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Some New Sufficient Conditions on p-Valency for Certain Analytic Functions

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Abstract: In the present paper, we develop some implications leading to Carathéodory functions in the open disk and provide some new conditions for functions to be p-valent functions. This work also extends the findings of Nunokawa and others.

Keywords: multivalent functions; p-valent functions; Carathéodory functions

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1. Introduction and Definitions

The notion of multivalent functions is a natural extension of the injective. A holomorphic function f in an arbitrary domain Ω, a subset of the complex-plane $\mathbb{C}$, is p-valent if it assumes every value a maximum of p-times, which means that the number of roots of the equation similar to $f(z) = w$ never exceeds in comparison of p. By the geometrical point of discussion, this leads to the fact that all points in the w-plane $\mathbb{C}$ lie, at most, p-times the corresponding Riemann surface, where $w = f(z)$ maps the domain Ω. If $p = 1$, then f is univalent in Ω. The p-valent mappings plays a vital role in the literature of the complex multivalent functions.

Suppose that m is the number of roots $f(z) = w$ in the set Ω and let p be a positive number. The function f is said to be p-valent in the mean of circles in the domain Ω, if for the number $\rho > 0$, we can write

$$\int_0^{2\pi} m\rho e^{i\phi} d\phi < 2\pi p. \tag{1}$$

From the geometric point of view, the inequality shows that the measure of the circle on the Riemann surface where f maps Ω, along with projecting $|w| = \rho$, never exceeds p-times the measure of this circle. A function f is termed p-valent in the mean over areas in the domain Ω, if we have

$$\int_0^{\rho} \left(\int_0^{2\pi} m\varrho e^{i\phi} d\phi \right) \varrho d\varrho < \pi p \rho^2. \tag{2}$$

This integral inequality implies that the area of a small segment on the Riemann surface where f takes points from Ω as well as projecting them on the region defined by $|w| < R$ and this never exceeds p-times the area of the region $|w| < R$. Multivalent functions have been under investigation in view of their distortion, as bounds for the coefficient estimates along with various other aspects; see, for example, [1–5].

Any convergent power series is used to represent a holomorphic mapping. If f is holomorphic at a point z_0, it is analytic everywhere else in some neighbourhood of z_0. Furthermore, if f is entire, then this domain is the finite complex plane. It is a difficult task to deal with the complicated domains in the entire complex plane. As a result, the open unit disc is often used for simplification due to the Riemann mapping theorem. Let $\mathcal{H}$ denote the family of holomorphic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A} \subseteq \mathcal{H}$ consist of holomorphic functions f satisfying $f'(0) = 1$ and $f(0) = 0$. Further, assume that $\mathcal{S} \subseteq \mathcal{A}$ consist of univalent functions. The analytic description of holomorphic mappings is coupled with the functions that map $\mathbb{D}$ to the right half-plane. Let $\mathcal{P}$ represent the family of functions q that is holomorphic in $\mathbb{D}$ with $q(0) = 1$ and $\Re q(\mathbb{D}) > 0$. The function $q \in \mathcal{P}$ is called Carathéodory function. It is known that the class $\mathcal{P}$ is compact and normal. In geometric function theory, the Carathéodory function is well-studied and has a lot of applications (see, for example, [6–9]).

Some known subfamilies of $\mathcal{S}$ are the families $\mathcal{S}^*$ and $\mathcal{K}$ of starlike and convex mappings, respectively; for detail and further investigations, see [10–15]. These families are related to the change in argument of the radius vector and tangent vector of the image of $re^{i\varphi}$ as non-decreasing functions of the angle φ, respectively.

Let $\mathcal{A}_p(n)$ denote the class of analytic functions f in the form

$$f(z) = z^p + \sum_{s=p+n}^{\infty} a_s z^s, \quad z \in \mathbb{D}. \tag{3}$$

In particular, $\mathcal{A}_p(1) = \mathcal{A}_p$, $\mathcal{A}_1(n) = A(n)$ and $\mathcal{A}_1(1) = \mathcal{A}$. A function $f \in \mathcal{A}_p$ is called p-valent in $\mathbb{D}$ if f for $\omega \in \mathbb{C}$, the equation $f(z) = \omega$ has, at most, p roots in $\mathbb{D}$ and there exists a $\omega_0 \in \mathbb{C}$ such that $f(z) = \omega_0$ has exactly p roots in $\mathbb{D}$.

A function $f \in \mathcal{A}_p$ is said to be p-valent starlike if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{D}. \tag{4}$$

It is known that the p-valent starlike function in $\mathcal{A}(p)$ is p-valent. For some investigations on properties of p-valent functions, we refer to [16–19].

It was proved that [20,21] if $f \in \mathcal{A}$ with $f' \in \mathcal{P}$; then, the function f is univalent in $\mathbb{D}$. Ozaki [22] further extended the above assertion. They conclude that if f is holomorphic in a convex domain $\Delta \subset \mathbb{C}$ and

$$\frac{e^{i\gamma} f^{(p)}(z)}{p!} \in \mathcal{P}, \quad z \in \Delta, \tag{5}$$

for some real γ, then f is, at most, p-valent in Δ. Thus, if $f \in \mathcal{A}_p$ with the condition

$$\Re\left\{f^{(p)}(z)\right\} > 0, \quad (z \in \mathbb{D}), \tag{6}$$

then we see that f is, at most, p-valent in $\mathbb{D}$.

Recently, Nunokawa et al. [23–26] found some interesting sufficient conditions for f to be a p-valent function, which improved Ozaki's condition. Motivated from these works, we aim to develop some new sufficient criteria for Carathéodory functions and obtain certain new conditions for functions to be p-valent.

The following lemmas will be required for our results.

Lemma 1 (See [27])**.** *Let $q(z)$ be holomorphic in $\mathbb{D}$ with $q(z) \neq 0$ and $q(0) = 1$. Suppose also that there is a point $z_0 \in \mathbb{D}$ such that $|\arg q(z)| < \frac{\pi}{2}\alpha$ for $|z| < |z_0|$ and $|\arg q(z_0)| = \frac{\pi}{2}\alpha$ for some $\alpha > 0$. Then,*

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik\alpha, \tag{7}$$

where $k \geq \frac{a+a^{-1}}{2}$ *when* $\arg q(z_0) = \frac{\pi}{2}\alpha$ *and* $k \leq -\frac{a+a^{-1}}{2}$ *when* $\arg q(z_0) = -\frac{\pi}{2}\alpha$, *with*

$$[q(z_0)]^{\frac{1}{\alpha}} = \pm ia, \quad a > 0. \tag{8}$$

Lemma 2 (See [28]). *Let* $f \in \mathcal{A}(p)$. *If there exists a* $(p-s+1)$*-valent starlike function* g *in the form of*

$$g(z) = z^{p-s+1} + \sum_{m=p-s+2}^{\infty} b_m z^m \tag{9}$$

such that

$$\Re\left\{\frac{zf^{(s)}(z)}{g(z)}\right\} > 0, \quad z \in \mathbb{D}, \tag{10}$$

then f *is* p*-valent in* $\mathbb{D}$.

2. Main Results

Theorem 1. *Let* q *be a holomorphic function in* $\mathbb{D}$ *with* $q(z) \neq 0$ *and* $q(0) = 1$. *Suppose also that*

$$\left|\frac{1}{n}\arg\left\{[q(z)]^n + n[q(z)]^{n-1}zq'(z) - \beta[q(z)]^{n-1}\right\}\right| < \frac{\pi}{2} + \frac{1}{n}\arctan\left(n\sqrt{\frac{n+2\beta}{n}}\right), \tag{11}$$

where $0 \leq \beta < 1$. *Then, we have*

$$|\arg\{q(z)\}| < \frac{\pi}{2}, \quad z \in \mathbb{D}, \tag{12}$$

or

$$\Re(q(z)) > 0, \quad z \in \mathbb{D}. \tag{13}$$

Proof. Suppose that we have a point z_0 with $|z_0| < 1$ in such a way that

$$|\arg\{q(z)\}| < \frac{\pi}{2}, \quad |z| < |z_0|, \tag{14}$$

and

$$|\arg\{q(z_0)\}| = \frac{\pi}{2}. \tag{15}$$

Then, by Lemma 1 with $\alpha = 1$, we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik. \tag{16}$$

For the case $\arg q(z_0) = \frac{\pi}{2}$, $q(z_0) = ia$ and $a > 0$, we have

$$\begin{aligned}
&\frac{1}{n}\arg\left\{[q(z_0)]^n + n[q(z_0)]^{n-1}z_0q'(z_0) - \beta[q(z_0)]^{n-1}\right\} \\
=&\frac{1}{n}\arg[q(z_0)]^n + \frac{1}{n}\arg\left\{1 + \frac{nz_0q'(z_0)}{q(z_0)} - \frac{\beta}{[q(z_0)]}\right\} \\
=&\frac{\pi}{2} + \frac{1}{n}\arg\left\{1 + nik - \frac{\beta}{ia}\right\} \\
=&\frac{\pi}{2} + \frac{1}{n}\arg\left\{1 + i\left(nk + \frac{\beta}{a}\right)\right\} \\
=&\frac{\pi}{2} + \frac{1}{n}\arg\left\{1 + i\frac{n}{2}\left(a + \frac{n+2\beta}{na}\right)\right\}.
\end{aligned}$$

Define

$$\vartheta(x) = \frac{n}{2}\left(x + \frac{n+2\beta}{nx}\right). \tag{17}$$

Then, this function ϑ assumes its minimum value for $x = \sqrt{\frac{n+2\beta}{n}}$. Therefore, in view of the above equality, we see that

$$\frac{1}{n}\arg\left\{[q(z_0)]^n + n[q(z_0)]^{n-1}z_0q'(z_0) - \beta[q(z_0)]^{n-1}\right\} \geq \frac{\pi}{2} + \frac{1}{n}\arctan\left(n\sqrt{\frac{n+2\beta}{n}}\right), \tag{18}$$

which contradicts the hypothesis (11). When $\arg q(z_0) = -\frac{\pi}{2}$, using the similar technique, we get that:

$$\frac{1}{n}\arg\left\{[q(z_0)]^n + n[q(z_0)]^{n-1}z_0q'(z_0) - \beta[q(z_0)]^{n-1}\right\} \geq -\frac{\pi}{2} - \frac{1}{n}\arctan\left(n\sqrt{\frac{n+2\beta}{n}}\right). \tag{19}$$

This above inequality also contradicts the hypothesis (11). The proof is thus completed. □

Corollary 1. *Let $p \geq 2$. If $f \in \mathcal{A}_p(n)$ satisfying that $f^{(p-1)} \neq 0$ in $\mathbb{D}$ and*

$$\left|\frac{1}{n}\arg\left\{\frac{f^{(p)}(z)}{p!} - \beta\left(\frac{f^{(p-1)}(z)}{p!z}\right)^{\frac{n-1}{n}}\right\}\right| < \frac{\pi}{2} + \frac{1}{n}\arctan\left(n\sqrt{\frac{n+2\beta}{n}}\right), \quad z \in \mathbb{D}, \tag{20}$$

where $0 \leq \beta < 1$, then f is p-valent in $\mathbb{D}$.

Proof. Assume that

$$[q(z)]^n = \frac{f^{(p-1)}(z)}{p!z}, \quad q(0) = 1. \tag{21}$$

Then on simplification, it follows that

$$\begin{aligned} &\left|\frac{1}{n}\arg\left\{[q(z)]^n + n[q(z)]^{n-1}zq'(z) - \beta[q(z_0)]^{n-1}\right\}\right| \\ =&\left|\frac{1}{n}\arg\left\{\frac{f^{(p)}(z)}{p!} - \beta\left(\frac{f^{(p-1)}(z)}{p!.z}\right)^{\frac{n-1}{n}}\right\}\right| \\ =&\frac{\pi}{2} + \frac{1}{n}\arctan\left(n\sqrt{\frac{n+2\beta}{n}}\right). \end{aligned}$$

From Theorem 1, we have

$$\Re\left\{\frac{f^{(p-1)}(z)}{z}\right\} > 0, \quad z \in \mathbb{D}. \tag{22}$$

This shows that the mapping f is p-valent in $\mathbb{D}$. □

Taking $n = 1$ and $\beta = 0$, we easily get the following result obtained by Nunokawa [29].

Corollary 2. *Let $p \geq 2$. If $f \in \mathcal{A}_p$ and*

$$\left|\arg\left\{f^{(p)}(z)\right\}\right| < \frac{3\pi}{4}, \quad z \in \mathbb{D}, \tag{23}$$

then f is p-valent in $\mathbb{D}$.

Theorem 2. *Let $q(z)$ be a holomorphic mapping in $\mathbb{D}$ with $q(0) = 1$ and $q(z) \neq 0$. Further, suppose that*

$$\left|\frac{1}{n}\arg\left\{[q(z)]^n + n[q(z)]^{n-1}\frac{zq'(z)}{q(z)} + \beta[q(z)]^{n-1}\right\}\right| < \frac{\pi}{2} - \frac{1}{n}\arctan\frac{\beta}{\sqrt{n(n+2)}}, \tag{24}$$

where $0 \leq \beta < \infty$. *Then*

$$|\arg q(z)| < \frac{\pi}{2}, \quad z \in \mathbb{D}. \tag{25}$$

Proof. We suppose that there is a point z_0 $(|z_0| < 1)$ such that

$$|\arg\{q(z)\}| < \frac{\pi}{2}, \quad |z| < |z_0|, \tag{26}$$

and

$$|\arg\{q(z_0)\}| = \frac{\pi}{2}. \tag{27}$$

Then, by using Lemma 1 with $\alpha = 1$, we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik, \tag{28}$$

For the case $\arg\{q(z_0)\} = \frac{\pi}{2}$ with $q(z_0) = ia$ and $a > 0$, we observe that

$$\begin{aligned}
&\frac{1}{n}\arg\left\{[q(z_0)]^n + n[q(z_0)]^{n-1}\frac{z_0 q'(z_0)}{q(z_0)} + \beta[q(z_0)]^{n-1}\right\} \\
=&\frac{1}{n}\arg[q(z_0)]^n + \frac{1}{n}\arg\left\{1 + \frac{nz_0 q'(z_0)}{q(z_0)}.\frac{1}{q(z_0)} + \frac{\beta}{q(z_0)}\right\} \\
=&\frac{\pi}{2} + \frac{1}{n}\arg\left\{1 + nk\frac{1}{a} - i\frac{\beta}{a}\right\} \\
=&\frac{\pi}{2} + \frac{1}{n}\arctan\left(\frac{-\beta}{a+nk}\right) \\
\geq&\frac{\pi}{2} - \frac{1}{n}\arctan\left(\frac{\beta}{a + \frac{n}{2}\left(a + \frac{1}{a}\right)}\right).
\end{aligned}$$

Let

$$\zeta(x) = x + \frac{n}{2}\left(x + \frac{1}{x}\right). \tag{29}$$

It is easy to note that ζ takes the minimum value for $x = \sqrt{\frac{n}{n+2}}$. Therefore, on some simple manipulation, the above equality leads to

$$\frac{1}{n}\arg\left\{[q(z_0)]^n + n[q(z_0)]^{n-1}\frac{z_0 q'(z_0)}{q(z_0)} + \beta[q(z_0)]^{n-1}\right\} \geq \frac{\pi}{2} - \frac{1}{n}\arctan\frac{\beta}{\sqrt{n(n+2)}},$$

which contradicts the hypothesis in (24). For the case $\arg q(z_0) = -\frac{\pi}{2}$, applying the same method as the above, we have

$$\begin{aligned}
&\frac{1}{n}\arg\left\{[q(z_0)]^n + n[q(z_0)]^{n-1}\frac{z_0 q'(z_0)}{q(z_0)} + \beta[q(z_0)]^{n-1}\right\} \\
\geq& -\left(\frac{\pi}{2} - \frac{1}{n}\arctan\frac{\beta}{\sqrt{n(n+2)}}\right),
\end{aligned}$$

which also contradicts the hypothesis as in (24). This completes the proof of Theorem 2. □

Theorem 3. *Let q be a holomorphic function in $\mathbb{D}$ with $q(0) = 1$ and $q(z) \neq 0$. Suppose that*

$$\Re\left\{\frac{1}{n}\sqrt{[q(z)]^n + n[q(z)]^{n-1}zq'(z)}\right\} > 0, \quad z \in \mathbb{D}. \tag{30}$$

Then we have

$$|\arg\{q(z)\}| < \frac{\pi}{2}\alpha_1, \quad z \in \mathbb{D}, \tag{31}$$

where α_1 is the positive zero or root of the equation

$$\alpha_1 + \frac{2}{n\pi}\arctan(n\alpha_1) = 2. \tag{32}$$

Proof. Assume that there is a point z_0 $(|z_0| < 1)$ such that

$$|\arg q(z)| < \frac{\pi}{2}\alpha_1, \quad (|z| < |z_0|) \tag{33}$$

and

$$|\arg q(z_0)| = \frac{\pi}{2}\alpha_1. \tag{34}$$

Then, by using Lemma 1 with $\alpha = \alpha_1$, we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = i\alpha_1 k, \tag{35}$$

For the case $\arg q(z_0) = \frac{\pi}{2}\alpha_1$, we have

$$\begin{aligned}
&\frac{1}{n}\arg\sqrt{[q(z_0)]^n + n[q(z_0)]^{n-1}z_0 q'(z_0)} \\
=&\frac{1}{2n}\left(\arg[q(z_0)]^n + \arg\left\{1 + \frac{z_0 q'(z_0)}{q(z_0)}\right\}\right) \\
=&\frac{1}{2n}\left(\frac{n\pi}{2}\alpha_1 + \arg\{1 + nik\alpha_1\}\right) \\
=&\frac{1}{2}\cdot\frac{\pi}{2}\left(\alpha_1 + \frac{2}{n\pi}\arctan(n\alpha_1)\right) \\
=&\frac{\pi}{2},
\end{aligned}$$

which implies that

$$\Re\frac{1}{n}\left\{\sqrt{[q(z_0)]^n + n[q(z_0)]^{n-1}z_0 q'(z_0)}\right\} \le 0, \tag{36}$$

and this contradicts the hypothesis as in (30). For $\arg q(z_0) = -\frac{\pi}{2}\alpha_1$, using the similar technique yields to

$$\frac{1}{n}\arg\sqrt{[q(z_0)]^n + n[q(z_0)]^{n-1}z_0 q'(z_0)} \le -\frac{\pi}{2}, \tag{37}$$

or

$$\Re\left\{\frac{1}{n}\sqrt{[q(z_0)]^n + n[q(z_0)]^{n-1}z_0 q'(z_0)}\right\} \le 0. \tag{38}$$

This also contradicts the hypothesis in (30) and, therefore, the assertion is concluded. □

Corollary 3. *Suppose that $p \ge 4$ and $f \in \mathcal{A}_p(n)$ satisfying that $f^{(k)}(z) \ne 0(k = p-1, p-2, p-3)$ in $\mathbb{D}$. If*

$$\left|\frac{1}{n}\arg\left\{f^{(p)}(z)\right\}\right| < \pi, \quad (z \in \mathbb{D}), \tag{39}$$

then the mapping f is p-valent in $\mathbb{D}$.

Proof. Assume that

$$[q_1(z)]^n = \frac{f^{(p-1)}(z)}{p!z}, \quad q_1(0) = 1. \tag{40}$$

Then a simple simplification leads to

$$[q_1(z)]^n + n[q_1(z)]^{n-1} z q_1'(z) = \frac{f^{(p)}(z)}{p!}. \tag{41}$$

In view of Theorem 3, we obtain that

$$\left| \frac{1}{n} \arg\left\{ \frac{f^{(p-1)}(z)}{z} \right\} \right| = \left| \frac{1}{n} \arg[q_1(z)]^n \right| < \frac{\pi}{2}\alpha_1, \quad z \in \mathbb{D}, \tag{42}$$

where α_1 is the positive zero or root of the above equation given by (32). Next, let us put

$$[q_2(z)]^n = \frac{2f^{(p-2)}(z)}{p!z^2}, \quad q_2(0) = 1. \tag{43}$$

Then a simple calculation leads to

$$2[q_2(z)]^n + n[q_2(z)]^{n-1} z q_2'(z) = \frac{2f^{(p-1)}(z)}{p!z}. \tag{44}$$

Let α_2 be a positive zero or root of the equation

$$\alpha + \frac{2}{n\pi} \arctan\left(\frac{n\alpha}{2}\right) = \alpha_1. \tag{45}$$

Suppose that there exists a point z_1 with $|z_1| < 1$ such that

$$|\arg q_2(z)| < \frac{\pi}{2}\alpha_2, \quad |z| < |z_1| \tag{46}$$

and $|\arg q_2(z_1)| = \frac{\pi}{2}\alpha_2$, then we write

$$\frac{z_1 q_2'(z_1)}{q_2(z_1)} = i\alpha_2 k. \tag{47}$$

For the choice of $\arg q_2(z_1) = \frac{\pi}{2}\alpha_2$, we have

$$\begin{aligned}
&\frac{1}{n} \arg\left\{ 2[q_2(z_1)]^n + n[q_2(z_1)]^{n-1} z_1 q_2'(z_1) \right\} \\
=&\frac{1}{n} \arg\left\{ \frac{f^{(p-1)}(z_1)}{z_1} \right\} \\
=&\frac{1}{n} \arg[q_2(z_1)]^n + \frac{1}{n} \arg\left\{ 2 + n\frac{z_1 q_2'(z_1)}{q_2(z_1)} \right\} \\
=&\frac{\pi}{2}\alpha_2 + \frac{1}{n} \arg\{2 + nik\alpha_2\} \\
=&\frac{\pi}{2}\alpha_2 + \frac{1}{n} \arctan(\frac{n\alpha_2}{2}) = \frac{\pi}{2}\alpha_1,
\end{aligned}$$

which contradicts the result in (42). For the assumption $\arg q_2(z_1) = -\frac{\pi}{2}\alpha_2$, we note that

$$\begin{aligned}
&\frac{1}{n} \arg\left\{ 2[q_2(z_1)]^n + n[q_2(z_1)]^{n-1} z_1 q_2'(z_1) \right\} \\
=&\frac{1}{n} \arg\left\{ \frac{2f^{(p-1)}(z_1)}{p!z_1} \right\} = \frac{1}{n} \arg\left\{ \frac{f^{(p-1)}(z_1)}{z_1} \right\} \leq -\frac{\pi}{2}\alpha_1.
\end{aligned}$$

This also contradicts (42). Hence, we have

$$\left|\frac{1}{n}\arg[q_2(z_1)]^n\right| = \left|\frac{1}{n}\arg\left\{\frac{f^{(p-2)}(z)}{z^2}\right\}\right| < \frac{\pi}{2}\alpha_2, \quad z \in \mathbb{D}, \tag{48}$$

where $\alpha_2 + \frac{2}{n\pi}\arctan\left(\frac{n\alpha_2}{2}\right) = \alpha_1$. Let

$$[q_3(z)]^n = \frac{6f^{(p-3)}(z)}{p!z^3}, \quad q_3(0) = 1. \tag{49}$$

Then we see that

$$3[q_3(z)]^n + n[q_3(z)]^{n-1}zq_3'(z) = \frac{6f^{(q-2)}(z)}{q!.z^2}. \tag{50}$$

Using the similar approach as adopted above, we note that

$$\begin{aligned}
&\left|\frac{1}{n}\arg\left\{3[q_3(z)]^n + n[q_3(z)]^{n-1}zq_3'(z)\right\}\right| \\
=&\left|\frac{1}{n}\arg[q_3(z)]^n + \frac{1}{n}\arg\left\{3 + n\frac{zq_3'(z)}{q_3(z)}\right\}\right| \\
=&\left|\frac{1}{n}\arg\left\{\frac{6f^{(p-2)}(z)}{p!z^2}\right\}\right| = \left|\frac{1}{n}\arg\left\{\frac{f^{(p-2)}(z)}{z^2}\right\}\right| < \frac{\pi}{2}\alpha_2.
\end{aligned}$$

This shows that

$$\left|\frac{1}{n}\arg\left\{\frac{zf^{(p-3)}(z)}{z^4}\right\}\right| = \left|\frac{1}{n}\arg\left\{\frac{zf^{(p-3)}(z)}{z^3}\right\}\right| < \frac{\pi}{2}\alpha_3 < \frac{\pi}{2}, \quad z \in \mathbb{D}, \tag{51}$$

or

$$\Re\left\{\frac{zf^{(p-3)}(z)}{z^4}\right\} > 0, \quad z \in \mathbb{D}. \tag{52}$$

Thus, we note that $g(z) = z^4$ is a four-valent starlike function in $\mathbb{D}$. Therefore, using the result in (52) and Lemma 2, we observe that f is p-valent in $\mathbb{D}$. This leads to the desired result in Corollary 3. $\square$

3. Conclusions

Analytic p-valent functions were intensively studied recently, as in [30–32]. In the present paper, we introduced several sufficient conditions for functions to be p-valent. Some simple criteria on p-valents are obtained. This generalizes some know results and may inspire more effective and concise univalent conditions in geometric function theory.

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References

1. Hayman, W.K. *Multivalent Functions*; Cambridge University Press: Cambrige, UK, 1958.
2. Leach, R.J. Coefficient estimates for certain multivalent functions. *Pac. J. Math.* **1978**, *74*, 133–142.
3. Khan, Q.; Arif, M.; Ahmad, B.; Tang, H. On analytic multivalent functions associated with lemniscate of Bernoulli. *AIMS Math.* **2020**, *5*, 2261–2271. [CrossRef]
4. Alb Lupaş, A. Applications of the q-Sălăgean differential operator involving multivalent functions. *Axioms* **2022**, *11*, 512. [CrossRef]
5. Lashin, A.M.Y.; El-Emam, F.Z. On certain classes of multivalent analytic functions defined with higher-order derivatives. *Mathematics* **2023**, *11*, 83. [CrossRef]
6. Mateljević, M.; Mutavdžić, N.; Örnek, B.N. Note on some classes of holomorphic functions related to Jack's and Schwarz's lemma. *Appl. Anal. Discret. Math.* **2022**, *16*, 111–131. [CrossRef]
7. Nunokawa, M.; Sokół, J. On the order of strong starlikeness and the radii of starlikeness for of some close-to-convex functions. *Anal. Math. Phys.* **2019**, *9* 2367–2378. [CrossRef]
8. Shi, L.; Arif, M.; Iqbal, J.; Ullah, K.; Ghufran, S.M. Sharp bounds of Hankel determinant on logarithmic coefficients for functions starlike with exponential function. *Fractal Fract.* **2022**, *6*, 645. [CrossRef]
9. Shi, L.; Srivastava, H.M.; Rafiq, A.; Arif, M.; Ihsan, M. Results on Hankel determinants for the inverse of certain analytic functions subordinated to the exponential function. *Mathematics* **2022**, *10*, 3429. [CrossRef]
10. Ronning, F. Uniformly convex functions and a corresponding class of starlike functions. *Proc. Am. Math. Soc.* **1993**, *118*, 189–196. [CrossRef]
11. Kanas, S.; Wiśniowska, A. Conic domains and starlike functions. *Rev. Roumaine Math. Pures Appl.* **2000**, *45*, 647–657.
12. Mateljević, M. Schwarz lemma and Kobayashi metrics for harmonic and holomorphic functions. *J. Math. Anal. Appl.* **2018**, *464*, 78–100. [CrossRef]
13. Khalfallah, A.; Mateljević, M.; Purtić, B. Schwarz–Pick Lemma for harmonic and Hyperbolic harmonic functions. *Results Math.* **2022**, *77*, 167. [CrossRef]
14. Mateljević, M.; Mutavdžić, N. The boundary Schwarz lemma for harmonic and pluriharmonic mappings and some generalizations. *Bull. Malays. Math. Sci. Soc.* **2022**, *45*, 3177–3195. [CrossRef]
15. Nunokawa, M. On the order of strongly starlikeness of strongly convex functions. *Proc. Jp. Acad. Ser. A* **1993**, *69*, 234–237. [CrossRef]
16. Arif, M.; Ayaz, M.; Iqbal, J.; Haq, W. Sufficient conditions for functions to be in a class of p-valent analytic functions. *J. Comput. Anal. Appl.* **2013**, *16*, 159–164.
17. Arif, M. Sufficiency criteria for a class of p-valent analytic functions of complex order. *Abstr. Appl. Anal.* **2013**, *2013*, 517296. [CrossRef]
18. Cho, N.E.; Aouf, M.K.; Srivastava, R. The principle of differential subordination and its application to analytic and p-valent functions defined by a generalized fractional differintegral operator. *Symmetry* **2019**, *11*, 1083. [CrossRef]
19. Hadid, S.B.; Ibrahim, R.W.; Momani, S. Multivalent functions and differential operator extended by the quantum calculus. *Fractal Fract.* **2022**, *6*, 354. [CrossRef]
20. Warschawski, S. On the higher derivatives at the boundary in conformal mappings. *Trans. Am. Math. Soc.* **1935**, *38*, 310–340. [CrossRef]
21. Noshiro, K. On the theory of schlicht functions. *J. Fac. Sci. Hokkaido Univ.* **1934**, *1*, 129–155. [CrossRef]
22. Ozaki, S. On the theory of multivalent functions. *Sci. Rep. Tokyo Bunrika Daigaku A* **1935**, *2*, 167–188.
23. Nunokawa, M.; Sokół, J. On the multivalency of certain analytic functions. *J. Ineq. Appl.* **2014**, *2014*, 357. [CrossRef]
24. Nunokawa, M.; Sokół, J. On some geometric properties of multivalent functions. *J. Ineq. Appl.* **2015**, *2015*, 300. [CrossRef]
25. Nunokawa, M.; Cho, N.E.; Kwon, O.S.; Sokół, J. An improvement of Ozaki's q-valent conditions. *Acta Math. Sin. Engl. Ser.* **2016**, *32*, 406–410. [CrossRef]
26. Nunokawa, M.; Sokół, J.; Trojnar-Spelina, L. On a sufficient condition for function to be p-valent close-to-convex. *Ramanujan J.* **2020**, *53*, 483–492. [CrossRef]
27. Nunokawa, M. On properties of non-Carathéodory functions. *Proc. Jpn. Acad. Ser. A* **1992**, *68*, 152–153. [CrossRef]
28. Nunokawa, M. On the theory of multivalent functions. *Tsukuba J. Math.* **1987**, *11*, 273–286. [CrossRef]
29. Nunokawa, M. A note on multivalent functions. *Tsukuba J. Math.* **1989**, *1*, 453–455. [CrossRef]
30. Khan, Q.; Dziok, J.; Raza, M.; Arif, M. Sufficient conditions for p-valent functions. *Math. Slovaca* **2021**, *71*, 1089–1102. [CrossRef]
31. Cotîrlă, L.I.; Juma, A.R.S. Properties of differential subordination and superordination for multivalent functions associated with the convolution operators. *Axioms* **2023**, *12*, 169. [CrossRef]
32. Liu, J.L.; Srivastava, R. Hadamard products of certain classes of p-valent starlike functions. *Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **2019**, *113*, 2001–2015. [CrossRef]