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One-Stage Multiple Comparisons of the Mean Lifetimes of k Treatments with the Average for Exponential Distributions under Heteroscedasticity

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Abstract: Making use of uniformly minimum-variance unbiased estimators for the parameters of two-parameter exponential distributions and the distribution of pivotal quantities, we propose one-stage multiple comparison procedures for k mean lifetimes with the average under heteroscedasticity. The multiple comparison procedures include one-sided and two-sided confidence intervals. These intervals can be applied to identify which treatment's mean lifetime is better than the average or worse than the average in terms of the mean lifetimes of all treatments. Critical values are obtained in order to assure users that the given confidence coefficient has been reached; they are organized in table format for practical and convenient use. An example is provided to demonstrate the proposed techniques, wherein the mean survival times of four different lung cancer categories are compared with the average.

Keywords: one-stage procedure; simulation; ranking and selection: multiple comparison procedures with the average; mean lifetimes

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1. Introduction

Bechhofer [1] and Gupta [2] are recognized as trailblazers in the application of the normal distribution model in the field of ranking and selection. In many cases of reliability analysis and lifetime testing, the lifetime of products does not follow a normal distribution. In this research, we focus on exponential distribution since it is one of the most frequently used lifetime distributions. Please see Lawless [3], Johnson et al. [4], Bain and Engelhardt [5], Lawless and Singhal [6], and Balakrishnan [7] for various applications of the exponential distribution, which is widely utilized for the purpose of measuring lifetime distribution. Balakrishnan and Joshi [8] proposed the product moments of order statistics from the doubly truncated exponential distribution. Balakrishnan and Sandhu [9] investigated the best linear unbiased and maximum likelihood estimators for the parameters of exponential distributions under general progressive type-II censored samples. Khan et al. [10] investigated the characterization of exponential distribution through the normalized spacing of generalized order statistics. In this paper, we examine k (≥ 2) independent populations $\pi_1, \ldots, \pi_k$ that conform to exponential distributions, represented by the notation $E(\theta_i, \sigma_i)$, $i = 1, \ldots, k$, where $\theta_1, \ldots, \theta_k$ are unknown location parameters and $\sigma_1, \ldots, \sigma_k$ are unknown and possibly unequal scale parameters. Under homoscedasticity (with k equal scale parameters), i.e., $\sigma_1 = \ldots = \sigma_k = \sigma$, Ng et al. [11] compared several location parameters with the control population by proposing multiple comparison procedures. Lam and Ng [12] developed two-stage multiple comparison methods with a control for location parameters when k scale parameters are unknown and there is a possibility of heteroscedasticity or inequality among them. The two-stage procedures are design-oriented and the sample size for the second stage is determined to attain the prespecified confidence length. Despite the potential benefits of the two-stage multiple comparison methods with a control for location parameters when dealing with unknown or unequal scale parameters, the increased sample size needed during the second stage may be large and therefore make the second stage

impractical in certain circumstances. Such a situation may arise due to budget constraints, limitations on available resources, or other factors that prevent the allocation of sufficient samples for the second stage of the experiment. To remedy this problem, Wu et al. [13] introduced an alternative solution to this issue by proposing data-analysis-based one-stage multiple comparison methods for comparing $k-1$ exponential location parameters with a control. Maurya [14] suggested a one-stage approach for comparing multiple exponential location parameters against multiple controls in the presence of heteroscedasticity. Maurya [15] proposed another one-stage multiple comparison procedure with a control under heteroscedasticity. In life testing experiments, it is crucial to compare not only k location parameters but also the mean lifetimes of k treatments, which hold significant importance. Therefore, Wu [16] proposed multiple comparison procedures to compare the mean lifetimes of k treatments with the control. Under double censoring, Wu [17] devised procedures for multiple comparisons with a control in terms of mean lifetimes under double censoring. In the context of mean lifetimes, Wu [18] introduced procedures for multiple comparisons involving multiple controls.

When the control population is not specified or not available, it is very important to compare the mean lifetimes of k treatments with their average in order to evaluate the ranking of k treatments. To compare k exponential location parameters with the average under heteroscedasticity, Wu [19] presented one-stage multiple comparison procedures.

The objective of our research paper is to present one-stage procedures for multiple comparisons, wherein we compare the mean lifetimes of k treatments with their average. To the best of the author's knowledge, no previous research on this particular topic has been conducted. During dose–response experiments, the mean effective duration of k drugs can be compared to the average. In reliability studies, the mean lifetimes of products manufactured by k assembly lines or production processes can be compared to the average. Section 2 of our paper outlines the procedures we propose for comparing k mean lifetimes with their average, utilizing Lam's [12,20] technique. We have included a table of critical values for the convenience of users who may wish to apply the proposed procedures in practice. In Section 3, we employ the survival data of patients with inoperable lung cancer as an example to demonstrate the proposed procedures for multiple comparisons with the average, focusing on exponential mean lifetimes under heteroscedasticity. Finally, the final section of our paper is dedicated to summarizing the conclusions we have drawn throughout our research.

2. One-Stage Multiple Comparisons with the Average for Exponential Mean Lifetimes of k Treatments

Refer to the i_{th} population π_i following the exponential distribution denoted by $E(\theta_i, \sigma_i)$, the mean lifetime is regarded as $\mu_i = \theta_i + \sigma_i$ for the i_{th} population, $i = 1, \ldots, k$. Take a one-stage random sample $X_{i1}, \ldots, X_{im}$ of size $m\ (\geq 2)$ from the i_{th} population. Let $Y_i = \min(X_{i1}, \ldots, X_{im})$ be the smallest order statistic and $S_i = \sum_{j=1}^{m} (X_{ij} - Y_i)/(m-1)$. Then Y_i and S_i are the uniformly minimum-variance unbiased estimators (UMVUEs) of θ_i and σ_i, respectively. Furthermore, $Y_i + S_i$ is the UMVUE of the mean lifetime $\mu_i = \theta_i + \sigma_i$ for the i_{th} population, $i = 1, \ldots, k$.

We define $\overline{\mu} = \sum_{i=1}^{k} \mu_i/k$ as the average of k treatment mean lifetimes. We develop the one-sided and two-sided confidence intervals for the difference between the i_{th} mean lifetime with the average denoted by $\mu_i - \overline{\mu}$, $i = 1, \ldots, k$. It appears that $Y_i - \overline{Y} + S_i - \overline{S}$ is the UMVUE of $\mu_i - \overline{\mu}$, $i = 1, \ldots, k$, where $\overline{Y} = \sum_{i=1}^{k} Y_i/k$ and $\overline{S} = \sum_{i=1}^{k} S_i/k$. The pivotal quantities for building our confidence intervals are given by $G_i = \frac{-mS_i/\sigma_i + m - m(Y_i - \theta_i)/\sigma_i}{S_i/\sigma_i}$, $i = 1, \ldots, k$. Making use of the UMVUE of $\mu_i - \overline{\mu}$ and these pivotal quantities, the one-sided and two-sided confidence intervals for $\mu_i - \overline{\mu}$, $i = 1, \ldots, k$ are proposed in the following theorem:

Theorem 1. *For a given confidence coefficient* $0 < P^* < 1$, *letting* $c_i^* = \max\left(\frac{S_i}{m}, \sum_{l\neq i}^{k} \frac{S_l}{m(k-1)}\right)$, *we have*

(a) $\mathrm{P}(\mu_i - \overline{\mu} \leq Y_i - \overline{Y} + S_i - \overline{S} + c_i^* s_U^*,\ i = 1, \ldots, k) \geq P^*$, *where* s_U^* *is the* $100P_{th}$ *percentile of the distribution of* $\max(-\widetilde{W}_i, G_i, G_i - \widetilde{W}_i, i = 1, \ldots, k)$ *multiplied by* $(k-1)/k$, *with* $\widetilde{W}_i = \min_{l\neq i} G_l$.
Thus, the upper confidence bound for $\mu_i - \overline{\mu}$ with confidence coefficient P^* is $(Y_i - \overline{Y} + S_i - \overline{S} + c_i^* s_U^*), i = 1, \ldots, k$.

(b) $\mathrm{P}(\mu_i - \overline{\mu} \geq Y_i - \overline{Y} + S_i - \overline{S} - c_i^* s_L^*,\ i = 1, \ldots, k) \geq P^*$, *where* s_L^* *is the* $100P_{th}$ *percentile of the distribution of* $\max(W_i, -G_i, W_i - G_i, i = 1, \ldots, k)$ *multiplied by* $(k-1)/k$, *with* $W_i = \max_{l\neq i} G_l$.
Thus, the lower confidence bound for $\mu_i - \overline{\mu}$ *with confidence coefficient* P^* *is* $(Y_i - \overline{Y} + S_i - \overline{S} - c_i^* s_L^*), i = 1, \ldots, k$.

(c) $\mathrm{P}(Y_i - \overline{Y} + S_i - \overline{S} - c_i^* s_t^* \leq \mu_i - \overline{\mu} \leq Y_i - \overline{Y} + S_i - \overline{S} + c_i^* s_t^*,\ i = 1, \ldots, k) \geq P^*$ *where* s_t^* *is the* $100P_{th}$ *percentile of the distribution of* $\max(|G_i|, W_i, W_i - G_i,\ -\widetilde{W}_i, G_i - \widetilde{W}_i,$ $i = 1, \ldots, k)$ *multiplied by* $(k-1)/k$ *with* $\widetilde{W}_i = \min_{l\neq i} G_l$ and $W_i = \max_{l\neq i} G_l$.
Thus, $(Y_i - \overline{Y} + S_i - \overline{S} \pm c_i^* s_t^*)$ *is the two-sided simultaneous confidence interval for* $\mu_i - \overline{\mu}$ *with confidence coefficient* $P^*, i = 1, \ldots, k$.

In order to prove Theorem 1, it is necessary to utilize the Lemma 1 provided by Lam [12,20].

Lemma 1. *Suppose that* X *and* Y *are two random variables, and that* a *and* b *are two positive constants, then we have* $[aX \geq bY - d\max(a, b)] \supseteq [X \geq -d,\ Y \leq d \text{ and } X \geq Y - d]$.

Proof of Theorem 1 : For (a), we have

$$
\begin{aligned}
&\mathrm{P}\left(\mu_i - \overline{\mu} \leq Y_i - \overline{Y} + S_i - \overline{S} + c_i^* s_U^*,\ i = 1, \ldots, k\right)\\
&= \mathrm{P}\left(\theta_i + \sigma_i - \overline{\theta} - \overline{\sigma} \leq Y_i + S_i - \overline{Y} - \overline{S} + c_i^* s_U^*,\ i = 1, \ldots, k\right)\\
&= \mathrm{P}\left(\tfrac{k-1}{k}(-S_i + \sigma_i + \theta_i - Y_i) \leq \sum_{l\neq i}^{k} \tfrac{(-S_l + \sigma_l + \theta_l - Y_l)}{k} + c_i^* s_U^*,\ i = 1, \ldots, k\right)\\
&= \mathrm{P}\left(\tfrac{S_i}{m}\tfrac{\sigma_i}{S_i}\tfrac{m(-S_i + \sigma_i + \theta_i - Y_i)}{\sigma_i} \leq \sum_{l\neq i}^{k} \tfrac{S_l}{m}\tfrac{\sigma_l}{S_l}\tfrac{m(-S_l + \sigma_l + \theta_l - Y_l)}{(k-1)\sigma_l} + \tfrac{k}{k-1} c_i^* s_U^*,\ i = 1, \ldots, k\right)\\
&= \mathrm{P}\left(\tfrac{S_i}{m} G_i \leq \sum_{l\neq i}^{k} \tfrac{S_l}{m}\tfrac{G_l}{(k-1)} + \tfrac{k}{k-1} c_i^* s_U^*,\ i = 1, \ldots, k\right)\\
&= \mathrm{P}\left(\sum_{l\neq i}^{k} \tfrac{S_l}{m}\tfrac{G_l}{(k-1)} \geq \tfrac{S_i}{m} G_i - \tfrac{k}{k-1} c_i^* s_U^*,\ i = 1, \ldots, k\right)\\
&\geq \mathrm{P}\left(\sum_{l\neq i}^{k} \tfrac{S_l}{m(k-1)} \widetilde{W}_i \geq \tfrac{S_i}{m} G_i - \tfrac{k}{k-1} \max\left(\tfrac{S_i}{m}, \sum_{l\neq i}^{k} \tfrac{S_l}{m(k-1)}\right) s_U^*,\ i = 1, \ldots, k\right)
\end{aligned}
$$

where $\widetilde{W}_i = \min_{l\neq i} G_l$

$$
\begin{aligned}
&= \mathrm{E}_{S_1, \ldots, S_k} \mathrm{P}\left(\sum_{l\neq i}^{k} \tfrac{S_l}{m(k-1)} \widetilde{W}_i \geq \tfrac{S_i}{m} G_i - \tfrac{k}{k-1} \max\left(\tfrac{S_i}{m}, \sum_{l\neq i}^{k} \tfrac{S_l}{m(k-1)}\right) s_U^*,\ i = 1, \ldots, k\right)\\
&\geq \mathrm{P}\left(\widetilde{W}_i \geq -\tfrac{k}{k-1} s_U^*,\ G_i \leq \tfrac{k}{k-1} s_U^*,\ \widetilde{W}_i - G_i \geq -\tfrac{k}{k-1} s_U^*,\ i = 1, \ldots, k\right) \text{(using Lemma 1)}\\
&= \mathrm{P}\left(-\widetilde{W}_i \leq \tfrac{k}{k-1} s_U^*,\ G_i \leq \tfrac{k}{k-1} s_U^*,\ G_i - \widetilde{W}_i \leq \tfrac{k}{k-1} s_U^*,\ i = 1, \ldots, k\right)\\
&= \mathrm{P}\left(\max(-\widetilde{W}_i, G_i, G_i - \widetilde{W}_i,\ i = 1, \ldots, k) \leq \tfrac{k}{k-1} s_U^*\right) = P^*
\end{aligned}
$$

Solving the above equation, we see that $ks_U^*/(k-1)$ is the $100P_{th}$ percentile of the distribution of $\max(-\widetilde{W}_i, G_i, G_i - \widetilde{W}_i,\ i = 1, \ldots, k)$. Thus, s_U^* is the $100P_{th}$ percentile of the

distribution of $\max(-\widetilde{W}_i, G_i, G_i - \widetilde{W}_i,\ i = 1, \ldots, k)$ multiplied by $(k-1)/k$ and the proof is thus complete.
For (b), we have

$$\begin{aligned}
&\mathrm{P}\big(\mu_i - \overline{\mu} \geq Y_i - \overline{Y} + S_i - \overline{S} - c_i^* s_L^*,\ i = 1, \ldots, k\big) \\
&= \mathrm{P}\big(\theta_i + \sigma_i - \overline{\theta} - \overline{\sigma} \geq Y_i + S_i - \overline{Y} - \overline{S} - c_i^* s_L^*,\ i = 1, \ldots, k\big) \\
&= \mathrm{P}\left(\tfrac{k-1}{k}(-S_i + \sigma_i + \theta_i - Y_i) \geq \sum_{l \neq i}^{k} \tfrac{(-S_l + \sigma_l + \theta_l - Y_l)}{k} - c_i^* s_L^*,\ i = 1, \ldots, k\right) \\
&= \mathrm{P}\left(\tfrac{S_i}{m}\tfrac{\sigma_i}{S_i}\tfrac{m(-S_i + \sigma_i + \theta_i - Y_i)}{\sigma_i} \geq \sum_{l \neq i}^{k} \tfrac{S_l}{m}\tfrac{\sigma_l}{S_l}\tfrac{m(-S_l + \sigma_l + \theta_l - Y_l)}{(k-1)\sigma_l} - \tfrac{k}{k-1} c_i^* s_L^*,\ i = 1, \ldots, k\right) \\
&= \mathrm{P}\left(\tfrac{S_i}{m} G_i \geq \sum_{l \neq i}^{k} \tfrac{S_l}{m}\tfrac{G_l}{(k-1)} - \tfrac{k}{k-1} c_i^* s_L^*,\ i = 1, \ldots, k\right) \\
&\geq \mathrm{P}\left(\tfrac{S_i}{m} G_i \geq \sum_{l \neq i}^{k} \tfrac{S_l}{m(k-1)} W_i - \tfrac{k}{k-1} \max\left(\tfrac{S_i}{m}, \sum_{l \neq i}^{k} \tfrac{S_l}{m(k-1)}\right) s_L^*,\ i = 1, \ldots, k\right)
\end{aligned}$$

where $W_i = \max_{l \neq i} G_l$

$$\begin{aligned}
&= \mathrm{E}_{S_1, \ldots, S_k} \mathrm{P}\left(\tfrac{S_i}{m} G_i \geq \sum_{l \neq i}^{k} \tfrac{S_l}{m(k-1)} W_i - \tfrac{k}{k-1} \max\left(\tfrac{S_i}{m}, \sum_{l \neq i}^{k} \tfrac{S_l}{m(k-1)}\right) s_L^*,\ i = 1, \ldots, k\right) \\
&\geq \mathrm{P}\Big(G_i \geq -\tfrac{k}{k-1} s_L^*,\ W_i \leq \tfrac{k}{k-1} s_L^*,\ G_i - W_i \geq -\tfrac{k}{k-1} s_L^*,\ i = 1, \ldots, k\Big) \text{(By Lemma 2)} \\
&= \mathrm{P}\Big(\max(W_i, -G_i, W_i - G_i, i = 1, \ldots, k) \leq \tfrac{k}{k-1} s_L^*\Big) = P^*
\end{aligned}$$

Solving the above equation, we see that $ks_L^*/(k-1)$ is the $100P_{th}$ percentile of the distribution of $\max(W_i, -G_i, W_i - G_i, i = 1, \ldots, k)$. Thus, s_L^* is the $100P_{th}$ percentile of the distribution of $\max(W_i, -G_i, W_i - G_i, i = 1, \ldots, k)$ multiplied by $(k-1)/k$ and the proof is thus complete.
For (c), combining (a) and (b), we have

$$\begin{aligned}
&\mathrm{P}\big(Y_i - \overline{Y} + S_i - \overline{S} - c_i^* s_t^* \leq \mu_i - \overline{\mu} \leq Y_i - \overline{Y} + S_i - \overline{S} + c_i^* s_t^*,\ i = 1, \ldots, k\big) \\
&\quad = \mathrm{E}_{S_1, \ldots, S_k} \mathrm{P}\Big(-\widetilde{W}_i \leq \tfrac{k}{k-1} s_t^*,\ G_i \leq \tfrac{k}{k-1} s_t^*,\ G_i - \widetilde{W}_i \leq \tfrac{k}{k-1} s_t^* \cap \\
&\qquad\qquad G_i \geq -\tfrac{k}{k-1} s_t^*,\ W_i \leq \tfrac{k}{k-1} s_t^*,\ G_i - W_i \geq -\tfrac{k}{k-1} s_t^*,\ i = 1, \ldots, k\Big) \\
&\geq \mathrm{P}\Big(\max(\big|G_i\big|, W_i, W_i - G_i, -\widetilde{W}_i, G_i - \widetilde{W}_i, i = 1, \ldots, k) \leq \tfrac{k}{k-1} s_t^*\Big) = P^*
\end{aligned}$$

Solving the above equation, we see that $ks_t^*/(k-1)$ is the $100P_{th}$ percentile of the distribution of $\max(\big|G_i\big|, W_i, W_i - G_i, -\widetilde{W}_i, G_i - \widetilde{W}_i, i = 1, \ldots, k)$. Thus, s_t^* is the $100P_{th}$ percentile of the distribution of $\max(\big|G_i\big|, W_i, W_i - G_i, -\widetilde{W}_i, G_i - \widetilde{W}_i, i = 1, \ldots, k)$ multiplied by $(k-1)/k$ and the proof is thus complete. □

To determine the distribution of G_i, we require three distributional results from Rousas [21] as follows:
(D1) $2(m-1)S_i/\sigma_i = Q_i$, $i = 1, \ldots, k$ follows a chi-squared distribution with $2m-2$ df denoted by χ^2_{2m-2}.
(D2) $m(Y_i - \theta_i)/\sigma_i = E_i$, $i = 1, \ldots, k$ follows a standard exponential distribution denoted by $Exp(1)$.
(D3) E_i and Q_i are two independent variables.

The distribution function of G_i is presented in the following theorem:

Theorem 2. *The cumulative distribution function (cdf) of* G_i *is* $F_{G_i}(x) = e^{-m}\left(\frac{\nu}{\nu - 2(x+m)}\right)^{m-1}$ *and the probability density function (pdf) of* G_i *is* $f_{G_i}(x) = \frac{2(m-1)}{\nu}\left(\frac{e(\nu - 2(x+m))}{\nu}\right)^{-m}$.

Proof of Theorem 2: Utilizing the above distributional results D1–D3, we have

$$G_i = \frac{-mS_i/\sigma_i + m - m(Y_i-\theta_i)/\sigma_i}{S_i/\sigma_i} = \frac{-m(2m-2)S_i/\sigma_i + (2m-2)(m-m(Y_i-\theta_i)/\sigma_i)}{2(m-1)S_i/\sigma_i}$$
$$= -m + \frac{\nu(m-E_i)}{Q_i},$$

where $Q_i \sim \chi^2_{2m-2}$, $E_i \sim Exp(1)$ and $\nu = 2m-2$.

We can find the cumulative distribution function of G_i as

$$\begin{aligned}
&F_{G_i}(x) = P(G_i \le x) = P(-m + \tfrac{\nu(m-E_i)}{Q_i} \le x) = EP(-m + \tfrac{\nu(m-E_i)}{Q_i} \le x | Q_i = y)\\
&= EP(-(x+m)y/\nu + m \le E_i | Q_i = y)\\
&= E(\exp((x+m)y/\nu - m) | Q_i = y)\\
&= e^{-m}\int_0^\infty \frac{\exp(-\left(\frac{\nu/2-(x+m)}{\nu}\right)y)y^{m-2}}{\Gamma(m-1)2^{m-1}}dy\\
&= e^{-m}\int_0^\infty \frac{\exp(-\left(\frac{\nu-2(x+m)}{2\nu}\right)y)y^{m-2}}{\Gamma(m-1)2^{m-1}}dy\\
&= e^{-m}\left(\frac{2\nu}{\nu-2(x+m)}\right)^{m-1}/2^{m-1}\\
&= e^{-m}\left(\frac{\nu}{\nu-2(x+m)}\right)^{m-1}
\end{aligned}$$

Taking the derivative of $F_{G_i}(x)$, we can find the probability density function (pdf) of G_i given by $f_{G_i}(x) = \frac{2(m-1)}{\nu}\left(\frac{e(\nu-2(x+m))}{\nu}\right)^{-m}$. The proof is thus complete. □

Using the probability integral transformation method, set $F_{G_i}(x) = e^{-m}\left(\frac{\nu-2(x+m)}{\nu}\right)^{-m+1} = U_i$, where $U_i \sim U(0,1)$. Solving this equation for x, we have $G_i = \left(\nu - 2m - \nu(e^m U_i)^{\frac{1}{1-m}}\right)/2$ coming from the distribution of G_i. The random variable G_i can be generated using the equation above.

In order to find the critical values s^*_U, s^*_L and s^*_t, we need to find the distributions of $\max(-\widetilde{W}_i, G_i, G_i - \widetilde{W}_i,\ i = 1,\ldots,k)$, $\max(W_i, -G_i, W_i - G_i, i = 1,\ldots,k)$ and $\max(|G_i|, W_i, W_i - G_i, -\widetilde{W}_i, G_i - \widetilde{W}_i, i = 1,\ldots,k)$. Since their distributions are very difficult to find, their empirical percentiles are found using the Monte Carlo method. The algorithm used to obtain the critical values s^*_U, s^*_L and s^*_t with confidence coefficient P^* is developed as follows:

Step 1: We need to generate k independent random variables $U_i \sim uniform(0,1)$, $i = 1,\ldots,k$ and then obtain k independent random variables $G_i = \left(\nu - 2m - \nu(e^m U_i)^{\frac{1}{1-m}}\right)/2$, $i = 1,\ldots,k$.

Step 2: From Theorem 1, the critical values s^*_U, s^*_L and s^*_t can be obtained by finding the $100P_{th}$ empirical percentiles of $\max(-\widetilde{W}_i, G_i, G_i - \widetilde{W}_i,\ i = 1,\ldots,k)$, $\max(W_i, -G_i, W_i - G_i, i = 1,\ldots,k)$ and $\max(\left|G_i\right|, W_i, W_i - G_i, -\widetilde{W}_i, G_i - \widetilde{W}_i,\ i = 1,\ldots,k)$, respectively, where $\widetilde{W}_i = \min_{l\neq i} G_l$ and $W_i = \max_{l\neq i} G_l$.

The critical values s^*_U, s^*_L and s^*_t for k = 3, 4, 5, 6, 7, 8, 9, m = 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20, 25, 30 and P^* = 0.75, 0.80, and 0.85 are listed in Table A1. The critical values for P^*=0.875, 0.90, and 0.925 are listed in Table A2. The critical values for P^* = 0.95 and 0.975 are listed in Table A3. The software we use to find the critical values is Fortran 90 and the programming manual refers to Mourik [22]. In Tables A1–A3, it can be seen that the approximate critical values s^*_U, s^*_L, and s^*_t are increasing while P^* is increasing for any given k and m or while k is increasing for any given P^* and m. Let L_1 be the length of the two-sided confidence intervals for $\mu_i - \overline{\mu}$, and we have the average length $L_1 = 2\bar{c}s^*_t$, where $\bar{c} = \sum_{i=1}^{k} c^*_t/k$. From the equation of $L_1 = 2\bar{c}s^*_t$, it is evident that, as P^* increases, the value of s^*_t increases and then the confidence length of L_1 increases for any given k and m. Furthermore, we can also see that, as the number of populations k increases, the value of s^*_t increases, and then the confidence length of L_1 increases for any given m and P^*.

3. A Biometrical Example

Referring to Maurya et al. [14], data comprising survival days of patients with four categories of inoperable lung cancer are utilized to illustrate our proposed multiple comparison procedures with the average, as proposed in Theorem 1. The four histological categories of tumor are squamous, small, adeno, and large. Table 1 presents the data on survival days of nine patients for each type of lung cancer.

Table 1. The survival times of four categories of lung cancer.

	Category	m	Survival Times								
1	Squamous	9	72	10	81	110	100	42	8	25	11
2	Small	9	30	13	23	16	21	18	20	27	31
3	Adeno	9	8	92	35	117	132	12	162	3	95
4	Large	9	177	162	553	200	156	182	143	105	103

According to Maurya et al. [15], the data in the four categories are obtained from exponential distributions with two parameters. To compare the mean survival days for the i_{th} category of lung cancer with the average survival days, the required statistics and critical values of s_i^* for P^* = 0.90, 0.95, and 0.975 are summarized in Table 2.

Table 2. The required statistics and critical values.

Statistics	Category 1	Category 2	Category 3	Category 4
Y_i	8	13	3	103
S_i	48.375	10.250	78.265	106.750
c_i^*	7.232	8.644	8.696	11.861
$Y_i - \overline{Y} + S_i - \overline{S}$	−36.285	−69.410	−11.395	117.090
P^*	$s_U^* = s_L^* = s_t^*$			
0.900	9.77			
0.950	11.78			
0.975	14.03			

Table 3 displays the upper confidence bounds and the lower confidence bounds for $\mu_i - \overline{\mu}$, $i = 1, \ldots, 4$ under confidence coefficients of 0.90, 0.95, and 0.975, which are obtained by utilizing (a) and (b) of Theorem 1. For all confidence bounds, only the lower bound for the mean survival time of the fourth category of inoperable lung cancer compared with the average is positive under confidence coefficient 0.90. Hence, we can infer that, for the confidence coefficient of 0.90, only this specific type of inoperable lung cancer has a mean survival time that surpasses the average. The results indicate that the mean survival time of this category of lung cancer is better than the average. The other three categories do not differ greatly from the average. Under confidence coefficients 0.95 and 0.975, all categories of inoperable lung cancer have mean survival times that do not differ greatly from the average.

Table 3. The 90%, 95% and 97.5% upper confidence bounds and lower confidence bounds for the mean survival times of four categories of lung cancer compared with the average.

Parameter	$(Y_i - \overline{Y} + S_i - \overline{S} + c_i^* s_U^*), (Y_i - \overline{Y} + S_i - \overline{S} - c_i^* s_L^*)$		
	90%	95%	97.5%
1. $\mu_1 - \overline{\mu}$	(34.372), (−106.942)	(48.908), (−121.478)	(65.180), (−137.750)
2. $\mu_2 - \overline{\mu}$	(15.043), (−153.863)	(32.417), (−171.237)	(51.87), (−190.686)
3. $\mu_3 - \overline{\mu}$	(73.566), (−96.356)	(91.045), (−113.835)	(110.61), (−133.40)
4. $\mu_4 - \overline{\mu}$	(232.973), (1.207)	(256.81), (−22.634)	(283.50), (−49.321)

Table 4 displays the two-sided simultaneous confidence intervals for $\mu_i - \overline{\mu}$, $i = 1, \ldots, 4$ under confidence coefficients of 0.90, 0.95, and 0.975, which are obtained by utilizing (c) of Theorem 1. For confidence coefficient 0.90, we can infer that only the fourth category of lung cancer has a longer mean survival time than the average since both limits of the related simultaneous two-sided confidence intervals are positive. For confidence coefficients 0.95 and 0.975, no categories have a mean survival time significantly different from the average.

Table 4. The 90%, 95% and 97.5% two-sided confidence intervals for the mean survival times of four categories of lung cancer compared with the average.

Parameter	$(Y_i - \overline{Y} + S_i - \overline{S} - c_i^* s_t^*, Y_i - \overline{Y} + S_i - \overline{S} + c_i^* s_t^*)$		
	90%	**95%**	**97.5%**
1. $\mu_1 - \overline{\mu}$	(−106.942, 34.372)	(−121.478, 48.908)	(−137.750, 65.180)
2. $\mu_2 - \overline{\mu}$	(−153.863, 15.043)	(−171.237, 32.417)	(−190.686, 51.87)
3. $\mu_3 - \overline{\mu}$	(−96.356, 73.566)	(−113.835, 91.045)	(−133.40, 110.61)
4. $\mu_4 - \overline{\mu}$	(1.207, 232.973)	(−22.634, 256.81)	(−49.321, 283.50)

4. Conclusions

In many practical applications, researchers would like to compare the k mean lifetimes with the average, especially when the control population is not identified or is unavailable. The shortcoming of two-stage multiple comparison procedures is that the additional sample required at the second stage can be so large that it is unavailable due to lack of budget for experimental work or other factors relevant to the experiment. In such cases, one-stage procedures should be considered as an alternative. We have proposed one-stage one-sided and two-sided multiple comparison procedures for comparing k treatments with the average. These procedures can be employed for multiple comparisons of the k treatment mean lifetimes with the average for exponential distribution models. The critical values are derived and are then tabulated using the Monte Carlo simulation method to allow practical and convenient use. In Tables A1–A3, we can see that, as the number of populations k increases, the value of the critical values increases; the confidence length then increases for any given m and P^*. As a final step, an example from biometrics is employed to illustrate our proposed multiple comparison procedures that utilize the mean lifetimes for comparisons with the average.

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Data Availability Statement: Data are available in a publicly accessible repository. The data presented in this study are openly available in Maurya et al. [14].

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Appendix A

Table A1. The critical values of $s_U^* = s_L^* = s_t^*$ for $P^* = 0.75$, 0.80, and 0.85.

		k						
P*	***m***	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0.75	2	8.92	13.82	18.69	23.74	28.85	33.40	38.26
	3	5.11	7.22	9.02	10.69	12.14	13.46	14.67
	4	4.68	6.42	7.86	9.07	10.18	11.10	11.95
	5	4.66	6.31	7.63	8.76	9.68	10.51	11.20
	6	4.80	6.41	7.72	8.78	9.69	10.45	11.20
	7	4.96	6.62	7.91	8.97	9.84	10.61	11.25
	8	5.15	6.83	8.11	9.22	10.05	10.85	11.47
	9	5.32	7.04	8.35	9.46	10.38	11.13	11.75
	10	5.50	7.30	8.64	9.72	10.66	11.42	12.08
	15	6.42	8.44	9.96	11.16	12.16	13.00	13.71
	20	7.23	9.46	11.15	12.49	13.58	14.46	15.30
	25	7.99	10.42	12.25	13.70	14.87	15.88	16.77
	30	8.66	11.30	13.30	14.83	16.12	17.19	18.11
0.80	2	11.46	17.68	24.24	30.26	36.81	43.17	49.02
	3	5.99	8.46	10.46	12.27	13.84	15.50	16.87
	4	5.34	7.24	8.80	10.13	11.36	12.41	13.26
	5	5.26	7.05	8.45	9.63	10.66	11.58	12.32
	6	5.36	7.09	8.53	9.58	10.51	11.34	12.06
	7	5.55	7.26	8.64	9.75	10.66	11.44	12.22
	8	5.68	7.48	8.85	9.96	10.86	11.67	12.37
	9	5.86	7.68	9.07	10.24	11.16	11.94	12.67
	10	6.06	7.92	9.35	10.49	11.42	12.23	12.90
	15	7.01	9.12	10.70	11.93	12.97	13.78	14.52
	20	7.84	10.22	11.94	13.31	14.41	15.32	16.10
	25	8.68	11.23	13.08	14.51	15.73	16.75	17.67
	30	9.41	12.11	14.14	15.69	16.99	18.06	19.04
0.85	2	15.71	24.43	32.57	41.18	50.24	59.25	66.50
	3	7.36	10.12	12.59	14.59	16.42	18.22	19.88
	4	6.26	8.47	10.15	11.64	12.93	14.04	15.15
	5	6.07	8.05	9.55	10.85	11.94	12.94	13.79
	6	6.11	8.07	9.48	10.73	11.73	12.58	13.40
	7	6.26	8.15	9.61	10.72	11.73	12.60	13.34
	8	6.42	8.29	9.76	10.96	11.89	12.78	13.50
	9	6.59	8.51	10.02	11.14	12.19	12.99	13.67
	10	6.78	8.78	10.23	11.49	12.41	13.21	14.01
	15	7.76	9.95	11.58	12.86	13.94	14.84	15.57
	20	8.66	11.05	12.85	14.26	15.38	16.33	17.24
	25	9.53	12.12	14.12	15.57	16.83	17.83	18.72
	30	10.28	13.11	15.19	16.76	18.07	19.22	20.17

Table A2. The critical values of $s_U^* = s_L^* = s_t^*$ for $P^* = 0.875$, 0.90, and 0.925.

		k						
P*	***m***	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0.875	2	19.47	29.50	40.22	49.24	59.63	70.68	80.54
	3	8.34	11.39	13.95	16.26	18.38	20.52	21.96
	4	6.91	9.20	11.13	12.68	14.04	15.25	16.43
	5	6.63	8.71	10.26	11.63	12.80	13.78	14.67
	6	6.64	8.57	10.18	11.42	12.45	13.36	14.22
	7	6.70	8.70	10.21	11.43	12.48	13.31	14.15
	8	6.91	8.88	10.35	11.59	12.60	13.43	14.17
	9	7.04	8.99	10.59	11.80	12.76	13.69	14.39
	10	7.25	9.25	10.86	12.01	13.02	13.94	14.61
	15	8.18	10.47	12.14	13.46	14.55	15.45	16.23
	20	9.13	11.64	13.42	14.87	16.04	17.05	17.80
	25	10.05	12.67	14.68	16.23	17.49	18.48	19.35
	30	10.83	13.71	15.83	17.47	18.81	19.92	20.88
0.90	2	24.30	37.05	48.67	62.96	75.91	88.74	100.46
	3	9.43	12.91	15.90	18.55	20.81	23.05	24.94
	4	7.68	10.25	12.28	13.95	15.40	16.60	17.86
	5	7.30	9.52	11.25	12.66	13.94	14.92	15.96
	6	7.21	9.30	10.94	12.29	13.41	14.38	15.31
	7	7.27	9.38	10.94	12.24	13.30	14.23	15.13
	8	7.39	9.48	11.09	12.28	13.38	14.31	14.99
	9	7.61	9.77	11.33	12.50	13.62	14.40	15.19
	10	7.77	9.94	11.50	12.80	13.72	14.65	15.42
	15	8.81	11.12	12.81	14.19	15.25	16.24	16.98
	20	9.75	12.23	14.13	15.65	16.81	17.77	18.55
	25	10.64	13.43	15.36	16.96	18.23	19.31	20.22
	30	11.49	14.47	16.62	18.26	19.65	20.74	21.66
0.925	2	33.10	50.61	67.37	85.87	100.34	120.16	137.23
	3	11.40	15.11	18.74	21.76	24.39	26.96	29.38
	4	8.86	11.64	13.83	15.70	17.42	18.90	20.17
	5	8.24	10.61	12.47	14.07	15.37	16.55	17.52
	6	8.09	10.34	12.04	13.52	14.68	15.70	16.55
	7	8.08	10.33	11.97	13.36	14.45	15.43	16.23
	8	8.16	10.31	12.10	13.43	14.51	15.39	16.25
	9	8.34	10.59	12.25	13.59	14.61	15.50	16.31
	10	8.49	10.79	12.36	13.77	14.77	15.80	16.51
	15	9.48	11.93	13.66	15.13	16.24	17.16	18.00
	20	10.48	13.17	15.05	16.54	17.69	18.71	19.63
	25	11.43	14.28	16.38	17.93	19.30	20.27	21.19
	30	12.30	15.31	17.49	19.21	20.62	21.75	22.69

Table A3. The critical values of $s_U^* = s_L^* = s_t^*$ for $P^* = 0.95$ and 0.975.

		k						
P*	***m***	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0.95	2	51.54	75.5	100.76	129.26	153.89	174.7	213.25
	3	14.57	19.25	23.27	27.37	30.45	33.50	36.48
	4	10.70	13.96	16.63	18.79	20.45	22.19	23.80
	5	9.69	12.31	14.51	16.30	17.49	18.78	20.13
	6	9.30	11.78	13.71	15.23	16.60	17.58	18.57
	7	9.24	11.64	13.43	14.96	16.23	17.25	18.19
	8	9.22	11.65	13.45	14.93	16.04	17.04	17.94
	9	9.38	11.78	13.47	14.98	16.11	17.06	17.96
	10	9.56	11.94	13.67	15.10	16.27	17.30	17.98
	15	10.56	13.04	14.95	16.32	17.57	18.56	19.40
	20	11.55	14.29	16.32	17.85	19.12	20.07	20.92
	25	12.50	15.49	17.65	19.18	20.58	21.75	22.66
	30	13.46	16.57	18.81	20.52	22.06	23.14	24.15
0.975	2	101.01	154.17	205.49	252.59	312.71	367.33	409.42
	3	21.18	28.17	34.28	39.45	43.91	47.80	53.06
	4	14.46	18.52	21.83	24.50	26.34	28.85	30.63
	5	12.47	15.65	18.48	20.47	22.19	23.54	24.74
	6	11.67	14.60	16.86	18.66	20.11	21.52	22.51
	7	11.23	14.19	16.16	17.82	19.12	20.44	21.52
	8	11.37	13.89	15.85	17.59	18.81	20.14	20.92
	9	11.26	14.03	15.91	17.54	18.73	20.07	20.87
	10	11.38	14.05	16.08	17.50	18.79	19.82	20.78
	15	12.22	15.05	17.06	18.78	19.86	20.81	21.89
	20	13.19	16.19	18.38	20.07	21.35	22.44	23.35
	25	14.37	17.51	19.80	21.47	22.98	23.98	25.05
	30	15.30	18.61	21.00	22.86	24.36	25.59	26.54

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