9).

Theorem 20 ([31]). *Assume that u_n is the sequence given by (9), and $L_\phi(u_n) \neq \emptyset$. If $\liminf_{n\to+\infty}\|u_n\| < +\infty$, then there is some $p \in (\nabla\phi)^{-1}(0)$, such that $u_n \rightharpoonup p$ as $n \to +\infty$, and if $p \notin \operatorname{Argmin}\phi$, the convergence is strong. If u_n is not bounded, then $\|u_n\| \to +\infty$, as $n \to +\infty$.*

Theorem 21 ([31]). *Let u_n be a bounded sequence, which satisfies (9), and let $L(u_n) \neq \emptyset$. If either one of the following assumptions holds, then u_n converges strongly to some point in $(\nabla\phi)^{-1}(0)$:*

(i) *Sublevel sets of ϕ are compact.*
(ii) $\operatorname{int} L_\phi(u_n) \neq \emptyset$.

Example 1. *Assume that $\phi : \mathbb{R} \to \mathbb{R}$ is defined by $\phi(x) = \arctan(x^3)$ and consider (9) with $\lambda_n = \frac{2}{3}n$ and $f_n \equiv 0$. Then, it is easy to see that all the assumptions of Theorem 21 are satisfied. In Table 1, we compare 1000 iterations u_n generated by (9) starting from two different initial points, namely $u_0 = -0.5$ and $u_0 = 1$. The numerical results show that for $u_0 = -0.5$, $u_n \to 0 \in (\nabla\phi)^{-1}(0)$, and for $u_0 = 1$, u_n slowly goes to infinity.*

Table 1. Comparing 1000 iterations u_n with different initial points.

n	u_n	u_n
0	−0.5	1
1	−0.00769231	2
10	−0.00404869	3.63765
20	−0.00171074	4.68854
30	−0.0008858	5.46951
40	−0.000533135	6.11128
50	−0.000354164	6.66517
60	−0.000251763	7.15741
70	−0.000187942	7.60348
80	−0.000145564	8.01339
90	−0.000116023	8.39404
100	−0.0000946225	8.7504
1000	-9.94968×10^{-7}	21.8786

9. Some New Results

As we have seen in Section 4, given a maximal monotone operator, expansive systems of the form (4) may be "strongly ill-posed" in general. In this section, we consider a special class of maximal monotone operators that induces well-posed expansive evolution systems. Motivated by this, we propose an expansive-type approach for the approximation of zeros of any maximal monotone operator.

9.1. Weak Convergence

We start with the following definition:

Definition 9. *Let $\lambda > 0$. The operator $A : H \to H$ is said to be λ-inverse strongly monotone if*

$$\lambda \|A(x) - A(y)\|^2 \leq \langle A(x) - A(y), x - y \rangle, \quad \forall x, y \in H.$$

Clearly, a λ-inverse strongly monotone operator is $\frac{1}{\lambda}$-Lipschitz.

Let $A : H \to H$ be a λ-inverse strongly monotone operator, such that $A^{-1}(0) \neq \emptyset$. Consider the following differential equation:

$$\begin{cases} \dot{u}(t) = Au(t), \\ u(0) = x \in H. \end{cases} \tag{11}$$

Since A is Lipschitz, then the Cauchy–Lipschitz theorem guarantees that there exists a unique solution to (11). The following Lemma is due to Z. Opial [32], and is an effective tool in the convergence analysis of curves in the weak topology.

Lemma 1. *Let $u : [0, +\infty) \to H$, and let $S \subset H$ be nonempty. Assume that*

(i) For every $y \in S$, $\lim_{t \to +\infty} \|u(t) - y\|$ exists;
(ii) Every sequential weak limit point of u belongs to S.

Then, there exists $p \in S$, such that $u(t) \rightharpoonup p$ as $t \to +\infty$.

Theorem 22. *Assume that u is a strong solution to (11). If u is unbounded, then $\|u(t)\| \to +\infty$, as $t \to +\infty$. If u is bounded, then there exists some $p \in A^{-1}(0)$, such that $u(t) \rightharpoonup p$ as $t \to +\infty$.*

Proof. Let $y \in A^{-1}(0)$ and $h_y(t) = \frac{1}{2}\|u(t) - y\|^2$. By the monotonicity of A we have:

$$\dot{h}_y(t) = \langle \dot{u}(t), u(t) - y \rangle = \langle Au(t), u(t) - y \rangle \geq 0.$$

Hence, $h_y(t)$ is nondecreasing. If $u(t)$ is unbounded then $h_y(t) \to +\infty$ is as $t \to +\infty$, which implies that $\|u(t)\| \to +\infty$ is as $t \to +\infty$. If $u(t)$ is bounded, then $\lim_{t \to +\infty} h_y(t)$ exists. Multiplying both sides of (11) by $u(t) - y$ and then using the fact that A is λ-inverse strongly monotone, we obtain:

$$\dot{h}_y(t) = \langle \dot{u}(t), u(t) - y \rangle = \langle Au(t), u(t) - y \rangle \geq \lambda \|Au(t)\|^2. \tag{12}$$

Replacing $Au(t)$ with $\dot{u}(t)$ in (12) and then integrating both sides of (12) on $[0, t]$, we obtain:

$$\lambda \int_0^t \|\dot{u}(\tau)\|^2 d\tau \leq h_y(t) - h_y(0).$$

Since $\lim_{t \to +\infty} h_y(t)$ exists, the above inequality implies that $\dot{u} \in L^2([0, +\infty), H)$. On the other hand, since u is bounded and A is Lipschitz, (11) yields $\dot{u}$ and is bounded, and hence u is Lipschitz. Now, since $\dot{u}$ is the composition of two Lipschitz mappings, $\dot{u}$ is Lipschitz too. This implies that $\dot{u}$ is uniformly continuous. This together with $\dot{u} \in L^2([0, +\infty), H)$ yields $\lim_{t \to +\infty} \dot{u}(t) = 0$ and hence by (11), $\lim_{t \to +\infty} Au(t) = 0$. Now, let q be a weak cluster point of $u(t)$. There exists a sequence $t_n \subset [0, +\infty)$, such that $t_n \to +\infty$ as $n \to +\infty$, and $u(t_n) \rightharpoonup q$ as $n \to +\infty$. From the maximality of A, we have $q \in A^{-1}(0)$. Now an easy application of Opial's Lemma concludes the proof. □

Let A be an arbitrary maximal monotone operator, and $\lambda > 0$. The resolvent of A of index λ is the single-valued operator $J_\lambda^A = (I + \lambda A)^{-1}$, which is nonexpansive and everywhere defined. The Yosida approximation of A of index λ is $A_\lambda = \frac{1}{\lambda}(I - J_\lambda^A)$. A straightforward calculation shows that the Yosida approximation of index λ is λ-inverse and strongly monotone, and $A_\lambda^{-1}(0) = A^{-1}(0)$. Therefore, the Cauchy–Lipschitz theorem implies that the differential equation

$$\dot{u}(t) = A_\lambda(u(t)), \tag{13}$$

with an initial condition $u(0) = u_0 \in H$ is well defined. Therefore, by Theorem 22, if $u_\lambda(t)$ is a solution to (13) that remains bounded, then $u_\lambda(t)$ converges weakly to a zero of A, otherwise $u_\lambda(t)$ goes to infinity in the norm as $t \to +\infty$.

9.2. Strong Convergence via Tikhonov Regularization

In this subsection, we propose well-posed dynamics that approximate zeros of an arbitrary maximal monotone operator A in strong topology. For this purpose, let us assume that $\alpha : [0, +\infty) \to (0, +\infty)$ is absolutely continuous on every finite interval, and define $A_t = A + \alpha(t)I$. Hence, A_t is onto, and due to the strong monotonicity of A_t, the zero set of A_t is a singleton. Let $\xi(t)$ denote the unique zero of A_t. We call $\xi(t)$ the central path of A corresponding to $\alpha(t)$.

Lemma 2 ([33])**.** *Let A be a maximal monotone operator, let $\alpha(t)$ be a positive function, and let $\xi(t)$ be the central path corresponding to A and $\alpha(t)$. If $A^{-1}(0) \neq \emptyset$, then $\xi(t)$ is bounded. Moreover, if $\lim_{t\to+\infty} \alpha(t) = 0$, then $\xi(t)$ converges strongly to the least norm element in $A^{-1}(0)$.*

Since $\xi(t) = J^A_{\frac{1}{\alpha(t)}}(0)$, by the resolvent identity, we have

$$\|\xi(t+\delta) - \xi(t)\| = \left\| J^A_{\frac{1}{\alpha(t)}}\left(\left(1 - \frac{\alpha(t)}{\alpha(t+\delta)}\right)\xi(t)\right) - J^A_{\frac{1}{\alpha(t)}}(0)\right\| \leq \left\|\left(1 - \frac{\alpha(t)}{\alpha(t+\delta)}\right)\xi(t)\right\|.$$

If $A^{-1}(0) \neq \emptyset$, then Lemma 2 implies that $\xi(t)$ is bounded. The boundedness of $\xi(t)$ and the absolute continuity of $\alpha(t)$ on every finite interval together with the above inequality implies that $\xi(t)$ is absolutely continuous on every finite interval, since $\alpha(t)$ does not take the value zero, therefore it is bounded away from zero. Hence, $\xi(t)$ is almost everywhere differentiable. Dividing both sides of the above inequality by δ and then letting $\delta \to 0$, we obtain

$$\|\dot{\xi}(t)\| \leq \frac{|\dot{\alpha}(t)|}{\alpha(t)}\|\xi(t)\|, \quad \text{a.e. } t \geq 0. \tag{14}$$

Theorem 23. *Let $\alpha : [0, +\infty) \to (0, +\infty)$ be absolutely continuous on every finite interval, such that*

(i) $\lim_{t\to+\infty} \alpha(t) = 0$;

(ii) $\lim_{t\to+\infty} \frac{\dot{\alpha}(t)}{\alpha(t)^2} = 0$;

(iii) $\int_0^{+\infty} \alpha(t)dt = +\infty$.

Let $A : H \rightrightarrows H$ be maximal monotone with a nonempty zero set. Then, every bounded (possible) solution to the following differential equation

$$\begin{cases} \dot{u}(t) = A(u(t)) + \alpha(t)u(t), \\ u(0) = u_0 \in H, \end{cases} \tag{15}$$

converges strongly to the zero of A with minimal norm.

Proof. Let $h(t) = \frac{1}{2}\|u(t) - \xi(t)\|^2$. We have

$$\dot{h}(t) = \langle \dot{u}(t) - \dot{\xi}(t), u(t) - \xi(t) \rangle,$$

hence

$$\dot{h}(t) + \langle \dot{\xi}(t), u(t) - \xi(t) \rangle = \langle A(u(t)) - A(\xi(t)), u(t) - \xi(t) \rangle + \alpha(t) \| u(t) - \xi(t) \|^2,$$

By applying the Cauchy–Schuartz inequality and the monotonicity of A, we obtain

$$2\alpha(t)h(t) \le \dot{h}(t) + M\|\dot{\xi}(t)\|, \tag{16}$$

where $M = \sup_{t>0} \|u(t) - \xi(t)\|$. Multiplying both sides of (16) by $e^{-E(t)}$, where $E(t) = \int_0^t \alpha(\tau)d\tau$, we get:

$$-M\frac{\|\dot{\xi}(t)\|}{\alpha(t)}\alpha(t)e^{-E(t)} \le e^{-E(t)}\dot{h}(t) - \alpha(t)e^{-E(t)}h(t),$$

Then,

$$M\frac{\|\dot{\xi}(t)\|}{\alpha(t)}\frac{d}{dt}\left(e^{-E(t)}\right) \le \frac{d}{dt}\left(e^{-E(t)}h(t)\right).$$

Integrating the above inequality on $[s, t]$, we get

$$m(s)\left(e^{-E(t)} - e^{-E(s)}\right) \le e^{-E(t)}h(t) - e^{-E(s)}h(s),$$

where $m(s) = M \inf_{t \ge s} \frac{\|\dot{\xi}(t)\|}{\alpha(t)}$. Letting $t \to +\infty$ in the above inequality, since $h(t)$ is bounded and $\lim_{t\to+\infty} E(t) = +\infty$, we obtain

$$e^{-E(s)}h(s) \le e^{-E(s)}m(s) \le e^{-E(s)}M\frac{\|\dot{\xi}(s)\|}{\alpha(s)}.$$

Multiplying the above inequality by $e^{E(s)}$, and applying (14), we obtain

$$h(s) \le M\frac{|\dot{\alpha}(s)|}{\alpha(s)^2}\|\xi(s)\|.$$

Now, letting $s \to +\infty$, we conclude the result by applying (ii) and Lemma 2. □

Remark 1. *By applying a nonautonomous version of the Cauchy–Lipschitz theorem, the following Tikhonov regularization system has a unique solution.*

$$\begin{cases} \dot{u} = A_\lambda(u(t)) + \alpha(t)u(t), \\ u(0) = u_0 \in H. \end{cases} \tag{17}$$

Therefore, by Theorem 23, the system (17) provides a continuous time-expansive method to approximate the zero with the least norm of any maximal monotone operator A.

Author Contributions: Writing—original draft, B.D.R. and M.R.P. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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