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Axiomatic Processes for Asymmetric Allocation Rules under Fuzzy Multicriteria Situations

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Abstract: The present paper is dedicated to investigating weighted allocation rules under fuzzy multicriteria situations. In real-world situations, participants may represent administrative departments of different natures in the management system; participants may be able to perform their duties in the economic model. In addition, participants may adopt relative operating levels to different situations and effectively engage multiple objectives under operational processes. Therefore, considering fuzzy behavior and multicriteria situations, it is reasonable to assign corresponding weights to participants and their relative behavior and to allocate efficiency according to weights in proportion to relative weights, even if it will lead to an asymmetrical situation. In existing studies on fuzzy allocation rules, weights are always given to "participants" or their "operating levels" and then the differences between participants and their operating levels are adjusted. Inspired by the above considerations, relative major results are as follows. (1) By simultaneously assigning weights to participants and their operating levels (strategies), this study seeks to use the supreme marginal variations among operating level vectors to define a new asymmetric allocation rule under fuzzy multicriteria situations. (2) This study further utilizes axiomatic results to illustrate the expedience for this weighted fuzzy asymmetric allocation rule. (3) Finally, an extended index is also proposed by replacing weights with the supreme marginal dedications.

Keywords: asymmetric allocation rule; fuzzy multicriteria situation; the supreme marginal dedication; axiomatic process

MSC: 91A12; 91A40; 91B06; 91B16

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1. Introduction

Under most interactive models, programs or systems (economic models, management programs, operational systems, etc.), attention is always paid to equilibrium or optimal states of relative allocation rules or processing concepts, usually called the rules. To point out the advantages of a rule, directly asserting how excellent it is does not necessarily lead to a majority of people approving the rule. Numerous mathematical results are often adopted to analyze and generate equilibrium or optimal states of these rules under *axiomatic processes*. Relative conception could be briefly described as follows. The axiomatic process is a mathematical notion that first adopts numerous mathematical results to model models, programs or systems and inevitable just, fair and well recognized properties relied upon theoretical bases. A corresponding rule is then analyzed, proposed and shown to be the unique rule that satisfies inevitable just, fair and well recognized properties. It is necessary that these properties are indispensable. Then, agreeing with these inevitable just, fair and well recognized properties is equivalent to agreeing with the rule. Axiomatic processes have been applied to various topics, such as game-theoretical analysis, operational research and structural optimization methodology. Related studies could be found in Bonnisseau and Iehlé [1], Lee et al. [2], Shapley [3] and so on.

Under traditional transferable-utility (TU) statuses, each participant is either entirely participated or completely outside of participation with other participants. By using the marginal dedications for the grand alliance, the *equal allocation of non-separable costs* (EANSC, Ransmeier [4]) is defined to be an allocation rule for distributing utility under traditional TU statuses. Later, Moulin [5] proposed the *complement-reduced status* and relative axiomatic processes to manifest that the EANSC presents a fair allocating rule.

However, participants might take different operating levels (strategies) to participate in many real-world situations. Under *fuzzy TU statuses*, each participant is permitted to manipulate with infinite operating levels. Thence, a fuzzy TU status could be treated as a generalization of a traditional TU status. The investigation of fuzzy TU statuses began with the research of Aubin [6,7] where the attitudes of a fuzzy TU status and the fuzzy core were analyzed. Many fuzzy rule concepts have been investigated extensively, e.g., Branzei et al. [8], Butnariu and Kroupa [9], Debnath and Mohiuddine [10], Gou et al. [11], Hwang [12], Masuya and Inuiguchi [13], Meng and Zhang [14], Tijs et al. [15] and so on. By determining overall values for a given participant on fuzzy TU statuses, Hwang [12], Liao et al. [16] and Wei et al. [17] provided different generalized rules and relative axiomatic processes for the core, the Shapley value and the EANSC, respectively.

Under different issues, from management sciences, environmental sciences and biomedical engineering, participants confront an increasing requirement to focus on multiple considerations effectively during operational procedures. Relative situations include analyzing allocation tradeoffs, choosing an optimal strategy or decision designs or other situations where one needs an efficient rule with tradeoffs among several goals. Under numerous cases, these real-world efficient situations might be constructed as mathematical multicriteria status. The rules of such situations require appropriate techniques to generate optimal outcomes that—unlike traditional conceptions—take several properties of the goals into account. Several pre-existing results considered fuzzy multicriteria status. For example, Atanassov et al. [18] provided some results on a multicriteria decision making approach in which intuitionistic fuzziness was applied. Based on interval type-2 fuzzy sets, Wu et al. [19] offered an integrated methodology to deal with the portfolio allocation problem based on multiple criteria group decision making.

In addition, participants might represent election divisions of distinct scales; participants might possess different haggling abilities. Lack of symmetry might be generated if different haggling abilities related to different participants are modeled. Based on the above interpretations, one would now desire that arbitrary utility could be distributed among the participants and its operating levels in proportion to *weights*. Weights turn up involuntary in the framework of utility allocating. For example, one might be dealing with an issue of utility allocating among investment projects. Thence, the weights could be associated to the profitability of the distinct options among all projects. Under the issue of allotting travel expenses among distinct areas arrived at, weights could be regarded as the amount of days visited at each one (cf. Shapley [20]). On the whole, weights might be allotted to the "participants" or the "levels" to discriminate the discrepancies among the participants or their operating levels, respectively. In the framework of multicriteria fuzzy statuses, Wei et al. [17] introduced two extensions of the EANSC by, respectively, assigning weights to the participants and their operating levels.

The incidents of symmetry exist in many natural situations, such as symmetry of shape, symmetry of functional property and symmetry of social status. In economic management, there is the so-called "equal pay for equal work". That is, in the economic management system, even for individuals engaged in different types of work, if they make symmetric contributions to the entire system, they should all be paid proportionally. Such relative symmetry seems reasonable. Nevertheless, such symmetry will vary from context to context. For example, doctors and accountants in a hospital have made symmetric contributions to their respective jobs and received corresponding job remuneration. However, surplus dividends will be allocated according to the resulting weighting relative to the importance. Since medical treatment is the primary source of income for hospitals, the relative weighting of doctors and accountants is naturally different, resulting in the asymmetry between the total reward plus bonuses and the relative contribution. However, such asymmetry is not unreasonable. After all, as mentioned above, the relative importance of doctors in hospitals is relatively high and the weighted ratio when allocating dividends should also be higher.

The above statements beget one motivation:

- Whether distinct allocating concepts might be generated via simultaneously assigning weights to the participants and their operating levels under multicriteria fuzzy situations.

The article is devoted to analyzing the motivation. The major consequences are as follows.

1. An asymmetric allocation rule different from previous ones, the *weighted fuzzy multicriteria index* (WFMI), is defined by simultaneously assigning weights to the participants and their operating levels in Section 2. This rule is a weighted generalization of the supreme EANSC (SEANSC) due to Wei et al. [17] under multicriteria fuzzy statuses.
2. To resolving the rationality of this weighted rule, a specific reduction and relative axiomatic processes are introduced to manifest that the WFMI is the only rule matching the properties of *multicriteria weighted standard* and *multicriteria stability* in Section 3.
3. Supreme marginal dedications instead of weights naturally in Section 4, the *interior fuzzy multicriteria index* (IFMI) and relative axiomatic processes are also generated under multicriteria fuzzy situations.

2. Preliminaries

Assume that UP is the universe of participants. For $s \in UP$ and $\theta_s \in (0,1]$, we define $\Theta_s = [0,\theta_s]$ to be the operating level (strategy) collection of participant s and $\Theta_s^+ = (0,\theta_s]$, where 0 represents no operation. Let $\Theta^P = \prod_{s\in P}\Theta_s$ be the product set of the operating level sets for all participants of P. For every $Q \subseteq P$, a participant-coalition Q corresponds in a standard mode to the fuzzy coalition $e^Q \in \Theta^P$, which is the vector satisfying $e^Q_s = 1$ if $s \in Q$ and $e^Q_s = 0$ if $s \in P \setminus Q$. Denote 0_P the zero vector of $\mathbb{R}^P$. For $n \in \mathbb{N}$, we also define 0_n to be the zero vector of $\mathbb{R}^n$ and $\Delta_n = \{1,2,\cdots,n\}$. Subsequently, we follow relative notations and terminology due to Wei et al. [17].

Definition 1 (Wei et al. [17])**.**

- *A* **fuzzy transferable-utility (TU) status** *(a fuzzy TU status (P,u^*) is originally defined by Aubin [6,7], where $u*$ is a mapping with $u^* : [0,1]^P \longrightarrow \mathbb{R}$ and $u^*(0_P) = 0$) is denoted by (P,θ,u), where $P \neq \emptyset$ is a finite collection of participants, $\theta = (\theta_s)_{s\in P} \in (0,1]^P$ is the vector that presents the number of operating levels for each participant and $u : \Theta^P \to \mathbb{R}$ is a mapping with $u(0_P) = 0$ which apportions to each operating level vector $\kappa = (\kappa_s)_{s\in P} \in \Theta^P$ the benefit that the participants can receive when each participant s operates at level κ_s.*
- *A* **multicriteria fuzzy TU status** *is denoted by (P,θ,U^n), where $n \in \mathbb{N}$, $U^n = (u^t)_{t\in\Delta_n}$ and (P,θ,u^t) is a fuzzy TU status for every $t \in \Delta_n$. We also denote the family of all multicriteria fuzzy TU statuses to be Λ.*
- *A* **rule** *is a mapping ρ assigning to each $(P,\theta,U^n) \in \Lambda$ an element*

$$\rho(P,\theta,U^n) = \big(\rho^t(P,\theta,U^n)\big)_{t\in\Delta_n},$$

where $\rho^t(P,\theta,U^n) = \big(\rho^t_s(P,\theta,U^n)\big)_{s\in P} \in \mathbb{R}^P$ and $\rho^t_s(P,\theta,U^n)$ is the remuneration of the participant s if s operates in (P,θ,u^t).

To introduce relative pre-existing results of rules, some more notations are needed. Let $(P,\theta,U^n) \in \Lambda$, $Q \subseteq P$ and $\kappa \in \mathbb{R}^P$; we set that $D(\kappa) = \{s \in P | \kappa_s \neq 0\}$ and $\kappa_K \in \mathbb{R}^Q$ to be the restriction of κ to Q. Given $s \in P$, we also define κ_{-s} to stand for $\kappa_{P\setminus\{s\}}$. Further, $\xi = (\kappa_{-s}, c) \in \mathbb{R}^P$ is defined by $\xi_{-s} = \kappa_{-s}$ and $\xi_s = c$.

Wei et al. [17] provided a fuzzy generalization of the EANSC under multicriteria situations as follows.

Definition 2 (Wei et al. [17])**.** *The* **supreme EANSC (SEANSC)***, $\overline{\tau}$, is defined for all $(P, \theta, U^n) \in \Lambda$, for all $t \in \Delta_n$ and for all $s \in P$,*

$$\overline{\tau_s^t}(P, \theta, U^n) = \tau_s^t(P, \theta, U^n) + \frac{1}{|P|} \cdot \left[u^t(\theta) - \sum_{k \in P} \tau_k^t(P, \theta, U^n)\right],$$

where $\tau_s^t(P, \theta, U^n) = \sup_{q \in \theta_s^+}\{u^t(\theta_{-s}, q) - u^t(\theta_{-s}, 0)\}$ is the **supreme marginal dedication** *of the participant s in (P, θ, u^t). (Here, we apply bounded fuzzy TU statuses, defined as the statuses (P, θ, u^t) such that there exists $N_u \in \mathbb{R}$ such that $u^t(\lambda) \leq N_u$ for every $\lambda \in \Theta^P$. We apply it to guarantee that $\tau_s(P, \theta, u^t)$ is well defined.) Under the notion of $\overline{\tau}$, all participants firstly receive their supreme marginal dedications and further allocate equally the rest of utility.*

A rule ρ matches **multicriteria efficiency (MECY)** if $\sum_{s \in P} \rho_s^t(P, \theta, U^n) = u^t(\theta)$ for every $(P, \theta, U^n) \in \Lambda$ and for every $t \in \Delta_n$. A rule ρ matches **multicriteria standard for statuses (MSFS)** if $\rho(P, \theta, U^n) = \overline{\tau}(P, \theta, U^n)$ for every $(P, \theta, U^n) \in \Lambda$ with $|P| \leq 2$. A rule ρ matches **multicriteria symmetry (MSYM)** if $\tau_s^t(P, \theta, U^n) = \tau_k^t(P, \theta, U^n)$ for every $(P, \theta, U^n) \in \Lambda$, for every $t \in \Delta_n$ and for some $s, k \in P$ implies that $\rho_s(P, \theta, U^n) = \rho_k(P, \theta, U^n)$.

MECY asserts that all participants allocate all the utility completely. MSFS is an extended analogue of the two-agent standard condition of Hart and Mas-Colell [21]. MSYM means that if any two participants make symmetric contributions to the entire status, they should both receive symmetric rewards.

Moulin [5] considered the reduced status as that in which each alliance in the subgroup could attain remunerations to its participants only if they agree with the original remunerations to "total" the participants out of the subgroup. A generalized Moulin reduction under multicriteria fuzzy TU statuses is considered by Wei et al. [17] as follows.

Definition 3 (Wei et al. [17])**.** *Let $(P, \theta, U^n) \in \Lambda$, $Q \subseteq P$ and ρ be a rule. The* **reduced status** *$(Q, \theta_Q, U^n_{Q,\rho})$ is defined by $U^n_{Q,\rho} = (u^t_{Q,\rho})_{t \in \Delta_n}$ and for every $\lambda \in \Theta^Q$,*

$$u^t_{Q,\rho}(\lambda) = \begin{cases} 0 & \text{if } \lambda = 0_Q, \\ u^t(\lambda, \theta_{P \setminus Q}) - \sum\limits_{s \in P \setminus Q} \rho_s^t(P, \theta, U^n) & \text{otherwise,} \end{cases}$$

Furthermore, a rule ρ matches **multicriteria stability (MSTA)** *if $\rho_s^t(Q, \theta_Q, U^n_{Q,\rho}) = \rho_s^t(P, \theta, U^n)$ for every $(P, \theta, U^n) \in \Lambda$, for every $t \in \Delta_n$, for every $Q \subseteq P$ with $|Q| = 2$ and for every $s \in Q$.*

As mentioned in the Introduction, weights turn up involuntary under the processes for utility allocating. For instance, one may be dealing with an issue of utility allocation among investment plans. Thence, the weights could be associated with the profitability of the different options among all plans. Weights are also contained in contracts approved by the proprietors of a townhouse and used to allocate the cost of maintaining or building common facilities. On the whole, weights might be allotted to the "participants" or the "levels" to discriminate the discrepancies among the participants or their operating levels, respectively. If $d : U \to \mathbb{R}^+$ is a positive mapping, then d is said to be a **weight mapping for participants**. If $w : \Theta^U \to \mathbb{R}^+$ is a positive mapping, then w is said to be a **weight mapping for levels**. Based on these two types of weight mapping, two weighted extensions of the SEANSC are generated by Wei et al. [17] as follows.

Definition 4 (Wei et al. [17])**.**

- *The* **1-Supreme weighted allocation of non-separable costs (1SWANSC)**, σ^d, *is considered by for all* $(P,\theta,U^n) \in \Lambda$, *for all weight mapping for participants* d, *for all* $t \in \Delta_n$ *and for all participant* $s \in P$,

$$\sigma_s^{d,t}(P,\theta,U^n) = \tau_s^t(P,\theta,U^n) + \frac{d(s)}{\sum\limits_{k\in P} d(k)} \cdot \left[u^t(\theta) - \sum_{k\in P} \tau_k^t(P,\theta,U^n)\right].$$

In accordance with the definition of σ^d, *all participators get firstly their supreme marginal dedications and further distribute the remaining utility proportionally by weight for participants.*

- *The* **2-Supreme weighted allocation of non-separable costs (2SWANSC)**, σ^w, *is considered for all* $(P,\theta,U^n) \in \Lambda$, *for all weight mapping for participants* w, *for all* $t \in \Delta_n$ *and for all participants* $s \in P$,

$$\sigma_s^{w,t}(P,\theta,U^n) = \tau_s^{w,t}(P,\theta,U^n) + \frac{1}{|P|} \cdot \left[u^t(\theta) - \sum_{k\in P} \tau_k^{w,t}(P,\theta,U^n)\right],$$

where $\tau_s^{w,t}(P,\theta,U^n) = \sup\limits_{q\in\theta_s^+} \{w(q) \cdot [u^t(\theta_{-s},q) - u^t(\theta_{-s},0)]\}$. *By definition of* $\sigma^{w,t}$, *all participants get firstly their supreme marginal dedications based on weights for levels and further distribute the remaining utility equally.*

A rule ρ matches **1-weighted standard for statuses (1WSFS)** if $\rho(P,\theta,U^n) = \sigma^d(P,\theta,U^n)$ for every $(P,\theta,U^n) \in \Lambda$ with $|P| \leq 2$ and for every weight mapping for participants d. A rule ρ matches **2-weighted standard for statuses (2WSFS)** if $\rho(P,\theta,U^n) = \sigma^w(P,\theta,U^n)$ for every $(P,\theta,U^n) \in \Lambda$ with $|P| \leq 2$ and for every weight mapping for levels w.

Several axiomatic processess of the SEANSC, the 1SWANSC and the 2SWANSC are proposed by Wei et al. [17] as follows.

Theorem 1 (Wei et al. [17])**.**

- *On* Λ, *the SEANSC, the 1SWANSC, the 2SWANSC satisfy MECY.*
- *On* Λ, *the SEANSC satisfies MSYM, but the 1SWANSC and the 2SWANSC violate MSYM.*
- *On* Λ, *the SEANSC is the only rule satisfying MSFS and MSTA.*
- *On* Λ, *the 1SWANSC is the only rule satisfying 1WSFS and MSTA.*
- *On* Λ, *the 2SWANSC is the only rule satisfying 2WSFS and MSTA.*

3. Different Weighted Extension

Here, we introduce a different extension of the SEANSC via simultaneously applying weights to the participants and their operating levels. Based on MSTA, we further axiomatize this weighted rule.

Definition 5. *The* **weighted fuzzy multicriteria index (WFMI)**, $\tau^{d,w}$, *is defined by*

$$\tau_s^{d,w,t}(P,\theta,U^n) = \tau_s^{w,t}(P,\theta,U^n) + \frac{d(s)}{\sum\limits_{k\in P} d(k)} \cdot \left[u^t(\theta) - \sum_{k\in P} \tau_k^{w,t}(P,\theta,U^n)\right] \tag{1}$$

for all weight mappings for participants d, *for all weight mappings for levels* w, *for all* $(P,\theta,U^n) \in \Lambda$, *for all* $t \in \Delta_n$ *and for all participants* $s \in P$. *By definition of* $\tau^{d,w,t}$, *all participants get firstly their supreme marginal dedications based on weights for levels and further distribute the remaining utility proportionally via weights for participants.*

Lemma 1. *The rule* $\tau^{d,w}$ *matches MECY.*

Proof. It is easy to complete this proof by Definition 5. Hence, we omit it. □

Lemma 2. *The rule $\tau^{d,w}$ matches MSTA.*

Proof. Let $(P,\theta,U^n) \in \Lambda$, $Q \subseteq P$, $t \in \Delta_n$, d be weight mapping for participants and w be weight mapping for levels. Let $|P| \geq 2$ and $|Q| = 2$. By Definition 5,

$$\begin{aligned} & \tau_s^{d,w,t}(Q,\theta_Q,U^n_{Q,\tau^{d,w}}) \\ = \; & \tau_s^{w,t}(Q,\theta_Q,U^n_{Q,\tau^{d,w}}) + \frac{d(s)}{\sum\limits_{k\in Q} d(k)} \cdot \left[u^t_{Q,\tau^{d,w}}(\theta_Q) - \sum_{k\in Q} \tau_k^{w,t}(Q,\theta_Q,U^n_{Q,\tau^{d,w}})\right] \end{aligned} \tag{2}$$

for all $s \in Q$ and for all $t \in \Delta_n$. Based on definitions of $\tau^{w,t}$ and $u^t_{Q,\tau^{d,w}}$,

$$\begin{aligned} \tau_s^{w,t}(Q,\theta_Q,U^n_{Q,\tau^{d,w}}) &= \sup_{q\in\theta_s^+}\{w(q)\cdot[u^t_{Q,\tau^{d,w}}(\theta_{Q\setminus\{s\}},q) - u^t_{Q,\tau^{d,w}}(\theta_{Q\setminus\{s\}},0)]\} \\ &= \sup_{q\in\theta_s^+}\{w(q)\cdot[u^t(\theta_{-s},q) - u^t(\theta_{-s},0)]\} \\ &= \tau_s^{w,t}(P,\theta,U^n). \end{aligned} \tag{3}$$

By Equations (2) and (3) and definitions of $u^t_{Q,\tau^{d,w}}$ and $\tau^{d,w}$,

$$\begin{aligned} & \tau_s^{d,w,t}(Q,\theta_Q,U^n_{Q,\tau^{d,w}}) \\ = \; & \tau_s^{w,t}(P,\theta,U^n) + \frac{d(s)}{\sum\limits_{k\in Q} d(k)} \cdot \left[u^t_{Q,\tau^{d,w}}(\theta_Q) - \sum_{k\in Q}\tau_k^{w,t}(P,\theta,U^n)\right] \\ = \; & \tau_s^{w,t}(P,\theta,U^n) + \frac{d(s)}{\sum\limits_{k\in Q} d(k)} \cdot \left[u^t(\theta) - \sum_{k\in P\setminus Q}\tau_k^{d,w,t}(P,\theta,U^n) - \sum_{k\in Q}\tau_k^{w,t}(P,\theta,U^n)\right] \\ = \; & \tau_s^{w,t}(P,\theta,U^n) + \frac{d(s)}{\sum\limits_{k\in Q} d(k)} \cdot \left[\sum_{k\in Q}\tau_k^{d,w,t}(P,\theta,U^n) - \sum_{k\in Q}\tau_k^{w,t}(P,\theta,U^n)\right] \\ & \textbf{(MECY of } \tau^{d,w}\textbf{)} \\ = \; & \tau_s^{w,t}(P,\theta,U^n) + \frac{d(s)}{\sum\limits_{k\in Q} d(k)} \cdot \left[\frac{\sum\limits_{k\in Q} d(k)}{\sum\limits_{p\in P} d(p)} \cdot \left[u^t(\theta) - \sum_{p\in P}\tau_p^{w,t}(P,\theta,U^n)\right]\right] \\ = \; & \tau_s^{w,t}(P,\theta,U^n) + \frac{d(s)}{\sum\limits_{p\in P} d(p)} \cdot \left[u^t(\theta) - \sum_{p\in P}\tau_p^{t}(P,\theta,U^n)\right] \\ = \; & \tau_s^{d,w,t}(P,\theta,U^n) \end{aligned}$$

for every $s \in Q$ and for every $t \in \Delta_n$. The proof is completed. □

Remark 1. *According to definitions of the weighted indexes (the 1SWANSC, the 2SWANSC and the WFMI) proposed in this paper, it can be clearly concluded that even if $\tau_s^t(P,\theta,U^n) = \tau_k^t(P,\theta,U^n)$ for every $(P,\theta,U^n) \in \Lambda$, for every $t \in \Delta_n$ and for some $s,k \in P$, the weighted indexes related to s and k might be not coincident when the weights among s, k and its levels are not the same. That is, an asymmetric allocating situation will appear. Still, as in the example related to doctors and accountants in a hospital mentioned in the Introduction, such an asymmetric distribution would not be unreasonable. Therefore, the axiomatic processes would be used to present the rationality of the weighted fuzzy allocation rules in the following sections of this paper.*

Inspired by Hart and Mas-Colell [21], we would like to use MSTA to axiomatize the WFMI. A rule ρ matches the **multicriteria weighted standard (MWS)** if $\rho(P,b,U^n) = \tau^{d,w}(P,b,U^n)$ for every $(P,b,U^n) \in \Lambda$ with $|P| \leq 2$, for every weight mapping for participants d and for every weight mapping for levels w.

Theorem 2. *On Λ, the WFMI is the only rule matching MWS and MSTA.*

Proof. By Lemma 2, the rule $\tau^{d,w}$ matches MSTA. Clearly, the rule $\tau^{d,w}$ matches MWS.

To manifest the uniqueness, suppose that ρ matches MWS and MSTA. By MWS and MSTA of ρ, it is easy to clarify that ρ also matches MECY; hence, we omit it. Let

$(P,\theta,U^n)\in\Lambda$, d be weight mapping for participants and w be weight mapping for levels. By MWS of ρ, $\rho(P,\theta,U^n)=\tau^{d,w}(P,\theta,U^n)$ if $|P|\leq 2$. The condition $|P|>2$: Let $s\in P$, $t\in\Delta_n$ and $Q=\{s,k\}$ with $k\in P\setminus\{s\}$.

$$\begin{aligned}
& \rho_s^t(P,\theta,U^n)-\tau_s^{d,w,t}(P,\theta,U^n) & \\
= \; & \rho_s^t(Q,\theta_Q,U^n_{Q,\rho})-\tau_s^{d,w,t}(Q,\theta_Q,U^n_{Q,\tau^{d,w}}) & \textbf{(MSTA of } \tau^{d,w,t} \textbf{ and } \rho\textbf{)} \\
= \; & \tau_s^{d,w,t}(Q,\theta_Q,U^n_{Q,\rho})-\tau_s^{d,w,t}(Q,\theta_Q,U^n_{Q,\tau^{d,w,}}). & \textbf{(MWS of } \rho\textbf{)}
\end{aligned} \tag{4}$$

Similar to Equation (3)

$$\tau_s^{w,t}(Q,\theta_Q,U^n_{Q,\rho})=\tau_s^{w,t}(P,\theta,U^n)=\tau_s^{w,t}(Q,\theta_Q,U^n_{Q,\tau^{d,w,}}). \tag{5}$$

By Equations (4) and (5),

$$\begin{aligned}
& \rho_s^t(P,\theta,U^n)-\tau_s^{d,w,t}(P,\theta,U^n) \\
= \; & \tau_s^{d,w,t}(Q,\theta_Q,U^n_{Q,\rho})-\tau_s^{d,w,t}(Q,\theta_Q,U^n_{Q,\tau^{d,w,}}) \\
= \; & \tfrac{d(s)}{d(s)+d(k)}\cdot\left[u^t_{Q,\rho}(\theta_Q)-u^t_{Q,\tau^{d,w}}(\theta_Q)\right] \\
= \; & \tfrac{d(s)}{d(s)+d(k)}\cdot\left[\rho_s^t(P,\theta,U^n)+\rho_k^t(P,\theta,U^n)-\tau_s^{d,w,t}(P,\theta,U^n)-\tau_k^{d,w,t}(P,\theta,U^n)\right].
\end{aligned}$$

Thus,

$$d(k)\cdot\left[\rho_s^t(P,\theta,U^n)-\tau_s^{d,w,t}(P,\theta,U^n)\right]=d(s)\cdot\left[\rho_k^t(P,\theta,U^n)-\tau_k^{d,w,t}(P,\theta,U^n)\right].$$

By MECY of $\tau^{d,w,t}$ and ρ,

$$\begin{aligned}
\left[\rho_s^t(P,\theta,U^n)-\tau_s^{d,w,t}(P,\theta,U^n)\right]\cdot\sum_{k\in P}d(k) &= d(s)\cdot\sum_{k\in P}\left[\rho_k^t(P,\theta,U^n)-\tau_k^{d,w,t}(P,\theta,U^n)\right] \\
&= d(s)\cdot\left[u^t(\theta)-u^t(\theta)\right] \\
&= 0.
\end{aligned}$$

Hence, $\rho_s^t(P,\theta,U^n)=\tau_s^{d,w,t}(P,\theta,U^n)$ for every $s\in P$ and for every $t\in\Delta_n$. □

4. Another Extension and Revised Stability

In Sections 2 and 3, several weighted generalizations are defined by applying weights to the participants and their operating levels (strategies) simultaneously. However, the weights to the participants and their operating levels (strategies) are apportioned artificially. It is reasonable that the weights could be replaced by supreme marginal dedications naturally.

Concerning "supreme marginal dedications" instead of "weights", a generalization could be considered as follows.

Definition 6. *The* **interior fuzzy multicriteria index (IFMI)**, Ψ, *is defined by*

$$\Psi_s^t(P,\theta,U^n)=\tau_s^t(P,\theta,U^n)+\frac{\tau_s^t(P,\theta,U^n)}{\sum_{k\in P}\tau_k^t(P,\theta,U^n)}\left[u^t(\theta)-\sum_{k\in P}\tau_k^t(P,\theta,U^n)\right] \tag{6}$$

for every $(P,\theta,U^n)\in\Lambda^*$, for every $t\in\Delta_n$ and for every participant $s\in P$, where $\Lambda^*=\{(P,\theta,U^n)\in\Lambda|\sum_{k\in P}\tau_k^t(P,\theta,U^n)\neq 0\;\forall\; t\in\Delta_n\}$. By definition of Ψ, all participants get firstly their supreme marginal dedications and further distribute the remaining utility proportionally via their supreme marginal dedications.

Remark 2. *According to the definition of the IFMI, it can be clearly concluded that if $\tau_s^t(P,\theta,U^n)=\tau_k^t(P,\theta,U^n)$ for every $(P,\theta,U^n)\in\Lambda$, for every $t\in\Delta_n$ and for some $s,k\in P$, $\Psi_s^t(P,\theta,U^n)=\Psi_k^t(P,\theta,U^n)$. That is, the IFMI is a symmetric allocating rule if the weights are replaced to be supreme marginal dedications.*

Subsequently, we would like to axiomatize the IFMI via applying stability. A rule ρ matches the **multicriteria interior standard (MIS)** if $\rho(P,b,U^n) = \Psi(P,b,U^n)$ for every $(P,b,U^n) \in \Lambda$ with $|P| \leq 2$.

It is trivial to verify that $\sum_{k\in Q} \tau_k^t(P,\theta,U^n) = 0$ for some $(P,\theta,U^n) \in \Lambda$, for some $Q \subseteq P$ and for some $t \in \Delta_n$, i.e., $\Psi^t(Q,\theta_Q,U^n_{Q,\Lambda})$ does not exist for some $(P,\theta,U^n) \in \Lambda$, for some $Q \subseteq P$ and for some $t \in \Delta_n$. Thus, one could focus on the *multicriteria revised stability* as follows. A rule ρ matches **multicriteria revised-stability (MRSTA)** if $(Q,\theta_Q,U^n_{Q,\rho})$ and $\rho(Q,\theta_Q,U^n_{Q,\rho})$ exist for some $(P,\theta,U^n) \in \Lambda$, for some $Q \subseteq P$ and for some $t \in \Delta_n$, it holds that $\rho_s(Q,\theta_Q,U^n_{Q,\rho}) = \rho_s(P,\theta,U^n)$ for every $s \in Q$.

Similar to Theorems 1 and 2, relative axiomatic results of Ψ could also be presented as follows.

Theorem 3.
1. *On Λ^*, the rule Ψ matches MECY.*
2. *On Λ^*, the rule Ψ matches MSYM.*
3. *On Λ^*, the rule Ψ matches MRSTA.*
4. *On Λ^*, the rule Ψ is the only rule satisfying MIS and MRSTA.*

Proof. Relative proofs are similar to Lemmas 1 and 2 and Theorems 1 and 2. □

Subsequently, we apply some examples to manifest that each of the properties adopted in Theorem 3 is independent of the rest of the properties.

Example 1. *Here, we consider the rule ρ as follows. For every $(P,\theta,U^n) \in \Lambda^*$, for every $t \in \Delta_n$ and for every participant $s \in P$,*

$$\rho_s^t(P,\theta,U^n) = \begin{cases} \Psi_s^t(P,\theta,U^n) & \text{if } |P| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, ρ matches MIS, but ρ violates MRSTA.

Example 2. *Here, we consider the rule ρ as follows. For every $(P,\theta,U^n) \in \Lambda^*$, for every $t \in \Delta_n$ and for every participant $s \in P$, $\rho_s^t(P,\theta,U^n) = 0$. Clearly, ρ matches MRSTA, but ρ violates MIS.*

5. Concluding Remarks

By applying simultaneously weights to the participants and corresponding operating levels under multicriteria fuzzy situations, the WFMI different from previous ones is proposed to present that such an allocation rule does not conform to relative symmetry. Subsequently, relative axiomatic processes are used to show that such an asymmetric allocation rule not only confirms the correctness in mathematics, but also confirms the rationality in practical applications.Concerning supreme marginal dedications instead of weights, naturally the IFMI and relative axiomatic processes are further introduced under multicriteria fuzzy situations. One should compare relative existing results with the results proposed throughout this article.

- The WFMI, the IFMI and related results are proposed initially under multicriteria fuzzy TU statuses.
- Rule concepts on traditional statuses have only discussed participation or non-participation of participants. Here, we propose two weighted fuzzy rules to investigate the asymmetric distributing mechanism under multicriteria situations.
- Different from the SEANSC, the 1SWANSC and the 2SWANSC defined by Wei et al. [17] on multicriteria fuzzy TU statuses, we propose the WFMI and the IFMI via applying simultaneously weights to the participants and their operating levels.
  - Under the SEANSC and the 2SWANSC, any additional fixed utility should be distributed equally among all participants.

– Under the 1SWANSC, all participants receive firstly their supreme marginal dedication and further distribute the remaining utility proportionally by weight for participants.
– Under the WFMI, all participants get firstly their supreme marginal dedications based on weights for levels and further distribute the remaining utility proportionally by weight for participants.
– Under the IFMI, all participants get firstly their supreme marginal dedications and further distribute the remaining utility proportionally via their supreme marginal dedications.
– The SEANSC and the IFMI are symmetric allocating rules. The 1SEANSC, the 2SEANSC and the WFMI are asymmetric allocating rules.

The results and relative comparisons proposed in this article give rise to some motivations.

1. Moulin [5] adopted the symmetry property to axiomatize the EANSC in the framework of traditional TU statuses. The multicriteria symmetry property is generalized analogue of the symmetry property under multicriteria fuzzy TU statuses.
    • Inspired by Moulin [5], the SEANSC and the IFMI might be axiomatized by means of the multicriteria symmetry property.
    • Since the 1SEANSC, the 2SEANSC and the WFMI are asymmetric, some more asymmetry properties might be introduced to axiomatize these three allocating rules.
2. Some more existing traditional rules and relative axiomatic processes might be generated by adopting simultaneously the supreme marginal dedications under fuzzy behavior and multicriteria situations.
3. The stability property of a rule is indispensable under axiomatic processes of existing studies. However, some rules might violate the stability property. In future studies, one could attempt to axiomatize these rules by reducing the stability property.

This is left to the readers.

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